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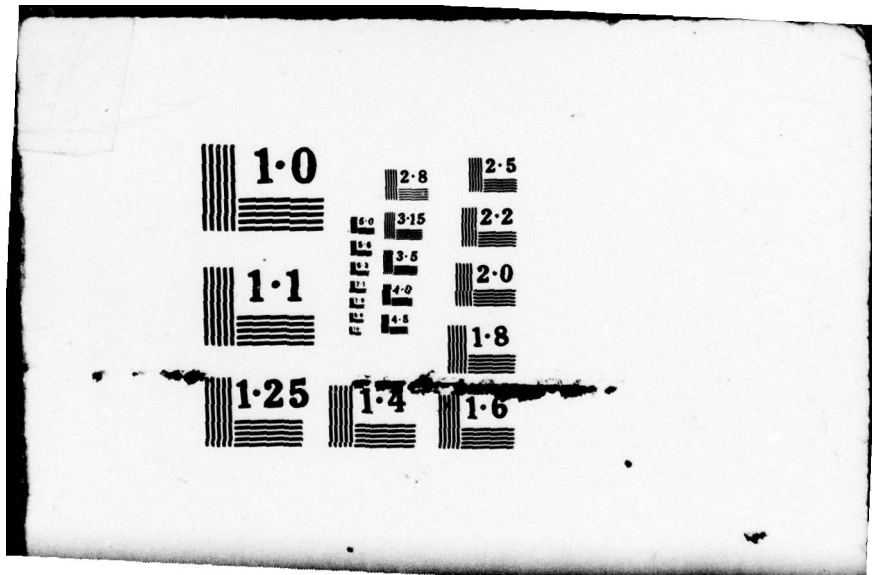
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DETECTION OF A COHERENT QUANTUM SIGNAL IN THERMAL NOISE

CW Helstrom and LB Stotts

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SUMMARY

OBJECTIVE

The objective of this study was to investigate the effect of thermal background noise on optimum quantum receivers. This study employed the methods of statistical hypothesis testing, estimation, and quantum decision theory.

RESULTS

Methods are presented for calculating the minimum attainable probability of error in detecting a coherent quantum signal, received amid thermal noise in the limits of low and high signal-to-noise ratio. In the latter limit, quantum-mechanical perturbation theory is applied to solving the detection operator equation approximately. Graphic results are furnished for binary signals transmitted with equal prior probabilities. Previous work on this problem is thus extended to a broader range of signal and noise strengths. The results are applied to the detection of antipodal two-photon-coherent-state signals.

RECOMMENDATIONS

1. Investigate alternate ways of solving this binary communications problem, in order to bridge the gap between high and low signal-to-noise ratio under medium-to-high thermal noise conditions.
2. Investigate the effect, on thermal noise, of the detection of more general two-photon-coherent-state signals.

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INTRODUCTION

The rapidly developing area of electro-optical (E/O) technology offers great potential for application to military command, control, and communications (C³) systems. Recent studies have shown that the replacement of electronic devices with E/O components, in the information transfer process, could result in a significant improvement in system size, weight, band width, covertness, and cost. It is likely that these and other advantages will lead to increased use of electro-optical technology in Navy C³ systems of the future. At present, a number of specific naval ship and aircraft applications (e.g., OPSATCOM and RED PERCH) have been identified and active research and development efforts, toward implementing this technology in these roles, have been initiated at NOSC and elsewhere.

The classical detection and estimation theories, which are generally utilized in optical system analysis, assume that information carrying fields and random noise can be measured with arbitrary accuracy. Unfortunately, the propagation and detection of optical signals is actually a very complex stochastic process; and, it must be treated by quantum mechanics rather than classical mechanics to be rigorously correct. This latter fact places certain constraints on the optical measurement process. These constraints show up in the failure of commutativity, between the electric and magnetic field components, at different space-time points on the same light-cone. Thus, in order to apply the principles of statistical hypothesis testing and estimation to optical signals, the theory must be compatible with the laws of quantum mechanics.

When we speak of a quantum "system" in connection with the detection and estimation of optical signals, we refer to the electromagnetic field at the aperture A of the observing optical instrument during the interval of observation (O,T). This field can be decomposed into spatio-temporal modes, ortho-normal over A and (O,T); and for these modes, creation and annihilation operators obeying the usual commutation rules can be defined. Thus, the establishment of the Hilbert space of state vectors, appropriate for the application of quantum mechanics, is permitted. In many problems only one, or a finite number, of these modes needs to be taken into account.

Unfortunately, few problems in quantum detection and estimation have been solved comprehensively. This paucity of solved problems has limited our understanding of the import of the new theory and has inhibited the development of intuitions for attacking more complicated problems. In detection, for instance, nearly all solutions apply to choices among pure quantum-mechanical states corresponding to the absence of thermal background light. In most of these, the optimum receiver attains error probabilities smaller, by an order of magnitude, than those attained by conventional detectors (examples are given in ref 1, ch VI). The question arises, how deleterious is a small amount of thermal noise? This is important because many actual external noise sources, such as the sun, can be modelled as thermal blackbodies.

1. Helstrom CW, Quantum Detection and Estimation Theory, chapt V, Academic Press, 1976.

The simplest problem in ordinary detection theory is to determine the optimum detector of a coherent signal of known form, received in the presence of Gaussian noise, and to calculate the minimum attainable probability of error. The corresponding problem in quantum detection theory has not been solved exactly. Approximation methods have been described by Yoshitani (ref 2) and by Klemm (ref 3), with comments by Helstrom (ref 4); and the problem has been reviewed in a recent book (ref 1). Here, we shall extend previous work over a broader range of signal and noise strengths. The principal novelty is the application of quantum-mechanical perturbation theory to a suitably reformed detection operator equation, which yields the asymptotic form of the error probability in the limit of large signal-to-noise ratio (SNR). The opposite limit of low SNR is also treated. Our results permit calculating the minimum error probabilities attainable in detecting antipodal two-photon-coherent-state (TCS) signals, received in the absence of background noise.

Before proceeding however, a review of the need for a quantum mechanical approach is in order.

BACKGROUND

In classical physics, a system can be characterized by performing a series of dynamic variable measurements designed to completely specify the state of this system. Each measurement has a particular probability density function associated with it. Thus, choices among various hypotheses about the system can be made by means of the optimum decision strategies prescribed by ordinary statistical hypothesis-testing theory. There are several monographs in the literature which describe this type of decision process in some detail.

Nature, on the other hand, precludes such a state characterization when dealing with a quantum system. In particular, the propagation and detection of optical signals is actually a very complex stochastic process, and must be treated by quantum mechanics to be rigorously correct. This latter fact, places certain constraints on the optical measurement process. These constraints show up in the failure of the commutativity between the electric and magnetic field components, at different space-time points on the same light-cone. Hence, in order to apply the principles of statistical hypothesis-testing to optical signals, the theory must be compatible with the laws of quantum mechanics. Thus, one is left not only with the problem of determining the best way to process the results of one's measurements, but also with the problem of which parameters should be measured in the first place.

