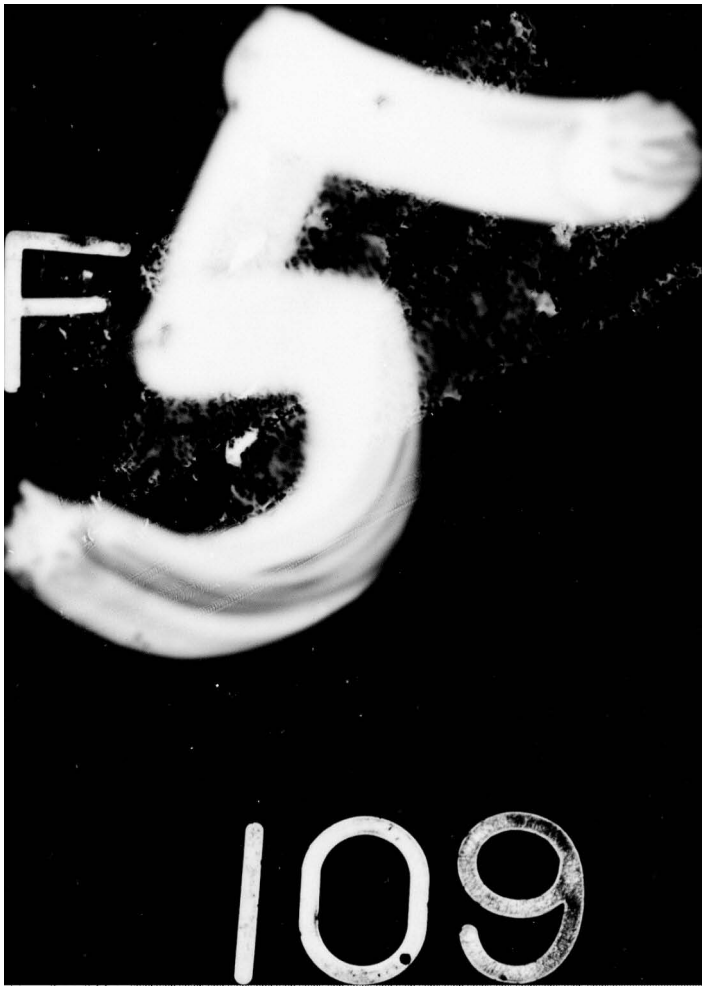
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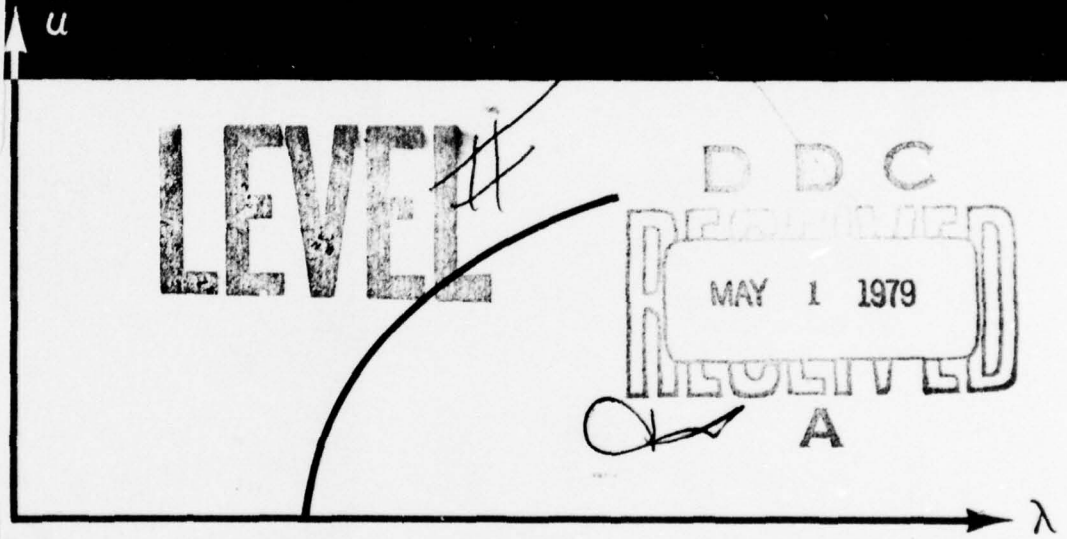
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Preface

The papers in this volume represent the proceedings of the Advanced Seminar on Applications of Bifurcation Theory held in Madison on October 27–29, 1976. With the exception of the survey by M. G. Crandall, the papers are published in the order in which they are presented.

The purpose of the Advanced Seminar was to survey a variety of applications of bifurcation theory to the sciences. There is a wealth of such applications and the papers contained in this volume discuss bifurcation problems in such areas as elasticity theory, fluid dynamics, geophysics, astrophysics, meteorology, statistical mechanics, and chemical kinetics. There are also two papers of a more mathematical nature: The volume begins with a discussion of analytical methods for bifurcation problems and concludes with a paper on numerical approaches to bifurcation questions.

We thank the authors for their efforts to meet our manuscript deadlines and thank Doris Whitmore, who put the volume together and compiled the index. We are especially grateful to Gladys Moran, who so ably attended to the details and paper work associated with the Conference. Finally, we are indebted to the Army Research Office under Contract Number DAAG29-75-C-0024, the National Science Foundation under Grant Number MCS76-12286, and the Office of Naval Research under Contract Number N00014-76-M-0036 for their financial support, which made possible the Advanced Seminar and this volume.

Paul H. Rabinowitz

An Introduction to Constructive Aspects of Bifurcation and the Implicit Function Theorem

M. G. Crandall

The goal of the lecture to which this article corresponds was to review some basic constructive results in bifurcation theory from the point of view used by the author and P. Rabinowitz in [3-5]. Since the existing literature seems quite adequate in this regard (see, e.g., [3-5], [16] and the review article [6]) there is little point in merely pursuing the same goal here. Thus a major portion of the current paper is devoted to presenting introductory material which may prove useful to certain prospective readers of the above literature. The contents of this paper are outlined below.

Section 1 attempts to describe in a general way what bifurcation theory is concerned about with emphasis on the parts of greatest interest to us. The style is informal and the attempt is addressed to both mathematicians and nonmathematicians.

Section 3 introduces the simple methods and some of the results of [3], [4] and [5]. For those who already know the terms, the main results concern bifurcation from "simple eigenvalues", Hopf bifurcation and stability. The exposition is brief and things are presented somewhat differently than elsewhere. The goal is to offer a sampling adequate for the reader to judge whether or not he wishes to pursue these topics further.

Section 2 is an attempt to explain and to sell the principal analytical tool used in Section 3 to potential users of this tool – the implicit function theorem in Banach spaces. Many people, appreciating the amazing scope and simplicity of this theorem, have tried to propogate its use with missionary zeal. (As Paul Rabinowitz commented after my lecture – "Nice sermon – er, lecture".) It is strange then that this cause is not further advanced a half-century after these tools became easily

available. Section 2 is completely independent of Section 1 and 3 and is our first written contribution (for better or worse) to the cause.

Lack of space and our limited goals precluded the discussion of serious concrete examples in this paper. We have restricted ourselves principally to general settings and a few illustrative examples. There is no lack of good examples. We suggest this volume, [3], [7], [12], [13], [14] and [22] as sources.

Other expository sources covering various aspects of bifurcation problems (many not touched upon here) are [16], [17], [18], [19], [20] and [24]. Another point of view is taken in [2].

The current paper is not an appropriate place to be very concerned about references. The sources listed above and the rest of this volume provide access to most of the literature.

Section 1. Introduction

The mathematical models of science and engineering may often be written in the form

$$\frac{dw}{dt} + G(w) = 0 \tag{1.1}$$

where w is the state of the system under consideration. We are thinking of (1.1) in a very general way. If the "system" under consideration is a bacteria culture, the "state" might be the number or mass of the bacteria. The simple model of this situation presented in beginning calculus reads $dw/dt = kw$ *for some constant* k, *so here* $G: \mathbb{R} \to \mathbb{R}$ *and* $G(w) = -kw$. In this example the state is a number. If one considers a system of particles, the state will be a list of numbers determining the positions and velocities of each particle. Many of the problems we are interested in have "states" which are not lists of numbers but functions. For example, in some models of a fluid confined in a region $\Omega \subseteq \mathbb{R}^3$ the state of the fluid at time t is taken to be a function $w(t, x) = (w_1(t, x), w_2(t, x), w_3(t, x))$, $x = (x_1, x_2, x_3)$, which tells the velocity of the fluid

at $x \in \Omega$ time t. Then (1.1) represents a complicated system of equations involving partial differentiations and other operations well as boundary conditions. We will not explicitly discuss such complicated models here – they appear elsewhere in this volume. But let us indicate how to rewrite a problem in partial differential equations in the form (1.1).

For this, consider the example problem

$$(1.2)\qquad \begin{cases} \text{(DE)} \quad \dfrac{\partial w}{\partial t} - \dfrac{\partial^2 w}{\partial x^2} + r(\lambda, w) = 0, & t > 0, \quad 0 < x < 1 \\ \text{(BC)} \quad \dfrac{\partial w}{\partial x}(0, t) = \dfrac{\partial w}{\partial x}(1, t) = 0 , & t > 0 \\ \text{(IC)} \quad w(x, 0) = w_0(x) , & 0 < x < 1 \end{cases}$$

consisting of the differential equation (DE), the boundary condition (BC) and the initial condition (IC). Here the unknown function $w(x, t)$ is sought and $r : \mathbb{R}^2 \to \mathbb{R}$ is a real-valued function of two real variables. The parameter λ is introduced for later use. Right now think of (1.2) as a problem for w with λ fixed. If we change λ we get a different problem. (1.2) corresponds to a system whose state at time t is $w(x, t)$ as a function of x. Let us think of this w as the mapping $t \to w(\cdot, t)$ where $w(\cdot, t)$ is the function $x \to w(x, t)$. We then write $w(t)$ rather than $w(x, t)$. Now $w(t)$ is itself a function for fixed t and $w(t)$ corresponds to $w(x, t)$ via $w(t)(x) = w(x, t)$. Next (for each λ) define a function $G(\lambda, \cdot)$ on functions of x by

$$(1.3)\qquad G(\lambda, w) = -\frac{d^2}{dx^2} w + r(\lambda, w) .$$

To each suitable function $w = w(x)$ (1.3) associates a new function $G(\lambda, w)$. Moreover, we incorporate the conditions (BC) into the definition of G. This is done by restricting the domain of G. Let us agree that

(1.4) domain of $G = D(G) = \{w \in C^2([0,1]): w'(0) = w'(1) = 0\}$.

Here $C^2([0,1])$ means the set of twice continuously differentiable functions on $[0,1]$. Corresponding to regarding $w(x,t)$ as $w(t)$, we rewrite $\frac{\partial}{\partial t} w(x,t)$ as $\frac{d}{dt} w(t)$. (To actually be precise about this we have to say what the allowable values $w(t)$ are, that is what states we are going to consider. Here we arbitrarily agree that states should be continuous functions of x, that is $w(t) \in C([0,1])$. Then $\frac{dw}{dt}(t_0) = f \in C([0,1])$ simply means that the function $h^{-1}(w(t_0+h)(x) - w(t_0)(x))$ converges to $f(x)$ uniformly for $x \in [0,1]$ as $h \to 0$. The whole problem (1.2) can formally be rewritten as

$$(1.5) \qquad \begin{cases} (DE) \quad \frac{dw}{dt} + G(\lambda, w) = 0, & t > 0 \\ (IC) \quad w(0) = w_0 \ . \end{cases}$$

The (DE) of (1.5) includes the (DE) of (1.2) and (BC) which is incorporated via (1.4). We agreed not to plug w into $G(\lambda, \cdot)$ unless it's derivative with respect to x vanishes at 0 and 1 .

Now, you may well (and should) complain that you don't know what (1.5) "means". To say precisely what (1.5) means is to define precisely what is meant by a solution of (1.5) (and the same is true for (1.2) to begin with). This is a more serious problem if one works with other state spaces in the current example (e.g., $L^1(0,1)$) or more complex equations (e.g., problems in fluid dynamics). We sidestep this question here by staying mostly in an abstract setting. This volume and the references herein provide access to an enormous spectrum of significant concrete models of all levels of complexity. We do caution the reader that only special choices of state spaces are either physically or mathematically reasonable in a given problem. It can also easily happen that a mathematically convenient choice may exclude physically significant possibilities.

It's time to discuss the appearance of parameters. Each mathematical model contains (implicitly or explicitly) parameters corresponding to the real world situation being described. For example, the outcome of a bacteria growing experiment will certainly depend on the size of the experimental apparatus, the temperature, the composition of the ambient atmosphere and other parameters, some of which may depend on time. To accomodate all of these we would have to have a model like $dw/dt + G(\lambda_1, \lambda_2, \ldots, \lambda_k, t, w) = 0$ with a large number k of parameters. However, even the biologists would be unable to figure out how G should depend on all these quantities. Thus they might try to fix as many as possible and perhaps vary one so as to see the effect of this. Say several experiments are run with all parameters fixed in each experiment and only the temperature varied between experiments. Then one is considering $dw/dt + G(T, w) = 0$. This is typical – there are in fact always many parameters. To get a tractable problem, it is assumed that many are fixed and a very few are varied between experiments. Whether this is reasonable in a given case is not a mathematical problem – it depends on the information being sought about the system and upon the agreement between theory and experiment. Many times several parameters in a model can be lumped into a single one by scaling procedures – these then often get names like Reynold's number, etc.

Thus we consider an "evolution problem"

$$\frac{dw}{dt} + G(\lambda, w) = 0 \tag{1.6}$$

which depends on parameters λ. We allow λ to be a vector and hence represent many parameters. The simplest solutions (1.6) can have are equilibrium solutions. These are time-independent solutions of (1.6), i.e. the states which satisfy $G(\lambda, w) = 0$. "Bifurcation theory" concerns itself with the study of the problem $G = 0$, the stability of equilibrium solutions of (1.7) and the study of (1.7) itself especially in neighborhoods of points where the structure of the set of solutions under consideration somehow changes. This is necessarily vague – "bifurcation

theory" is not a very unified subject of study. The results of Section 3 are motivated by examples in which the "change of structure" happens as described below.

Assume there is a known curve $w = \varphi(\lambda)$ of solutions of $G(\lambda, w) = 0$. This means that $\varphi(\lambda)$ is an admissible state for each λ and $G(\lambda, \varphi(\lambda)) = 0$. Next change variables in (1.6) according to $u + \varphi(\lambda) = w$:

$$\frac{du}{dt} = \frac{dw}{dt} = -G(\lambda, w) = -G(\lambda, u + \varphi(\lambda))$$

or

$$\frac{du}{dt} + F(\lambda, u) = 0 \tag{1.7}$$

where $F(\lambda, u) = G(\lambda, u + \varphi(\lambda))$ satisfies

$$F(\lambda, 0) = 0 . \tag{1.8}$$

For example, if G is given by (1.3), (1.4) and $r(\lambda, w) = -\lambda(w-\lambda) + (w-\lambda)^2 w$, then $\varphi(\lambda)$ could be the constant function λ (constant states are admissible in (1.4)). If $u = w - \lambda$, then $F(\lambda, u) = -\frac{d^2u}{dx^2} - \lambda u + u^2(u+\lambda)$ on the domain of F.

We call the solutions $u = 0$ of (1.7) the trivial solutions. It is always interesting to study the stability of a known solution of an evolution problem. Roughly, 0 is a stable solution of (1.7) (λ is now regarded as fixed) provided that we can keep the solution of (1.7) as close to 0 as we desire for all $t \geq 0$ by choosing $w(0)$ sufficiently small (in some sense) and 0 is an asymptotically stable solution of (1.7) if it is stable and we can also guarantee that $u(t) \to 0$ as $t \to \infty$ by choosing $u(0)$ sufficiently small. If we are in finite dimensions (i.e. $u \in \mathbb{R}^N$ and $F: \mathbb{R}^k \times \mathbb{R}^N \to \mathbb{R}^N$) for some k and N, it is well-known that the zero solution of (1.7) is asymptotically stable if the eigenvalues of $F_u(\lambda, 0)$ (the Jacobian matrix $(\partial F_i/\partial u_j)$ evaluated at λ, $u = 0$) all have positive real parts and is unstable if $F_u(\lambda, 0)$ has an eigenvalue with a negative real part. If we are in "infinite dimensions", as is the case when (1.7)

corresponds to a problem in partial differential equations, analogues of this remain true. One has a "linearized operator" $F_u(\lambda, 0)$ (see the next section) at the trivial solutions and its eigenvalues (or, more precisely, spectrum) often corresponds in the above way to the stability of the trivial solution. The assumption that this is so is called the "principle of linearized stability".

Assume now that λ is one real parameter and there is a critical parameter value λ_0 such that $u = 0$ is stable for $\lambda < \lambda_0$ and is unstable for $\lambda > \lambda_0$. This information may be pictured as being obtained via the principle of linearized stability and knowledge of the spectrum of the linearized operator or from experiments in which $u = 0$ is observed for $\lambda < \lambda_0$ but is not observable for $\lambda > \lambda_0$. The problem discussed in Section 3 is to figure out a little about what things look like near $\lambda = \lambda_0$ and $u = 0$. Since $F_u(\lambda, 0)$ has spectrum with positive real part for $\lambda < \lambda_0$ and some of its spectrum has nonpositive real part if $\lambda > \lambda_0$, it will in general have some spectrum with zero real part if $\lambda = \lambda_0$. When 0 (respectively, $\pm i$) is in the spectrum of $F_u(\lambda_0, 0)$ and certain "simplicity" conditions are satisfied, the methods of Section 3 show that there are equilibrium (respectively, periodic) solutions of (1.7) near $\lambda = \lambda_0$, $u = 0$ which can be constructively obtained and which account for the stability lost by the trivial solution as λ passes through λ_0.

Section 2. Implicit Functions – A Discussion for Potential Users

The number of methods which are available to solve nonlinear problems is small, especially if one refuses to regard a nonconstructive existence proof as a "solution" of the problem being treated. Perhaps the two methods of broadest scope and success are perturbation expansions and iterations. (We hope the reader has seen at least one power series expansion and one iteration scheme somewhere before.) There are three main steps in trying either method on a particular problem. The first is to decide what scheme (i.e. form of expansion or iterations) to try. Given a candidate scheme, the next problem is to show it is

defined (i.e. the successive iterates or terms in the expansion can be found). Finally, there is the error estimation problem (i.e., to show how close the iterates or partial sums of the expansion are to a true solution). All of these steps may be substantial. It is unfortunate that abstract mathematics tends to ignore the first one. We teach the contraction mapping principle, but do little to explain how to convert given problems into contraction mapping problems. One of our main points in this section is that the implicit function theorem in Banach spaces can be regarded in this way: It says "If you can verify this condition, the following iterates will work by the contraction mapping principle." It contains instructions on how to convert problems into contraction mapping problems. Theorems that do this are worth knowing. Of course it says much more. One of the other things it says is that if the problem is analytic and certain conditions hold, then the solution is analytic. Hence perturbation expansions will converge. It also says that one can compute the expansion coefficients by implicit differentiations, which is an efficient way to keep track of the algebra. And the proof of all this is quite simple. The plan below is to first restate and expand some of the above remarks in terms of a simple example in an attempt to make things seem more concrete. After this we discuss differentiation, compute some typical derivatives and then informally state and prove the implicit function theorem in Banach spaces.

Say we are lead to the simple model

$$(2.1)\qquad \begin{cases} u' - u^2 = \cos x + \cos^2 x - 1, & 0 \le x \le \pi \ , \\ u(0) - u(\pi) = 0 \ , \end{cases}$$

while considering some problem. Someone cleverly observes that $u = \sin x$ is a solution. Assume that perturbing the parameters which led us to (2.1) yields

$$(2.2)\qquad \begin{cases} u' - (1+\mu)u^2 = \cos x + \cos^2 x - 1 + \delta & 0 \le x \le \pi \\ u(0) = u(\pi) = 0 \end{cases}$$

and we want to know about "the solution" of (2.2) for (μ, δ) near $(0,0)$ if there is such a solution. The existence is not entirely obvious. For example, if $\delta > 2$ and $\mu \geq -1$ then (2.2) implies $u' > 0$ on $0 \leq x \leq \pi$ and $u(0) = u(\pi)$, which is impossible. In fact, there are no solutions in the range $\delta > \frac{1}{2}$, $\mu \geq -1$.

A standard way to look for a solution is to make the ansatz

$$(2.3)\quad u = u_0 + \mu u_{1,0} + \delta u_{0,1} + \mu^2 u_{2,0} + \mu\delta u_{1,1} + \delta^2 u_{0,2} + \dots$$

with $u_0 = \sin x$. Plugging (2.3) in (2.2) and equating coefficients of like monomials $\mu^j \delta^k$ on either side leads to equations for the $u_{\ell,m}$. To first order these are

$$(2.4)\quad \begin{cases} u'_{1,0} - 2u_0 u_{1,0} = u_0^2 + f_0 & 0 \leq x \leq \pi \\ u'_{0,1} - 2u_0 u_{0,1} = 1 & 0 \leq x \leq \pi \end{cases}$$

where $f_0 = \cos x + \cos^2 x - 1$. Each of these linear problems is to be solved subject to the requirement $u_{\ell,m}(0) = u_{\ell,m}(\pi)$. This is easily done and $u_{1,0}$, $u_{0,1}$ can be expressed explicitly in terms of integrals. Moreover, it is not hard to see that the remaining $u_{\ell,m}$ can be obtained in a similar way by solving problems

$$(2.5)\quad \begin{cases} Lu_{\ell,m} = u'_{\ell,m} - 2u_0 u_{\ell,m} = f_{\ell,m}, & 0 \leq x \leq \pi, \\ u_{\ell,m}(0) = u_{\ell,m}(\pi), \end{cases}$$

where $f_{\ell,m}$ is a function involving f_0 and the $u_{j,k}$ for $j + k < \ell + m$, which are determined at previous steps. This method (a very simple perturbation expansion) yields a formal series (2.3) for the solution of (2.2). If the series converges well enough it will certainly solve (2.2). Does it converge? In what region? Considerable work is involved in studying these questions directly. If the answers to them are satisfactory, we

may want to know more, for example the parameter range in which the estimate $|u - (u_0 + \mu u_{1,0} + \delta u_{0,1})| < .01$ for $0 \leq x \leq \pi$ is valid.

There are many other possibilities we could try. For example, define the iterations

$$u_{j+1}(x) = c_{j+1} + \int_0^x (f_0(s) + \delta + (1+\mu)\, u_j(s)^2)\, ds \tag{2.6}$$

where $u_0 = \sin x$, $c_0 = 0$ and the c_j's are to be determined by the conditions

$$\int_0^\pi (f_0(s) + \delta + (1+\mu)(u_j(s))^2)\, ds = 0 \tag{2.7}$$

which are necessary in order that $u_{j+1}(0) = u_{j+1}(\pi)$. Next it is necessary to see if this iteration scheme is defined, that is if (2.7) can be solved at each stage in some parameter range. If so, it still remains to study the convergence.

Another iteration scheme can be set up as follows: Let L be the operator corresponding to (2.1) linearized at $u_0 = \sin x$, i.e.

$$\begin{cases} Lu = u' - 2u_0 u \\ u(0) = u(\pi) \ . \end{cases} \tag{2.8}$$

We can always uniquely solve $Lu = f$, $u(0) = u(\pi)$ when f is given on $[0, \pi]$. You give the formula. (We do quickly note that $Lu = 0$, $u(0) = u(\pi)$ implies $u \equiv 0$. Indeed $u(0) = u(\pi)$ implies $u'(x_0) = 0$ for some $x_0 \in (0, \pi)$. Then $Lu = 0$ implies $u'(x_0) - 2(\sin x_0)\, u(x_0) = 0$, so $u(x_0)$ is also 0. By uniqueness of solutions of initial value problems, $u \equiv 0$.) Denote the solution of $Lu = f$, $u(0) = u(\pi) = 0$ by $u = Kf$. Then consider the iterations

$$u_{j+1} = u_j - K(u_j' - (1+\mu)u_j^2 - f_0 - \delta) \tag{2.9}$$

with u_0 as before.

There are several points to be made. The first is that there is no unique approach to (2.2). Next the program (2.3)-(2.5) requires the functions involved to have derivatives of all orders. In this case it will converge in some (μ, δ) region containing $(0, 0)$, but to prove it and get estimates would be messy. We haven't checked what happens with the scheme (2.6)-(2.7). The iteration scheme (2.9) involves inverting the same linearized operator as (2.3)-(2.5). It will converge in some (μ, δ) region. The method of (2.9), which we have not yet identified, does not require the existence of higher order derivatives. So what is the "method"? Iterations are used – but how are these iterations chosen? What guarantees the convergence? The way in which many iteration schemes converge is well summarized by the contraction mapping principle, but this principle does not explain how to set up an iteration scheme involving contraction mappings in a given problem. The implicit function theorem in fact does. The implicit function theorem, as applied to (2.2) at the known solution $\mu = \delta = 0$, $u_0 = \sin x$ tells one to use the iterations (2.9), that the right hand side of (2.9) is a contraction mapping in u_j in a suitable metric space and gives estimates all at once. Moreover, (in appropriate versions), it also tells one how to compute the $u_{i,j}$ of (2.3) in an efficient way and a sense in which the series there will converge.

In order to use this very efficient packaging of contraction mappings, iteration schemes and simple perturbation expansions by the implicit function theorem one must supplement the "linearized" gadgets of applied mathematics with a little more information to be able to call them Frechét derivatives. This is not a big deal, since to upgrade a linearized gadget to a Frechét derivative one must only give adequate quantitative information about the sense in which it is a good linear approximation. When one knows how to do this, what the implicit function theorem says and some facts of linear analysis, he discovers an incredible array of very useful results which may be understood quickly in terms of the implicit function theorem. A few of these relevant for this article are

bifurcation from simple eigenvalues, the Liapunov - Schmidt procedure, perturbation of simple eigenvalues and Hopf bifurcation. (There are many more – Picard existence theorem, conditional stability, etc., etc.) Frechét derivatives and the implicit function theorem are explained below.

We originally thought to begin with the definition of a Banach space and work as quickly as possible to the implicit function theorem. This is a surprisingly short program – but it does not seem optimally useful. The presentation in [21] is close to this if one reads it ignoring the distracting array of definitions (assume every space in sight is a Banach space, which is defined there). See also [9]. (We like the discussion in [1] even if it is not so quick.) What is lacking in [21] is examples. So we have compromised by assuming the definition of a Banach space and then defining Frechét derivatives, discussing the calculation of Frechét derivatives in applications, informally stating the implicit function theorem and *giving its proof.*

If $f: \mathbb{R} \to \mathbb{R}$, the derivative f' of f is the function defined by

$$f'(t_0) = \lim_{h \to 0} \frac{f(t_0+h) - f(t_0)}{h} \tag{2.10}$$

provided the limit exists. Equivalently, $\ell = f'(t_0)$ provided that

$$f(t_0 + h) = f(t_0) + \ell h + o(h) \tag{2.11}$$

where $o(h)$ stands for any function satisfying $\lim_{h \to 0} o(h)/|h| = 0$. The definition (2.10) is computational – it provides a way to compute f'. The definition (2.11) is often easier to work with (e.g., in proving the chain and product rules) and introduces the idea that derivatives are "linear approximations". For example, the proof of the chain rule reads

$$f(g(t_0 + h)) = f(g(t_0) + g'(t_0)h + o(h))$$

$$= f(g(t_0)) + f'(g(t_0))\,(g'(t_0)h + o(h))$$

$$+ o(g'(t_0)h + o(h))$$

$$= f(g(t_0)) + f'(g(t_0))\,g'(t_0)h + o(h)$$

since $f'(g(t_0))\,o(h) + o(g'(t_0)h + o(h)) = o(h)$.