As an example, imagine an optical communication system transmitting messages that have been coded into an alphabet of M symbols. The message source emits a new symbol every T seconds. The transmitter can produce M different optical signals, one for each symbol in the alphabet. They are modulations of the electromagnetic field, produced perhaps by a laser, and last no longer than T seconds. When the message source emits the j^{th} symbol of the alphabet, the transmitter sends out the j^{th} signal.

2. Yoshitani R, On the Detectability Limit of Coherent Optical Signals in Thermal Radiation, *J Statist Phys*, vol 2, no 4, p 347-378, 1970.
3. Klemm RA, Detection of a Coherent Signal of Known Phase, *Info and Control*, vol 28, p 35-53, May 1975.
4. Helstrom CW, Comments on Detection of a Coherent Signal of Known Phase by R. A. Klemm, *Info and Control*, vol 35, p 114-118, October 1977.

The receiver collects all the light incident on its aperture A. We suppose it to be so synchronized with the transmitter that, taking account of the time needed for a light signal to pass from transmitter to receiver, it can identify the beginning and end of each signaling interval of duration T seconds. During each interval, it observes the incident light and on the basis of the observation decides which of the M signals are transmitted.

A typical observation begins with filtering the light to cut out as much background radiation as possible. Light from a local laser may be added to the incident beam, as to a heterodyning component, and the combined beam then focused on a photosensitive surface. The electrical signal so generated is amplified and applied to a bank of M filters, each appropriately matched to one of the M possible signals. Whichever filter has the largest output, when sampled at a certain moment, indicates a decision that the corresponding signal was the one transmitted (ref 5). Any procedure of this kind is subject to error, because of noise created by the background and by the physical processes of the receiving system itself. One would like to design the entire procedure so that the average probability of error is minimal; or, when costs are assigned to the various kinds of error, the average cost per decision is as small as possible.

Whatever procedure is adopted is basically a processing of the electromagnetic field at the aperture A during the observation interval (O,T). It must be compatible with the laws of quantum mechanics. It is convenient, as suggested by Takahasi (ref 6), to imagine the incident light to have been admitted to a lossless box or cavity behind the aperture A, which is opened at $t=0$ and closed at $t=T$. The field within the cavity is identified as a quantum system S. Its physical properties, which determine the structure of its Hilbert Space H_S , has recently been described (ref 1). There, one finds the nature of the signals and the background light determining a set of M density operators ($\rho_1, \rho_2, \dots, \rho_M$), which respectively specify the state of the field in each of the M possible transmitted signals. It is necessary to choose one of these density operators that, in some sense, best represents the cavity field. For this purpose, the field is observed at a time T, or later, by an instrument I that applies to it a probability-operator measure (p.o.m.) (ref 7). This measure will have M components, denoted by $\Pi_1, \Pi_2, \dots, \Pi_M$.

The lossless cavity is called an *ideal cavity*. The artifice of observing its electromagnetic field at a single instant t, T is adopted in order to avoid the complication of analyzing quantum-mechanical measurements made at a succession of times in (O,T); each of these would interfere with the field in an unpredictable, random manner. No generality is lost for the cavity field at $t \geq T$ and the evolution of the aperture field during the interval (O,T) are uniquely related, the one containing all the information provided by the other. Moreover, no scheme for processing the aperture field can yield a lower average cost than that attained by the optimum p.o.m. applied to the field in the ideal receiver. In order to show how the optimum p.o.m. is determined, we consider the general problem of deciding among M density operators of an arbitrary quantum system.

5. Pratt WK, *Laser Communication Systems*, Wiley, 1968.

6. Takahasi H, "Information Theory of Quantum-Mechanical Channels," *Advances in Communication Systems*, ed A Balakrishnan, vol 1, p 227-310, Academic Press, 1965.

7. Helstrom CW, *Quantum Detection and Estimation Theory*, chapt III, Academic Press, 1976.

There are M hypotheses about the state of the system S , of which the j th is the proposition that its density operator is ρ_j . In our example, it implies that the j th signal was transmitted. The p.o.m. affecting the decision needs only M components, $\Pi_1, \Pi_2, \dots, \Pi_M$, which we shall call detection operators. These nonnegative-definite Hermitian operators sum to the identity

$$\sum_{j=1}^M \Pi_j = I, \quad (1)$$

and specify, according to reference 7, the conditional probabilities

$$\Pr(j | k) = \text{Tr}(\rho_k \Pi_j), \quad (j,k) = 1, 2, \dots, M, \quad (2)$$

that the instrument I chooses hypothesis H_j when H_k is true. In order to reach that decision, the instrument may have registered certain meter or counter readings and processed them in some way, perhaps by a computer that ultimately prints out one of the integers $1, 2, \dots, M$. These detection operators Π_j are not necessary projectors, but rather form a generalized resolution of the identity (ref 7).

An instrument, for example, that simply guesses which hypothesis might be true, selecting an arbitrary one with probability $1/M$ without even interacting with the system S , in effect applies the p.o.m. composed of the operators

$$\Pi_j \equiv M^{-1} I, \quad j = 1, 2, \dots, M.$$

These are certainly not projectors.

Let ξ_j denote the prior probability of hypothesis H_j in our communication system. This is the relative frequency with which the j th signal is transmitted. The cost of choosing hypothesis H_j when H_j is true, is again C_{ij} . Then, the average cost of an observational strategy specified by the p.o.m. $\{\Pi_\alpha\}$ is

$$\bar{C} = \sum_{i=1}^M \sum_{j=1}^M \xi_j C_{ij} \text{Tr}(\rho_j \Pi_i) = \text{Tr} \sum_{i=1}^M W_i \Pi_i, \quad (3)$$

where the Hermitian risk operators W_i are defined by

$$W_i = \sum_{j=1}^M \xi_j C_{ij} \rho_j. \quad (4)$$

We must minimize equation (3) under the constraints that equation (1) hold and that the detection operators Π_j be nonnegative definite, $\Pi_j \geq 0$, and Hermitian.

This problem resembles one of linear programming, except that operators are involved instead of functions. Its structure is much like the problem of minimizing the Bayes cost of a decision strategy in ordinary statistical theory. The equations for the solution are,

$$(W_i - T)\Pi_i = \Pi_i(W_i - T) = 0, \quad i = 1, 2, \dots, M, \quad (5)$$

$$W_i - T \geq 0, \quad i = 1, 2, \dots, M, \quad (6)$$

with the operator

$$\Upsilon = \sum_{j=1}^M \Pi_j W_j = \sum_{j=1}^M W_j \Pi_j \quad (7)$$

required to be Hermitian. In equation (6) the inequality sign means that the operator $W_i - \Upsilon$ must be nonnegative definite. This solution to the problem of optimum quantum hypothesis testing was first published by Holevo (ref 8). Equations (5) - (7) were reported by Yuen as solving a problem in vector-space optimization theory later recognized as the equivalent (ref 9). A demonstration, based on the principle of duality in vector-space optimization theory, can be found in a paper by Yuen, Kennedy, and Lax (ref 10).