If $f: \mathbb{R} \to V$ where V is a Banach space, both (2.10) and (2.11) make perfect sense and remain equivalent. The limit in (2.10) and $o(h)$ in (2.11) now refer to the way things are measured by the norm $\|\ \|_V$ of V. If $f: U \to V$ where U and V are Banach spaces we cannot write (2.10) directly. A natural generalization is the notion of directional derivative. Define the derivative of f at $u_0 \in U$ in the direction $\hat{u} \in U$ by

$$f'(u_0 ; \hat{u}) = \lim_{\substack{h \in \mathbb{R} \\ h \to 0}} \frac{f(u_0 + h\hat{u}) - f(u_0)}{h} \tag{2.12}$$

provided the limit exists. This notion is used in defining "Gateaux derivatives". It has the weakness that it does not involve the way things are measured in U. The limit on the right-hand side is taken in V, so $\|\ \|_V$ is involved, but $\|\ \|_U$ is not. The natural generalization of (2.11) involves both $\|\ \|_U$ and $\|\ \|_V$.

Definition. A continuous linear mapping $T: U \to V$ is the Frechét derivative of the (nonlinear) function $f: U \to V$ at $u_0 \in U$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\hat{u} \in U$ and $\|\hat{u}\|_U < \delta$ implies $\|f(u_0 + \hat{u}) - (f(u_0) + T\hat{u})\|_V < \varepsilon \|\hat{u}\|_U$.

Be patient – the definition will gradually become clearer and more transparent. We summarize it by saying that T is the Frechét derivative of f at u_0 provided that $f(u_0 + \hat{u}) = f(u_0) + T\hat{u} + o(\hat{u})$. One says "$f'(u_0)$ exists" or "f is Frechét differentiable at u_0" if there is such

a T and writes $T = f'(u_0)$, f' is the mapping $u_0 \to f'(u_0)$ defined on the set of $u_0 \in U$ such that $f'(u_0)$ exists. The values of f' are continuous linear mappings. Implicit in these remarks is the easily checked fact that f can have at most one Frechét derivative at any u_0. In fact, clearly $f'(u_0)\hat{u} = f'(u_0;\hat{u})$ so $f'(u_0)$ can be computed by (2.12) provided $f'(u_0)$ exists.

Now we consider the computation of derivatives. In applications in analysis there are two common types of things we need to differentiate. These are linear mappings and substitution operators. The differentiation of continuous linear mappings is easy. If $T: U \to V$ is continuous and linear, then $T(u_0 + \hat{u}) = Tu_0 + T\hat{u}$, so $T'(u_0) = T$ for every u_0. The Frechét derivative of a continuous linear mapping is itself at each point. This needs to be reconciled with the one dimensional case. If $f(t) = 2t$ for $t \in \mathbb{R}$ we usually write $f' = 2$. f' is the constant function 2, which does not seem to be f. But the derivative of a continuous linear function is itself – what is going on? Go back to (2.10) and (2.11). The Frechét derivative of $f(t) = 2t$ at t_0 is the linear mapping $h \to f'(t_0)h = 2h$, which is f at each t_0. More generally, if $f: \mathbb{R} \to V$ then $f'(t_0)$ can be thought of as an element of V computed via (2.10) or as the linear mapping $h \to hf'(t_0)$ defined by this element of V. As a V-valued function f' can be constant even if its values regarded as linear mappings are not constant mappings. Both ways of thinking are useful in this special case (i.e. f' has values in V or in the continuous linear mappings from $\mathbb{R}$ into V) and we use them interchangeably.

Let us go on to substitution operators. Assume now that $g: \mathbb{R}^n \to \mathbb{R}^m$. Given a set Ω and a function $u: \Omega \to \mathbb{R}^n$ we define a new function $G(u): \Omega \to \mathbb{R}^m$ by

$$G(u)(x) = g(u(x)) \quad \text{for} \quad x \in \Omega . \tag{2.13}$$

G is a substitution operator. Assume the first partial derivatives of g are continuous and let $g_u(u)$ denote the Jacobian matrix $(\frac{\partial}{\partial u_j} g_i(u_1, \ldots, u_n))$ evaluated at $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$. By calculus (Taylor's theorem),

if $u, \hat{u}: \Omega \to \mathbb{R}^n$ and $x \in \Omega$ are fixed, then

$$\lim_{h\to 0} \frac{1}{h}(G(u + h\hat{u})(x) - G(u)(x))$$

$$= \lim_{h\to 0} \frac{g(u(x) + h\hat{u}(x)) - g(u(x))}{h}$$

$$= g_u(u(x))\, \hat{u}(x) .$$

Thus we expect the Frechét derivative of G at some u_0 to be the linear operator "multiply $\hat{u}(x)$ by $g_u(u_0(x))$" . That is, we expect

$$(G'(u_0)\hat{u})(x) = g_u(u_0(x))\, \hat{u}(x) . \tag{2.14}$$

Now is the time to make the following point: There is no doubt that (2.14) is the right way to linearize G at u_0 . There is almost never any doubt about what the "linearized operator" should be. The formulas for "linearized operators" do not depend on functional a sis or what Banach spaces are involved at all. One finds the formulas easily. But to say a linearized operator is a Frechét derivative is a statement about how small errors are, and the usual ways of measuring errors introduce the norms of Banach spaces. To claim $G'(u_0)$ exists and is given by (2.14) we must say what the spaces are and verify estimates. Let Ω be a closed bounded subset of some Euclidean space. Let $C(\Omega: \mathbb{R}^k)$ denote the Banach space of continuous functions $u: \Omega \to \mathbb{R}^k$ equipped with the norm

$$\|u\|_{C(\Omega:\mathbb{R}^k)} = \sup_{x\in\Omega} \|u(x)\|_{\mathbb{R}^k} .$$

Regard G as a map from $C(\Omega:\mathbb{R}^n)$ to $C(\Omega:\mathbb{R}^m)$. If the first partials of g are continuous, then G' is given by (2.14). To prove this we must show that given $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\sup_{x\in\Omega} \|g(u_0(x) + \hat{u}(x)) - (g(u_0(x)) + g_u(u_0(x))\,\hat{u}(x))\|_{\mathbb{R}^m}$$

$$< \varepsilon \sup_{x\in\Omega} \|\hat{u}(x)\|_{\mathbb{R}^n}$$

provided $\sup_{x\in\Omega} \|\hat{u}(x)\|_{\mathbb{R}^n} < \delta$. Since $u_0(x)$ ranges over a compact subset of $\mathbb{R}^n$ as x ranges over Ω, this follows easily from Taylor's theorem.

For example, if $G: C([0,1]; \mathbb{R}^2) \to C([0,1]:\mathbb{R})$ is given by $G(u_1, u_2)(x) = u_1(x)\,u_2(x) + \sin u_1(x)$, then $(G'(u_{01}, u_{02})\hat{u})(x) = u_{01}(x)\,\hat{u}_2(x) + \hat{u}_1(x)\,u_{02}(x) + (\cos u_1(x))\,\hat{u}_1(x)$ where $u_0 = (u_{01}, u_{02})$ and $\hat{u} = (\hat{u}_1, \hat{u}_2)$ are in $C([0,1]; \mathbb{R}^2)$. (Exercise: differentiate $G(u)(x) = g(x, u(x))$, $g: \Omega \times \mathbb{R}^n \to \mathbb{R}^m$).

Working in other spaces be more subtle. For example, if $G: L^2(0,1) \to L^2(0,1)$ is given by $G(u)(x) = \sin u(x)$, then $G'(u_0)$ ought to be given by $(G'(u_0)\hat{u})(x) = (\cos u_0(x))\,\hat{u}(x)$. However, even though $\hat{u} \to (\cos u_0)\,\hat{u}$ is a bounded linear operator on $L^2(0,1)$, $G'(u_0)$ does <u>not</u> exist. We <u>cannot</u> guarantee that

$$\int_0^1 (\sin(u_0(x) + \hat{u}(x)) - (\sin u_0(x) + (\cos u_0(x))\,\hat{u}(x))^2\,dx < \varepsilon^2 \int_0^1 (\hat{u}))^2\,dx$$

by requiring $\int_0^1 \hat{u}(x)^2\,dx < \delta^2$ no matter how small $\delta > 0$ is nor what u_0 is. Thus the linearized operators of applications usually define Frechét derivatives in some spaces and not in others. Further information on substitution operators, etc., may be found in [23].

Consider now the mapping $K: C([0,1]; \mathbb{R}) \to C([0,1]: \mathbb{R})$ given by

$$K(u)(x) = \int_0^x \sin(u(s^2))ds \quad . \tag{2.15}$$

K is built up from three mappings as follows: If

$$H(u)(x) = u(x^2),\ G(u)(x) = \sin u(x),\ F(u)(x) = \int_0^x u(s)ds$$

then $K(u) = F(G(H(u)))$. We have two linear and continuous mappings

(F and H) and the substitution operator G . Their derivatives (as mappings of $C([0,1]; \mathbb{R})$ into itself) are known. The derivative of K can therefore be computed by the chain rule which says: If $A: U \to V$ is Frechét differentiable at u_0 and $B: V \to Z$ is Frechét differentiable at $A(u_0)$, then $C(u) = B(A(u))$ is Frechét differentiable at u_0 and $C'(u_0) = B'(A(u_0))\, A'(u_0)$. The proof is <u>exactly</u> the same as in the scalar case we reviewed above. Applying this twice to compute K' we find

$$K'(u_0) = F'(G(H(u_0)))\, G'(H(u_0))\, H'(u_0) \ .$$

Recalling what the derivatives of F, G, H are yields the formula

$$(K'(u_0)\hat{u})(x) = \int_0^x (\cos u_0(s^2))\, \hat{u}(s^2) ds \ .$$

While this formula is no surprise, we now know that this is a Frechét derivative. Adding to our tool kit (presently consisting of differentiation formulas for continuous linear mappings, substitution operators and the chain rule) the obvious formulas for derivatives of sums and linear combinations, we can handle most problems one encounters.

We will use the notation $B(U;V)$ for the Banach space of continuous and linear mappings $T: U \to V$ equipped with the norm $\|T\| = \sup\{\|Tu\|_V : u \in U \text{ and } \|u\|_U \leq 1\}$. The principle estimate in the proof of the implicit function theorem simply says

$$(2.16) \quad \|F(u_1) - F(u_2)\|_V \leq \sup_{0 \leq t \leq 1} \|F'(u_1 + t(u_2 - u_1))\| \, \|u_1 - u_2\|_U$$

whenever $F: U \to V$ and the indicated quantities exist. Thus bounds on derivatives are Lipschitz constants. The proof of (2.16) is sketched later. Let Λ, U, V be Banach spaces and $F: \Lambda \times U \to V$. Let $(\lambda_0, u_0) \in \Lambda \times U$ and consider the problem

$$(2.17) \qquad F(\lambda, u) = F(\lambda_0, u_0) \ .$$

Assume that F is continuous and that for $\lambda \in \Lambda$ the mapping $u \to F(\lambda, u)$

is Fréchet differentiable with the derivative denoted by $F_u(\lambda, u)$. Assume further that $(\lambda, u) \to F_u(\lambda, u)$ is continuous and $F_u(\lambda_0, u_0) \in B(U, V)$ has an inverse $F_u(\lambda_0, u_0)^{-1} \in B(V, U)$. The implicit function theorem (stated informally) says that under these assumptions (2.17) has a unique solution $u(\lambda)$ near u_0 for each λ near λ_0. Moreover $u(\lambda)$ is the uniform limit of $\{u_k(\lambda)\}$ where $u_0(\lambda) \equiv u_0$ and

$$(2.18) \quad u_{k+1}(\lambda) = u_k(\lambda) - (F_u(\lambda_0, u_0))^{-1} (F(\lambda, u_k(\lambda)) - F(\lambda_0, u_0)) .$$

The proof (given (2.16)) is nearly trivial. $F(\lambda, u) = F(\lambda_0, u_0)$ is, under the current assumptions, equivalent to

$$(2.19) \qquad u = u - (F_u(\lambda_0, u_0))^{-1} (F(\lambda, u) - F(\lambda_0, u_0)) = T(\lambda, u)$$

where (2.19) is the definition of $T(\lambda, u)$. Now

$$(2.20) \quad \begin{cases} \text{(i)} & T(\lambda_0, u_0) = u_0 , \\ \text{(ii)} & T(\lambda, u) \text{ and } T_u(\lambda, u) \text{ are continuous in } (\lambda, u) , \\ \text{(iii)} & T_u(\lambda_0, u_0) = 0 . \end{cases}$$

By (2.20) (ii) and (iii), there is an $r_0 > 0$ such that if $\|\lambda - \lambda_0\|_\Lambda$, $\|u - u_0\|_U < r_0$, then $\|T_u(\lambda, u)\| < \frac{1}{2}$ (the choice of $\frac{1}{2}$ is arbitrary, any $c < 1$ is all right). By (2.16), a bound on $\|T_u\|$ is a Lipschitz constant and so

$$(2.21) \quad \begin{cases} \|T(\lambda, u_1) - T(\lambda, u_2)\|_U \le \frac{1}{2} \|u_1 - u_2\|_U \\ \text{if } \|\lambda - \lambda_0\|_\Lambda, \ \|u_i - u_0\|_U \le r_0 \text{ for } i = 1, 2 . \end{cases}$$

One easily checks that $\|T(\lambda, u) - u_0\|_U \le r$ if $\|u - u_0\|_U \le r$ provided that $r \le r_0$, $\|\lambda - \lambda_0\|_\Lambda \le r_0$ and $2\|T(\lambda, u_0) - u_0\|_U \le r$. Thus the contraction mapping principle guarantees the convergence of $\{u_k(\lambda)\}$ uniformly on $\{\lambda: \|\lambda - \lambda_0\|_\Lambda \le r_0$ and $2\|T(\lambda, u_0) - u_0\| \le r\}$. (If you do not know the contraction mapping principle you should learn it as soon as possible.) Everything follows at once.

We have not yet really used that the parameter λ lies in a Banach space. When it does and F_λ is continuous, differentiating (2.18) implies

$$u'_{k+1}(\lambda) = B_k(\lambda)\, u'_k(\lambda) + A_k(\lambda) \tag{2.21}$$

where

$$\begin{cases} B_k(\lambda) = (I - (F_u(\lambda_0, u_0))^{-1} F_u(\lambda, u_k(\lambda))) \\ A_k(\lambda) = -(F_u(\lambda_0, u_0))^{-1} F_\lambda(\lambda, u_k(\lambda)) \ . \end{cases} \tag{2.22}$$

Iterating (2.21) yields

$$u'_{k+1} = B_k B_{k-1} \cdots B_1 A_0 + B_k B_{k-1} \cdots B_2 A_1 + \cdots + B_k A_{k-1} + A_k \ . \tag{2.23}$$

It is then easy to show, using the uniform convergence of u_k, (2.22) and (2.23) that as $k \to \infty$ $u'_k(\lambda)$ converges uniformly in a neighborhood of λ_0 to

$$\sum_{n=0}^{\infty} B(\lambda)^n A(\lambda) = (I - B(\lambda))^{-1} A(\lambda)$$

$$= -F_u(\lambda, u(\lambda))^{-1} F_\lambda(\lambda, u(\lambda))$$

where $B = \lim_{k\to\infty} B_k$, $A = \lim_{k\to\infty} A_k$. Thus $u(\lambda)$ is continuously differentiable and its derivative is $-F_u(\lambda, u(\lambda))^{-1} F_\lambda(\lambda, u(\lambda))$, which is what one finds by differentiating $F(\lambda, u(\lambda)) = F(\lambda_0, u_0)$ and solving for $u'(\lambda)$. If F has higher derivatives, so will u and this can be proved by differentiating (2.18) repeatedly and showing the resulting expressions converge. This is an important point: If F has continuous derivatives with respect to (λ, u) of orders up through m inclusively, then sequence obtained by differentiating $\{u_k(\lambda)\}$ at most m-times converges in the same neighborhood of λ_0 that we showed $\{u_k(\lambda)\}$ converged in.

Finally, if F is analytic in its arguments (which we haven't and won't define), then so is each u_k. Moreover, everything in sight can be expanded in power series and can therefore be defined on the complexifications of Λ, U, V (if the scalar field was $\mathbb{C}$ to start with, this isn't necessary). By the above proof (which works over the reals or complexes) $u(\lambda)$ will be C^1 from complexified Λ to complexified U, hence analytic. Another argument goes: $\{u_k(\lambda)\}$ converges uniformly to $u(\lambda)$ in a complexified neighborhood of λ_0, so $u(\lambda)$ is analytic.

Further details about analyticity can be found in, e.g., [10] and a good source for higher derivatives, applications, etc. is [1]. The article [15] is also useful reading. (The author of [15] expresses the pleasure of realizing how neat the implicit function theorem is. The notion he touts there is a perturbation of old ones. See e.g., [8, Theorem 3] and [1, Prop. 4.3.1].)

Consider (2.2) again. Let U be the Banach space of once continuously differentiable functions $u: [0, \pi] \to \mathbb{R}$ such that $u(0) = u(\pi)$ equipped with the norm

$$\|u\|_U = \max_{x \in [0,1]} (|u(x)| + |u'(x)|) .$$

Let $\Lambda = \mathbb{R}^2$ and $F: \Lambda \times U \to V = C([0, \pi]: \mathbb{R})$ be defined by $F(\mu, \delta, u) = u' - (1+\mu)u^2 - (\cos x + \cos^2 x - 1 + \delta)$. F and F_u are continuous and F_u is given by

$$F_u(\mu, \delta, u)\hat{u} = \hat{u}' - 2(1+\mu)u\hat{u} .$$

Indeed, $U \ni u \to u' \in V$ is linear and continuous. ($\sup|u'| = \|u'\|_V \leq \sup(|u| + |u'|) = \|u\|_U$) while $u \to -(1+\mu)u^2$ is a substitution operator. (We did not explicitly differentiate the mapping $u \to u^2$ with this domain U before, but one can think of it as the composition $u \to u \to u^2$ from U to V to V. The first linear map is continuous and $u \to u^2$ from V to V we did differentiate above.) You check that F is continuous. We note that

$$\| F_u(\mu, \delta, u)\hat{u} - F_u(\bar{\mu}, \bar{\delta}, \bar{u})\hat{u} \|_V = \| 2[(1+\mu)u - (1+\bar{\mu})\bar{u}]\hat{u} \|_V$$

$$\leq 2(|\mu-\bar{\mu}| \|u\|_U + |(1+\bar{\mu})| \|u - \bar{u}\|_U) \|\hat{u}\|_U$$

which implies that F_u is continuous. If $u_0(x) = \sin x$, $F(0, 0, u_0) = 0$ while $F_u(0, 0, u_0)$ is <u>nonsingular</u> (i.e. has a continuous inverse). The implicit function theorem applied here says that (2.9) converges and the resulting function $u(\mu, \delta)$ is analytic (with values in $C^1([0, \pi])$) in (μ, δ) near $(0, 0)$. Thus (2.3)-(2.5) also converges. In fact the $u_{j,k}$ can be obtained by successive implicit differentiations of (2.2) with respect to μ and δ. This leads again to (2.5), but the computations are easier. What about the region of convergence of (2.3)? Perhaps this can be estimated explicitly here, owing to the simplicity of (2.2), but usually it is too difficult to bother with. By contrast, it would not be hard to keep track of the contraction constants in (2.9) and obtain explicit estimates as outlined in the proof of the implicit function theorem.

We close this section with some remarks about verifying the hypotheses of the implicit function theorem and then the proof of (2.16). First of all, if U and V are concrete Banach spaces (e.g. $C(\Omega: \mathbb{R}^n)$, $L^2(0, 1)$, etc.) and $T: U \to V$ is a concrete linear operator defined <u>on all of U</u> (that is, explicit instructions for computing Tu given u are available), then T is continuous. You cannot "write down" a discontinuous linear operator $T: U \to V$, so it is unnecessary to worry about continuity of linear operators of this sort in applications. Similarly, if $T: U \to V$ is a concrete linear operator, then $T^{-1}: V \to U$ will exist and be continuous when T is one-to-one and onto V. Thus to use the implicit function theorem it is only necessary to show $F_u(\lambda, u)$ exists (i.e., is a Frechét derivative), depends continuously on (λ, u) and that $F_u(\lambda_0, u_0)$ is one-to-one and onto. Next, it is easy to "generalize" the implicit function theorem. One doesn't need $F_u(\lambda, u)$ to exist, etc. It is only necessary to find a linear mapping A such that contraction mapping theorem can be used to solve $u = u - A(F(\lambda, u) - F(\lambda_0, u_0))$. But

this sort of generalization is not very worthwhile and messes up any presentation unless the problem being discussed really justifies it. Otherwise do the smooth case – we all know now how to generalize it. (Again, [8, Thm. 3] and [15] are relevant.)

Proof of (2.16). Define $g(t,\tau) = \|F(u_1 + t(u_2 - u_1)) - F(u_1 + \tau(u_2 - u_1))\|_V$. If $\|F'(u_1 + t(u_2 - u_1))\| < M$ for $t \in [0,1]$, we need to show $g(1,0) \leq M\|u_2 - u_1\|_M$. Assume $u_2 \neq u_1$. The triangle inequality yields

$$g(t + h, 0) \leq g(t, 0) + g(t + h, t)$$

for $0 \leq t \leq t + h \leq 1$, while the definition of F' implies

$$g(t + h, t) = \|F'(u_1 + t(u_2 - u_1))h(u_2 - u_1) + o(h)\|_V < M\|u_1 - u_2\|_U h$$

if $h > 0$ is sufficiently small. Using these facts it follows easily that the set $\mathcal{S} = \{t \in [0,1]: g(t,0) \leq tM\|u_2 - u_1\|_U\}$ contains 0, contains $t + h$ if $t \in \mathcal{S} \cap [0,1)$ and $h > 0$ is sufficiently small. Thus $\max \mathcal{S} = 1$, and we are done.

Section 3. Implicit Functions and Bifurcation

We rather briefly cover some of the types of results discussed in [3], [4] and [5]. First, curves of nontrivial solutions bifurcating from particular trivial solutions are constructed in certain situations. A simple application to an equilibrium problem is presented and the application to Hopf bifurcation is very quickly sketched. Then the stability of equilibrium solutions obtained this way is treated in the finite dimensional setting. Analogous stability results for the Hopf bifurcation are reviewed in [6] and proved in [5], which covers the infinite dimensional case as well. Finally, the Liapunov-Schmidt procedure is recalled.

Constructing Bifurcating Curves

Let $F: \Lambda \times U \to V$ where Λ, U, V are Banach spaces. The discussion of Section 1 leads us to study the problem $F(\lambda, u) = 0$ when there are known trivial solutions $u = 0$. A point $(\lambda_0, 0) \in \Lambda \times U$ is a

bifurcation point for $F = 0$ (with respect to the trivial solutions) if every neighborhood of $(\lambda_0, 0)$ in $\Lambda \times U$ contains nontrivial solutions of $F = 0$. Assume all derivatives appearing below depend continuously on their arguments. If $F_u(\lambda_0, 0)$ is nonsingular (i.e. has a continuous inverse), then $F(\lambda, u) = 0$ is uniquely soluable near $(\lambda_0, 0)$ for $u = u(\lambda)$. Since $u(\lambda) \equiv 0$ works, the uniqueness implies that $(\lambda_0, 0)$ is not a bifurcation point. If $F_u(\lambda_0, 0)$ is singular, anything can happen. In certain simple situations additional curves of nontrivial solutions of $F = 0$ which pass through $(\lambda_0, 0)$ can be constructed. The proof of this is sketched below.

Since $F(\lambda, 0) \equiv 0$, given $\lambda_0 \in \Lambda$ F can be written in the form

$$F(\lambda, u) = Lu + B(\lambda - \lambda_0, u) + p(\lambda, u) \tag{3.1}$$

where $L = F_u(\lambda_0, 0)$, $B: \Lambda \times U \to V$ is a continuous bilinear mapping and p satisfies

$$\begin{cases} \text{(i)} \quad p(\lambda, 0) \equiv 0 \\ \text{(ii)} \quad p(\lambda_0, 0) = 0, \quad p'(\lambda_0, 0) = 0 \text{ and } p_{\lambda u}(\lambda_0, 0) = 0 \ . \end{cases} \tag{3.2}$$

In (3.1), B is the second mixed partial derivative $F_{\lambda u}(\lambda_0, 0)$ interpreted as a bilinear form. (To compute $B(\hat{\lambda}, \hat{u})$ differentiate $\lambda \to F_u(\lambda, 0)\hat{u}$ at λ_0. $F_u(\lambda_0 + \hat{\lambda}, 0)\hat{u} = F_u(\lambda_0, 0)\hat{u} + B(\hat{\lambda}, \hat{u}) + o(\hat{\lambda})$ for fixed $\hat{u}$. In a similar way, all higher derivatives may be regarded as multilinear mappings.)

Theorem 1. Let F be of class C^2 (i.e., have two continuous derivatives), $\lambda_0 \in \Lambda$ and (3.1), (3.2) hold. Assume: $u_0 \in U$, $u_0 \neq 0$, $Lu_0 = 0$, Z is a closed subspace of U and that every $v \in V$ can be uniquely written in the form $v = L\hat{z} + B(\hat{\lambda}, u_0)$ for some $\hat{z} \in Z$ and $\hat{\lambda} \in \Lambda$. Then there are continuously differentiable functions $\lambda: (-a, a) \to \Lambda$, $\psi: (-a, a) \to Z$ defined on some interval $(-a, a)$ such that $\lambda(0) = \lambda_0$, $\psi(0) = 0$, and $F(\lambda(s), s(u_0 + \psi(s))) = 0$ for $s \in (-a, a)$. Moreover, all the solutions of $F = 0$ in $\Lambda \times \{\text{span}\{u_0\} + Z\}$ near $(\lambda_0, 0)$ have the form $(\lambda(s), s(u_0 + \psi(s)))$ for some $s \in (-a, a)$.

Sketch of Proof: In the theorem, span $\{u_0\} + Z$ denotes the set of vectors $\alpha u_0 + z$ with α a scalar and $z \in Z$. Define $G: \mathbb{R} \times \Lambda \times Z \to V$ by $G(s, \beta, z) = s^{-1} F(\beta, s(u_0 + z))$ if $s \neq 0$ and extend by continuity to $s = 0$. Assume first that $p \equiv 0$ in (3.1). Then $G(s, \lambda, z) = L(u_0 + z) + B(\lambda - \lambda_0, u_0 + z)$ is independent of s. Now $G(0, \lambda_0, 0) = Lu_0 = 0$ and the derivative of $(\lambda, z) \to G(s, \lambda, z)$ at $(0, \lambda_0, 0)$ is

$$(\hat{\lambda}, \hat{z}) \to L\hat{z} + B(\hat{\lambda}, u_0) \quad . \tag{3.3}$$

This mapping is nonsingular by assumption. Hence the implicit function theorem implies that $G = 0$ is solvable for (λ, z) in terms of s near $(0, \lambda_0, 0)$. Say $\lambda = \lambda(s)$, $z = \psi(s)$ is the solution. (When $p \equiv 0$, $\lambda(s) \equiv \lambda_0$, $\psi(s) \equiv 0$ works.) In the general case, all the arguments just made apply. (3.2) guarantees that (3.3) is still the Frechét derivative of G with respect to (λ, z) at $(0, \lambda_0, 0)$. This presentation emphasizes that the solutions of $F = 0$ we have just found can be thought of as arising from "bending" the curve $\lambda = \lambda_0$, $u = su_0$ associated with the linear problem appropriately near $s = 0$.

Remark: If Λ, U, V are complex Banach spaces, s can be a complex parameter and $(-a, a)$ replaced by $|s| < a$.

The uniqueness assertion of the theorem follows from the uniqueness in the implicit function theorem and an additional estimate. More precisely, let $\lambda = \lambda(s)$, $z = \psi(s)$ be the solution of $G = 0$ just obtained. If the uniqueness of solutions of $G = 0$ holds in the region $\|\lambda - \lambda_0\|_\Lambda < r$, $\|z\|_U < r$, $|s| < r$, then any solution of $F(\lambda, su_0 + w) = 0$ with $w \in Z$ must satisfy $\lambda = \lambda(s)$, $w = s\psi(s)$ provided that $\|w\|_U < |s|r$, $|s| < r$ and $\|\lambda - \lambda_0\|_\Lambda < r$. Now if $\lambda \in \Lambda$, $w \in Z$ there is a $c > 0$ such that

$$Lw + B(\lambda - \lambda_0, u_0) = v \text{ implies } \|w\|_U + \|\lambda - \lambda_0\|_\Lambda \leq c\|v\|_V \tag{3.4}$$

because (3.3) is nonsingular. Moreover, given $\varepsilon > 0$ (3.2) implies there is a $\delta > 0$ such that

$$(3.5)\quad \begin{cases} \|w\|_U,\ |s|,\ \|\lambda - \lambda_0\|_\Lambda < \delta \text{ implies } \|p(\lambda, su_0 + w)\|_V < \\ \\ \varepsilon(|s| + \|w\|_U)\ . \end{cases}$$

This follows from (3.3), (2.16) and $p(\lambda, su_0 + w) = p(\lambda, su_0 + w) - p(\lambda, 0)$. Rewriting $F(\lambda, su_0 + w) = 0$ as

$$Lw + B(s(\lambda - \lambda_0), u_0) = -B(\lambda - \lambda_0, w) + p(\lambda, su_0 + w)$$

and using (3.4), (3.5) yields

$$(3.6)\quad \|w\|_U + |s|\,\|\lambda - \lambda_0\|_\Lambda \le c(\|B\|\,\|\lambda - \lambda_0\|_\Lambda\,\|w\|_U + \varepsilon(|s| + \|w\|_U))$$

where $\|B\| = \sup\{\|B(\hat\lambda, \hat u)\|: \|\hat\lambda\|_\Lambda,\ \|\hat u\|_U \le 1\}$. Assuming $c\|B\|\,\|\lambda - \lambda_0\|_\Lambda + c\varepsilon \le \frac{1}{2}$, (3.6) implies $\|w\|_U + |s|\,\|\lambda - \lambda_0\|_\Lambda \le 2c\varepsilon|s|$. But $\varepsilon > 0$ is at our disposal, and the local uniqueness follows.

Example 1. Let $\Lambda = \mathbb{R}$, $U = \{u \in C^2([0,\pi]): u'(0) = u'(\pi) = 0\}$ under the norm

$$\|u\|_U = \max_{0 \le x \le \pi} (|u(x)| + |u'(x)| + |u''(x)|)\ ,$$

$V = C([0,\pi])$ with the usual norm and $F(\lambda, u) = u'' + \lambda u - u^2(u+\lambda)$. (Via the discussion following (1.8) the problem $F = 0$ corresponds to equilibrium solutions of (1.2).) Now $F_u(\lambda, 0)\hat u = \hat u'' + \lambda\hat u$. The problem $\hat u'' + \lambda\hat u = v$, $\hat u'(0) = \hat u'(\pi) = 0$, always has a unique solution unless $\lambda = n^2$ for $n = 0, 1, \ldots$, in which case it is solvable exactly when $\int_0^\pi v(x)\cos nx\,dx = 0$. Therefore the possible bifurcation points correspond to $\lambda_0 = 0, 1, 4, \ldots$. Let $\lambda_0 = n^2$. The decomposition (3.1) holds with $Lu = u'' + n^2u$, $B(\lambda, u) = \lambda u$. Set $u_0(x) = \cos nx$ and let Z be a subspace of U. We want the problem $L\hat z + B(\hat\lambda, u_0) = \hat z'' + n^2\hat z + \hat\lambda\cos nx = v$ to be uniquely solvable for $\hat z$, $\hat\lambda$. Choosing

$Z = \{u \in U: \int_0^\pi u(x)\, u_0(x)dx = 0\}$, this is the case. Indeed, $\hat{\lambda} = \frac{2}{\pi} \int_0^\pi (\cos nx)\, v(x)dx$ (if $n = 1, 2, \ldots$) or $\hat{\lambda} = \frac{1}{\pi} \int_0^\pi v(x)dx$ if $n = 0$, and you can work out the formula for $\hat{z}$. Let $L\hat{z} + B(\hat{\lambda}, u_0) = v$ have the solution $\hat{z} = K_1 v$, $\hat{\lambda} = K_2 v$. To solve $G = 0$ the implicit function theorem offers the prescription: $z_{k+1} = z_k - K_1 G(s, \lambda_k, z_k)$, $\lambda_{k+1} = \lambda_k - K_2 G(s, \lambda_k, z_k)$ with $z_0 = 0$, $\lambda_0 = \lambda_0$. Expressed in terms of $u_k = s(u_0 + z_k)$ this scheme is $u_{k+1} = u_k - K_1 F(\lambda_k, u_k)$, $s\,\lambda_{k+1} = s\lambda_k - K_2 F(\lambda_k, u_k)$ or

$$\begin{cases} s\,\lambda_{k+1} = s\lambda_k - \left(\int_0^\pi u_0^2\, dx\right)^{-1} \int_0^\pi (u_k'' + \lambda_k u_k - u_k^2(u_k + \lambda)u_0\, dx \\ u_{k+1}'' + n^2 u_{k+1} = (n^2 - \lambda_k)u_k + u_k^2(u_k + \lambda_k) - s(\lambda_{k+1} - \lambda_k)u_0 \\ u_{k+1}'(0) = u_{k+1}'(\pi) = 0 \\ \int_0^\pi u_{k+1}\, u_0\, dx = s \int_0^\pi u_0^2\, dx\ . \end{cases}$$

As an exercise compute a few of the derivatives of the solution (λ, u) with respect to s at $s = 0$. Observe also that the entire analysis is correct if the "$u^2(u+\lambda)$" term is replaced by anything suitably small at $u = 0$. For example $xu(x)^3 + \sin(u'(x))^2 + (u''(\pi/2))^4 + \int_0^\pi (u(.7\tau))^2 d\tau$ is quite acceptable.

Example 2. (Hopf Bifurcation)

Consider the problem

$$\frac{du}{dt} + f(\mu, u) = 0\ . \tag{3.7}$$

(We have in mind (1.7), but have changed notation for convenience.) Assume $f(\mu, 0) \equiv 0$ and let $\pm i$ be simple eigenvalues of the Jacobian matrix $f_u(\mu_0, 0)$. Assuming a little more we find nontrivial periodic solutions of (3.7) near $\mu = \mu_0$, $u = 0$. If $\rho\tau = t$, (3.7) becomes

$$\frac{du}{d\tau} + \rho f(\mu, u) = 0 \tag{3.8}$$

and we look for 2π-periodic solutions of (3.8) (or $2\pi\rho$-periodic solutions of (3.7)). Let $U = \{u: \mathbb{R} \to \mathbb{R}^n:$ u is once continuously differentiable and 2π-periodic$\}$, $V = \{v: \mathbb{R} \to \mathbb{R}^n:$ v is continuous and 2π-periodic$\}$ with the usual norms. Let $\Lambda = \mathbb{R}^2 = \{\lambda = (\rho, \mu): \rho, \mu \in \mathbb{R}\}$. Define

$$F(\lambda, u) = \frac{du}{d\tau} + \rho f(\mu, u) \quad . \tag{3.9}$$

Let L_0, L_1 be the matrices $F_u(\mu_0, 0)$ and $F_{\mu u}(\mu_0, 0)$ respectively. Then L and B in (3.1) are

$$Lu = \frac{du}{d\tau} + L_0 u, \quad B(\lambda, u) = \rho L_0 u + \mu L_1 u \quad ,$$

where $\lambda = (\rho, \mu)$. Let $L_0 a = ia$ and $u_0 = \text{Re } e^{-i\tau} a$, $u_1 = \text{Im } e^{-i\tau} a$. Choose $Z = \{u \in U: \int_0^{2\pi} (u(\tau), u_i(\tau))_{\mathbb{R}^n} d\tau = 0$ for $i = 0, 1\}$. The mapping $(\hat{\lambda}, \hat{z}) \to L\hat{z} + B(\hat{\lambda}, u_0)$ is one-to-one and onto exactly when

$$\begin{cases} \text{(i) } ki \text{ is not an eigenvalue of } L_0 \text{ for } k = 0, 2, \ldots \\ \text{(ii) If } \beta(\mu) \text{ is the eigenvalue of } L_0 + \mu L_1 \text{ near } i \\ \quad \text{for } \mu \text{ near } \mu_0, \text{ then } \text{Re } \beta'(\mu_0) \neq 0 \quad . \end{cases} \tag{3.10}$$

The assumptions (3.10) were made by Hopf and the current proof simplifies and generalizes his. It is an interesting (and somewhat extended) exercise to verify the preceding assertion relating Hopf's assumptions to the mapping $(\hat{\lambda}, \hat{z}) \to L\hat{z} + B(\hat{\lambda}, u_0)$. See [5] for details.

Simple Eigenvalues and Stability in $\mathbb{R}^n$

Consider (1.7) with $\Lambda = \mathbb{R}$, $U = \mathbb{R}^n$. Now $F: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. Let us consider the conditions under which Theorem 1 applies at some $\lambda_0 \in \mathbb{R}$. Let L_0, L_1 be the matrices $L_0 = F_u(\lambda_0, 0)$, $L_1 = F_{\lambda u}(\lambda_0, 0)$. Then L and B in (3.1) are given by

$$\begin{cases} Lu = L_0 u \\ B(\lambda - \lambda_0, u) = (\lambda - \lambda_0) L_1 u \quad . \end{cases} \tag{3.11}$$

Since L is to be singular, there is a $u_0 \neq 0$ such that $L_0 u_0 = 0$. When can we choose a subspace Z of $\mathbb{R}^n$ so that

$$\mathbb{R} \times Z \ni (\hat{\lambda}, \hat{z}) \to L_0 \hat{z} + \hat{\lambda} L_1 u_0 \in \mathbb{R}^n \tag{3.12}$$

is one-to-one and onto $\mathbb{R}^n$? Clearly this requires $u_0 \notin Z$ since L_0 must be 1-1 on Z. Thus the dimension of $Z \leq n-1$. If (3.12) is to be onto $\mathbb{R}^n$ the dimension of Z clearly must be $(n-1)$. It follows that $N(L_0)$ (the null space of L_0) must be $\text{span}\{u_0\}$. Thus 0 is a geometrically simple eigenvalue of L_0 (i.e. $\dim N(L_0) = 1$). When this holds (3.12) is onto for any choice of Z satisfying $u_0 \notin Z$ and $\dim Z = n-1$ provided $L_1 u_0 \notin R(L_0)$ (the range of L_0). It is easy to construct examples in which these conditions hold and the algebraic multiplicity of u_0 (i.e. $\dim \bigcup_1^\infty N(L_0^k)$) is greater than one. It is useful to define 0 to be an L_1-simple eigenvalue of L_0 when the above conditions ($N(L_0) = \text{span}\{u_0\}$, $L_1 u_0 \notin R(L)$) hold. If I is the identity, then the I-simple eigenvalues are precisely the algebraically simple eigenvalues (checking this is an easy exercise). We further define κ to be an L_1-simple eigenvalue of L_0 if 0 is an L_1-simple eigenvalue of $L_0 - \kappa L_1$. (At this point we are thinking of L_0, L_1 as arbitrary $n \times n$ matrices.)

Theorem 2. Let $L_0, L_1 \in B(\mathbb{R}^n; \mathbb{R}^n)$ and $\kappa_0 \in \mathbb{R}$ be an L_1-simple eigenvalue of L_0. Then every $L \in B(\mathbb{R}^n; \mathbb{R}^n)$ near L_0 has a unique L_1-simple eigenvalue $\kappa(L)$ near κ_0. If $N(L_0) = \text{span}\{u_0\}$, $\dim Z = n-1$ and $u_0 \notin Z$, the corresponding eigenvector can be uniquely written in the form $u_0 + z(L)$ where $z(L) \in Z$. $L \to \kappa(L)$ and $L \to z(L)$ are analytic functions.

Sketch of proof: Define

$$\mathfrak{F}: B(\mathbb{R}^n; \mathbb{R}^n) \times \mathbb{R} \times Z \to \mathbb{R}^n$$

by $\mathfrak{F}(L, \kappa, z) = L(u_0 + z) - \kappa L_1(u_0 + z)$. Then $\mathfrak{F}(L_0, \kappa_0, 0) = L_0 u_0 - \kappa_0 L_1 u_0 = 0$

and the derivative of $(\kappa, z) \to \mathfrak{F}(L, \kappa, z)$ at $(L_0, \kappa_0, 0)$ is

$$(\hat{\kappa}, \hat{z}) \to (L_0 - \kappa_0 L_1)\hat{z} + \hat{\kappa} L_1 u_o \ .$$

This is nonsingular by assumption, so $\mathfrak{F} = 0$ can be solved by the implicit function theorem for $\kappa(L)$, $z(L)$ near $(\kappa_0, 0)$ when L is near L_0. One next checks the uniqueness and simplicity. (See [4].)

Theorem 3. Let $F: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $F(\lambda, 0) \equiv 0$, $\lambda_0 \in \mathbb{R}$, $L_0 = F_u(\lambda_0, 0)$, $L_1 = F_{\lambda u}(\lambda_0, 0)$ and 0 be an I-simple and an L_1-simple eigenvalue of L_0. Let $N(L_0) = \text{span}\{u_0\}$ and $\mathbb{R}^n = Z \times \text{span}\{u_0\}$. Let $(\lambda, u) = (\lambda(s), s(u_0 + \psi(s))$ be the curve of solutions of $F = 0$ constructed on Theorem 1. Let $\kappa(s)$ be the unique I-simple eigenvalue of $F_u(\lambda(s), s(u_0 + \psi(s)))$ near 0 for s near 0 constructed in Theorem 2. Then there is a continuous real-valued function A of s such that

$$\kappa(s) = s\lambda'(s)\, A(s) \ . \tag{3.14}$$

for s near 0 and $A(0)$ is determined by

$$A(0)\, u_0 + L_1 u_0 \in R(L_0) \ . \tag{3.15}$$

Before proving the result, let us consider its utility as regards the stability question. If the eigenvalues of $F_u(\lambda, 0)$ have positive real parts for $\lambda < \lambda_0$ (so the trivial solution is stable) and the only purely imaginary eigenvalue for $\lambda = \lambda_0$ is 0, then $F_u(\lambda(s), u(s))$ has eigenvalues with positive real part except, perhaps, one corresponding to the perturbed zero eigenvalue of $F_u(\lambda_0, 0) = F_u(0)$. This is $\kappa(s)$. In this situation then, $\kappa(s) > 0$ implies $su_0 + s\psi(s)$ is a stable solution of (1.7) with $\lambda = \lambda(s)$. If $\kappa(s) < 0$, then this solution is unstable. Theorem 2 explains how to compute $\kappa(s)$ (by the implicit function theorem). However, this computation requires detailed knowledge of $\lambda(s)$, $\psi(s)$. Theorem 3 says, in fact, that for small s it is enough to know the signs of $s\lambda'(s)$ and $A(0)$. No additional computations are necessary. Recalling that the iterative approximations $\{\lambda_k(s)\}$ to $\lambda(s)$ provided by

the implicit function theorem have the property that $\{\lambda_k'(s)\}$ converges to $\lambda'(s)$ (see Section 2), this is quite pleasant. If we are in the analytic case, (3.14) implies that the leading terms in the series of $\kappa(s)$ and $s\lambda'(s)\,A(0)$ are equal. Finally, let $\gamma(\lambda)$ be defined by $\gamma(\lambda_0) = 0$ and

$$F_u(\lambda, 0)\,(u_0 + w(\lambda)) = \gamma(\lambda)\,(u_0 + w(\lambda)) \tag{3.16}$$

where $w(\lambda) \in Z$, $w(0) = 0$. Differentiating (3.16) at $\lambda = \lambda_0$ yields

$$L_0 w'(\lambda_0) + L_1 u_0 = \gamma'(\lambda_0)u_0 \ . \tag{3.17}$$

Thus $A(0) = -\gamma'(\lambda_0)$. In the situation we just described, $\gamma(\lambda) > 0$ for $\lambda < \lambda_0$ and $\gamma(\lambda) < 0$ for $\lambda > \lambda_0$. Hence $\gamma'(\lambda_0) < 0$. (Note $\gamma'(\lambda_0) = 0$ is impossible since $L_1 u_0 \notin R(L_0)$.) Combining these observations with (3.14), we have $\kappa(s) > 0$ is $s\lambda'(s) > 0$ and $\kappa(s) < 0$ if $s\lambda'(s) < 0$. This is a precise version of the statement "supercritical bifurcations are stable, subcritical bifurcations are unstable".

Proof of Theorem 3. Let $F_u(s) = F_u(\lambda(s),\ su_0 + s\psi(s))$, etc., and $F_u(s)(u_0 + z(s)) = \kappa(s)\,(u_0 + z(s))$ $(z(s) = z(F_u(s))$ where $z(L)$ is the function of Theorem 2.) We have

$$\begin{cases} \text{(i)}\ \ F_u(s)(u_0 + (s\psi(s))') + (s\lambda'(s))\ s^{-1}\ F_\lambda(s) = 0 \\ \text{and} \\ \text{(ii)}\ \ F_u(s)(u_0 + z(s)) - \kappa(s)(u_0 + z(s)) = 0\ , \end{cases} \tag{3.18}$$

where (3.18) (i) follows from differentiating $F(\lambda(s),\ su_0 + s\psi(s)) = 0$. Try the ansatz

$$\begin{cases} \kappa(s) = s\lambda'(s)\ A(s) \\ z(s) = (s\psi(s))' + s\lambda'(s)\ B(s) \end{cases} \tag{3.19}$$

where A has values in $\mathbb{R}$ and B in Z. Subtracting (3.18) (i) from (3.18) (ii) and using (3.19) we find that if $\mathfrak{F} = 0$ where $\mathfrak{F}: \mathbb{R} \times \mathbb{R} \times Z \to \mathbb{R}^n$

is given by

$$\mathfrak{F}(s, A, B) = F_u(s)B + (s^{-1}F_\lambda(s) + A(u_0 + (s\psi(s))' + s\lambda'(s)B \tag{3.20}$$

then κ and z as given by (3.19) solve (3.18) (ii). Now

$$\mathfrak{F}(0, A, B) = L_0 B - (L_1 u_0 + Au_0) . \tag{3.21}$$

Choosing A_0, B_0 in $\mathbb{R}$ and Z respectively so that $L_0 B_0 - A_0 u_0 = L_1 u_0$ we have $\mathfrak{F}(0, A_0, B_0) = 0$. Moreover the derivative of $\mathfrak{F}$ with respect to (A, B) at $(0, A_0, B_0)$ is

$$(\hat{A}, \hat{B}) \to L_0 \hat{B} + \hat{A} u_0$$

which is nonsingular. Everything follows now by the implicit function theorem.

We want to emphasize the following distinction. Assume that $N(L_0) = \text{span}\{u_0\}$ is one dimensional, $L_0 = F_u(\lambda_0, 0)$ and $L_1 = F_{\lambda u}(\lambda_0, 0)$. In order to construct bifurcating curves as in Theorem 1 we need $L_1 u_0 \notin R(L_0)$. *In order to use Theorem 3 for a stability criterion we need* $u_0 \notin R(L_0)$. These are different conditions. One easily sees, from (3.16), (3.17) that <u>if</u> (3.16) has a smooth solution $\gamma(\lambda)$, $w(\lambda)$ then $\gamma'(\lambda_0) \neq 0$ is <u>equivalent</u> to $L_1 u_0 \notin R(L_0)$ <u>or</u> $u_0 \notin R(L_0)$. However, it may happen (see [3]) that $L_1 u_0 \notin R(L_0)$ and (3.16) has no smooth solutions and likewise (3.18) (ii) has no smooth solutions. In this case, stability <u>cannot</u> be studied by assuming (3.18) (ii) has solutions $\kappa(s)$, $z(s)$ given by power series. (As has occurred.)

For further reading in this direction (extension to Hopf bifurcation, infinite dimensions, etc.) and relevant historical remarks see [4], [5] and the review article [6]. The information that the sign and zeros of $\kappa(s)$ could locally be read off from the same information about $s\lambda'(s)$ was first proved in [4]. The more elegant "factored form" of Theorem 3 was remarked in [11].

<u>Liapunov - Schmidt Procedure</u>

Let $G: X \to V$ where X and V are Banach spaces. (X corresponds

to $\Lambda \times U$ above.) Assume $G(x_0) = 0$. We want to study $G = 0$ near u_0. Assume X and V can be written in the form

$$(3.22) \qquad \begin{cases} \text{(i)} \quad X = N(G'(x_0)) \oplus Z \\ \text{(ii)} \quad V = R(G'(x_0)) \oplus W \end{cases}$$

where Z and W are closed subspaces of X and V respectively. This assumes that $R(G'(x_0))$ is closed. The decomposition (3.22) is always possible if $N(G'(x_0))$ and $V/R(G'(x_0))$ are finite dimensional or if $R(G'(x_0))$ is closed and X, V are Hilbert spaces. Given $x \in X$ we write $x = x_N + x_Z + x_0$ where $x_N \in N(G'(x_0))$ and $x_Z \in Z$. Let P project V on $R(G'(x_0))$ along W. The problem $G(u) = 0$ can be rewritten as

$$(3.23) \qquad \begin{cases} \text{(i)} \quad H(x_N, x_Z) = P(G(x_N + x_Z + x_0)) = 0 \\ \text{(ii)} \quad (I-P)\, G(x_N + x_Z) = 0 \ . \end{cases}$$

Now $H(0, 0) = 0$ and the derivative of $Z \ni x_Z \to H(x_N, x_Z) \in R(G'(x_0))$ at $x_N = 0$, $x_Z = 0$ is $\hat{x}_Z \to PG'(x_0)\hat{x}_Z$ which is one-to-one and onto. Thus the implicit function theorem says that (3.23) (i) can be solved for $x_Z = x_Z(x_N)$ and $G = 0$ is locally equivalent to $(I - P)\, G(x_N + x_Z(x_N)) = 0$, an equation "in $N(G'(x_0))$".

Example: $X = \mathbb{R} \times U$, $V = U$, $T: U \to U$ compact linear, $p: R \times V \to U$ satisfies (3.2) at $\lambda_0 \in \mathbb{R}$ and $G(\lambda, u) = u - \lambda Tu + P(\lambda, u)$. Then $G'(\lambda_0, 0)(\hat{\lambda}, \hat{u}) = (I - \lambda_0 T)\hat{u}$ has the null space $\{(\hat{\lambda}, \hat{u}): \hat{u} \in N(I - \lambda_0 T)\}$ of dimension $n + 1$ where $n = \dim N(I - \lambda_0 T)$. $R(I - \lambda_0 T)$ is also of co-dimension n (Riesz-Schauder theory). The above procedure then leads to "$n+1$ equations in n unknowns". These are (if $n > 1$) usually attacked by formal expansion methods. In the analytic case one is confronted with problems of real analytic geometry. See [24].

In situations where the construction of Theorem 1 applies the Liapunov-Schmidt procedure can be used to first reduce the "dimensionality". This does not seem advantageous, since it requires more

iterating (once for Liapunov-Schmidt and then once more) than our original presentation.

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Imperfect Bifurcation

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Abstract

Bifurcation, or branching of solutions, is a prominent feature of many nonlinear stability problems. In elastic stability, bifurcation is called buckling, and in hydrodynamic stability bifurcations are called transitions. In many applications, the bifurcations are perturbed by imperfections, impurities, and other inhomogeneities in the physical system. These perturbations are frequently singular. A new method for analyzing singular perturbations of bifurcations is described. The method of matched asymptotic expansions is employed in the analysis. Applications of the method to the buckling of crooked columns, and to forced nonlinear oscillations are presented.

1. Introduction.

The buckling of an elastic column by an axially compressive thrust was one of the first bifurcation problems to be studied. The theory, which is called the Elastica, was derived and analyzed by Euler. In this theory, it is assumed that the axis of the column is a straight line before

the application of the thrust. The axis remains straight until the thrust reaches a critical value that is called the Euler buckling load. The column bends into a buckled equilibrium state when the thrust exceeds this critical value.

We assume that the x axis coincides with the initial axis of the column. It is normalized by the length of the column, so that $0 \leq x \leq 1$. The horizontal and vertical displacements of the buckled axis are denoted by $u(x)$ and $w(x)$ respectively. Furthermore, $\phi(x)$ is the angle between the tangent to the column's axis and the x axis. Then the Elastica boundary value problem is given by:

$$\phi'' + \lambda \sin \phi = 0 , \qquad \phi'(0) = \phi'(1) = 0 ; \tag{1.1}$$

$$\begin{aligned} u' &= \cos \phi - 1 , & u(0) &= 0 , \\ w' &= \sin \phi , & w(0) &= w(1) = 0 . \end{aligned} \tag{1.2}$$

A prime denotes differentiation, and $\lambda \geq 0$ is a parameter proportional to the thrust. The boundary conditions in (1.1) and (1.2) imply that the ends of the column are pinned (simply supported).

The solutions of (1.1) and (1.2) have been obtained explicitly in terms of elliptic functions [1-3]. The results are summarized in the "bifurcation diagram" given in Figure 1.1. We observe that $\phi = u = w = 0$ is a solution of (1.1) and (1.2) for all values of λ. This is the straight state of the column. Buckled states branch from the straight state at the buckling loads

$$\lambda = \lambda_n \equiv (n\pi)^2 , \qquad n = 1,2,\ldots . \tag{1.3}$$

The minimum buckling load λ_1 is the Euler buckling load. The buckled state branching from λ_n exists for all $\lambda \geq \lambda_n$. Two of these states are sketched in Figure 1.1. In more general discussions of bifurcation, the straight state, the buckled states, and the buckling loads (1.3) are called, the basic solution, the bifurcation branches, and the bifurcation points respectively. We shall adhere to this terminology.

The solutions of the Elastica demonstrate one of the main features of bifurcation...the sharp transition in solution multiplicity at the bifurcation points. However, buckling experiments show that bifurcation does not occur for real columns. The experimental results are qualitatively similar to the heavy curves in Figure 1.2. The sharp transition of bifurcation is replaced by the smooth transition of reality.