The operator Υ in equations (5) - (6) plays the role of a Lagrange multiplier taking account of the constraint of equation (1); which, involving Hermitian operators, requires Υ to be Hermitian. We call Υ the Lagrange operator. The minimum Bayes cost \bar{C}_{\min} is (refs 1, 10):

$$\bar{C}_{\min} = \text{Tr } \Upsilon. \quad (8)$$

BINARY DETECTION USING THE BAYES CRITERION

When there are only two hypotheses between which to choose, it is customary to label them H_0 and H_1 . The system S has density operators ρ_0 and ρ_1 under H_0 and H_1 , respectively, and their prior probabilities are ξ_0 and ξ_1 , $\xi_0 + \xi_1 = 1$. We shall see that the optimum p.o.m. for deciding between the two hypotheses is projection-valued and can be defined in terms of the eigenvectors of a linear combination of ρ_0 and ρ_1 .

There are now only two detection operators, Π_0 and Π_1 , and

$$\Pi_0 + \Pi_1 = 1. \quad (9)$$

From this condition,

$$\Pi_0(\Pi_0 + \Pi_1) = (\Pi_0 + \Pi_1)\Pi_0$$

yields

$$\Pi_0 \Pi_1 = \Pi_1 \Pi_0; \quad (10)$$

or, the detection operators commute. The Lagrange operator Υ is now

$$\Upsilon = W_0 \Pi_0 + W_1 \Pi_1 \quad (11)$$

8. Holevo AS, Statistical Problems in Quantum Physics, Proc 2nd Japan-USSR Symp Probability Theory, Kyoto vol 1, p 22-40, August 1972.
9. MIT Res Lab Electron TR 482. Communication Theory of Quantum Systems, HP Yuen, 30 August 1971.
10. Yuen, HP, Kennedy RS, and Lax M, Optimum Testing of Multiple Hypotheses in Quantum Detection Theory, IEEE Trans, IT-21, P 125-134, March 1975.

and

$$W_0 - T = W_0 - W_0 \Pi_0 - W_1 \Pi_1 = (W_0 - W_1) \Pi_1. \quad (12)$$

Now, equation (5) yields

$$(W_0 - W_1) \Pi_1 \Pi_0 = 0; \quad (13)$$

where, by equation (4),

$$\begin{aligned} W_0 - W_1 &= \zeta_1 (C_{01} - C_{11}) \rho_1 - \zeta_0 (C_{10} - C_{00}) \rho_0 \\ &= \zeta_1 (C_{01} - C_{11}) (\rho_1 - \lambda \rho_0), \end{aligned} \quad (14)$$

with

$$\lambda = \frac{\zeta_0 (C_{10} - C_{00})}{\zeta_1 (C_{01} - C_{11})}. \quad (15)$$

This corresponds to the classical decision level Λ_0 . We can confine our operations to the subspace of the Hilbert space \mathcal{H}_S spanned by the eigenvectors of the density operators ρ_0 and ρ_1 , corresponding to nonzero eigenvalues. That subspace may coincide with the entire space \mathcal{H}_S . In it, the operator $W_0 - W_1$ will not in general vanish, and therefore, $\Pi_1 \Pi_0 = 0$; thus, the operators Π_0 and Π_1 are projectors onto orthogonal subspaces of \mathcal{H}_S .

The nonnegativity of $W_0 - T$ yields

$$W_0 - T = (W_0 - W_1) \Pi_1 = \zeta_1 (C_{01} - C_{11}) (\rho_1 - \lambda \rho_0) \Pi_1 \geq 0.$$

Since $C_{01} - C_{11}$ represents the net cost of an error of the second kind, it is positive, and

$$\Pi_1 (\rho_1 - \lambda \rho_0) = (\rho_1 - \lambda \rho_0) \Pi_1 \geq 0. \quad (16)$$

The eigenvectors $|\eta_i\rangle$ and the eigenvalues η_i of the Hermitian operator $\rho_1 - \lambda \rho_0$ are defined by

$$(\rho_1 - \lambda \rho_0) |\eta_i\rangle = \eta_i |\eta_i\rangle. \quad (17)$$

In general, some of the eigenvalues η_i will be positive, or zero, and the rest negative. For each negative eigenvalue, form by equation (16),

$$\langle \eta_i | (\rho_1 - \lambda \rho_0) \Pi_1 | \eta_i \rangle = \eta_i \langle \eta_i | \Pi_1 | \eta_i \rangle \geq 0.$$

With $\eta_i < 0$, this implies

$$\langle \eta_i | \Pi_1 | \eta_i \rangle \leq 0,$$

but because Π_1 must be nonnegative definite, this requires

$$\langle \eta_i | \Pi_1 | \eta_i \rangle = 0, \quad \eta_i < 0. \quad (18)$$

Similarly,

$$\langle \eta_i | \Pi_0 | \eta_i \rangle = 0, \quad \eta_i > 0. \quad (19)$$

Thus, we find that Π_1 projects onto the subspace of \mathcal{H}_S spanned by the eigenvectors $|\eta_i\rangle$ corresponding to positive eigenvalues; and Π_0 projects onto the subspace spanned by $|\eta_i\rangle$ for $\eta_i < 0$. The subspace spanned by any eigenvectors with $\eta_i = 0$ contributes nothing to the average cost. We can write the optimum detection operators as

$$\Pi_0 = \sum_i U(-\eta_i) |\eta_i\rangle\langle\eta_i|, \quad \Pi_1 = \sum_i U(\eta_i) |\eta_i\rangle\langle\eta_i|, \quad (20)$$

where, $U(x)$ is the unit step-function,

$$U(x) = \begin{cases} 1, & x > 0, \\ 1/2, & x = 0, \\ 0, & x < 0. \end{cases} \quad (21)$$

The optimum observation strategy can be described as the "measurement," in the conventional sense, of the operator $\rho_1 - \lambda\rho_0$ (ref 11). If a positive eigenvalue results, choose H_1 ; if a negative one, choose H_0 . If there exists a zero eigenvalue, and it turns up in the measurement, either hypothesis may be selected. We have arbitrarily picked a strategy that chooses randomly between them and with equal probabilities. The minimum Bayes cost is

$$\bar{C}_{\min} = \text{Tr } \Upsilon = \zeta_0 C_{00} + \zeta_1 C_{01} - \zeta_1 (C_{01} - C_{11}) \sum_i \eta_i U(\eta_i). \quad (22)$$

Unfortunately, finding the eigenvalues and eigenvectors of the operator $\rho_1 - \lambda\rho_0$ may be very difficult.

BINARY DECISIONS IN ZERO THERMAL NOISE

Given two pure states $|\psi_0\rangle$ and $|\psi_1\rangle$, we have that

$$\rho_0 = |\psi_0\rangle\langle\psi_0| \quad (23)$$

and

$$\rho_1 = |\psi_1\rangle\langle\psi_1|. \quad (24)$$

As this is a two dimensional space, we also have

$$\Pi_j = |\omega_j\rangle\langle\omega_j| \quad (25)$$

with $j = 0, 1$. The closure or completeness relation tells us that

$$\underline{I} = \sum_{j=0}^1 |\omega_j\rangle\langle\omega_j| = \Pi_0 + \Pi_1. \quad (26)$$

If we now expand $|\psi_k\rangle$, $k = 0, 1$, in terms of the $|\omega_k\rangle$'s, we obtain

11. Helstrom CW, *Detection Theory and Quantum Mechanics, Info and Control*, vol 10, p 254-291, March 1967.

$$|\psi_0\rangle = x_{00}|\omega_0\rangle + x_{10}|\omega_1\rangle \quad (27)$$

and

$$|\psi_1\rangle = x_{01}|\omega_0\rangle + x_{11}|\omega_1\rangle, \quad (28)$$

where $x_{ij} = \langle \omega_i | \psi_j \rangle$. Let us associate $|\omega_0\rangle$ with the negative eigenvalue η_0 of $(\rho_1 - \lambda\rho_0)$ and $|\omega_1\rangle$ with the positive eigenvalue η_1 . From

$$(\rho_1 - \lambda\rho_0)|\eta\rangle = \eta|\eta\rangle, \text{ we have}$$

$$|\psi_1\rangle\langle\psi_1|\eta\rangle - \lambda|\psi_0\rangle\langle\psi_0|\eta\rangle = \eta|\eta\rangle. \quad (29)$$

Taking the scalar product of equation (29) with $\langle\psi_0|$, and then with $\langle\psi_1|$, yields

$$\gamma^* \langle\psi_1|\eta\rangle - (\eta + \lambda) \langle\psi_0|\eta\rangle = 0 \quad (30)$$

and

$$(1 - \eta) \langle\psi_1|\eta\rangle - \lambda\gamma \langle\psi_0|\eta\rangle = 0 \quad (31)$$

with $\gamma = \langle\psi_1|\psi_0\rangle$. The above simultaneous equations will have a non-zero solution if

$$\begin{vmatrix} \gamma^* & -(\eta + \lambda) \\ (1 - \eta) & -\lambda\gamma \end{vmatrix} = 0$$

This implies that

$$\eta_1 = \frac{(1 - \lambda) + \sqrt{(1 - \lambda)^2 + 4\lambda(1 - |\gamma|^2)}}{2} \geq 0 \quad (32)$$

and

$$\eta_0 = \frac{(1 - \lambda) - \sqrt{(1 - \lambda)^2 + 4\lambda(1 - |\gamma|^2)}}{2} \leq 0 \quad (33)$$

Therefore, we can write

$$\begin{bmatrix} \gamma^* & -(\eta_k + \lambda) \\ (1 - \eta_k) & -\lambda\gamma \end{bmatrix} \begin{bmatrix} \langle\psi_1|\eta_k\rangle \\ \langle\psi_0|\eta_k\rangle \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (34)$$

Thus,

$$\langle\psi_1|\eta_k\rangle = A_k \gamma \lambda \quad (35)$$

and

$$\langle\psi_0|\eta_k\rangle = (1 - \eta_k) A_k \quad (36)$$

with

$$|A_k|^{-2} = 2R(\eta_k - g)$$

$$R = \left\{ \left[\frac{1}{2} (1-\lambda) \right]^2 + \lambda g \right\}^{1/2}$$

$$g = 1 - |\gamma|^2$$

for $k = 0, 1$.

The false-alarm and detection probabilities are equal to

$$\begin{aligned} Q_0 &= \Pr \{ H_1 | H_0 \} = \text{Tr} (\rho_1 \Pi_0) \\ &= \frac{(1-\eta_1)^2 [\eta_1 + \lambda \cdot g]}{\lambda^2 |\gamma|^2 R} \end{aligned}$$

and

$$\begin{aligned} Q_d &= \Pr \{ H_1 | H_1 \} = \text{Tr} (\rho_1 \Pi_1) \\ &= \frac{[\eta_1 + \lambda \cdot g]}{R} \end{aligned}$$

respectively.

COHERENT DETECTION IN THE PRESENCE OF THERMAL NOISE

In the preceding subsection we discussed the decision between two pure states in the absence of any noise. Unfortunately, this is not very realistic since most optical systems must contend with some sort of noise mechanism affecting the decision process. The intent of these sections is to investigate binary "on-off" coherent and two-photon coherent detection in the presence of thermal noise. In particular, we will expand the approach developed above to take into account the effects of a thermal noise source in the decision process.

DETECTION OF BINARY ON-OFF SIGNALS

In an on-off binary communication system, a coherent optical signal, such as the output of an ideal laser, is transmitted for each '1', and nothing is transmitted for each '0'. Symbols occur at regular intervals, '1's with prior probability ξ_1 ; '0's with prior probability ξ_0 ; $\xi_0 + \xi_1 = 1$. In the absence of background noise, we found that the signal induces a pure coherent state $|\mu\rangle$ in the matched mode of the aperture field of the receiver; with μ being the complex amplitude of the signal. When thermal noise of effective absolute temperature T is also present, the density operator of the matched mode under hypothesis H_1 (signal present) is

$$\rho_1 = \rho(\mu) = (1 - v) \exp [-w(a^\dagger - \mu^* \underline{1})(a - \mu \underline{1})] . \quad (37)$$

Here, a and a^\dagger are the photon annihilation and creation operators, $\underline{1}$ is the identity operator in the Hilbert space for the quantum states of the modal field, and

$$w = h\nu/KT, v = e^{-w},$$

h = Planck's constant, ν = signal carrier frequency, and K = Boltzmann's constant. The mean number of photons supplied by the signal is $S = |\mu|^2$, and the mean number supplied by the background noise is $N = \nu/(1-\nu)$. Without loss of generality, we take the amplitude μ as real and positive. Under hypothesis H_0 (signal absent), the density operator of the mode is

$$\rho_0 = \rho(0) = (1 - \nu) \exp(-\nu a^\dagger a), \nu = e^{-w}. \quad (38)$$

The effective SNR D^2 is defined as

$$D^2 = 4S/(2N+1); \quad (39)$$

which, in the limit $w \ll 1$, $\nu = 1$, becomes the usual SNR $2E/KT$ in terms of the energy E of the received signal.

The optimum detection strategy and the minimum probability P_e of error again, depends on the eigenvalues η_k and the eigenvectors $|\eta_k\rangle$ of the operator $\rho_1 - \lambda\rho_0$, $\lambda = \xi_0/\xi_1$,

$$(\rho_1 - \lambda\rho_0)|\eta_k\rangle = \eta_k|\eta_k\rangle, \quad (40)$$

as was shown in the previous subsections. The optimum quantum receiver measures the projector

$$\Pi_1 = \sum_{\eta_k > 0} |\eta_k\rangle\langle\eta_k|, \quad (41)$$

choosing hypothesis H_m , $m = 0, 1$, when the outcome of the measurement is m (ref 1). The minimum probability of error is

$$P_e = \xi_1 \left(1 - \sum_{\eta_k > 0} \eta_k\right) \quad (42)$$

$$\text{or, } P_e = \xi_0 \left(1 + \lambda^{-1} \sum_{\eta_k < 0} \eta_k\right). \quad (43)$$

It is convenient to label the negative eigenvalues with negative indices. When $\xi_0 = \xi_1 = 1/2$, they occur in pairs of opposite sign and we can put $\eta_k = -\eta_{-k}$. This problem can be solved exactly, in both the extreme quantum limit ($\nu = N = 0$) and the classical limit ($\nu = 1$). For $\xi_0 = \xi_1 = 1/2$,

$$P_e = 1/2 \left\{1 - (1 - e^{-S})^{1/2}\right\}, \nu = 0, \quad (44)$$

and

$$P_e = \text{erfc}(D/2) = (2\pi)^{-1/2} \int_{D/2}^{\infty} \exp(-t^2/2) dt, \nu = 1 \quad (45)$$