This discrepancy between experiments and theory is attributed to small imperfections that are always present in experimental specimens. The small deviations of the axis of the experimental column from the assumed straight line of the Elastica is a dominant imperfection. Experimental results show that, in general, the smaller the imperfections the closer the agreement between experiments and theory. Thus the column corresponding to experimental curve 2 in Figure 1.2 has less imperfections than column 1.

Similar phenomena occur in the buckling of elastic plates and shells. The classical theory for the buckling of

axially compressed, cylindrical elastic shells [4,5] gives a bifurcation problem. For many ranges of parameters, the bifurcation from the minimum bifurcation point λ_1 is as shown in Figure 1.3. In this theory, the midsurface of the shell is assumed to be a perfect, right circular cylinder. One of the principal imperfections in shells is the waviness of the initial midsurface of experimental specimens. Representative experimental results are sketched in Figure 1.3. A maximum thrust λ_e is reached at which the shell jumps to a large amplitude equilibrium state. The dashed segment of the curve in the figure is conjectured, because it is not observed in experiments. The value of λ_e is the experimental buckling load. It may be 1/2 or as small as 1/5 of λ_1. The difference $\lambda_1 - \lambda_e$ is decreased for experiments on carefully manufactured specimens. Thus imperfections have a more dramatic effect in shell buckling than column buckling.

For column buckling, as shown in Figure 1.1, the bifurcation branches exist near the bifurcation points only to the right of the bifurcation points. This is called supercritical bifurcation. Subcritical bifurcation occurs if the bifurcation solutions branch only to the left near the bifurcation points; see Figure 5.1. For the bifurcation shown in Figure 1.3, there is a bifurcation branch for values of λ above and below λ_1. This is called transcritical bifurcation.*

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*This term is due to T. B. Benjamin.

Imperfections are significant in other nonlinear stability problems, such as the thermal convection of a viscous fluid between two parallel plates. The bottom plate is uniformly heated and the top plate is uniformly cooled. The Rayleigh number λ is the basic parameter. It is proportional to the temperature difference between the top and bottom plates. The Boussinesq theory [6] is used to describe the convection mathematically. It yields a bifurcation problem. The basic solution is the conduction state, in which the heat is transmitted from the bottom to the top plate by conduction only (no fluid motion). Steady convection solutions bifurcate supercritically from the conduction state at an infinite sequence of bifurcation points. The lowest bifurcation point is called the critical Rayleigh number. However, careful experiments, see e.g. [7], show that bifurcation does not occur. Instead, there is a smooth transition similar to the buckling of imperfect elastic columns, as shown in Figure 1.2. Experimental and theoretical attempts, see e.g. [7,8], to ascribe this discrepancy to "non-Boussinesq" effects, such as variation of the fluid viscosity with temperature, have not been successful. Small imperfections such as, thermal noise in the plates [9], or lateral variations in the distance between the plates, are sufficient to qualitatively explain the deviation, as we shall show in a future publication.

To analyze mathematically the perturbations of bifurcations caused by imperfections, inhomogeneities, and other

impurities, we modify the classical bifurcation theory by introducing a small parameter δ which characterizes the magnitudes of these imperfections. Thus we consider the general nonlinear problem

$$F[y;\lambda,\delta] = 0 , \tag{1.4}$$

where $y = y(\lambda,\delta)$ is the solution vector. The corresponding bifurcation problem is

$$F[y;\lambda,0] = 0 \tag{1.5}$$

with solutions $y(\lambda) = y(\lambda,0)$. We refer to (1.4) as the perturbed problem, and to the solutions of (1.4) and (1.5) as the perturbed and bifurcation branches, respectively. We assume that as $\delta \to 0$, the perturbed branches $y(\lambda,\delta) \to$ the bifurcation branches $y(\lambda)$, and the perturbed problem (1.4) $\to$ the bifurcation problem (1.5).

The principal methods for analyzing perturbed problems are due to Koiter [10], Keener and H. Keller [11], J. Keller [12], Chow, Hale and Mallet-Paret [13], and Matkowsky and Reiss [14]. The methods in [10-13] are local analyses near bifurcation points. Koiter's method employs a variational principle and uses the Ritz method. It is a formal procedure that is widely used by structural engineers. The review [15] contains references to applications of the method. Keener and H. Keller's [11] method is a rigorous iteration procedure. It is applicable apparently only to perturbed branches that have a "nose", such as occurs in shell buckling (Figure 1.3), and not to perturbed branches as shown in

Figure 1.2. The nonlinear forced oscillation of an acoustic waveguide was treated by J. Keller [12] as a perturbed problem by a formal perturbation expansion near the bifurcation points of periodic solutions of the free oscillation problem. A modification of the Lyapunov-Schmidt method of classical bifurcation theory [16], and a theorem of Thom [17] on the generic forms at the singularities of nonlinear mappings, were used by Chow, Hale and Mallet-Paret [13] to give a rigorous analysis of the perturbed problem near bifurcation points. The analysis of Matkowsky and Reiss [14] will be briefly described in Section 2. It employs the method of matched asymptotic expansions [18-20] to obtain formal asymptotic expansions of the perturbed branches as $\delta \to 0$ near bifurcation branches which are known either exactly or approximately. Thus when bifurcation branches are known globally, for example by numerical computation, the method [14] gives global representations of the perturbed branches. Applications of the method to the buckling of crooked columns, and to the forced response of a nonlinear oscillator are given in Sections 3-5. Additional applications to, the buckling of imperfect plates, thermal convection, the robustness of population dynamics models, panel flutter, and other problems, will be presented in future publications. Generalizations of the method to multiple bifurcation points, and to imperfections and secondary bifurcation are presently being investigated.

2. Singular Perturbations of Bifurcations.

To motivate the method of Matkowsky and Reiss [14], we consider the algebraic equation

$$F[y;\lambda,\delta] \equiv y^2 + (1 - \lambda)y - \lambda\delta = 0 . \tag{2.1}$$

The solutions of the corresponding bifurcation problem ($\delta = 0$) are, $y(\lambda) \equiv 0$, and $y(\lambda) = \lambda - 1$. They are sketched in Figure 2.1 with the solutions of (2.1) for representative positive and negative values of δ. Thus $\lambda = 1$, $y = 0$ is the only bifurcation point of the bifurcation branches.

As $\delta \to 0$, the perturbed branches approach the bifurcation branches. The perturbed branches $y(\lambda,\delta)$ are smooth functions of λ for each δ, but the bifurcation branches $y(\lambda)$ have a "corner" at the bifurcation point. Thus $y(\lambda,\delta) \to y(\lambda)$ as $\delta \to 0$ non-uniformly in λ, in any interval containing $\lambda = 1$. Consequently, (2.1) is a singular perturbation of the corresponding bifurcation problem. This immediately suggests using the method of matched asymptotic expansions to obtain uniformly valid asymptotic expansions of the perturbed branches as $\delta \to 0$.

We now briefly describe this method for the general nonlinear equation

$$F[y;\lambda,\delta] = 0 . \tag{2.2}$$

The corresponding bifurcation problem is

$$F[y;\lambda,0] = 0 . \tag{2.3}$$

We assume that the solutions of (2.2) and (2.3) are

elements of a real Hilbert space, and that F is a nonlinear operator on that space. The parameters λ and δ are assumed to be real, and the bracket notation $\langle \cdot, \cdot \rangle$ is used for the inner product.

We first define the singular points of any bifurcation branch $y(\lambda)$ as the values $\lambda = \lambda_o$ and the corresponding values $y(\lambda_o)$ for which the linearized problem

$$F_y[y(\lambda);\lambda,0]\phi = 0 , \qquad \langle \phi,\phi \rangle = 1 , \tag{2.4}$$

has solutions. The linear operator $F_y[y(\lambda);\lambda,0]$ in (2.4), is the Frechet derivative of $F[y;\lambda,0]$. The significance of the singular points, in the present context, is that they are the points at which non-uniform convergence is possible. All bifurcation points are singular points. There are singular points, such as the limit points shown in Figure 2.2, that are not bifurcation points. With reference to Figure 2.2, we define the sub-branches of a bifurcation branch $y(\lambda)$ as the segments connecting two successive singular points of $y(\lambda)$. Thus the bifurcation branch shown in Figure 2.2 has three sub-branches.

The Outer Expansions. Let $y_0(\lambda)$ denote a sub-branch, and λ_o a singular point of y_0. We seek asymptotic expansions of the perturbed branches near y_0 in the form

$$y(\lambda,\delta) = \sum_{j=0}^{\infty} y_j(\lambda)\delta^j . \tag{2.5}$$

By substituting (2.5) into (2.2) and equating to zero the coefficient of each power of δ, we obtain the recursive system of linear equations,

$$
\begin{aligned}
&F_y^o(\lambda)y_1(\lambda) = -F_\delta^o(\lambda) \equiv R_1(\lambda) \ , \\
&F_y^o(\lambda)y_j(\lambda) = R_j(\lambda) \ , \qquad j = 2,3,\ldots \ .
\end{aligned} \tag{2.6}
$$

The superscript zero denotes the operator evaluated at $\delta = 0$, e.g. $F_y^o(\lambda) \equiv F_y[y_0(\lambda),\lambda;0]$.

We observe that $F_y^o(\lambda)$ is invertible except at a singular point $\lambda = \lambda_o$. In [14], we studied (2.6) under the assumption that there is an element ϕ^* in the null-space of the operator that is adjoint to $F_y^o(\lambda_o)$, such that,

$$
I \equiv \langle F_\delta^o(\lambda_o),\phi^* \rangle \neq 0 \ . \tag{2.7}
$$

Then it follows from (2.6) that $y_1(\lambda)$, and hence the expansion (2.5), is not defined at λ_o. In fact, $|y(\lambda,\delta)| \to \infty$ as $\lambda \to \lambda_o$, for many operators F. Thus we refer to (2.5) as the outer expansion corresponding to $y_0(\lambda)$.

Perturbations caused by imperfections that satisfy (2.7) "destroy" bifurcation, as we shall show. If $I = 0$, then the imperfections need not destroy bifurcation. This is demonstrated in Section 4, where it is shown for a special problem, that the imperfections distort the bifurcation branches without destroying bifurcation. In this section we shall retain assumption (2.7). A similar analysis is required if (2.7) is violated.

The Inner Expansions. We obtain expansions of the perturbed branches that are valid near λ_o as $\delta \to 0$ by first introducing the "stretching" transformation

$$
\lambda = \lambda_o + \xi\mu^a + \sum_{i=2}^{\infty} \xi_i(\mu^a)^i \ , \tag{2.8}
$$

where μ is a new small parameter that is defined by

$$|\delta(\mu)| = \mu^b \ . \tag{2.9}$$

The positive constants, a and b, are to be determined. Then we obtain asymptotic expansions of $y(\lambda,\delta)$ that are valid near λ_o in the form

$$z(\mu) \equiv y(\lambda(\mu),\delta(\mu)) = \sum_{j=0}^{\infty} z_j \mu^j \ , \tag{2.10}$$

where $z_0 = y(\lambda_o)$. The coefficients in the inner expansion (2.10) are determined in the usual way by inserting (2.10) into (2.2). This leads to the recursive system of linear problems

$$F_y^o(\lambda_o) z_j = \rho_j \ , \qquad j = 1,2,\ldots \ . \tag{2.11}$$

The inhomogeneous terms ρ_j depend on a, b, $z_1, z_2, \ldots, z_{j-1}$, and the derivatives of λ and δ with respect to μ, evaluated at $\mu = 0$.

Since the operator $F_y^o(\lambda_o)$ is singular, the ρ_j must satisfy the solvability conditions,

$$\langle \rho_j, \phi^* \rangle = 0 \ , \qquad j = 1,2,\ldots \ , \tag{2.12}$$

where ϕ^* is in the null space of the adjoint of $F_y^o(\lambda_o)$. We have assumed that the null space is one dimensional and spanned by ϕ^*. Multidimensional null spaces will be considered in a subsequent publication. An analysis of (2.12) and the linear problems (2.11) is given in [14]. We shall summarize the results for operators F which satisfy the conditions

$$F^o_\lambda(\lambda_o) = F^o_{\lambda\lambda}(\lambda_o) = \cdots = 0 \ . \tag{2.13}$$

The more general case is discussed in [14]. The conditions (2.13) are satisfied in many problems at the bifurcation points, but not at limit points.

The precise form of the inner expansion then depends on the nonlinearity at the singular point. If

$$Q \equiv \langle (F^o_{zz}(\lambda_o)\phi)\phi, \phi^* \rangle \neq 0 \ , \tag{2.14}$$

then $|\delta| = \mu^2$, and

$$\begin{aligned} z &= z_0 + A(\xi)\phi|\delta|^{1/2} + 0(\delta) \ , \\ \lambda &= \lambda_o + \xi|\delta|^{1/2} + 0(\delta) \ , \end{aligned} \tag{2.15}$$

where the amplitude $A(\xi)$ is a real root of the quadratic equation

$$\begin{aligned} QA^2 &+ 2\xi\langle F^o_{z\lambda}(\lambda_o)\phi, \phi^* \rangle A \\ &+ 2(\operatorname{sgn}\delta)\langle F^o_\delta(\lambda_o), \phi^* \rangle = 0 \ , \end{aligned} \tag{2.16}$$

where sgn is the sign function. Thus there are inner expansions (2.15) if and only if the discriminant of (2.16) is non-negative, i.e.

$$\xi^2 \geq \frac{2(\operatorname{sgn}\delta)Q\langle F^o_\delta(\lambda_o), \phi^* \rangle}{\langle F^o_{z\lambda}\phi, \phi^* \rangle} \ . \tag{2.17}$$

Furthermore, inner expansions occur in pairs, and they coalesce when the equality holds in (2.17). Condition (2.14) characterizes the nonlinearity near λ_o as quadratic.

Higher order terms in the expansions can be obtained by analyzing (2.11) and (2.12). The amplitudes A_j of ϕ in each

z_j, for $j > 1$, are then solutions of linear inhomogeneous algebraic equations. The solvability conditions for these equations determine the parameters $\xi_2, \xi_3, \ldots$.

If $Q = 0$ and

$$C \equiv \langle \{ ((F^o_{zzz}(\lambda_o)\phi)\phi)\phi + 4(F^o_{zz}(\lambda_o)\phi)z_2 + 2(F^o_{zz}(\lambda_o)z_2)\phi \}, \phi^* \rangle \neq 0, \tag{2.18}$$

where z_2 is defined by

$$z_2 \equiv -\frac{1}{2} F^o_z(\lambda_o)^{-1}[(F^o_{zz}\phi)\phi] , \tag{2.19}$$

then the nonlinearity near λ_o is cubic. Then $|\delta| = \mu^3$ and the inner expansions are given by

$$\begin{aligned} z &= z_0 + A(\xi)\phi|\delta|^{1/3} + 0(\delta^{2/3}) , \\ \lambda &= \lambda_o + \xi|\delta|^{2/3} + 0(\delta) . \end{aligned} \tag{2.20}$$

The amplitudes $A(\xi)$ are the real roots of the canonical cubic equation,

$$CA^3 + 6\xi\langle (F^o_{z\lambda}(\lambda_o)\phi), \phi^* \rangle A + 6(\operatorname{sgn}\delta)\langle F^o_\delta(\lambda_o), \phi^* \rangle = 0 . \tag{2.21}$$

Thus there may be one, two, or three inner expansions at a singular point with cubic nonlinearity, depending on the values of ξ.

If $C = 0$, then quartic and higher order nonlinearities near λ_o must be analyzed to determine the inner expansions. Thus, a secondary result of this method for constructing asymptotic expansions of the perturbed branches, is a formal classification of the singularities of general nonlinear

equations. Thom [17] has rigorously classified the singularities of a special class of equations, using different methods.

We observe from (2.15) and (2.20) that $\lambda - \lambda_o$ is $0(|\delta|^{1/2})$ for quadratic nonlinearities, and $0(|\delta|^{2/3})$ for cubic nonlinearities. Thus, the effects of imperfections are more severe for quadratic than cubic nonlinearities. Since quadratic nonlinearities usually correspond to transcritical bifurcation, this conclusion is in qualitative agreement with experimental observations.

Matching. There are several sub-branches of the bifurcation problem emanating from each singular point. For each sub-branch there is an outer expansion, and there are several inner expansions for each of its singular points. To obtain uniform asymptotic expansions of the perturbed branches, it is necessary to appropriately connect the outer and inner expansions. These connections are obtained systematically by the matching procedure of the method of matched asymptotic expansions [18-20]. The matching will be illustrated for the specific problem in Section 3. A discussion of the procedure for the general problem (2.2) is given in [14].

Finally, we wish to remark that the inner and outer expansions are completely determined by the analysis. This is in contrast to the usual applications of the method of matched asymptotic expansions, where the inner and outer expansions contain undetermined constants or functions. The

matching conditions then evaluate these undetermined quantities.

Sketches of typical inner and outer expansions at the bifurcation point λ_1 are given in Figure 3.1 for a cubic nonlinearity and supercritical bifurcation. The curves denoted by B are bifurcation sub-branches. The heavy curves denoted by O are the outer expansions. The dashed curves show the inner expansions. The composite expansions are sketched in Figure 3.2. These results are obtained in Section 3.

3. Buckling of Crooked Columns.

We now generalize the Elastica theory to account for imperfections in the column. We assume that the initial unstressed axis is a specified curve. The angle between the tangent to this curve and the horizontal is $\phi_I(x)$. Then, as we can show, the boundary value problem (1.1) is modified to

$$\phi'' + \lambda \sin \phi = \delta g(x) , \quad \phi'(0) = \phi'(1) = 0 , \tag{3.1}$$

where $\delta g(x) \equiv \phi_I''(x)$ and $\phi_I'(0) = \phi_I'(1) = 0$. We shall not present the boundary value problems for u and w corresponding to (1.2). Thus the imperfection appears as a specified "forcing function" in the differential equation. Here, δ is the magnitude of the imperfections, e.g. $\delta = \max_{0 \le x \le 1} |\phi_I''(x)|$, and $g(x)$ is the imperfection shape function. For simplicity, we shall assume $\delta \geq 0$.

We obtain the bifurcation problem (1.1) by setting $\delta = 0$ in (3.1). It can be solved explicitly in terms of elliptic functions [1-3]. We shall be concerned only with the solution branching from the lowest bifurcation point $\lambda = \lambda_1$. The bifurcation points (1.3) are the eigenvalues of the bifurcation problem linearized about $\phi \equiv 0$. The normalized eigenfunctions are $\phi_n(x) \equiv \sqrt{2} \cos n\pi x$.

There are four bifurcation sub-branches emanating from λ_1. They are the two sub-branches $\phi_0 \equiv 0$, to the left and right of λ_1, and the two supercritical bifurcation sub-branches $\phi_0 = \pm\Psi$, branching above and below λ_1; see Figure 1.1.

To demonstrate the application of the method [14] to problems where the bifurcation branches cannot be obtained explicitly, we approximate them near λ by the standard perturbation method; see e.g. [21]. If we define the amplitude parameter ε by

$$\varepsilon^2 = \int_0^1 \Psi^2 dx \tag{3.2}$$

then we obtain

$$\begin{aligned} \lambda &= \lambda_1\left[1 + \frac{\varepsilon^2}{4} + 0(\varepsilon^3)\right] , \\ \Psi &= \varepsilon\phi_1 + 0(\varepsilon^3) . \end{aligned} \tag{3.3}$$

Let $\phi_0(x;\lambda)$ denote any one of the four bifurcation sub-branches. Then the outer expansions of the perturbed branches are

$$\phi(x;\lambda,\delta) = \sum_{j=0}^{\infty} \phi_j(x;\lambda)\delta^j . \tag{3.4}$$

By substituting (3.4) into (3.1), we find, for example, that ϕ_1 is determined by

$$(3.5) \qquad \phi_1'' + \lambda \cos(\phi_0)\phi_1 = g(x) , \qquad \phi_1'(0) = \phi_1'(1) = 0 .$$

We solve (3.5) by an eigenfunction expansion, using the normalized eigenfunctions v_m and eigenvalues σ_m, that are defined for $m = 1,2,\ldots,$ by

$$(3.6) \qquad v_m'' + \lambda(\cos \phi_0)v_m = \sigma_m v_m , \qquad v_m'(0) = v_m'(1) = 0 .$$
$$\langle v_m, v_m \rangle = 1 .$$

Then by assuming that $g(x)$ has the representation

$$(3.7) \qquad g(x) = \sum_{j=1}^{\infty} g_m v_m(x) , \qquad g_m \equiv \langle g, v_m \rangle ,$$

we obtain the solution of (3.5) as

$$(3.8) \qquad \phi_1 = \sum_{m=1}^{\infty} \frac{g_m}{\sigma_m} v_m .$$

For the two sub-branches $\phi_0 \equiv 0$, we obtain from (3.6) that

$$(3.9) \qquad \sigma_m = \lambda - \lambda_m , \quad \text{and} \quad v_m = \phi_m(x) , \quad m = 1,2,\ldots .$$

Thus, by (3.8) and (3.9), the outer expansions (3.4) for the sub-branches $\phi_0 \equiv 0$ are

$$(3.10) \qquad \phi = \left(\sum_{m=1}^{\infty} \frac{g_m^o}{\lambda - \lambda_m} \phi_m \right) \delta + O(\delta^2) ,$$

where the coefficients g_m^o are given by

$$(3.11) \qquad g_m^o \equiv \langle g, \phi_m \rangle , \qquad m = 1,2,\ldots .$$

Thus if $g_1^o \neq 0$, which is equivalent to the condition (2.7), then the outer expansions (3.10) are unbounded as $\lambda \to \lambda_1$.

In particular, if $g_1^o < 0$, then $\phi \to \infty$ as $\lambda \to \lambda_1$ from below. The sign of ϕ is switched if $\lambda \to \lambda_1$ from above. Sketches of these outer expansions are indicated by the curves O_o in Figure 3.1.

For the sub-branches $\phi_0 = \pm\Psi$ given by (3.3), the coefficients in the differential equations (3.6) for v_m are analytic functions of ε. Therefore we determine the eigenvalues and eigenfunctions in the form

$$(3.12)\quad v_m(x;\varepsilon) = \sum_{j=0}^{\infty} v_{m,j}(x)\varepsilon^j\ , \qquad \sigma_m(\varepsilon) = \sum_{j=0}^{\infty} \sigma_{m,j}\varepsilon^j\ ,$$
$$m = 1,2,\ldots\ .$$

By analyzing the resulting system of linear problems, we obtain

$$v_m = \phi_m + O(\varepsilon^2)\ , \qquad m = 1,2,\ldots\ ,$$
$$(3.13)\quad \sigma_1 = -\frac{\lambda_1}{2}\varepsilon^2 + O(\varepsilon^3)\ ,$$
$$\sigma_m = \lambda_1 - \lambda_m + O(\varepsilon)\ , \qquad m = 2,3,\ldots\ .$$

Substituting these results in (3.7) and (3.8), and then into (3.4), we finally obtain,

$$(3.14)\quad \phi = \pm\Psi + \left[-\frac{2g_1^o\phi_1}{\lambda\varepsilon^2} + \sum_{m=2}^{\infty}\frac{g_m^o}{\lambda_1(1-m^2)}\phi_m + O(\varepsilon)\right]\delta$$
$$+ O(\delta^2)\ .$$

The two outer expansions given by (3.14) are unbounded as $\varepsilon \to 0$ ($\lambda \to \lambda_1$) if $g_1^o \neq 0$. They are sketched in Figure 3.1 for $g_1^o < 0$, and they are labelled as O_1. The bifurcation branch is the light curve.

The Inner Expansions. In the notation of Section 2, $F_{zz} = -\lambda \sin z$, and $F_{zzz} = -\lambda \cos z$, for the modified Elastica (3.1). Thus it follows from (2.14) and (2.18) that $Q = 0$, $C \neq 0$, and the nonlinearity is cubic at the bifurcation point $\lambda = \lambda_1$. Then, the inner expansions are of the form

$$\lambda = \lambda_1 + \xi\mu^2 + 0(\mu^3) , \tag{3.15a}$$

$$\phi(x;\lambda(\mu),\delta(\mu)) \equiv z(\mu) = \sum_{j=1}^{\infty} z_j \mu^j , \tag{3.15b}$$

where $\delta = \mu^3$. By inserting (3.15) into (3.1), we get

$$Lz_1 \equiv z_1'' + \lambda_1 z_1 = 0 , \qquad z_1'(0) = z_1'(1) = 0 ; \tag{3.16a}$$

$$Lz_2 = 0 , \qquad z_2'(0) = z_2'(1) = 0 ; \tag{3.16b}$$

$$Lz_3 = \frac{1}{6}\left[\lambda_1 z_1^3 - 6\xi z_1 + 6g\right] , \qquad z_3'(0) = z_3'(1) = 0 . \tag{3.16c}$$

.....

Equations (3.16a,b) imply that

$$z_1 = A\phi_1 , \qquad z_2 = B\phi_2 . \tag{3.17}$$

From the solvability condition for (3.16c), the amplitude A must satisfy the cubic,

$$A^3 - \frac{4}{\lambda_1}\xi A + 4\frac{g_1}{\lambda_1} = 0 . \tag{3.18}$$

This equation has a real root $A_1(\xi)$ for all ξ. It is positive (negative) if g_1 is negative (positive). Furthermore we have

$$\begin{aligned} |A_1(\xi)| &\to \infty , \quad \text{as} \quad \xi \to \infty \\ A_1(\xi) &\to 0 , \quad \text{as} \quad \xi \to -\infty . \end{aligned} \tag{3.19}$$

$A_1(\xi)$ is the unique root of (3.18) if

$$\xi < \xi_o \equiv 3\left(\frac{\lambda_1 g_1^2}{16}\right)^{1/3} .$$

If ξ exceeds ξ_o, then there are two additional roots $A_2(\xi)$ and $A_3(\xi)$. Their sign is opposite to the sign of A_1. In addition, these roots satisfy

$$\begin{aligned} &A_2(\xi) \to 0 , \quad \text{and} \quad |A_3(\xi)| \to \infty , \quad \text{as } \xi \to \infty , \\ &A_2(\xi_o) = A_3(\xi_o) . \end{aligned} \tag{3.20}$$

Thus, for each real root of (3.18) we have, from (3.15), (3.17), and $\delta = \mu^3$, an inner expansion

$$\begin{aligned} \lambda &= \lambda_1 + \xi\delta^{2/3} + 0(\delta) , \\ z_{(i)} &= A_i(\xi)\phi_1\delta^{1/3} + 0(\delta^{2/3}) , \quad i = 1,2,3 . \end{aligned} \tag{3.21}$$

Matching Conditions. There are four outer expansions given by (3.10) and (3.14), and one, two, or three inner expansions given by (3.21) corresponding to the bifurcation point λ_1. The matching conditions of the method of matched asymptotic expansions determine how these expansions are connected.

The matching conditions are obtained by expressing the outer expansions, given generically by (3.4), near λ_1 in terms of the inner variables $\lambda(\mu)$, and $\delta(\mu) = \mu^3$. This gives

$$\phi(x;\lambda(\mu),\delta(\mu)) = \sum_{j=0}^{\infty} \phi_j(x;\lambda(\mu))\mu^{3j} . \tag{3.22}$$

Then (3.22) is expanded in a power series in μ as

$$\phi = \sum_{k=0}^{\infty} \eta_k \mu^k . \tag{3.23}$$

This gives a representation of the outer expansions near λ_1. For the outer expansions (3.10), $\eta_1 = (g_1^o/\xi)\phi_1$, and for the outer expansions (3.14), we have

$$\eta_1 = \left[\pm 2(\xi/\lambda_1)^{1/2} - g_1^o/(2\xi)\right]\phi_1 \ . \tag{3.24}$$

We have used (3.3) to obtain (3.24).

The matching conditions are derived by assuming that there is a common interval in λ, that approaches λ_1 as $\mu \to 0$, in which the outer expansions (3.23) and the inner expansions (3.15b) are valid. It then follows that

$$\lim_{|\xi|\to\infty} (\eta_k - z_k) = 0 \ , \qquad \text{for } k = 1,2,\ldots \ . \tag{3.25}$$

Discussions of matching are given in [14,18,19], and elsewhere.

To apply (3.25) we require the asymptotic expansions as $|\xi| \to \infty$ of $z_k(\xi)$. Since the amplitude A of z_1 is a root of (3.18), we have, using (3.19) and (3.20), that

$$\begin{aligned} &\lim_{\xi\to\infty} A_1 = \lim_{\xi\to\infty} A_3 = 2(\xi/\lambda_1)^{1/2} \ , \\ &\lim_{\xi\to\infty} A_2 = \lim_{\xi\to-\infty} A_1 = g_1/\xi \ . \end{aligned} \tag{3.26}$$

The matching condition (3.25) with k = 1 then shows that the inner and outer expansions match as indicated in Figure 3.1. The inner expansions are represented by dashed lines.

The uniform expansions are sketched in Figure 3.2. The upper branch in Figure 3.2 is in qualitative agreement with column buckling experiments.

4. Weak Imperfections.

The analysis in the preceding section is not valid if $g_1^o = 0$; that is, if the imperfections have no projection into the eigenfunction of the bifurcation point. This is equivalent to the condition (2.7) for the general nonlinear equation analyzed in Section 2. We now modify the analysis of Section 3, and consider imperfections with $g_1^o = 0$.

For $g_1^o = 0$, the four outer expansions are given by, c.f. (3.10) and (3.14),

$$\phi = U(x;\lambda)\delta + 0(\delta^2) , \qquad \text{for } \phi_0 \equiv 0 , \tag{4.1}$$

$$\phi = \pm\Psi + U(x;\lambda_1)\delta + 0(\delta^2) , \qquad \text{for } \phi_0 = \pm\Psi , \tag{4.2}$$

where U is defined by

$$U(x;\lambda) \equiv \sum_{m=2}^{\infty} \frac{g_m^o}{\lambda - \lambda_m} \phi_m(x) . \tag{4.3}$$

Since U is bounded as $\lambda \to \lambda_1$, the leading term of the outer expansion (4.1) is bounded. The higher order terms may be unbounded as $\lambda \to \lambda_1$.

An analysis of the inner expansions and the matching conditions, which we do not present, shows that $\delta = \mu$. Furthermore $\lambda(\mu)$ and $z(x;\mu)$ have the power series representations

$$\begin{aligned} \lambda(\mu) &= \lambda_1 + \xi\delta + 0(\delta^2) , \\ z(x;\mu) &= U(\lambda_1)\delta + 0(\delta^2) . \end{aligned} \tag{4.4}$$

There is only one inner expansion (4.4), and it matches with the two outer expansions (4.1) corresponding to the

bifurcation solution $\phi_0 \equiv 0$. Then the composite expansion of (4.1) and (4.4) is (see Section 2),

$$\phi = \phi^* \equiv U(x;\lambda)\delta + O(\delta^2) \ . \tag{4.5}$$

If $g_2^o \neq 0$, then the outer expansion (4.1) is unbounded as $\lambda \to \lambda_2$. Perturbations of bifurcations occur at λ_2 similar to those at λ_1 when $g_1^o \neq 0$.

We now demonstrate that when $g_1^o = 0$, there are solutions of the perturbed problem (3.1) that bifurcate from the perturbed solution (4.5). That is perturbations of bifurcations of the Elastica with $g_1^o = 0$ maintain the bifurcation. This result may not be true for more general nonlinear problems. We linearize (3.1) about the perturbed solution $\phi^*(x;\lambda,\delta)$ given by (4.5) to obtain

$$v'' + \lambda(\cos \phi^*)v = 0 \ , \qquad v'(0) = v'(1) = 0 \ , \tag{4.6}$$

$$\langle v,v \rangle = 1 \ .$$

The bifurcation points of ϕ^* are the values of $\lambda(\delta)$ for which (4.6) has solutions $v(x;\delta) \not\equiv 0$. We observe that for $\delta = 0$, $\phi^* \equiv 0$, and (4.6) is reduced to the linearized problem of the Elastica. Consequently, we have $\lambda(0) = \lambda_1$ and $v(x;0) = \phi_1(x)$. Thus for $\delta \neq 0$, we seek solutions of (4.6) near $\lambda = \lambda_1$ in the form

$$\begin{aligned} \lambda &= \lambda_1 + \Lambda_1\delta + \frac{\Lambda_2}{2}\delta^2 + \ldots \ , \\ v &= \phi_1 + v_1\delta + \frac{v_2}{2}\delta^2 + \ldots \ . \end{aligned} \tag{4.7}$$

We insert (4.7) into (4.5) and (4.6) to obtain sequence of linear problems to determine the coefficients in (4.7).

The first two problems are,

$$(4.8a)\qquad Lv_1 \equiv v_1'' + \lambda_1 v_1 = -\Lambda_1\phi_1 \ , \quad v_1'(0) = v_1'(1) = 0 \ ,$$

$$<v_1,\phi_1> = 0 \ ;$$

$$(4.8b)\qquad Lv_2 = -\Lambda_2\phi_1 + \lambda_1\phi_1[U(x;\lambda)]^2 - 2\Lambda_1 v_1 \ ,$$

$$v_2'(0) = v_2'(1) = 0 \ , \quad <\phi_1,v_2> + <v_1,v_1> = 0 \ .$$

The solvability condition applied to the inhomogeneous term in (4.8a) yields $\Lambda_1 = 0$, and hence (4.8a) implies that $v_1 \equiv 0$. Then the solvability condition applied to (4.8b) gives

$$(4.9)\qquad \Lambda_2 = \lambda_1 \int_0^1 U^2(x;\lambda_1)\phi_1^2(x)dx > 0 \ .$$

The coefficient Λ_2 can be explicitly evaluated in terms of the imperfection coefficients g_n^o, by inserting U from (4.3) into (4.9) and evaluating the resulting integrals.

Thus the bifurcation point $\lambda = \lambda_1$ of the Elastica theory is shifted to

$$(4.10)\qquad \lambda(\delta) = \lambda_1 + (\Lambda_2/2)\delta^2 + 0(\delta^3) \ ,$$

which is a bifurcation point of the perturbed solution (4.5). Since $\Lambda_2 > 0$, it is shifted to the right of λ_1. The perturbed solution that bifurcates from (4.10) can be obtained by the standard perturbation method in a new amplitude parameter. Equivalently it can be obtained by an appropriate analysis of the outer expansion (4.2).

5. Forced Oscillations as a Singular Perturbation of Bifurcation.

The forced nonlinear oscillations of many one-degree-of-freedom systems are described by

$$\lambda^2 y'' + y + f(y) - \delta g(t) = 0 \ , \tag{5.1}$$

where $\delta g(t)$ is the forcing function, and the restoring force $f(y)$ is a nonlinear function that satisfies

$$f(0) = f'(0) = 0 \ . \tag{5.2}$$

The amplitude δ of the forcing function is assumed to be small, $g(t)$ is a prescribed periodic function of period 2π, t is a scaled time variable, and λ is the forcing frequency in the original (unscaled) time. The forcing function is treated as an imperfection of the bifurcation problem

$$\lambda^2 y'' + y + f(y) = 0 \ , \tag{5.3}$$

which describes the free, nonlinear oscillations of the system. We apply the method outlined in Section 2 to obtain periodic solutions of (5.1) of period 2π that are near periodic solutions of period 2π of (5.3). Thus we solve (5.1) and (5.3) subject to the periodicity conditions,

$$y(0) = y(2\pi) \ , \qquad y'(0) = y'(2\pi) = 0 \ . \tag{5.4}$$

For simplicity, we shall assume that $\delta \geq 0$.

We observe that $y \equiv 0$ is a periodic solution of (5.3) for all values of λ. Other periodic solutions of period 2π of (5.3) branch from $y \equiv 0$. They are obtained by standard perturbation methods as

$$y = \varepsilon \cos t + \varepsilon^2 \frac{f''(0)}{6}[-3 + 2\cos t + \cos 2t] + 0(\varepsilon^3)$$

(5.5)

$$\lambda = 1 + \lambda_{(2)}\varepsilon^2 + 0(\varepsilon^3)$$

where the small parameter ε is defined by $\varepsilon \equiv y(0)$, and $\lambda_{(2)}$ is defined by

$$\lambda_{(2)} = \frac{1}{24}\{-5[f''(0)]^2 + 3f'''(0)\} . \tag{5.6}$$

The bifurcation is supercritical (subcritical) from the bifurcation point $\lambda = 1$ if $\lambda_{(2)} > 0$ (< 0). If $\lambda_{(2)} = 0$, then higher order terms must be calculated to determine the direction of bifurcation. For subcritical bifurcation, the frequency λ of the free oscillations decreases with amplitude, i.e. the nonlinearity is "soft".

There are four bifurcation sub-branches y_0 emanating from the bifurcation point. There is an outer expansion corresponding to each sub-branch. They are obtained by applying the method discussed in Sections 2 and 3. If we assume that the forcing function is given by,

$$g(t) = \sum_{m=1}^{\infty} g_m \cos mt , \qquad g_m \equiv \frac{1}{\pi}\int_0^{2\pi} g(t)\cos mt\,dt , \tag{5.7}$$

then the outer expansions are given by

$$y = \left(\sum_{m=1}^{\infty} \frac{g_m}{1 - \lambda^2 m^2}\cos mt\right)\delta + 0(\delta^2) , \tag{5.8}$$

for $y_0 \equiv 0$, and by

$$y = \varepsilon \cos t + 0(\varepsilon^2) + \left[\frac{g_1}{\lambda_{(2)}\varepsilon^2}\cos t + 0(1)\right]\delta + 0(\delta^2) \tag{5.9}$$

for y_0 given by (5.5). Thus the outer expansions are unbounded at $\lambda = 1$ if $g_1 \neq 0$. We shall assume that $g_1 \neq 0$.

If $g_1 = 0$, an analysis analogous to Section 4 is required.

Similarly, it can be shown that the inner expansions are given by

$$\delta = \mu^3 , \quad \lambda = 1 + \xi\delta^{2/3} + 0(\delta) , \tag{5.10}$$
$$z = (A(\xi)\cos t)\delta^{1/3} + 0(\delta^{2/3}) ,$$

where $A(\xi)$ is a real root of the cubic

$$A^3 - \frac{4\xi}{\lambda_{(2)}} A - \frac{2g_1}{\lambda_{(2)}} = 0 . \tag{5.11}$$

The matching conditions then determine how to connect the inner and outer expansions. The resulting uniform expansions are sketched in Figure 5.1 for subcritical bifurcation with $g_1 < 0$. The singular point $\lambda = 1$, $y = 0$ of the free oscillations is shifted to the singular point at $\lambda = 1 + \xi_o\delta^{2/3} + 0(\delta)$ of the perturbed problem.

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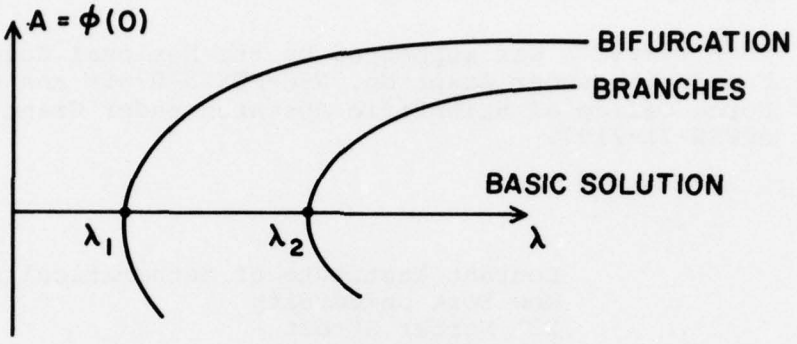


Figure 1.1

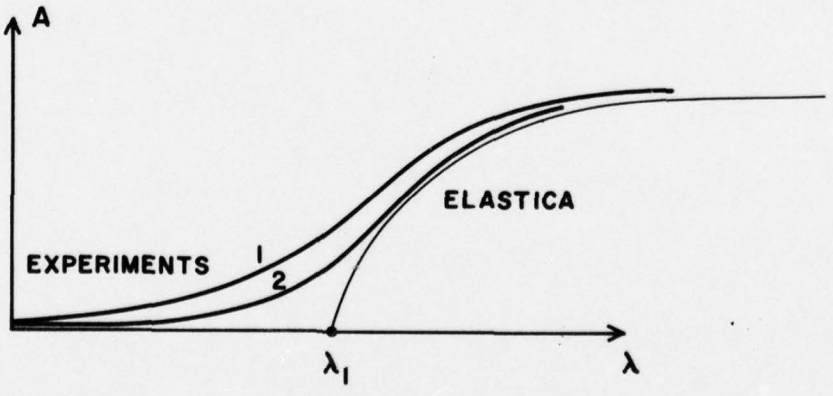


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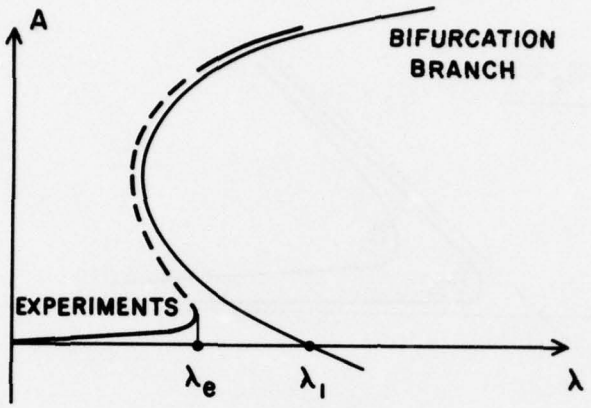


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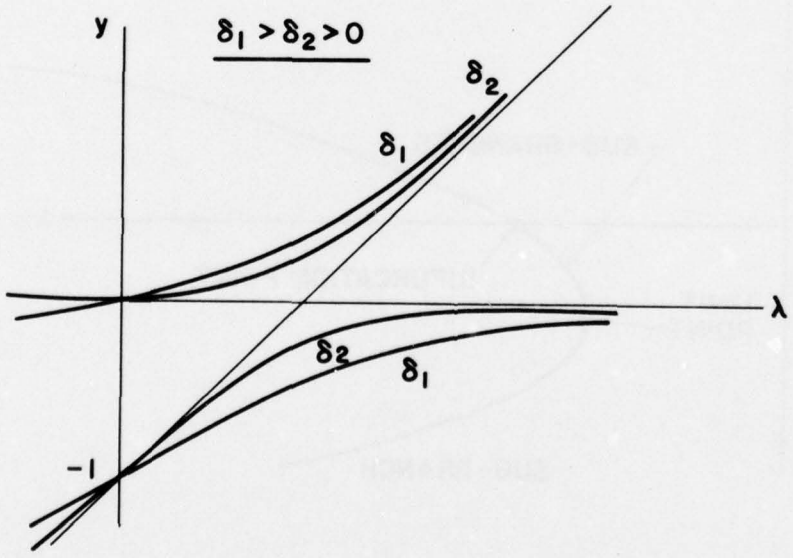


Figure 2.1a

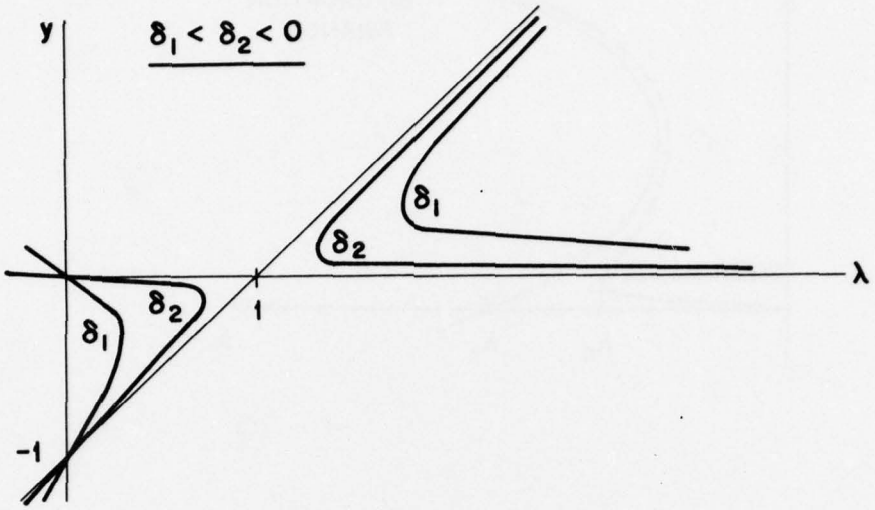


Figure 2.1b

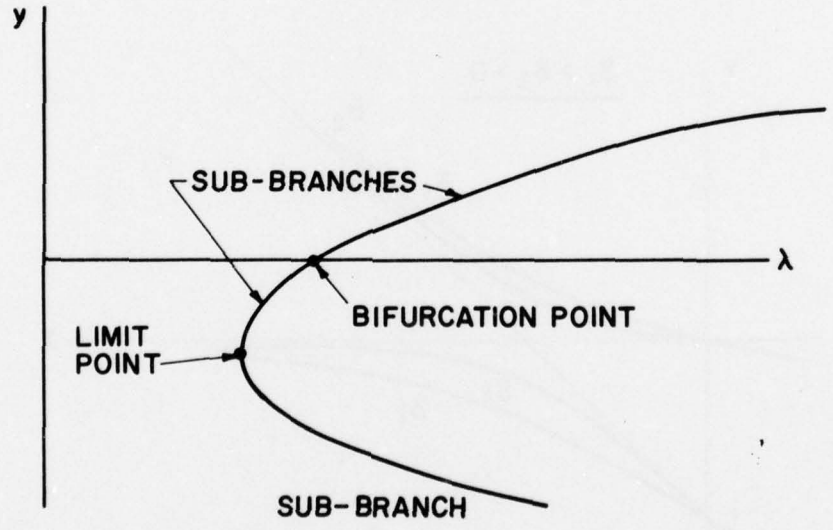


Figure 2.2

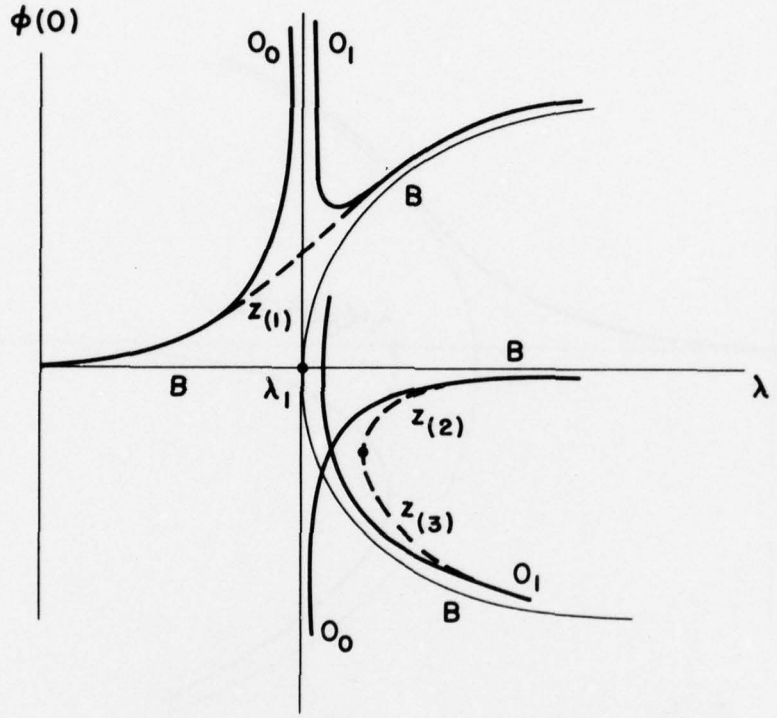


Figure 3.1

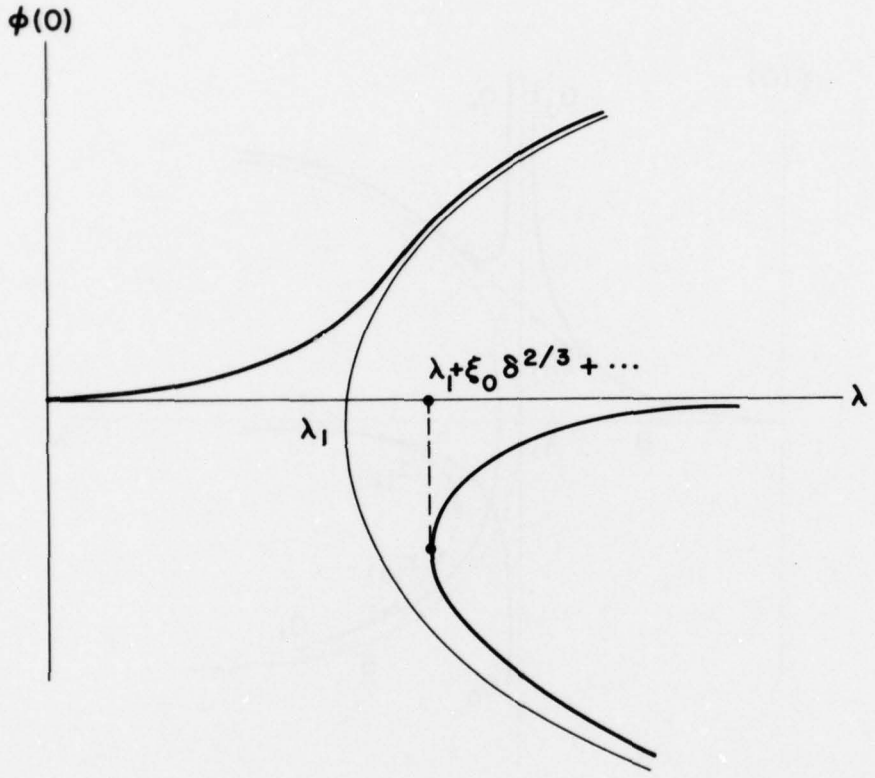


Figure 3.2

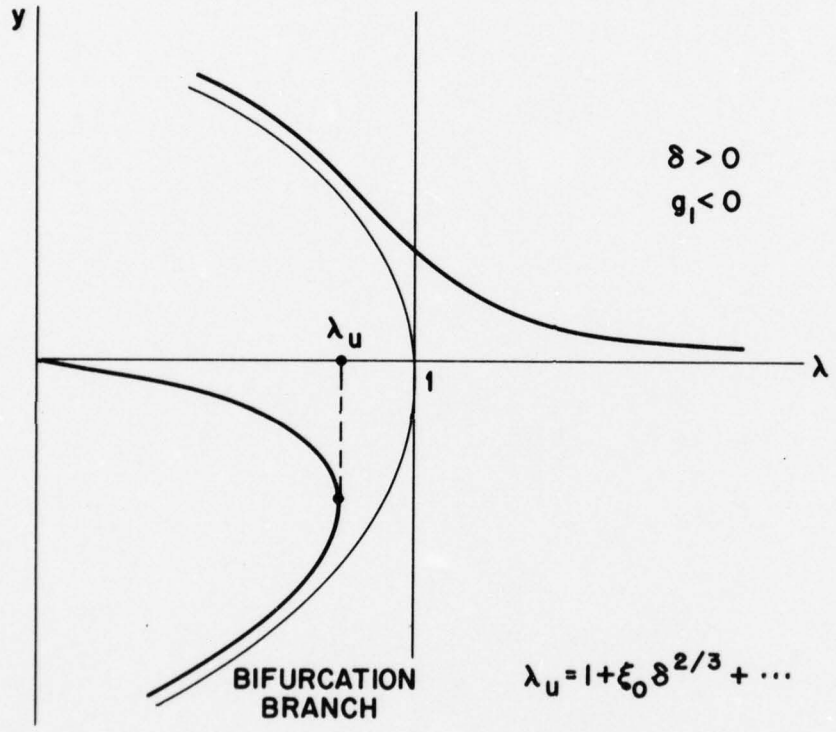


Figure 5.1

Bifurcation Problems for Nonlinearly Elastic Structures

Stuart S. Antman

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1. Introduction

In this article we examine both the local and global behavior of bifurcating solutions of boundary value problems for the quasilinear systems of ordinary differential equations that describe the equilibrium states of nonlinearly elastic rods and shells. We are particularly interested in examining the critical role of material behavior in determining the nature of solutions.

We limit our attention to geometrically exact theories of nonlinearly elastic structures. To explain what is meant by such theories and to distinguish them from the far more intensively cultivated nonlinear technical theories of elastic structures, it is first worthwhile to trace the highlights of the historical development of theories of elastic structures.

The elastica is the simplest geometrically exact model for the planar equilibrium of a flexible rod. The configuration of an elastica is described by an inextensible, smooth plane curve. The theory of the elastica, which was developed by Jas. Bernoulli, D. Bernoulli, and Euler in the first half of the eighteenth century (cf. [59]), rests upon the fundamental law that the resultant force and moment on every piece of the rod must be zero for the rod to be in equilibrium and upon the constitutive assumption that the bending moment is proportional to the curvature. If the only loads applied to the rod are compressive terminal forces (which must be collinear and of equal magnitude for equilibrium), then the equilibrium of the elastica is governed by the semilinear ordinary differential equation

$$[B(s)\theta']' + p \sin\theta = 0, \tag{1.1}$$

where s is the arc length parameter of plane curve specifying the elastica's configuration, the prime denotes differentiation with respect to s, $\theta(s)$ is the angle between the tangent to the curve at s and the direction of the applied load, p is the prescribed magnitude of the applied

load, and B is the prescribed bending stiffness $(B > 0)$. That no geometrical approximations are made in the formulation of this equation is responsible for the presence of the nonlinear function sin in (1.1) and for the characterization of (1.1) as geometrically exact. Although (1.1) is nonlinear, it does not describe the behavior of a nonlinearly elastic rod because the constitutive equation $M(s) = B(s)\theta'(s)$, which gives the bending moment M as a function of the curvature θ', is linear. If we replace this relation with the nonlinear constitutive equation $M(s) = \hat{M}(\theta'(s),s)$ where $\hat{M}$ is prescribed, then corresponding to (1.1) we have the quasilinear equation

$$[\hat{M}(\theta',s)]' + p \sin\theta = 0, \tag{1.2}$$

which describes a nonlinearly elastic rod. It is interesting to observe that equations like (1.2) were not studied in mechanics until the last decade. As we shall show in Section 2, the solutions of boundary value problems for equations like (1.2) may differ markedly from those for (1.1).

The great influence of the elastica theory embodied in (1.1) may be attributed to several factors:

(i) Euler's demonstration in 1744 of the tractability of (1.1) by his brilliant qualitative analysis [27] of all the solutions of this equation when B = const.