We plot our results in terms of the parameter

$$y(\nu, D) = D^2 \left\{-4 \ln [4P_e(1 - P_e)]\right\}^{-1}, \quad (46)$$

which is the ratio of the SNR D^2 required for a given probability P_e of error, to that required when no noise is present ($v = 0$). Thus, $y(0, D) = 1$, and

$$y(1, 0) = \pi/2 \leq y(1, D) \leq y(1, \infty) = 2.$$

Figure 1 displays the parameter $y(v, D)$ in decibels as a function of D for various values of v , with dashed lines indicating regions where our results lack precision. Figure 2 displays $y(v, 0)$ and $y(v, \infty)$, again in decibels, as functions of $v = N/(N+1)$. In terms of D and $y(v, D)$, the probability of error is, for $\xi_0 = \xi_1 = 1/2$,

$$P_e = \frac{1}{2} \left\{ 1 - [1 - \exp(-D^2/4y)]^{1/2} \right\}; \quad (47)$$

and for large values of the SNR D^2 , it is asymptotically proportional to $\exp[-1/4 D^2/y(v, \infty)]$.

The error probability so calculated applied not only to on-off signaling, but also to decisions between signals of the same form and arbitrary complex amplitudes μ_0 and μ_1 , with $\mu = |\mu_1 - \mu_0|$. The new density operators are related to those in equations (37) and (38) through the unitary transformation,

$$\rho_1' = D(\mu_0)\rho_1 D^\dagger(\mu_0), \quad \rho_0' = D(\mu_0)\rho_0 D^\dagger(\mu_0); \quad (48)$$

where, $D(\mu_0)$ is a displacement operator whose form is given in section 3 of reference 1. Antipodal coherent signals are included in this class, with $\mu_0 = -\mu/2$. Furthermore, when antipodal two-photon-coherent-state (TCS) signals, as defined by Yuen and Shapiro (ref 12), are received in the absence of background noise, the density operators under the two hypotheses are related to those in equation (48), with $\mu_0 = -\mu/2$ by a second unitary transformation. A unitary transformation of both density operators does not change the spectrum of eigenvalues in equation (41) and, hence, leaves the probability P_e of error the same.

In the limit $\mu \rightarrow 0$ the density operator under hypothesis H_1 is approximately

$$\rho_1 \doteq \rho_0 + \frac{1}{2}(\rho_0 L + L \rho_0)\mu, \quad (49)$$

where

$$L = 2(a^\dagger + a)/(2N + 1) \quad (50)$$

is the symmetrized logarithmic derivative of $\rho(\mu)$ with respect to μ , evaluated at $\mu = 0$. In the number representation based on the eigenvectors $|n\rangle$ of the number operator $a^\dagger a$, the operator $\rho_1 - \lambda \rho_0$ whose eigenvalues we seek becomes the approximate matrix,

$$\begin{aligned} \langle n | (\rho_1 - \lambda \rho_0) | m \rangle &\doteq (1 - \lambda) e_n \delta_{nm} \\ &+ \mu (2N + 1)^{-1} \langle n | [(a^\dagger + a)\rho_0 + \rho_0(a^\dagger + a)] | m \rangle \end{aligned} \quad (51)$$

with

$$e_n = (1 - v)v^n \quad (52)$$

12. Yuen HP, and Shapiro JH, "Optical Communication with Two-Photon Coherent States - Part I: Quantum State Propagation and Quantum Noise Reduction," Working Papers.

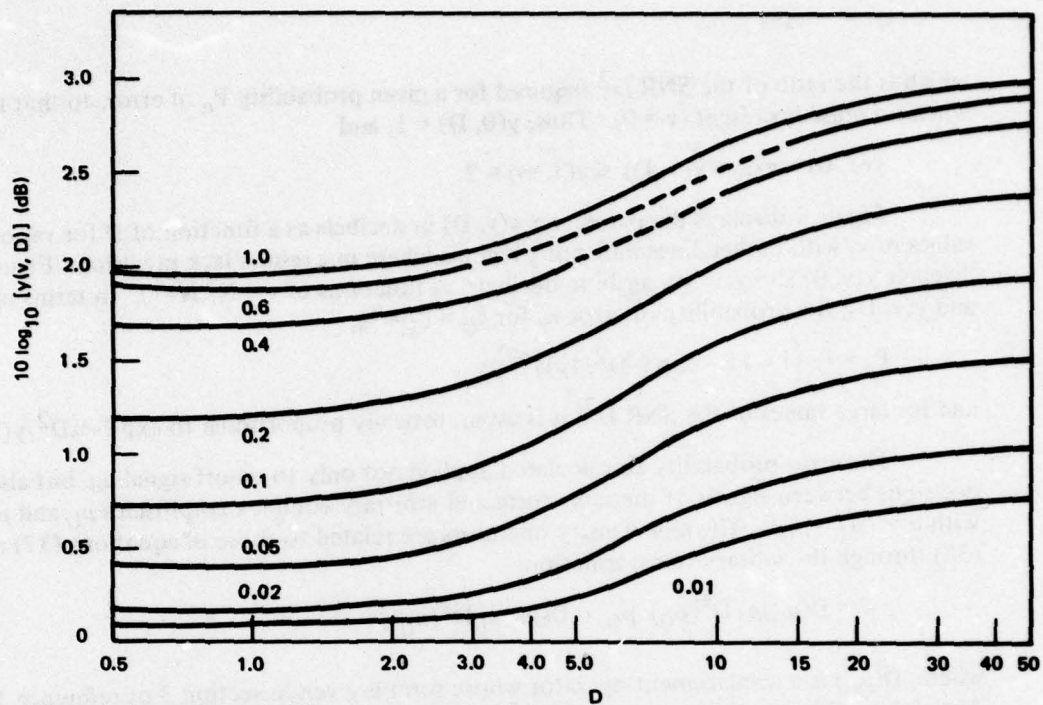


Figure 1. Detection probability parameter $y(v, D)$ in dB versus the effective SNR D . Curves are labeled with the value of $v = N/(N + 1)$.

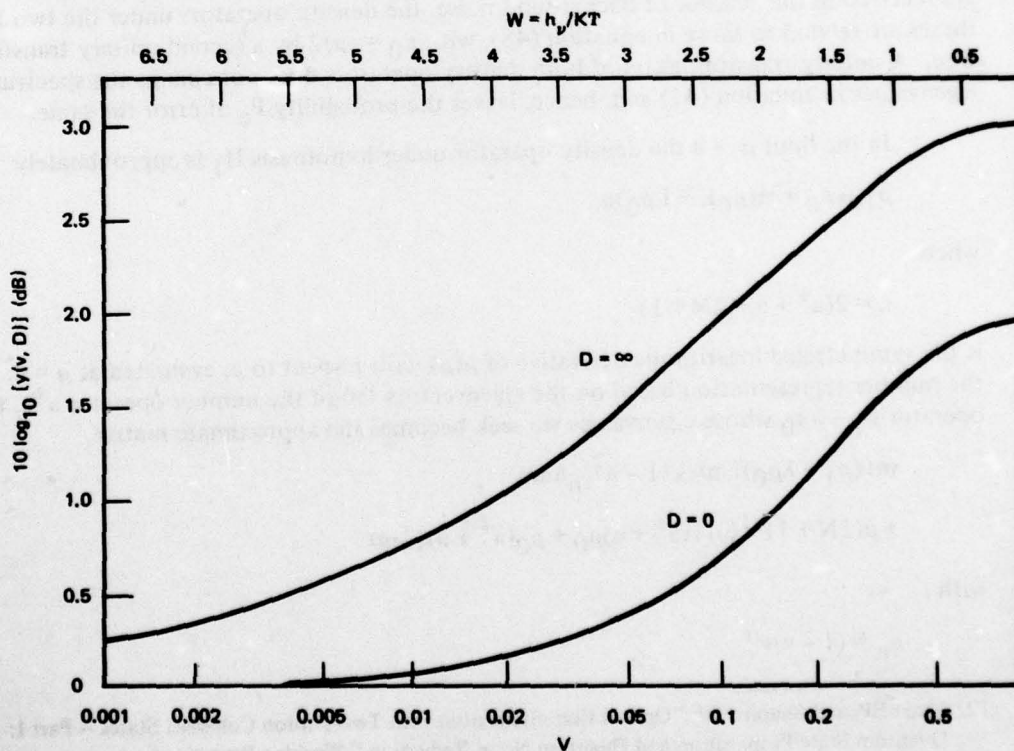


Figure 2. Detection probability parameter $y(v, D)$ in dB for $v = 0$ and $v = 1$.

the n th eigenvalue of ρ_0 . This matrix is tridiagonal; its diagonal elements are $(1 - \lambda)e_n$ and the elements of the adjacent subdiagonals are (ref 2),

$$\langle n | (\rho_1 - \lambda \rho_0) | n + 1 \rangle \doteq \mu(1 - v)^2 (n + 1)^{1/2} v^n, \quad n \geq 0, \quad (53)$$

$$\langle n | (\rho_1 - \lambda \rho_0) | n - 1 \rangle \doteq \mu(1 - v)^2 n^{1/2} v^{n-1}, \quad n < 0. \quad (54)$$

The eigenvalues of a tridiagonal matrix can be calculated by iterative evaluation of a continued fraction into which the determinantal equation can be converted (ref 13).

In particular, when $\zeta_0 = \zeta_1 = 1/2$, $\lambda = 1$, the eigenvalues are

$$\eta_k = \mu(1 - v)^2 \xi_k; \quad (55)$$

with ξ_k the eigenvalues of the tridiagonal matrix $||M_{ij}||$ whose elements are

$$M_{n,n+1} = M_{n+1,n} = (n + 1)^{1/2} v^n,$$

$$M_{ij} \equiv 0, \quad |i - j| \neq 1. \quad (56)$$

From equation (43),

$$\begin{aligned} 4P_e(1 - P_e) &= 1 - \left[\sum_{k=0}^{\infty} \eta_k \right]^2 \\ &\doteq 1 - \mu^2(1 - v)^4 \left[\sum_{k=0}^{\infty} \xi_k \right]^2; \end{aligned} \quad (57)$$

which implies, from equation (46), that

$$y(v, 0) = (1 - v^2)^{-1} (1 - v)^{-2} \left[\sum_{k=0}^{\infty} \xi_k \right]^{-2} \quad (58)$$

and is the signal amplitude cancelling out in this limit. The lower curve in figure 2 was calculated in this manner.

By suitable algebraic manipulation, equation (41) can be rewritten in the form

$$\begin{aligned} [\rho_0 - \rho_0^{1/2} \rho_1 (\rho_1 - \eta_k)^{-1} \rho_0^{1/2}] | \eta_k \rangle' \\ = -\lambda^{-1} \eta_k | \eta_k \rangle', \end{aligned} \quad (59)$$

with $| \eta_k \rangle = \rho_0^{1/2} | \eta_k \rangle'$. In the number representation the operator on the left-hand side of equation (59) becomes the matrix $\tilde{T}(\eta_k)$, where $\tilde{T}(\eta)$ designates a matrix whose elements are

13. Erdélyi A, et al eds, Higher Transcendental Functions, vol 2, p 189, equa 10.12(20), McGraw-Hill, 1953.

$$T_{nm}(\eta) = \langle n | [\rho_0 - \rho_0^{1/2} \rho_1 (\rho_1 - \eta)^{-1} \rho_0^{1/2}] | m \rangle$$

$$= e_n \delta_{nm} - S_{nm}(\eta), \quad (60)$$

with

$$S_{nm}(\eta) = (e_n e_m)^{1/2} \langle n | \rho_1 (\rho_1 - \eta)^{-1} | m \rangle$$

$$= (e_n e_m)^{1/2} \sum_{r=0}^{\infty} e_r (e_r - \eta)^{-1} \langle n | w_r \rangle \langle w_r | m \rangle. \quad (61)$$

Here, e_n is given by equation (52) and the $|w_r\rangle$ are the eigenvectors of the density operator ρ_1 ; that is,

$$\rho_1 |w_r\rangle = e_r |w_r\rangle$$

with

$$e_r = (1 - v)v^r, \quad |w_r\rangle = D(\mu) |r\rangle, \quad (62)$$

in which

$$D(\mu) = \exp(\mu a^\dagger - \mu^* a)$$

is the displacement operator. Equation (62) follows from

$$\rho_1 = D(\mu) \rho_0 D^\dagger(\mu). \quad (63)$$

Furthermore, with $S = |\mu|^2$,

$$\langle n | w_r \rangle = \langle n | D(\mu) | r \rangle$$

$$= (n!/r!)^{1/2} (-\mu^*)^{r-n} e^{-S/2} L_n^{(r-n)}(S)$$

$$= (r!/n!)^{1/2} \mu^{n-r} e^{-S/2} L_n^{(n-r)}(S).$$

In terms of the associated Laguerre polynomials, a result can be derived by expanding the coherent-state representation of the displacement operator

$$D(\alpha^*, \beta; \mu) = \exp \frac{1}{2} (|\alpha|^2 + |\beta|^2) \langle \alpha | D(\mu) | \beta \rangle$$

$$= \exp (\alpha^* \beta + \alpha^* \mu - \mu^* \beta - \frac{1}{2} |\mu|^2). \quad (64)$$

In a double power series in α^* and β , and using the formula of Mollow and Glauber (ref 14),

$$D(\alpha^*, \beta; \mu) = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} (n!/r!)^{-1/2} \alpha^{*n} \beta^r \langle n | D(\mu) | r \rangle. \quad (65)$$

14. Mollow B, and Glauber RJ, Quantum Theory of Parametric Amplification, Phys Rev, vol 160, p 1076-1096, 25 August 1967.

The matrix elements can be calculated recursively by means of the recurrent relations for the associated Laguerre polynomials.