(ii) The ability of nineteenth and twentieth century scientists to solve boundary value problems for (1.1) with B = const. in terms of elliptic functions.

(iii) The central role played by linearized versions of (1.1) in structural engineering.

In 1859, Kirchhoff [36] showed how to generalize (1.1)

to account for the combined flexure and torsion of an inextensible rod in space. The resulting theory yields a system of equations completely analogous to those governing the motion of a heavy rigid body with one point fixed. This theory, called Kirchhoff's kinetic analogy, is certainly tractable when it reduces to any of the classical integrable cases. The most important integrable case corresponds to that in which the cross-section of the rod has two axes of symmetry. In this case the governing equations may be solved in terms of elliptic functions. Moreover this case (as well as major generalizations of it) can be treated qualitatively along the lines laid down by Euler (cf. [12]). Kirchhoff's kinetic analogy is geometrically exact as is strongly suggested by the welter of trigonometric functions that appear in its governing equations.

The historical development of nonlinear plate and shell theories has not been as felicitous as that of rod theories up to the creation of Kirchhoff's kinetic analogy. Kirchhoff established a satisfactory linear theory of plates begun by Navier. The development of a geometrically exact theory was no doubt hampered by difficulties of describing surface geometry and strain that do not arise in rod theories, by the inappropriateness of constraints such as inextensibility for plate and shell theories, and by a lack of interest occ sioned by the likelihood thac the resulting nonlinear equations would be mathematically intractable. (Until recently, this pessimism was well justified.)

A popular nonlinear plate theory was finally developed by v. Kármán [32] in 1910. In this theory, the vertical deflection w and the stress function φ of a plate subject to a planar load on its edge with a load parameter λ is governed by the semilinear system of partial differenial equations

$$\begin{aligned} \Delta\Delta w &= \lambda[f,w] + [\varphi,w], \\ \Delta\Delta\varphi &= -[w,w], \end{aligned} \tag{1.3}$$

where $\Delta\Delta$ is the biharmonic operator, $[u,v] \equiv u_{xx}v_{yy} + u_{yy}v_{xx} - 2u_{xy}v_{xy}$, and f is a given function. These equations are "derived" by introducing into the standard procedures of mechanics a number of geometric approximations that are roughly analogous to the replacement of the sin in (1.1) by its cubic approximation. Consequently this theory is not geometrically exact (as is suggested by the absence of trigonometrical functions from (1.3)). Moreover (1.3) is based upon a certain "linear" stress-strain law. For these reasons, this theory has never been embedded in a simple, rational (albeit formal) approximation scheme. Nevertheless the appearance of the familiar biharmonic and Monge-Ampère operators have caused this equation and its various related shell theories to attract nearly all the serious mathematical attention bestowed on nonlinear problems of plates and shells in recent years. The popularity of this theory is somewhat paradoxical: Its inadequacy as a rational theory of mechanics is not compensated for by analytic simplicity. Indeed many studies of (1.3) have demanded considerable mathematical innovation as well as technical expertise. (Cf. [19,21,62], e.g.) Because of the geometrical approximations made in equations like (1.3) their global analyses are devoid of physical interest.

During the first half of the twentieth century the problem of developing nonlinear rod theories that account for such effects as extensibility and shear in addition to flexure has not resulted in refinements of the beautiful theories of Euler and Kirchhoff, but rather to developments along the lines laid out by v. Kármán. (A survey of such theories has been given in [26].)

Advances in continuum mechanics since the Second World War have led to the development in the last few years of geometrically exact models of nonlinearly elastic rods, plates, and shells. (Cf. [3,8,44]. Like (1.2) these models are characterized by quasilinear differential equations involving arbitrary constitutive functions. Recent advances

in analysis have made at least some of these problems tractable (in perhaps a somewhat different sense from that understood up to the middle of the twentieth century.) As we shall show, the underlying physics prevents the resulting mathematical problems from being readily subsumed under the available nonlinear operator theories.

The arbitrary constitutive functions that appear in the equations for nonlinearly elastic structures (like $\hat{M}$ in (1.2)) may be regarded as parameters in an infinite dimensional space. (In (1.1) with B = const. there is but one essential parameter: p/B. Geometric and material parameters have been suppressed in (1.3).) By allowing the constitutive functions to vary within physically reasonable limits, we can use the bifurcation process not only as an end in itself but also to illuminate the nature of material behavior. Indeed, we shall show that solutions of problems for nonlinearly elastic structures may well exhibit important new effects, which more than compensate for the analytic difficulties presented by these theories.

2. Planar Buckling of Straight Rods

In this section we generalize the elastica theory to account for extension and shear as well as flexure and to permit a large class of nonlinear constitutive relations. We show that the richness of this theory is reflected in the richness of the modes of behavior of the solutions.

Let $\{\underset{\sim}{i},\underset{\sim}{j},\underset{\sim}{k}\}$ be an orthonormal basis for $\mathbb{E}^3$. The configuration of a rod undergoing a planar deformation is given by the pair of vector functions

$$(2.1)\quad s \in [0,1] \longmapsto \underset{\sim}{r}(s), \quad \underset{\sim}{b}(s) \in \operatorname{span}\{\underset{\sim}{i},\underset{\sim}{j}\}, \quad |\underset{\sim}{b}(s)| = 1.$$

(s may be taken as the (scaled) arc length parameter of the line of centroids of the rod in its reference configuration, $\underset{\sim}{r}$ as giving the deformed configuration of this curve, and $\underset{\sim}{b}$ as characterizing the deformed configuration

of a cross-section. See Fig. 2.1.) We denote differentiation with respect to s by a prime. We define θ and $\underset{\sim}{a}$ by

$$\underset{\sim}{a}(s) = \cos\theta(s)\underset{\sim}{i} + \sin\theta(s)\underset{\sim}{j}, \tag{2.2a}$$

$$\underset{\sim}{b}(s) = -\sin\theta(s)\underset{\sim}{i} + \cos\theta(s)\underset{\sim}{j}. \tag{2.2b}$$

(This θ gives the orientation of $\underset{\sim}{b}$; it need not be the same as the θ of (1.1).) We define strains ξ and η by

$$\underset{\sim}{r}' = (1+\xi)\underset{\sim}{a} + \eta\underset{\sim}{b}. \tag{2.3}$$

We require that

$$1 + \xi \equiv \underset{\sim}{r}'\cdot\underset{\sim}{a} \geq 0 \tag{2.4}$$

so that $|\underset{\sim}{r}'| > 0$ and so that $\underset{\sim}{b}(s)$ is never collinear with $\underset{\sim}{r}'(s)$. (More stringent restrictions are geometrically appropriate. Cf. [8,10]. For simplicity we do not describe them here.) The <u>strains</u> for our problem are

$$\underset{\sim}{w} = (\xi,\eta,\mu \equiv \theta') \in E \equiv (-1,\infty)\times\mathbb{R}\times\mathbb{R}. \tag{2.5}$$

Let $\underset{\sim}{n}(s)$ and $\underset{\sim}{m}(s)$ denote the resultant force and resultant couple applied to $[0,s]$ by $(s,1]$. If the only loads applied to the rod are terminal, then the equilibrium equations are

$$\underset{\sim}{n}' = \underset{\sim}{0}, \qquad \underset{\sim}{m}' + \underset{\sim}{r}'\times\underset{\sim}{n} = \underset{\sim}{0}. \tag{2.6a,b}$$

In consonance with the assumption of planarity we decompose $\underset{\sim}{n}$ and $\underset{\sim}{m}$ thus:

$$\underset{\sim}{n}(s) = \Xi(s)\underset{\sim}{a}(s) + H(s)\underset{\sim}{b}(s), \qquad \underset{\sim}{m}(s) = M(s)\underset{\sim}{k}. \tag{2.7a,b}$$

Then (2.6a) implies that

$$\underset{\sim}{n}(s) = \underset{\sim}{n}(0) = \underset{\sim}{n}(1) \equiv -\lambda\underset{\sim}{i} + \nu\underset{\sim}{j}, \tag{2.8}$$

which when coupled with (2.7c) yields

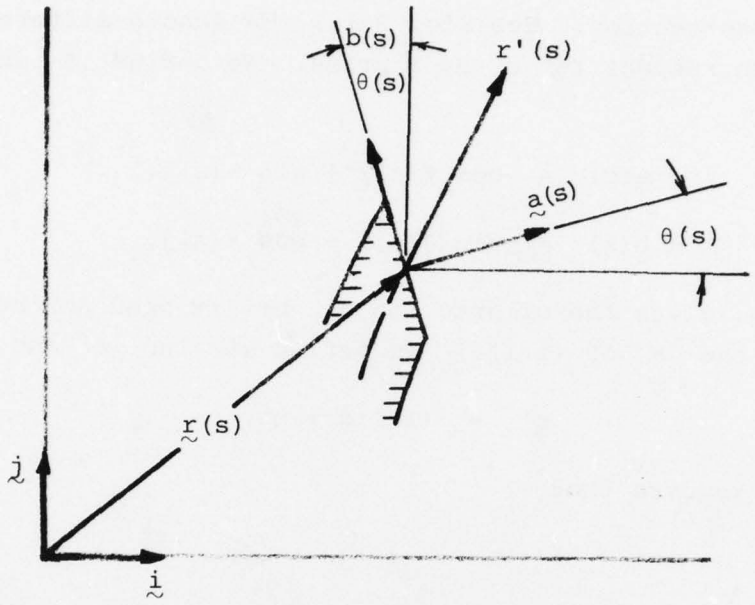


Figure 2.1

(2.9a) $\quad \Xi(s) = -\lambda \cos\theta(s) + \nu \sin\theta(s),$

(2.9b) $\quad H(s) = \lambda \sin\theta(s) + \nu \cos\theta(s).$

The substitution of (2.3), (2.7), (2.9) into (2.6b) reduces the latter to

(2.10) $\quad M' + (\lambda \sin\theta + \nu \cos\theta)(1+\xi) + (\lambda \cos\theta - \nu \sin\theta)\eta = 0.$

We introduce the constitutive functions

(2.11) $\quad (\underset{\sim}{w},s) \in E \times [0,1] \longmapsto (\hat{\Xi}(\underset{\sim}{w},s),\hat{H}(\underset{\sim}{w},s),\hat{M}(\underset{\sim}{w},s)) \in \mathbb{R}^3$

with $\hat{\Xi}$, $\hat{H}$, $\hat{M}$ in $C^4(E \times [0,1])$. Then our constitutive equations are

(2.12a,b) $\quad \Xi(s) = \hat{\Xi}(\underset{\sim}{w}(s),s), \qquad H(s) = \hat{H}(\underset{\sim}{w}(s),s),$

(2.12c) $\quad M(s) = \hat{M}(\underset{\sim}{w}(s),s).$

Denoting partial differentiation by subscripts, we require

that

$$\begin{pmatrix} \hat{\Xi}_\xi & \hat{\Xi}_\eta & \hat{\Xi}_\mu \\ \hat{H}_\xi & \hat{H}_\eta & \hat{H}_\mu \\ \hat{M}_\xi & \hat{M}_\eta & \hat{M}_\mu \end{pmatrix} \tag{2.13}$$

be positive-definite. This condition ensures that an increase in tensile force corresponds to an increase in extension, an increase in M corresponds to an increase in $\mu = \theta'$, etc. (This condition is intimately related to the strong ellipticity condition of the three-dimensional theory, cf. [8].) We also require that extreme strains must correspond to extreme resultants by

$$\hat{\Xi}(\xi,\eta,\mu,s) \longrightarrow \begin{cases} \infty & \text{as } \xi \to \infty \\ -\infty & \text{as } \xi \to -1 \end{cases}, \tag{2.14a}$$

$$\hat{H}(\xi,\eta,\mu,s) \longrightarrow \pm\infty \quad \text{as } \eta \to \pm\infty, \tag{2.14b}$$

$$\hat{M}(\xi,\eta,\mu,s) \longrightarrow \pm\infty \quad \text{as } \mu \to \pm\infty. \tag{2.14c}$$

We also assume that

$$\hat{H}(\xi,0,\mu,s) = 0, \qquad \hat{M}(\xi,\eta,0,s) = 0. \tag{2.15}$$

Relations (2.14) and the positive-definiteness of (2.13) imply that if Ξ, H, M, s are prescribed in $\mathbb{R}^3 \times [0,1]$, then the algebraic equations

$$\hat{\Xi}(\xi,\eta,\mu,s) = \Xi, \qquad \hat{H}(\xi,\eta,\mu,s) = H, \tag{2.16a,b}$$

$$\hat{M}(\xi,\eta,\mu,s) = M \tag{2.16c}$$

have a unique solution (ξ,η,μ) that we denote by

$$\xi = \tilde{\xi}(\Xi,H,M,s), \qquad \eta = \tilde{\eta}(\Xi,H,M,s), \tag{2.17a,b}$$

(2.17c) $$\mu = \tilde{\mu}(\Xi, H, M, s)$$

and that if Ξ, H, μ, s are prescribed, then (2.16a,b) has a unique solution (ξ,η) that we denote by

(2.18a,b) $$\xi = \hat{\xi}(\Xi, H, \mu, s), \qquad \eta = \hat{\eta}(\Xi, H, \mu, s).$$

We assume that

(2.19a) $$\xi(\Xi, H, \mu, s) \longrightarrow \begin{cases} \infty & \text{as } \Xi \to \infty \\ -1 & \text{as } \Xi \to -\infty \end{cases},$$

(2.19b) $$\hat{\eta}(\Xi, H, \mu, s) \longrightarrow \pm\infty \quad \text{as } H \to \pm\infty.$$

(Cf. (2.14).)

Let us substitute (2.12a,b) into (2.9). Then (2.18) implies that the resulting expressions are equivalent to

(2.20a)

$$\xi(s) = \hat{\xi}(-\lambda \cos\theta(s) + \nu \sin\theta(s),\ \lambda \sin\theta(s) + \nu \cos\theta(s),\ \theta'(s),\ s),$$

(2.20b)

$$\eta(s) = \hat{\eta}(-\lambda \cos\theta(s) + \nu \sin\theta(s),\ \lambda \sin\theta(s) + \nu \cos\theta(s),\ \theta'(s),\ s).$$

The substitution of (2.12c) and (2.20) into (2.10) yields

(2.21) $$[\hat{M}(\hat{\xi},\hat{\eta},\theta',s)]' + \lambda[(1+\hat{\xi})\sin\theta + \hat{\eta}\cos\theta] + \nu[(1+\hat{\xi})\cos\theta - \hat{\eta}\sin\theta] = 0,$$

where the arguments of $\hat{\xi}$ and $\hat{\eta}$ are those listed in (2.20).

We examine the boundary conditions

(2.22) $$\theta(0) = 0 = \theta(1),$$

(2.23) $$\underset{\sim}{r}(0)\cdot\underset{\sim}{j} = 0,$$

(2.24) $$\underset{\sim}{n}(1)\cdot\underset{\sim}{i} = -\lambda,$$

and either

(2.25) $$\underset{\sim}{r}(1)\cdot\underset{\sim}{j} = 0$$

or

(2.26) $$\underset{\sim}{n}(1)\cdot\underset{\sim}{j} \equiv \nu = 0.$$

Note that ν, which appears in (2.21), is not prescribed by the boundary conditions (2.22)-(2.25). (This is a "statically indeterminate" problem.) To use (2.21) in this case, we note that (2.3), (2.23), (2.25) imply that

(2.27) $$\hat{y}[\theta,\lambda,\nu] \equiv \int_0^1 [(1+\hat{\xi})\sin\theta + \hat{\eta}\cos\theta]ds = 0.$$

Now

(2.28)

$$\hat{y}_\nu[\theta,\lambda,\nu] =$$

$$\int_0^1 \{[\hat{\xi}_\Xi \sin\theta + \hat{\xi}_H \cos\theta]\sin\theta + [\hat{\eta}_\Xi \sin\theta + \hat{\eta}_H \cos\theta]\cos\theta\}ds > 0$$

as a consequence of the positive-definiteness of (2.13). Relations (2.19) imply that

(2.29) $$\hat{y}[\theta,\lambda,\nu] \to \pm\infty \quad \text{as} \quad \nu \to \pm\infty.$$

Relations (2.28), (2.29) support a global and a local implicit function theorem to the effect that (2.27) has a unique continuously differentiable solution

(2.30a) $$(\theta,\lambda) \in C^1([0,1]) \times [0,\infty) \longmapsto \hat{\nu}[\theta,\lambda]$$

with

(2.30b) $$\hat{\nu}[0,\lambda] = 0$$

by (2.15). Let us define $\hat{\nu} = 0$ when (2.26) holds. Then our boundary value problem consists of the quasilinear, second-order, ordinary-functional differential equation (2.21) with ν replaced by $\hat{\nu}[\theta,\lambda]$ subject to (2.22). (Compare this with (1.1).) This problem has the trivial solution pair $(0,\lambda)$.

We wish to determine to what extent the solution branches of our boundary value problem inherit the properties of the eigenfunctions of the linearization of this boundary value problem about the trivial solution. We accordingly examine this linearization. It is given by

(2.31a) $$[P(s,\lambda)\psi']' + \lambda Q(s,\lambda)\psi = -Q(s,\lambda)\nu_\theta[0,\lambda]\psi,$$

(2.31b) $$\psi(0) = 0 = \psi(1),$$

where

(2.31c) $$P(s,\lambda) = \hat{M}_\mu(\hat{\xi}(-\lambda,0,0,s),0,0,s),$$

(2.31d) $$Q(s,\lambda) = 1 + \hat{\xi}(-\lambda,0,0,s) + \lambda\hat{\eta}_H(-\lambda,0,0,s),$$

(2.31e) $$\nu_\theta[0,\lambda]\psi = -\int_0^1 Q(s,\lambda)\psi(s)\,ds \Big/ \int_0^1 \hat{\eta}_H(-\lambda,0,0,s)\,ds$$

if (2.25) holds,

(2.31f) $$\nu_\theta[0,\lambda]\psi = 0 \quad \text{if (2.26) holds.}$$

Let us first examine (2.31a,b) when (2.31f) holds. With an eye toward applying the Sturmian theory (cf. [31], e.g.) we multiply (2.31) by $\lambda^{-\alpha}$ where $\alpha > 0$ and note the following results.

i) If $\hat{\xi} = 0$, $\hat{\eta} = 0$ so that the rod is inextensible and unshearable, then P is independent of λ and $Q = 1$. (In this case (2.31a) reduces to the linearization of

(1.2) about $\theta = 0$.) The Sturmian theory then implies that the problem has eigenvalues $\lambda_0 < \lambda_1 < \ldots$ with $\lambda_0 > 0$ and corresponding eigenvectors $\psi_0, \psi_1, \ldots$ with ψ_k having exactly k simple zeros in $(0,1)$.

ii) If the rod is extensible, then the three-dimensional interpretation of our resultants $\hat{\Xi}$, $\hat{H}$, $\hat{M}$ (which we do not describe here) implies that

(2.32) $\lambda[1 + \hat{\xi}(-\lambda,0,0,s)] \big/ P(s,\lambda) \to 0$ as $\lambda \to \infty$,

(2.33) if there is a number $\Lambda \geq 0$ such that $[\lambda^{-\alpha} P(s,\lambda)]_\lambda \leq 0$ for $\lambda \geq \Lambda \geq 0$, then $\alpha \geq 1$,

(2.34) if $\alpha \geq 1$, then $\lambda^{1-\alpha}[1 + \hat{\xi}(-\lambda,0,0,s)] \to 0$ as $\lambda \to \infty$.

We then have the following alternatives:

a) If $\hat{\eta} = 0$, then $\lambda^{1-\alpha} Q(s,\lambda) \to 0$ and $\lambda Q(s,\lambda)/P(s,\lambda) \to 0$ as $\lambda \to \infty$. The Sturmian theory implies that the eigenvalues are confined to a bounded interval of $(0,\infty)$. Mild restrictions on $\hat{\Xi}$ prevent the existence of an infinite number of eigenvalues in this case. When such conditions obtain, there are but a finite number of corresponding eigenfunctions. The number of nodes of these eigenfunctions does not correspond to the ordering of the eigenvalues. Thus the relaxation of the constraint $\xi = 0$ markedly changes the character of the spectrum and the nodal structure of solutions.

b) If $\hat{\eta} \neq 0$, then the behavior of $\lambda^{1-\alpha} Q(s,\lambda)$ and $\lambda Q(s,\lambda)/P(s,\lambda)$ as $\lambda \to \infty$ (and hence the character of the spectrum and the nodal structure of solutions) devolves upon the nature of the shear

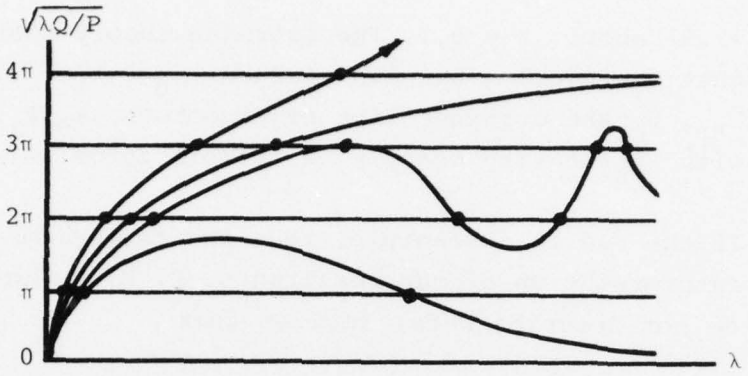


Figure 2.2. Possible forms of $\sqrt{\lambda Q/P}$. The value of λ at which $\sqrt{\lambda Q/P} = n\pi$ is an eigenvalue corresponding to an eigenfunction with $n-1$ interior zeros.

response embodied in $\lambda \mapsto \hat{\eta}_H(-\lambda,0,0,s)$. There are neither theoretical nor experimental results available to restrict the behavior of this function and so there is a rich collection of physically acceptable possibilities for the nodal structure of the eigenfunctions. This variety of response is readily observed in the constant coefficient case. The eigenvalues are solutions λ of $\lambda Q(\lambda)/P(\lambda) = n^2\pi^2$ for n a positive integer. (See Fig. 2.2.) To each solution λ of this equation (there may be many or none) there corresponds just one eigenfunction $s \mapsto \sin n\pi s$. We also note that (2.31a,b) may have negative eigenvalues. These correspond to shear instabilities, which we discuss in a more general setting in Section 6.

These results indicate that the nature of the collection of solution branches for our general quasilinear problem when (2.31f) holds may be markedly different from that for (1.1). These differences may become more pronounced

when (2.31e) holds. To gain some insight into the source of the difficulties in this case, we introduce the Prüfer transformation

$$\psi = \rho \sin \omega + \zeta, \qquad \lambda^{-\alpha} P \psi' = \rho \cos \omega, \tag{2.35a,b}$$

where

$$\zeta = \frac{\int_0^1 Q(s,\lambda)\psi(s)\, ds}{\lambda \int_0^1 \hat{\eta}_H(-\lambda,0,0,s)\, ds}$$

(2.35c)

$$= - \frac{\int_0^1 Q(s,\lambda)\rho(s) \sin \omega(s)\, ds}{\int_0^1 [1 + \hat{\xi}(-\lambda,0,0,s)]\, ds}$$

if (2.31e) holds and $\zeta = 0$ if (2.31f) holds. Then (2.31a,b) is equivalent to

$$\rho' = (\lambda^{\alpha} P^{-1} - \lambda^{1-\alpha} Q)\rho \sin \omega \cos \omega, \tag{2.36a}$$

$$\omega' = \lambda^{\alpha} P^{-1} \cos^2 \omega + \lambda^{1-\alpha} Q \sin^2 \omega, \tag{2.36b}$$

$$\rho(0) \sin \omega(0) = -\zeta = \rho(1) \sin \omega(1). \tag{2.36c}$$

By analyzing (2.36a,b) in the usual way (cf. [23], e.g.) we find that phase-plane trajectories of (2.31a) may have the general form indicated in Fig. 2.3. Candidates for solutions of (2.31a,b) correspond to trajectories beginning and ending on the line $\psi = 0$. Note that the number of zeros of ψ on the trajectory AF, which is four, does not correspond to the number of zeros of ψ', which is seven, or the number of zeros of $\psi - \zeta$, which is eight. Moreover as λ changes so does the shape of the trajectory and, in particular, the value of ζ. Thus it can well

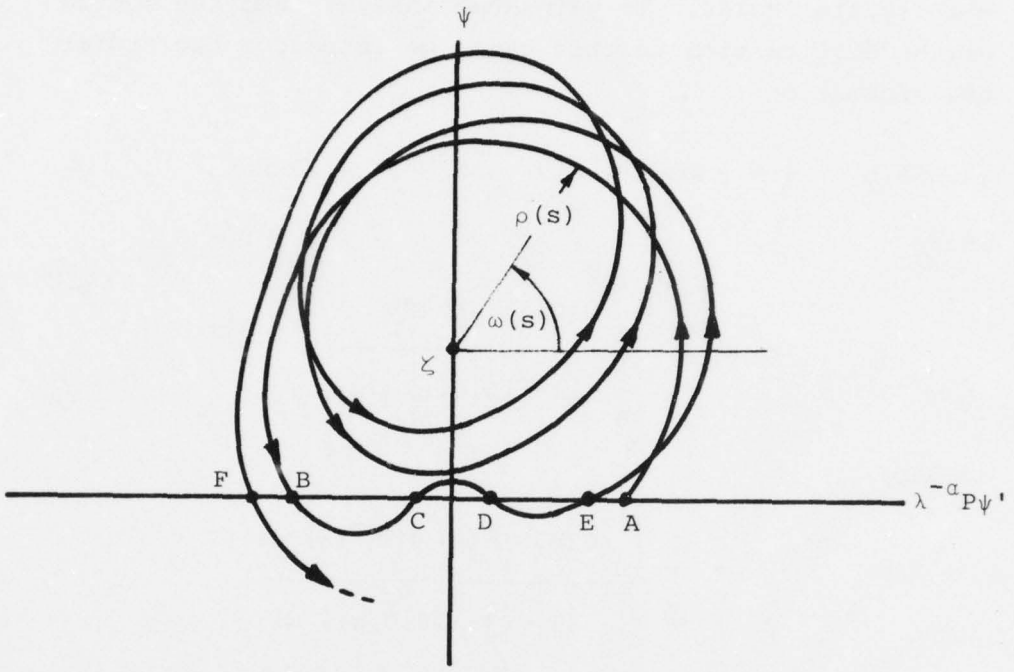


Figure 2.3. Typical trajectory of (2.31a)

happen that the number of interior zeros of ψ on AF can change, although the number of interior zeros of ψ' and $\psi - \zeta$ would not change. Now consider the boundary value problem in which (2.31b) is replaced by the conditions $\psi'(0) = 0 = \psi'(1)$. Then candidates for solutions correspond to trajectories beginning and ending on the ψ-axis. Our same analysis indicates that the zeros of ψ are singularly inappropriate for the characterization of eigenfunctions. This difficulty becomes even more pronounced if (2.31b) is replaced by mixed conditions. Note that these difficulties arise solely from the fact that ζ need not be zero. Indeed, the collection of eigenfunctions of such problems can nearly be characterized by collections of J-tuples of integers, each integer representing the number of zeros of a function such as ψ, ψ', $\psi - \zeta$,..., that is associated with the eigenfunction ψ. (When mixed boundary

conditions are used, this list of associated functions should be supplemented with appropriate linear combinations of ψ and ψ' that are related to the boundary conditions.) We call each such J-tuple a label for the eigenfunction.