When the signal is strong $S \gg 1$ the second term in equation (60), proportional to e^{-S} , is much smaller than the first; and, we can expect the negative eigenvalues of equations (41) and (51) to be approximately

$$\eta_{-k} \cong -\lambda e_k = -\lambda(1-v)v^k. \quad (66)$$

This approximation can be steadily improved in the following way. For the eigenvalue of interest, say η_{-k} , we substitute $-\lambda e_k$ into the matrix $\underline{T}(\eta)$ of equation (60), truncated to a manageable but sufficiently large number of rows and columns; then, we use a standard computer algorithm to calculate the eigenvalues of the matrix $\underline{T}(-\lambda e_k)$. The output from this computation will be a list of eigenvalues $\tilde{\eta}_{-j}^{(k)}$, $j = 1, 2, \dots, M$, where M is the dimension of the matrix and

$$|\tilde{\eta}_{-1}^{(k)}| > |\tilde{\eta}_{-2}^{(k)}| > \dots > |\tilde{\eta}_{-j}^{(k)}| > \dots > |\tilde{\eta}_{-M}^{(k)}| > 0.$$

The k th of these eigenvalues $\tilde{\eta}_{-j}^{(k)}$, when thus arranged by descending magnitude, is a second approximation η''_{-k} to η_{-k} . It is used to generate a revised list of eigenvalues by diagonalizing the matrix $\underline{T}(\eta''_{-k})$; and from this list, the k th in order of descending magnitude is again selected. This procedure is reiterated, until the value of the approximation to η_{-k} ceases to change significantly. The eigenvalues η_{-j} , $j = 1, 2, \dots, M$, resulting from this procedure are substituted into equation (43) to determine the probability P_e of error.

Diagonalizing the matrix $\langle n | (\rho_1 - \lambda \rho_0) | m \rangle$, representing the operator in equation (41) yields as many eigenvalues as the number M of rows and columns to which it is truncated. Some of these will be positive - half, when $\lambda = 1$, and the rest negative. However, the iterative method just described, by determining M negative characteristic roots, requires less computing time for evaluating the probability of error to a given accuracy. We have found that this technique converges quickly, even for weak input signals. In addition, it does not require the extensive algebraic manipulations involved in Yoshitani's and Klemm's approaches (refs 2-4). The solid curves in the left-hand side of figure 1 were obtained in this way.

Because the matrix elements $S_{nm}(\eta)$ of the operator $\rho_0^{1/2} \rho_1 (\rho_1 - \eta)^{-1} \rho_0^{1/2}$ appearing on the left-hand side of equation (59) are proportional to e^{-S} , they will be much smaller than the eigenvalues e_n of the density operator ρ_0 when $S \gg 1$. The former operator can therefore be treated as a perturbation of ρ_0 in equation (59); and, as the eigenvalues and eigenvectors of ρ_0 are known,

$$\rho_0 |n\rangle = e_n |n\rangle, e_n = (1-v)v^n, n \geq 0, \sum_{n=0}^{\infty} e_n = 1, \quad (67)$$

standard quantum-mechanical perturbation theory can be applied (ref 15). It yields for the eigenvalues, through second order terms, the approximation

15. Dirac PAM, Quantum Mechanics, 3rd ed, p 168-172, Clarendon Press, 1947.

$$-\lambda^{-1}\eta_{-n} \doteq e_n - S_{nn}(\eta_{-n}) + \sum_{m \neq n} \frac{|S_{nm}(\eta_{-n})|^2}{e_n - e_m} \quad (68)$$

with $S_{nm}(\eta)$ given by equation (61). The average probability P_e of error is now, by equation (43), approximately

$$\begin{aligned} P_e &= \zeta_0 \sum_{n=0}^{\infty} (e_n + \lambda^{-1}\eta_{-n}) \\ &\cong \zeta_0 \sum_{n=0}^{\infty} \left[S_{nn}(\eta_{-n}) - \sum_{m \neq n} \frac{|S_{nm}(\eta_{-n})|^2}{e_n - e_m} \right]. \end{aligned} \quad (69)$$

To terms of first order we can use the first term of this expression, replacing η_{-n} with its approximate value $-\lambda^{-1}e_n$ as in equation (66),

$$\begin{aligned} P_e &\cong P_e^{(1)}(\lambda) \\ &= \zeta_0 \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{e_n e_r |\langle n | D(\mu) | r \rangle|^2}{e_r + \lambda e_n} \\ &= \frac{1}{2}(1 + \lambda)\zeta_0 \sum_{n=0}^{\infty} \sum_{r=n}^{\infty} \frac{(2 - \delta_{rn})(e_n + e_r) e_n e_r}{e_r + \lambda e_n (e_n + \lambda e_r)} |\langle n | D(\mu) | r \rangle|^2 \\ &= \frac{1}{2}\lambda(1 - v)e^{-S} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(2 - \delta_{rn})v^r (1 + v^{r-n})n!}{(v^{r-n} + \lambda)(1 + \lambda v^{r-n})r!} \\ &\quad \times S^{r-n} [L_n^{(r-n)}(S)]^2 \\ &= \frac{1}{2}\lambda(1 - v)e^{-S} \sum_{p=0}^{\infty} \frac{(2 - \delta_{p0})v^p (1 + v^p)}{(v^p + \lambda)(1 + \lambda v^p)} \\ &\quad \times S^p \sum_{n=0}^{\infty} \frac{n!v^n}{(p+n)!} [L_n^{(p)}(S)]^2 \\ &= \frac{1}{2}\lambda \exp[-(2N + 1)S] \\ &\quad \times \sum_{p=0}^{\infty} \frac{(2 - \delta_{p0})v^{p/2} (1 + v^p)}{(v^p + \lambda)(1 + \lambda v^p)} I_p\left(\frac{2Sv^{1/2}}{1 - v}\right) \end{aligned} \quad (70)$$

Here, we have used the symmetry of $|\langle n | D(\mu) | r \rangle|^2$ in rearranging the terms of the summation, and we have applied the formula (ref 16):

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{n!}{(n+p)!} L_n^{(p)}(x) L_n^{(p)}(y) v^n \\ &= (1-v)^{-1} \exp\left[-\frac{v(x+y)}{1-v}\right] (xyv)^{-p/2} I_p\left(\frac{2(xyv)^{1/2}}{1-v}\right). \end{aligned} \quad (71)$$

By summing equation (69) on a digital computer and equation (70) on a programmable calculator, we have found that the second-order perturbation term in equation (69) becomes much smaller than the first for $D \geq 8$; and, that the results of equations (69) and (70) agree within the accuracy of plotting our graph for $D \geq 10$. The solid curves in the right-hand half of figure 1 were calculated by means of equations (69) and (70).

For $D \gg 1$, the argument for the modified Bessel functions in equation (70) is so significant that we can replace these functions with their asymptotic form,

$$I_p(x) \sim (2\pi x)^{-1/2} e^x,$$

in all significant terms of the series. The error probability is then asymptotically proportional to

$$\exp\left[-(2N+1)S + \frac{2Sv^{1/2}}{1-v}\right] = \exp\left[-\frac{D^2}{4} \frac{(1+v)}{(1+v^{1/2})^2}\right] \quad (72)$$

so that for $D \gg 1$, our parameter $y(v, D)$ in equation (46) becomes

$$y(v, \infty) = (1 + v^{1/2})^2 (1 + v)^{-1}.$$

This is plotted in dB versus v in figure 2.