Let us now return to our nonlinear boundary value problem. The global theory of bifurcation introduced by Crandall & Rabinowitz [25] and further developed by Rabinowitz (cf. [49-52]) may be applied to our problem. This theory implies that if $\bar{\lambda}$ is a nonnegative eigenvalue of (2.31a,b) of odd algebraic multiplicity, then there is a "branch" of solution pairs $(\theta,\lambda) \in C^1([0,1]) \times \mathbb{R}$ bifurcating from $(0,\bar{\lambda})$ that is unbounded in $C^1([0,1]) \times \mathbb{R}$ and/or returns to the trivial solution branch at $(0,\mu)$, where μ is another eigenvalue of (2.31a,b). If $\bar{\lambda}$ is a simple eigenvalue of (2.31a,b) and if (θ,λ) lies near $(0,\bar{\lambda})$ on the branch through $(0,\bar{\lambda})$, then θ behaves like the eigenfunction $\bar{\psi}$ of (2.31a,b) corresponding to $\bar{\lambda}$. In particular, the numbers of simple interior nodes of θ and of θ' or $M(\hat{\xi},\hat{\eta},\theta',\cdot)$ agree with those for $\bar{\psi}$ and $\bar{\psi}'$. Now if the number of simple interior nodes of θ should change, then (2.22) implies that θ must have a double zero at some $t \in [0,1]$. A standard uniqueness theorem then implies that $\theta = 0$. Thus the number of simple interior nodes of θ can only change at the trivial solution. To examine the nodal structure of M, we first set

$$\delta[\theta,\lambda] = \arctan(\nu[\theta,\lambda]/\lambda), \quad \gamma(s) = \theta(s) + \delta,$$
$$N[\theta,\lambda] = (\lambda^2 + \nu[\theta,\lambda]^2)^{1/2}. \tag{2.37}$$

For $\lambda > 0$, $\delta \in (-\pi/2, \pi/2)$. Then (2.21) implies that γ satisfies

$$[\hat{M}(\hat{\xi},\hat{\eta},\gamma',s)]' + N[(1+\hat{\xi})\sin\gamma + \hat{\eta}\cos\gamma] = 0, \tag{2.38a}$$

where the arguments of $\hat{\xi}$ and $\hat{\eta}$ are

(2.38b) $\qquad -N\cos\gamma, \quad N\sin\gamma, \quad \gamma', \quad s.$

If M should change the number of its internal nodes, then (2.22) and the positive-definiteness of (2.13) imply that either $\theta(0) = \theta'(0) = 0$, or $\theta(1) = \theta'(1) = 0$, or there is a $t \in (0,1)$ such that $\gamma'(t) = M(t) = M'(t) = 0$. In the first two cases, our previous argument shows that θ must equal zero. In the last case (2.37) and (2.38) imply that

(2.39a) $\qquad (1+\hat{\xi})\sin\gamma(t) + \hat{\eta}\cos\gamma(t) = 0$

where the arguments of $\hat{\xi}$ and $\hat{\eta}$ are

(2.39b) $\qquad -N\cos\gamma(t), \quad N\sin\gamma(t), \quad 0, \quad t.$

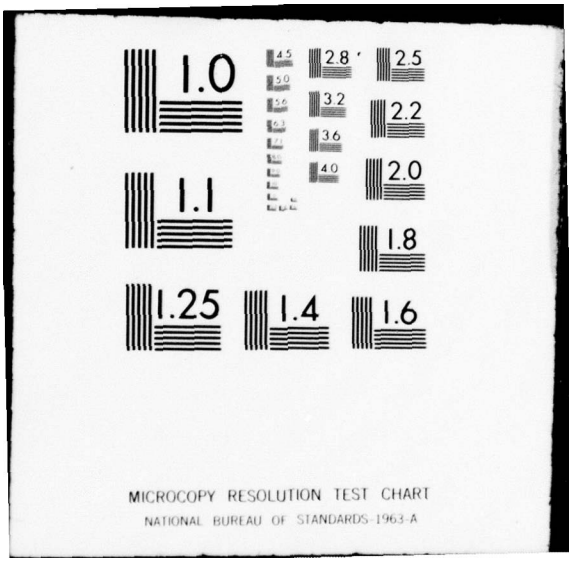
A careful analysis of (2.39) in conjunction with either (2.26) or (2.27) yields that if $|\theta(t)| \le \pi/2$ then $\theta(t) = 0$. Since $\theta'(t) = 0$, we get that $\theta = 0$ as before. We may thus conclude that on the portion of the branch from $(0,\bar{\lambda})$ to its first contact with $\{(\varphi,\lambda) : \|\varphi\|_{C^0} = \pi/2\}$ the function M has the same nodal structure as $\bar{\psi}$. (Where $|\theta|$ exceeds $\pi/2$, the rod is in tension and may suffer a shear instability. The shear instability [11] may be responsible for the break-down of the nodal structure. See Section 6.) Similar results hold for other functions depending on θ. (These are harder to prove.) Thus it is possible to show that the nodal structure of a solution branch through $(0,\bar{\lambda})$, which is characterized by a label, is inherited from that of $\bar{\psi}$ and is preserved in some large neighborhood of the trivial solution.

A simple perturbation analysis (cf. [33]), which can be readily justified (cf. [34,54,60]), shows that whether or not λ increases on a branch as it leaves a bifurcation point $(0,\bar{\lambda})$ depends on the first three derivatives of $\underset{\sim}{w} \mapsto \hat{\Xi}(\underset{\sim}{w},s)$, $\hat{H}(\underset{\sim}{w},s)$, $\hat{M}(\underset{\sim}{w},s)$ evaluated at the trivial solution. Thus the nature of material behavior can be an

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important contributor to the process by which the straight state loses its stability. (Cf. [46].)

Bibliographical Remarks. The results described in this section are based upon the much more extensive development of [16]. Modern treatments of bifurcation problems for (1.1) under statically determinate conditions are given by [24, 25,34,40,53]. Bifurcation problems for nonlinearly elastic rods having governing equation intermediate in complexity between (1.2) and (2.21), (2.29) have been treated by [29, 46,47,55].

3. Spatial Buckling of Straight Rods

In this section we indicate how the theory described in Section 2 can be extended to treat the spatial buckling of straight rods under terminal thrust and twist (Greenhill's problem, cf. [42]). These nonlinearly elastic rods can suffer flexure, torsion, extension, and shear.

In place of (2.1), we characterize the configuration of a rod by a triple of vector functions

$$(3.1) \qquad s \in [0,1] \longmapsto \underset{\sim}{r}(s),\ \underset{\sim}{b}_1(s),\ \underset{\sim}{b}_2(s) \in \mathbb{E}^3$$

with $\underset{\sim}{b}_1$ and $\underset{\sim}{b}_2$ forming an orthonormal pair. We set $\underset{\sim}{b}_3 = \underset{\sim}{b}_1 \times \underset{\sim}{b}_2$ so that $\{\underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_3\}$ form a basis for $\mathbb{E}^3$. We may represent these vectors by Euler angles θ, ψ, φ (cf. [42, §253]). We define the strains $\{y_k, u_k,\ k = 1,2,3\}$ by

$$(3.2) \qquad \underset{\sim}{r}' = y_k \underset{\sim}{b}_k,$$

$$(3.3) \qquad \underset{\sim}{b}_k' = (u_\ell \underset{\sim}{b}_\ell) \times \underset{\sim}{b}_k.$$

Here and below we sum repeated Latin indices over the range 1, 2, 3. The form of (3.3) reflects the orthonormality of $\{b_k\}$. We require that

$$(3.4) \qquad y_3 \equiv \underset{\sim}{r}' \cdot \underset{\sim}{b}_3 > 0.$$

(Cf. (2.4).)

If the only loads are terminal, then the equilibrium equations are given by (2.6), which we rewrite:

$$\text{(3.5a,b)} \qquad \underset{\sim}{n}' = \underset{\sim}{0}, \qquad \underset{\sim}{m}' + \underset{\sim}{r}' \times \underset{\sim}{n} = \underset{\sim}{0}.$$

These equations immediately imply that

$$\text{(3.6a,b)} \quad \underset{\sim}{n}(s) = \underset{\sim}{n}(1), \qquad \underset{\sim}{m}(s)\cdot\underset{\sim}{n}(1) = \underset{\sim}{m}(1)\cdot\underset{\sim}{n}(1).$$

We set

$$\text{(3.7a,b)} \quad \underset{\sim}{n}(s) = n_k(s)\underset{\sim}{b}_k(s), \qquad \underset{\sim}{m}(s) = m_k(s)\underset{\sim}{b}_k(s).$$

From (3.6) we then get

$$\text{(3.8)} \quad n_k(s) = \underset{\sim}{n}(1)\cdot\underset{\sim}{b}_k(s), \qquad m_k(s)\underset{\sim}{n}(1)\cdot\underset{\sim}{b}_k(s) = \underset{\sim}{m}(1)\cdot\underset{\sim}{n}(1).$$

(The first three equations of (3.8) are analogous to (2.9). The right sides of these equations depend upon the Euler angles.) By substituting (3.7) into (3.5), we get two more equations, which are independent of (3.8):

$$\text{(3.9)} \qquad m_3' = u_2m_1 - u_1m_2 + y_2n_1 - y_1n_2,$$

$$\text{(3.10)} \qquad u_k[m_k' + e_{k\ell m}y_\ell n_m] = 0.$$

Here $e_{k\ell m}$ is the alternating symbol.

We introduce constitutive functions

$$\text{(3.11)} \qquad (y_\ell, u_\ell, s) \longmapsto \hat{n}_k(y_\ell, u_\ell, s),\ \hat{m}_k(y_\ell, u_\ell, s)$$

so that our constitutive equations are

$$\text{(3.12)} \qquad \begin{aligned} n_k(s) &= \hat{n}_k(y_\ell(s), u_\ell(s), s), \\ m_k(s) &= \hat{m}_k(y_\ell(s), u_\ell(s), s). \end{aligned}$$

We require that the 6×6 matrix

$$(3.13)\qquad \begin{pmatrix} (\partial \hat{n}_k/\partial y_\ell) & (\partial \hat{n}_k/\partial u_\ell) \\ (\partial \hat{m}_k/\partial y_\ell) & (\partial \hat{m}_k/\partial u_\ell) \end{pmatrix}$$

be positive definite, that $(\hat{n}_1,\hat{n}_2)$ and $(\hat{m}_1,\hat{m}_2)$ each be isotropic 2-vector functions of the pair of 2-vectors (y_1,y_2) and (u_1,u_2), that

$$(3.14)\qquad \begin{aligned} n_2 &= 0 \text{ if and only if } y_2 = 0, \\ m_2 &= 0 \text{ if and only if } u_2 = 0, \end{aligned}$$

and that these constitutive functions satisfy growth conditions like (2.14), (2.19). The isotropy condition and (3.14) imply that the substitution of (3.12) into (3.9) reduces the latter to

$$(3.15)\qquad \hat{m}_3(y_\ell(s),u_\ell(s),s) \equiv m_3(1).$$

This equation and the four obtained by substituting (3.12) into (3.8) represent a collection of five integrals of the governing equations. The positive-definiteness of (3.13) and the growth conditions justify a global implicit function theorem to the effect that these five equations can be solved for the strains $\{y_k\}$ and two Euler angles ψ, φ in terms of the third Euler angle θ (the nutation angle). The substitution of these solutions into (3.10) reduce it to a single second-order, quasilinear ordinary differential equation for θ, depending upon the parameters $\underset{\sim}{n}(1)$ and $\underset{\sim}{m}(1)$. This equation is a complicated generalization of (2.21). After boundary conditions are imposed, we can study our buckling problem by imitating the development of Section 2 that follows (2.26). Not only does this buckling problem differ from that of Section 2 in the complexity of the governing equations, but it also has two eigenvalue parameters, the thrust and twisting moment, in place of λ, and it has up to four statically indeterminate reactions in place of ν. The presence of two eigenvalue parameters

means that solutions branch off in sheets from the eigencurves of the linearized problems. The theory of Crandall and Rabinowitz can be modified to show that these sheets may be characterized by their nodal structure.

Bibliographic Remarks. The formulation of this section is based on [7,12]. Also see [42]. The full analysis of the global behavior of buckled states is given in [13]. A buckling problem for Kirchhoff's theory is studied by [39], which shows the existence of some interesting secondary bifurcations. It should be noted that the presence of the twisting makes the connection between the bifurcation analysis and stability of motion more tenuous than usual because the twisting couple need not be conservative (cf. [22,37, 48]).

There is a related problem that leads to a somewhat more complicated set of equations than those of this section: Find all the travelling waves that can be sustained in a homogeneous infinite rod with hyperelastic constitutive equations of the form (3.12). The homogeneity and the hyperelasticity imply that there is a function $(y_\ell, u_\ell) \longmapsto W(y_\ell, u_\ell)$ such that

$$\hat{n}_k = \partial W/\partial y_k, \qquad \hat{m}_k = \partial W/\partial u_k. \tag{3.16}$$

If the rod is doubly-symmetric, then the equations of motion have the form

$$\underset{\sim}{n}_s = A\underset{\sim}{r}_{tt}, \qquad \underset{\sim}{m}_s + \underset{\sim}{r}_s \times \underset{\sim}{n} = J(\underset{\sim}{b}_1 \times \underset{\sim}{b}_{1_t} + \underset{\sim}{b}_2 \times \underset{\sim}{b}_{2_t})_t, \tag{3.17}$$

where A and J are given positive numbers. In a travelling wave all the fields are taken to depend on s, t through the combination $\zeta = s - ct$. Denoting derivatives with respect to ζ by a prime, one can show that (3.16) - (3.18) imply that

$$\underset{\sim}{n} - c^2 A\underset{\sim}{r}' = \underset{\sim}{a}(\text{const.}), \tag{3.18}$$

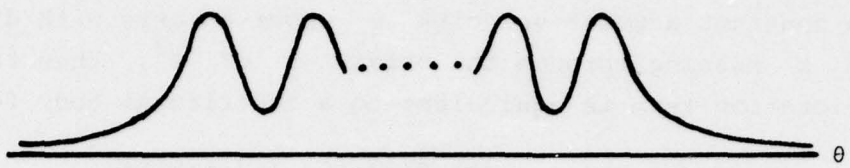


Figure 3.1

$$(3.19)\qquad \underset{\sim}{m}\cdot\underset{\sim}{a} - c^2 J(\underset{\sim}{b}_1 \times \underset{\sim}{b}_1' + \underset{\sim}{b}_2 \times \underset{\sim}{b}_2')\cdot a = \alpha(\text{const.}),$$

$$(3.20)\qquad m_3 - 2c^2 J u_3 = \beta(\text{const.}),$$

$$(3.21)\qquad y_k \frac{\partial W}{\partial y_k} + u_k \frac{\partial W}{\partial u_k} - W - \frac{c^2}{2}\left[A y_k y_k + J(u_1^2+u_2^2+2u_3^2)\right] = h(\text{const.}).$$

Thus the problem is "completely integrable" and would be reducible to the study of phase-plane diagrams were it not for the fact that the terms with c^2 can destroy the monotonicity of the left sides of (3.18)-(3.20). Thus the solvability of (3.18)-(3.20) depends critically upon the size of c, the asymptotic or average values of the loads, the size of solutions, and the behavior of the constitutive functions. One can exploit these variables to prove a variety of existence and nonexistence theorems. The qualitative behavior may be determined by multiple-valued phase-plane trajectories. Under certain critical conditions one can show that there are pulses of the form of Fig. 3.1 with the number of bays arbitrary! This travelling wave problem is treated in [14].

4. Buckling of Rotating Rods

In this section we merely formulate a particular problem that describes the buckling of a rotating rod and indicate some of the interesting effects that can arise. We employ the theory of Section 3. If the rod is rotating

with constant angular velocity ω about an axis with direction $\underset{\sim}{k}$ passing through the origin $\underset{\sim}{0}$ of $\mathbb{E}^3$, then the acceleration term is equivalent to a centrifugal body force

$$(4.1) \qquad \underset{\sim}{f}(\underset{\sim}{r}(s),s) = \rho(s)\omega^2\{\underset{\sim}{r}(s) - [\underset{\sim}{r}(s)\cdot\underset{\sim}{k}]\underset{\sim}{k}\}.$$

Here $\rho(s)$ is the mass per unit length at s in the reference configuration. We shall assume that $\rho = \text{const}$. In place of (3.5), we have

$$(4.2a,b) \qquad \underset{\sim}{n}' + \underset{\sim}{f}(\underset{\sim}{r}) = \underset{\sim}{0}, \qquad \underset{\sim}{m}' + \underset{\sim}{r}' \times \underset{\sim}{n} = \underset{\sim}{0}.$$

Let us suppose that the reference configuration of the rod is given by

$$(4.3) \qquad \underset{\sim}{r}(s) = s\underset{\sim}{i}, \qquad \underset{\sim}{b}_1(s) = \underset{\sim}{j}, \qquad \underset{\sim}{b}_2(s) = \underset{\sim}{k}, \qquad s \in [1-L,1].$$

Here $(\underset{\sim}{i},\underset{\sim}{j},\underset{\sim}{k})$ is an orthonormal basis, s is the arc length parameter of the reference axis, and L is the length of this axis in the reference configuration.

We suppose that the end $s = 1$ of the rod is welded to a rigid ring that is constrained to lie in a plane perpendicular to the $\underset{\sim}{k}$-axis and to rotate about it with constant angular velocity ω. We assume that $(\underset{\sim}{i},\underset{\sim}{j})$ are fixed to this ring. We assume that the end $s = 1 - L$ is free. Then our boundary conditions are

$$(4.4) \qquad \underset{\sim}{r}(1) = \underset{\sim}{i}, \qquad \underset{\sim}{b}_1(1) = \underset{\sim}{j}, \qquad \underset{\sim}{b}_2(1) = \underset{\sim}{k},$$

$$(4.5) \qquad \underset{\sim}{n}(1-L) = \underset{\sim}{0}, \qquad \underset{\sim}{m}(1-L) = \underset{\sim}{0}.$$

(There are but six scalar equations in (4.4) because $\underset{\sim}{b}_1$ and $\underset{\sim}{b}_2$ are unit vectors and because the problem is invariant under translations along $\underset{\sim}{k}$.) Our boundary value problem consists of the equilibrium equations (4.2), the "strain-displacement" equations (3.2), (3.3), the constitutive equations (3.12), and the boundary conditions (4.4), (4.5). ω^2 plays the role of the eigenvalue parameter.

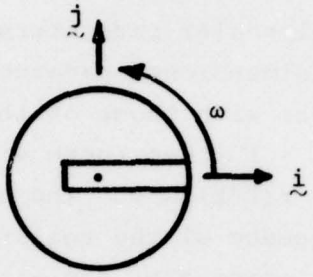


Figure 4.1. Trivial state of the rod

A trivial solution is one for which the rod remains unbent, unsheared, and untwisted. As is typical for geometrically exact theories of nonlinear elasticity, a trivial solution may not be simple: In the trivial state $\underset{\sim}{r}$ has the form

(4.6) $$\underset{\sim}{r}(s) = r(s)\underset{\sim}{i} \quad \text{with} \quad r'(s) = y_3$$

so that (3.14) implies that $\underset{\sim}{n}$ has the form

(4.7) $$\underset{\sim}{n}(s) = \Xi(s)\underset{\sim}{i}.$$

Moreover, in this state

(4.8) $$u_1 = u_2 = u_3 = 0 \quad \text{so that} \quad m_1 = m_2 = m_3 = 0.$$

Thus the boundary value problem for the trivial solution reduces to

(4.9) $$\hat{\Xi}(r')' + \rho\omega^2\underset{\sim}{r} = 0, \qquad s \in (1-L,1),$$

(4.10a,b) $$r(1) = 1, \qquad \hat{\Xi}(r'(1-L)) = 0.$$

Here $\hat{\Xi}(y_3) = \hat{n}_3(0,0,y_3,0,0,0)$.

That $\hat{\Xi}$ is a strictly monotone function from $(0,\infty)$ onto $(-\infty,\infty)$ and that (4.9) is autonomous give us powerful

machinery with which to attack (4.9), (4.10). The analysis, which we omit, is nevertheless tricky because the problem has two essential scalar parameters ω^2 and L, as well as the infinite-dimensional parameter $\hat{\Xi}$. The results of the analysis agree with those of the following heuristic argument: If $L \leq 1$ then there will be a compressed configuration with $r(1-L) > 0$. The existence of this solution is a consequence of the requirement that $\hat{\Xi}(y_3) \searrow -\infty$ as $y_3 \searrow 0$. This means that the material is strong enough to resist any compression due to the centrifugal force. There might also be a trivial state with part of the rod in tension if the deformed length exceeds 1. This configuration would occur if ω^2 can be so chosen so that the centrifugal forces can balance the tensile stresses. In particular, if $\hat{\Xi}$ is superlinear for large values of its argument, then one can expect the existence of solutions with $r(1-L) < 0$ for ω^2 sufficiently large, whereas if $\hat{\Xi}$ is sublinear for large values of its argument, then existence of such solutions could only be expected for small ω^2. These same considerations give rise to somewhat different results when $L > 1$. In short, there can be zero, one, or more trivial solutions for each value of ω^2, with this number depending on L and $\hat{\Xi}$.

Local bifurcation analyses for the full system of equations can be carried out in the neighborhood of eigenvalues of the linearizations about each trivial branch, especially the trivial branch containing the completely unstrained state for $\omega = 0$.. Although the nonlinear equations do not reduce to a second order ordinary differential equation, there is evidence suggesting that a global qualitative analysis along the lines of Sections 2 and 3 may nevertheless be possible.

The strange behavior of the trivial solutions is reflected in that of bifurcated states. This behavior is a manifestation of the coercivity problem caused by the centrifugal force: The centrifugal force behaves like a

linear functional of the strains. Coercivity conditions like (2.14a) show that the rod can always resist large compressive stresses but might fail to resist large tensile stresses if the constitutive functions are not ultimately superlinear. When (3.16) holds, the problem is variational. If

$$U = \int_{1-L}^{1} W(y_k(s), u_k(s))\, ds,$$

$$V = \frac{1}{2}\int_{1-L}^{1} \rho |\underset{\sim}{r}(s) - \underset{\sim}{r}(s)\cdot\underset{\sim}{k}\,\underset{\sim}{k}|^2\, ds, \tag{4.11}$$

then the governing equations are equivalent to the Euler-Lagrange equations for the functional $U - \omega^2 V$. The problem of minimizing this functional immediately leads to difficulty with coercivity described above. The coercivity problem evaporates if we consider the problem of minimizing U on the manifold defined by $V = \text{const}$. But then the multiplier rule gives us equations that are equivalent to the governing equations except that the parameter ω^2 is replaced by a multiplier λ. The difficulty is thus shifted to that of verifying that λ is positive. Similar remarks apply to the problem of minimizing or maximizing V on the manifold defined by $U = \text{const}$. (See [3,9] for technical details on the variational problems.) These difficulties are just like those for the travelling wave problem described at the end of Section 3 and are closely related to those that can arise for the problems of Section 5.

Bibliographic Remarks. This and other problems for rotating bars are being investigated in [15]. A problem for a rotating elastica was analyzed in [45]. Since the elastica is inextensible, it cannot suffer many of the effects described in this section. These problems may be regarded as successors to problems for rotating strings. The results of [38] constitute one of the most beautiful global

analyses in the applications of bifurcation theory. In this connection, see [56].

5. Buckling of Rings and Arches

We consider the planar buckling of a nonlinearly elastic ring under hydrostatic pressure. We employ the same theory as that used in Section 2. The hydrostatic pressure has intensity p per unit actual length of the deformed ring. Then in place of (2.7), (2.9) and (2.10), we have the scalar system of equilibrium equations

$$\Xi' - H\theta' - p\eta = 0, \tag{5.1a}$$

$$H' + \Xi\theta' + p(1+\xi) = 0, \tag{5.1b}$$

$$M' + H(1+\xi) - \Xi\eta = 0. \tag{5.1c}$$

We assume that the reference configuration is a uniform ring of radius 1. Thus s is restricted to the interval $[0,2\pi]$ and the constitutive equations (2.11) are independent of s. We seek solutions of (5.1), (2.12) for which ξ and η have period 2π and for which

$$\theta(s+2\pi) = \theta(s) + 2\pi. \tag{5.2}$$

We set

$$\bar{\mu} = \theta' - 1 \tag{5.3}$$

so that (5.2) is equivalent to

$$\bar{\mu}(s+2\pi) = \bar{\mu}(s), \qquad \int_0^{2\pi} \bar{\mu}(s)\,ds = 0. \tag{5.4a,b}$$

Rather than getting a second-order ordinary (-functional) differential equation for θ as in Sections 2 and 3, we instead obtain something equivalent to a second-order equation for $\bar{\mu}$. This appears to be a nonlinear analog of the fact that classical linear theories of curved

rods are governed by systems of higher order than those for straight rods. From (5.1) and (2.11) we get

$$D[\hat{\Xi}^2 + \hat{H}^2 - 2p\hat{M}] = 0, \tag{5.5}$$

$$\xi D\hat{\Xi} + \eta D\hat{H} + (1+\bar{\mu})D\hat{M} = 0, \tag{5.6}$$

where

$$D \equiv \xi' \frac{\partial}{\partial \xi} + \eta' \frac{\partial}{\partial \eta} + \bar{\mu}' \frac{\partial}{\partial \bar{\mu}} . \tag{5.7}$$

Suppose that $\hat{\Xi}$ and $\hat{M}$ are even functions of η and that $\hat{H}$ is an odd function of η. We set

$$2\omega = \eta^2. \tag{5.8}$$

These symmetry results enable us to introduce new constitutive functions for Ξ, H/η, M, that depend on ω. We substitute these into (5.5) and (5.6) and then divide the resulting equations by $\bar{\mu}'$. We obtain a pair of ordinary differential equations for ξ and ω as functions of $\bar{\mu}$. The positive-definiteness of (2.13) enables us to solve these equations in a suitably large neighborhood of $\bar{\mu} = 0$ (which corresponds to a uniformly compressed circular state) and also to solve (5.1), (2.11) for $\bar{\mu}'$ as a function of ξ, η, μ. Combining these two results with (5.8) we find that $\bar{\mu}$ satisfies a pair of equations of the form

$$\bar{\mu}' = \pm F(\bar{\mu}), \tag{5.9}$$

where F depends on $\hat{\Xi}$, $\hat{H}$, $\hat{M}$ and their first derivatives. The two signs come from the two roots η of (5.8). These equations give the phase plane trajectories for $\bar{\mu}$. By studying these trajectories in conjunction with (5.4), one can prove the following results: Under mild constitutive restrictions solutions preserve their least periods inherited from the eigenfunctions linearized about the trivial solution in a large neighborhood of the trivial solution.

In this neighborhood, every simple configuration of the ring must have at least two axes of symmetry. The number of axes of symmetry are likewise preserved in this neighborhood.

We note that the proof of these results did not require the apparatus of Crandall and Rabinowitz because the periodicity conditions (5.4) greatly simplify the analysis. On the other hand, bifurcation problems for circular arches under hydrostatic pressure would require an analysis along the lines of that of Section 2. In place of (2.21), we would have the second order ordinary differential equation corresponding to (5.9).