For large SNR D one could determine the false-alarm probability Q_0 and the detection probability Q_d as functions of the parameter λ , by using the approximation $P_e^{(1)}(\lambda)$ in the formulas

$$Q_0 = P_e + (1 + \lambda) \frac{dP_e}{d\lambda}, \quad Q_d = 1 - P_e + \lambda(1 + \lambda) \frac{dP_e}{d\lambda}.$$

These could also be applied in conjunction with previously described methods, but the amount of computation required would, except for small values of the parameter v , be prohibitive.

16. Helstrom CW, Optimum Quantum Decision between a Pure and Mixed State, with Application of TCS Signals, Working Papers.

DETECTION OF ANTIPODAL TCS SIGNALS

In order to transmit a TCS signal, the radiating mode of the field in the transmitting aperture is placed in a right-eigenstate of the non-self-adjoint operator $\mu_0 a + \nu_0 a^\dagger$; where, a and a^\dagger are the usual photon annihilation and creation operators, and μ_0 and ν_0 are parameters related by (ref 12)

$$|\mu_0|^2 - |\nu_0|^2 = 1.$$

Here, we assume that μ_0 and ν_0 are real. When such a signal arrives at a distant receiver, in the absence of background light, it places the matched mode of the receiver aperture into a mixed quantum state whose density operator is a particular unitary transformation of the density operator ρ_1 in equation (37) (ref 16). The parameters μ and $\nu = e^{-w}$ of this density operator are related to those of the TCS signal by the following formulas, in which N_T is the average total number of photons in the received signal and f is the fraction of energy in the core of the transmitted TCS signal:

$$\nu = N/(N+1),$$

$$N = \frac{1}{2} \{ [4\kappa'^2 N_T (1-f) + 1]^{1/2} - 1 \},$$

$$\mu = (N_T f)^{1/2} (m+n),$$

$$m = \left[\frac{N_T (1-f) + N + 1}{2N + 1} \right]^{1/2}, \quad n = (m^2 - 1)^{1/2}.$$

Here, κ'^2 is the fraction of the signal energy lost during transmission to the receiver, whose aperture intercepts only a part of the transmitted field. In reference 16 our μ , m , n , and N were denoted by δ , μ , ν , and N respectively. Setting $f = 1$ corresponds to transmitting a pure coherent signal by putting the radiating mode of the transmitter aperture in a pure coherent state $|\alpha\rangle$, i.e., transmitting a right-eigenstate of the annihilation operator a . The matched mode of the receiver aperture is then, in the absence of background light, also in a pure coherent state $|\mu\rangle$, and $N = 0$. For $0 < f < 1$, what we called the nimbus of the transmitted TCS signal partially deteriorates during transmission into noise, whose strength is represented by the parameter N . Close to the transmitter, where this deterioration is not excessive, the probability P_e of error in detecting a TCS signal can be significantly smaller than that of an ordinary coherent signal conveying the same number N_T of photons, provided the fraction f is properly chosen (refs 12, 15).

Antipodal TCS signals induce two density operators at the receiver aperture that are the same unitary transformation of ρ_1 in equation (37), but with complex amplitudes $+\mu$ and $-\mu$, respectively. We assume that the signals are transmitted with equal prior probabilities $\xi_0 = \xi_1 = 1/2$. The average probability of error will then be the same as what was calculated in the foregoing sections, except that we replace $S = |\mu|^2$ by $4|\mu|^2$ for the mean number of photons in the equivalent received coherent signal.

In order to determine how this error probability P_e depends on the fraction f of energy in the core of the TCS signal, we have calculated P_e from equation (70) for various values of the fraction κ'^2 of energy lost during transmission. The mean number N_T of

photons received was set equal to 5, and the resulting values of the equivalent SNR D were, in all cases, large enough so that equation (70) is a good approximation. The values of the error probability for these antipodal TCS signals are plotted in figure 3. It is apparent from the figure that transmitting antipodal TCS signals, rather than ordinary coherent signals ($f = 1$), is advantageous only when less than about six-tenths of the signal energy is lost during transmission. The same conclusion was reached in the limit of large SNR for which equation (70) is valid.

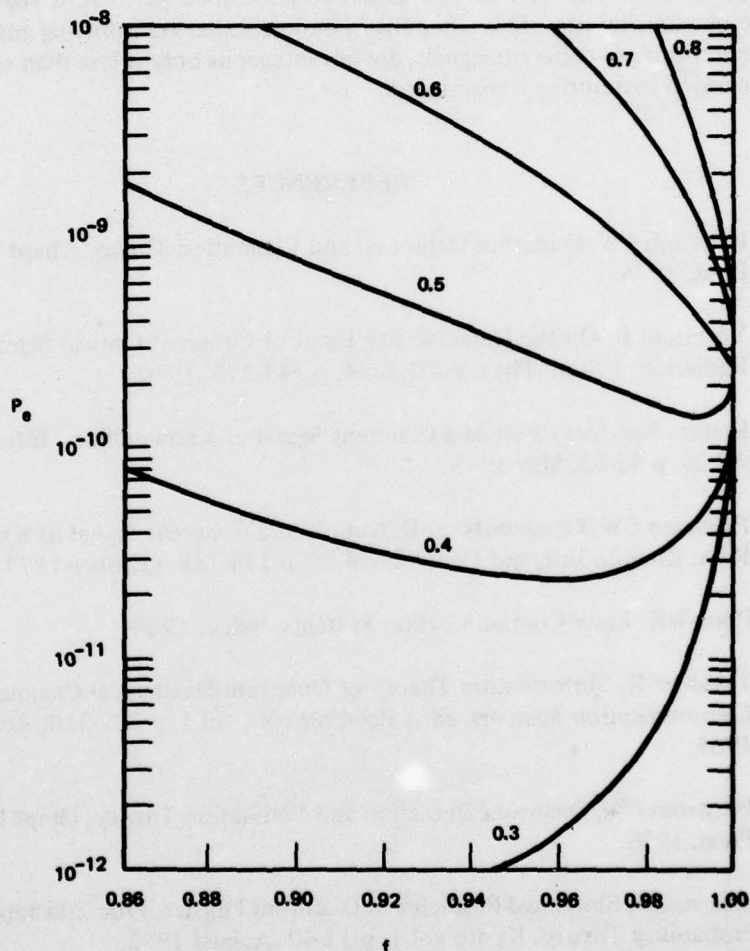


Figure 3. Probability P_e of error in detecting antipodal TCS signals of prior probability in absence of background, versus the fraction f of energy in the core of the signals; $N_T = 5$. Curves are labeled with the fraction κ^2 of energy lost in transmission.

CONCLUSION

Methods were presented for calculating the minimum attainable probability of error in detecting a coherent quantum signal received amid thermal noise in the limits of low and high signal-to-noise ratio. In the latter limit, quantum-mechanical perturbation theory was applied to solving the detection operator equation approximately. Graphic results are furnished for binary signals transmitted with equal prior probabilities. Thus, previous work on this problem is extended to a broader range of signal and noise strengths. The results were then applied to the detection of antipodal two-photon-coherent-state signals, and compared with the previous results. This comparison indicates that transmitting antipodal TCS signals, rather than ordinary coherent signals, are advantageous only if less than six-tenths of the signal energy is lost during transmission.

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