Bibliographic Remarks. The results of this section are based on [1,5,6]. The first two of these references examine unshearable rods in detail, [1] giving a local bifurcation analysis. Local and global results for the elastica case are developed from a modern viewpoint in [57]. The contact problem for a buckled elastica ring is treated in [28]. The global qualitative analysis for nonlinearly elastic arches is still open. Such problems should exhibit some interesting effects that depend on the shallowness of the arches. In particular, linearized analyses of simpler engineering models (cf. [58], e.g.) suggest that buckling is initiated in an asymmetric mode, although the stable state of large deflection is symmetric. These factors are closely related to the phenomenon of snapping-through.

6. Tensile Instabilities of Rods

In Section 2 we found that the qualitative behavior of solution branches could change markedly when parts of the buckled rod are in tension. (Cf. the paragraph following (2.39).) Moreover, an examination of the linear eigenvalue problem (2.31) shows that it may well possess negative eigenvalues. In this section we subsume the study of such tensile instabilities for the theory of Section 2 under

that for a richer theory in which the rod can also suffer transverse contraction. We again define the configuration of the rod by the two planar vector functions $\underset{\sim}{r}$ and $\underset{\sim}{b}$ of (2.1), except we now suspend the requirement that $\underset{\sim}{b}$ be a unit vector. We define

$$(6.1) \qquad 1 + \rho(s) \equiv |\underset{\sim}{b}(s)|, \qquad \sigma(s) \equiv \rho'(s).$$

In addition to (2.4) we demand that

$$(6.2) \qquad 1 + \rho > 0.$$

The strains for this theory are

$$(6.3) \qquad \underset{\sim}{w} = (\xi, \eta, \rho, \sigma, \mu).$$

We let Ξ, H, P, Σ, M denote the generalized forces corresponding to ξ, η, ρ, σ, μ (according to a suitable principal of virtual work. Ξ, H, M are the resultants defined in Section 2; P and Σ can be defined as certain stress integrals [3,11]). Then the equilibrium equations consist of (2.6) and

$$(6.4) \qquad \Sigma' - M(1+\rho)^{-1}\theta' - P = 0.$$

If the rod is homogeneous, then it has constitutive equations of the form

$$(6.5) \quad \Xi(s) = \hat{\Xi}(\underset{\sim}{w}(s)), \quad H(s) = \hat{H}(\underset{\sim}{w}(s)), \quad P(s) = \hat{P}(\underset{\sim}{w}(s)),$$

$$\Sigma(s) = \hat{\Sigma}(\underset{\sim}{w}(s)), \quad M(s) = \hat{M}(\underset{\sim}{w}(s)).$$

The analog of the strong ellipticity condition is that

$$\text{(6.6a)}\qquad \begin{pmatrix} \hat{\Xi}_\xi & \hat{\Xi}_\eta & \hat{\Xi}_\sigma & \hat{\Xi}_\mu \\ \hat{H}_\xi & \hat{H}_\eta & \hat{H}_\sigma & \hat{H}_\mu \\ \hat{\Sigma}_\xi & \hat{\Sigma}_\eta & \hat{\Sigma}_\sigma & \hat{\Sigma}_\mu \\ \hat{M}_\xi & \hat{M}_\eta & \hat{M}_\sigma & \hat{M}_\mu \end{pmatrix}$$

be positive definite and

$$\text{(6.6b)}\qquad \hat{P}_\rho > 0.$$

In addition to (2.14), we require that

$$\text{(6.7a)}\qquad \hat{P}(\underset{\sim}{w}) \longrightarrow \left\{ \begin{matrix} \infty & \text{as} & \rho \to \infty \\ -\infty & \text{as} & \rho \to -1 \end{matrix} \right\},$$

$$\text{(6.7b)}\qquad \hat{\Sigma}(\underset{\sim}{w}) \longrightarrow \pm\infty \quad \text{as} \quad \sigma \to \pm\infty.$$

The requirements that it be no easier to shear or bend the rod in one sense than in the opposite sense and that the same effort is needed to produce σ as to produce $-\sigma$ is made precise by the following conditions in which the arguments ξ and ρ are suppressed:

$$\text{(6.8a)}\qquad \hat{\Xi}(\eta,\sigma,\mu) = \hat{\Xi}(\eta,-\sigma,-\mu) = \hat{\Xi}(-\eta,\sigma,-\mu) = \hat{\Xi}(-\eta,-\sigma,\mu),$$

$$\text{(6.8b)}\qquad \hat{H}(\eta,\sigma,\mu) = \hat{H}(\eta,-\sigma,-\mu) = -\hat{H}(-\eta,\sigma,-\mu) = -\hat{H}(-\eta,-\sigma,\mu),$$

$$\text{(6.8c)}\qquad \hat{P}(\eta,\sigma,\mu) = \hat{P}(\eta,-\sigma,-\mu) = \hat{P}(-\eta,\sigma,-\mu) = \hat{P}(-\eta,-\sigma,\mu),$$

$$\text{(6.8d)}\qquad \hat{\Sigma}(\eta,\sigma,\mu) = -\hat{\Sigma}(\eta,-\sigma,-\mu) = \hat{\Sigma}(-\eta,\sigma,-\mu) = -\hat{\Sigma}(-\eta,-\sigma,\mu),$$

$$\text{(6.8e)}\qquad \hat{M}(\eta,\sigma,\mu) = -\hat{M}(\eta,-\sigma,-\mu) = -\hat{M}(-\eta,\sigma,-\mu) = \hat{M}(-\eta,-\sigma,\mu).$$

We consider the boundary conditions

$$\text{(6.9a)}\qquad \underset{\sim}{r}(0) = \underset{\sim}{0}, \qquad \underset{\sim}{r}(1) = (1+\gamma)\underset{\sim}{i} \quad \text{with} \quad \gamma > 0,$$

(6.9b) $\Sigma(0) = 0, \qquad \Sigma(1) = 0,$

(6.9c) $\theta(0) = 0, \qquad \theta(1) = 0.$

Condition (6.9a) prescribes the elongation γ, which serves as the eigenvalue parameter in place of the tensile force λ. (It is technically more convenient to use γ instead of λ.) The remaining two conditions ensure that there are trivial solutions to the governing equations.

Our boundary value problem consists of (2.2) with $\underset{\sim}{b}$ replaced by $\underset{\sim}{b}/|\underset{\sim}{b}|$, (2.3), the three scalar equilibrium equations implicit in (2.6), the equilibrium equation (6.4), the constitutive equation (6.5), and the boundary conditions (6.9). By using (2.8), one can reduce the equations to a fourth-order, quasilinear system of ordinary differential equations. Since the order is not 2, it would be very difficult to extend to this problem our results on nodal structure of Section 2, because these results rely on Sturmian theory and uniqueness theorems suitable only for second order systems. Consequently we limit our attention primarily to the local analysis of bifurcation.

Using (6.6b), (6.7a), we can readily show that our boundary value problem possesses a trivial solution

(6.10) $\xi = \gamma, \qquad \eta = 0, \qquad \rho = \bar{\rho}(\gamma), \qquad \theta = 0,$

where $\bar{\rho}(\gamma)$ is the unique solution of $\hat{P}(\gamma,0,\rho,0,0) = 0$. This solution describes an elongated rod with uniformly contracted and unsheared cross-sections.

We study our boundary value problem by a perturbation scheme [33] that converges at least for branches emanating from simple eigenvalues of the problem linearized about (6.10). (The convergence may be proved by the Poincaré shooting method [34] or by the method of Lyapunov and Schmidt [54,60].) We introduce an amplitude parameter α and assume that on a solution branch γ and the geometric variables have representations of the form

$$\gamma(\alpha) = \gamma_0 + \gamma_1\alpha + \gamma_2\alpha^2/2! + \gamma_3\alpha^3/3! + O(\alpha^4),$$

$$\xi(s,\alpha) = \xi_0 + \xi_1(s)\alpha + \xi_2(s)\alpha^2/2! + \xi_3(s)\alpha^3/3! + O(\alpha^4), \quad \text{etc.} \tag{6.11}$$

We substitute (6.11) into our boundary value problem. We get conditions for $\gamma_0,\gamma_1,\gamma_2,\ldots,\xi_1,\xi_2,\ldots$, etc., by repeatedly differentiating these equations with respect to α and then setting $\alpha = 0$. The conditions of (6.8) reduce the determination of these unknowns to a sequence of third-order (rather than sixth-order) linear ordinary differential equations with constant coefficients. These equations are readily analyzed. We find that there are two different kinds of bifurcation processes corresponding to two different kinds of eigenvalues of the linearized problems. Let

$$p(\gamma) \equiv \hat{\Xi}[\hat{\Xi} - (1+\gamma)\hat{H}_\eta] \,/\, \hat{H}_\eta\hat{M}_\mu, \tag{6.12}$$

where the constitutive functions are evaluated at the trivial solution (6.10). Then one set of eigenvalues called eigenvalues of shear type, are the real, non-zero solutions γ of the equations

$$p(\gamma) = 4m^2\pi^2 \quad \text{for } m \text{ a positive integer,} \tag{6.13a}$$

$$\cot(\sqrt{p(\gamma)}/2) = -2\sqrt{p(\gamma)}\hat{M} \,/\, (1+\gamma)\hat{\Xi} \tag{6.13b}$$

with the constitutive functions evaluated at (6.10). It is not hard to see that the smallest positive root γ_p of (6.13), if it exists, is a root of (6.13b) with $p(\gamma_p) \in (\pi^2, 4\pi^2)$. If this root is simple and distinct from the roots of (6.16) below, then the leading terms of (6.11) corresponding to the branch emanating from γ_p are given by

$$(6.14)\qquad \begin{cases} \gamma_1 = 0, \\ \xi_1(s) = 0, \quad \rho_1(s) = 0, \\ (\eta_1(s), \varphi_1(s)) \\ \quad = \text{const.}\,(1, -\hat{H}_\eta/\hat{\Xi}) \sin \dfrac{\sqrt{p(\gamma_p)}}{2}\, s \sin \dfrac{\sqrt{p(\gamma_p)}}{2}\,(1-s). \end{cases}$$

(Indeed, if γ_p is any simple root of (6.13b), then (6.14) holds.) Let

$$(6.15)\qquad q(\gamma) = (\hat{P}_\xi \hat{\Xi}_\rho - \hat{P}_\rho \hat{\Xi}_\xi)/\hat{\Xi}_\xi \hat{\Sigma}_\sigma,$$

where the constitutive functions are evaluated at (6.10). The second set of eigenvalues, called eigenvalues of necking type, are the real, non-zero solutions γ of

$$(6.16)\qquad q(\gamma) = n^2\pi^2 \quad \text{for } n \text{ a positive integer.}$$

Let γ_q be the smallest positive root of (6.16). It satisfies $q(\gamma_q) = \pi^2$. If this root is simple and distinct from the roots of (6.13), then on the branch emanating from γ_q the leading terms of (6.11) are given by

$$(6.17)\qquad \begin{cases} \gamma_1 = 0 \\ (\xi_1(s), \rho_1(s)) = \text{const.}\,(-\hat{\Xi}_\rho/\hat{\Xi}_\xi, 1) \cos n\pi s, \\ \eta_1(s) = 0, \quad \varphi_1(s) = 0. \end{cases}$$

(The same formulae apply if γ_q is any simple root of (6.16).)

The perturbation analysis shows the following results: The higher order terms for the branch emanating from γ_p have nonconstant values for ξ_k, ρ_k, whereas those for the branch emanating from γ_q have $\eta_k = \varphi_k = 0$. Thus the development of a "shear instability" from γ_p is accompanied by nonuniform longitudinal stretching and transverse contraction. On the other hand, there is no shearing

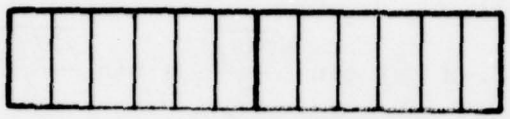


Figure 6.1a. The undeformed configuration of rod.

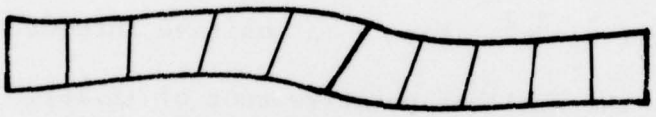


Figure 6.1b. The shear instability, which is a deformed configuration on the branch emanating from γ_p.

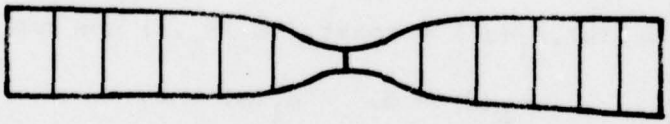


Figure 6.1c. The necking instablility, which is a deformed configuration on the branch emanating from γ_q.

accompanying the initiation of the necking at γ_q. In each case the value of γ_2, and in particular its sign, is determined by derivatives up to order 3 of the constitutive functions evaluated at the trivial solution. From the first few computed terms of (6.11) we can sketch the deformed configurations of the rods. This is done in Fig. 6.1. These shapes are like those of the photographs given in [43]. The pure necking problem can also be given a global qualitative analysis [5] under arbitrary boundary conditions.

The same perturbation results enable us to determine the tensile load $\Lambda = [\hat{\Xi}\underset{\sim}{a} + \hat{H}\underset{\sim}{b}]\cdot\underset{\sim}{i}$ as a function of γ. This is the information recorded on a tension test. We illustrate the typical results in Fig. 6.2, which conform to those found experimentally. Note that the positive definiteness of (6.6a) does not require that Λ corresponding to the trivial solution and a fortiori Λ corresponding to a bifurcating solution to be a monotonically increasing function of γ. The elementary theory of elastic-perfectly plastic materials inspired by curves like Fig. 6.2 describes these instabilities by replacing the strictly monotone function $\xi \mapsto \hat{\Xi}(\underset{\sim}{w})$ by one in which $\hat{\Xi}_\xi = 0$ on an interval of the form $[Y,\infty)$ with $Y > 0$. (Y corresponds to γ_p.)

If the material is hyperelastic, then the problem is variational. By using the theorems of [3,9], we can show that the potential energy functional has a smooth global minimizer for each value of γ. It can be shown that the second variation of this functional at the trivial solution is not positive-definite for γ exceeding the smaller of γ_p and γ_q. Thus the trivial solution is unstable for γ in this range. This notion of stability can be made precise [22].

As noted above, it can well happen that $\gamma_2 < 0$. This together with the instability results strongly suggest that these problems will exhibit hysteresis when γ is slowly

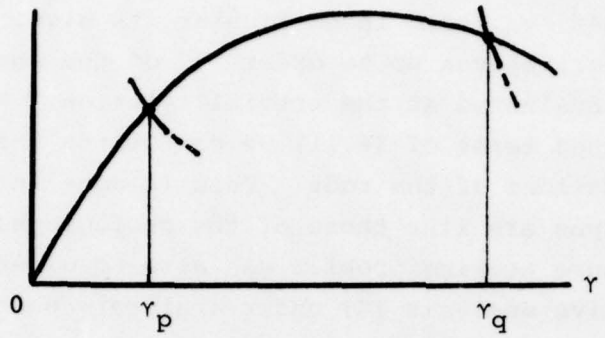


Figure 6.2. The effect of bifurcation on the load-extension curve. The heavy curve corresponds to the trivial solution. The curves joining it at the heavy dots correspond to the bifurcating branches.

oscillated in a suitable open interval containing $[0, \min(\gamma_p, \gamma_q)]$.

Thus our geometrically exact theory of nonlinearly elastic rods has solutions that can exhibit some of the main effects of plastic rods: shear (or yield-point) instabilities, necking, and hysteresis.

Bibliographic Remarks. The results described here are based on the extensive development of [4,5,11]. Paper [4] gives an analytic treatment of the necking problem by the Poincaré shooting method, paper [5] examines the global behavior of the necking problem by use of phase-plane methods. Paper [11] gives a detailed treatment of the material described here except that it replaces (6.9c) with the conditions $M(0) = M(1) = 0$. (By the positive-definiteness of (6.6a) and by (6.8), these are equivalent to $\theta'(0) = \theta'(1) = 0$.)

7. Buckling of Circular Plates

We now examine the axisymmetric buckling of circular plates under a compressive edge thrust. Our approach bears a marked resemblance to that of Section 2, but the analytic problems are strikingly different.

Let $(\underset{\sim}{i},\underset{\sim}{j},\underset{\sim}{k})$ be an orthonormal basis for $\mathbb{E}^3$ and let (s,φ) be polar coordinates for the $(\underset{\sim}{i},\underset{\sim}{j})$-plane. Set

$$(7.1)\quad \underset{\sim}{e}_1(\varphi) = \cos\varphi\,\underset{\sim}{i} + \sin\varphi\,\underset{\sim}{j}, \qquad \underset{\sim}{e}_2(\varphi) = -\sin\varphi\,\underset{\sim}{i} + \cos\varphi\,\underset{\sim}{j}.$$

The axisymmetric configuration of a circular plate that can suffer flexure, extension, and shear is determined by the pair of vector functions

$$(7.2a)\quad (s,\varphi) \in [0,1] \longmapsto \underset{\sim}{r}(s,\varphi),\ \underset{\sim}{b}(s,\varphi), \qquad |\underset{\sim}{b}(s,\varphi)| = 1$$

with

$$(7.2b,c)\quad \underset{\sim}{r}(s,\varphi)\cdot\underset{\sim}{e}_2(\varphi) = 0, \qquad \underset{\sim}{b}(s,\varphi)\cdot\underset{\sim}{e}_2(\varphi) = 0,$$

$$(7.2d.e)\quad \underset{\sim}{r}(0,\varphi)\cdot\underset{\sim}{e}_1(\varphi) = 0, \qquad \underset{\sim}{b}(0,\varphi) = \underset{\sim}{k}.$$

We set

$$(7.3a)\quad \underset{\sim}{a}(s,\varphi) = \cos\theta(s)\underset{\sim}{e}_1(\varphi) + \sin\theta(s)\underset{\sim}{k},$$

$$(7.3b,c)\quad \underset{\sim}{b}(s,\varphi) = -\sin\theta(s)\underset{\sim}{e}_1(\varphi) + \cos\theta(s)\underset{\sim}{k}, \qquad \theta(0) = 0,$$

$$(7.4a,b)\quad \underset{\sim}{r}(s,\varphi)\cdot\underset{\sim}{e}_1(\varphi) = r(s), \qquad r(0) = 0,$$

$$(7.5)\quad \frac{\partial \underset{\sim}{r}}{\partial s}(s,\varphi) = [1+\xi(s)]\underset{\sim}{a}(s,\varphi) + \eta(s)\underset{\sim}{b}(s,\varphi),$$

$$(7.6)\quad 1 + \tau(s) = r(s)/s,$$

$$(7.7)\quad \sigma(s) = \sin\theta(s)/s,$$

$$(7.8)\quad \mu(s) = \theta'(s).$$

The strains for our problem are

$$\underset{\sim}{w} = (\tau, \xi, \eta, \sigma, \mu). \tag{7.9}$$

The interpretation of ξ, η, μ parallels that of Section 2: ξ and η measure extension and shear in the radial direction; μ measures the flexure about $\underset{\sim}{e}_2$. τ measures the circumferential extension, σ (which bears no relation to the σ of Section 6) measures flexure about $\underset{\sim}{e}_1$. We require that

$$1 + \xi > 0, \qquad 1 + \tau > 0. \tag{7.10}$$

(More sophisticated conditions are appropriate. See [8,10].)

Let

$$\underset{\sim}{n}^1(s_0,\varphi_0) \equiv \Xi(s_0)\underset{\sim}{a}(s_0,\varphi_0) + H(s_0)\underset{\sim}{b}(s_0,\varphi_0) \tag{7.11}$$

be the force per unit reference length and let

$$\underset{\sim}{m}^1(s_0,\varphi_0) \equiv M(s_0)\underset{\sim}{e}_2(\varphi_0) \tag{7.12}$$

be the couple per unit reference length of $\{(s,\varphi) : s = s_0\}$ exerted by $\{(s,\varphi) : s > s_0\}$ on $\{(s,\varphi) : s \le s_0\}$ at (s_0,φ_0). Let

$$\underset{\sim}{n}^2(s_0,\varphi_0) \equiv T(s_0)\underset{\sim}{e}_2(\varphi_0) \tag{7.13}$$

be the force per unit reference length and let

$$\underset{\sim}{m}^2(s_0,\varphi_0) \equiv -\Sigma(s_0)\cos\theta(s_0)\underset{\sim}{e}_1(\varphi_0) + \underset{\sim}{m}^2(s_0,\varphi_0)\cdot\underset{\sim}{k}\,\underset{\sim}{k} \tag{7.14}$$

be the couple per unit reference length of $\{(s,\varphi) : \varphi = \varphi_0\}$ exerted by $\{(s,\varphi) : \varphi_0 < \varphi < \varphi_0 + \pi\}$ on $\{(s,\varphi) : \varphi_0 - \pi \le \varphi \le \varphi_0\}$ at (s_0,φ_0). The representations (7.11)-(7.14) embody the requirement that the resultants be axisymmetric. Assume that the only external load applied to the body consists of reactions to geometric boundary conditions at the edge and a normal pressure applied to the edge:

$$\text{(7.15)} \qquad \underset{\sim}{n}^{1}(1,\varphi) \;=\; -\lambda g(r(1))\underset{\sim}{e}_{1}(\varphi).$$

(We permit this normal pressure to have a hydrostatic character by permitting its amplitude to depend on $r(1)$.) The classical versions of the equilibrium equations have the form

$$\text{(7.16a)} \qquad s\Xi(s) \;=\; -[\lambda g(r(1)) + \int_{s}^{1} T(t)\,dt]\cos\theta(s),$$

$$\text{(7.16b)} \qquad sH(s) \;=\; [\lambda g(r(1)) + \int_{s}^{1} T(t)\,dt]\sin\theta(s),$$

$$\text{(7.17)} \qquad (sM)' - \Sigma\cos\theta + s[\lambda g(r(1)) + \int_{s}^{1} T(t)dt][(1+\xi)\sin\theta + \eta\sin\theta] = 0.$$

Compare this system with (2.9), (2.10). There is no vertical edge reaction ν in (7.16), (7.17) because its presence would be inconsistent with axisymmetry and equilibrium. Thus the delicate analytic questions arising from the nonvanishing of ν in Section 2 are not present for our plate problem. On the other hand, the explicit presence of the radial coordinate s and the replacement of the scalar λ of (2.9) by the bracketed expression of (7.16) are the source of considerable analytic difficulty that we describe below.

We assume that the material of the plate is homogeneously elastic (implicit in which is the notion that the plate is of uniform thickness). We introduce constitutive functions

$$\text{(7.18)} \qquad \underset{\sim}{w} \longmapsto \hat{\underset{\sim}{T}}(\underset{\sim}{w}), \quad \hat{\Xi}(\underset{\sim}{w}), \quad \hat{H}(\underset{\sim}{w}), \quad \hat{\Sigma}(\underset{\sim}{w}), \quad \hat{M}(\underset{\sim}{w})$$

such that

$$\text{(7.19a,...,e)} \qquad T(s) \;=\; \hat{T}(\underset{\sim}{w}(s)),\ldots,M(s) \;=\; \hat{M}(\underset{\sim}{w}(s)).$$

We require that

(7.20a,b) $$\begin{pmatrix} \hat{T}_\tau & \hat{T}_\sigma \\ \hat{\Sigma}_\tau & \Sigma_\sigma \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \hat{\Xi}_\xi & \hat{\Xi}_\eta & \hat{\Xi}_\mu \\ \hat{H}_\xi & \hat{H}_\eta & \hat{H}_\mu \\ \hat{M}_\xi & \hat{M}_\eta & \hat{M}_\mu \end{pmatrix}$$

be positive-definite. This is the plate-theoretic analog of the strong ellipticity condition. It is not unreasonable to supplement this condition with the requirement that

(7.21) $$\begin{pmatrix} \hat{T}_\tau & \hat{T}_\xi & \hat{T}_\eta \\ \hat{\Xi}_\tau & \hat{\Xi}_\xi & \hat{\Xi}_\eta \\ \hat{H}_\tau & \hat{H}_\xi & \hat{H}_\eta \end{pmatrix}$$

be positive-definite. This is a restricted plate-theoretic version of the Coleman-Noll inequality [61]. We finally require that the constitutive functions meet coercivity conditions like (2.14) and invariance conditions like (6.8).

We choose the boundary condition

(7.22) $$\theta(1) = 0.$$

Thus our boundary value problem consists of the "strain-displacement" relations (7.5)-(7.8), the equilibrium equations (7.16), (7.17), the constitutive equations (7.19), and the geometric boundary conditions (7.3c), (7.4b), (7.22).

Our plan is to solve the system consisting of (7.5), (7.6), (7.16), (7.19a,b,c) for τ, ξ, η as functions of θ and λ and then substitute the resulting expressions and (7.19d,e) into (7.17) to get a second order functional-differential equation for θ to which we can apply the theory of Crandall & Rabinowitz [25] supported by the full power of the Sturmian theory. Thus we must solve the

system

$$(7.23a)\quad s\hat{\Xi}(\underset{\sim}{w}(s)) = -[\lambda g(r(1)) + \int_s^1 \hat{T}(\underset{\sim}{w}(t))\,dt]\cos\theta(s),$$

$$(7.23b)\quad s\hat{H}(\underset{\sim}{w}(s)) = [\lambda g(r(1)) + \int_s^1 \hat{T}(\underset{\sim}{w}(t))\,dt]\sin\theta(s),$$

$$(7.24)\quad r(s) \equiv s[1+\tau(s)] = r(1) - \int_s^1 \{[1+\xi(t)]\cos\theta(t) + \eta(t)\sin\theta(t)\}\,dt,$$

subject to the strict inequalities (7.10), when θ is a given function in $C^1((0,1)) \cap C^0([0,1])$ satisfying (7.3c), (7.22). We do not employ the alternative integral representation

$$(7.25)\quad r(s) = \int_0^s \{[1+\xi(t)]\cos\theta(t) + \eta(t)\sin\theta(t)\}\,dt$$

that incorportates (7.46) for (7.24) because it presumes that r is absolutely continuous on $[0,1]$, while the singular character of (7.23) puts such a presumption in doubt. The system (7.23), (7.24) generalizes that of Section 2 leading the solution (2.20). The problem of Section 2 is just a pair of algebraic equations that are resolved by a global implicit function theorem relying on the monotonicity of the constitutive functions. In contrast, we are now confronted by a bona fide system of integral equations. Using the positive-definiteness of (7.20b) or (7.21) we could convert (7.23), (7.24) to a variety of equivalent forms.

Our first inclination in attacking (7.23) would be to formulate (7.23) as an operator equation of the form $(\xi,\eta,r) = F(\xi,\eta,r)$ on the Banach space of continuous functions and then employ Schauder's fixed point theorem. But

it seems impossible to choose F so that it both meets the requirements of Schauder's theorem and is compatible with (7.10) and the linearity of (7.24). We are consequently led to an alternative formulation of the problem in the context of montone or pseudo-monotone operators on Sobolev spaces. In such a setting the disadvantages of (7.25) become more pronounced. We accordingly formulate a suitable weak form of the equations without (7.4b). Then in place of (7.4b), the regularity theory produces the natural boundary condition

$$\lim_{s\to 0} s\hat{\underset{\sim}{\mathfrak{E}}}(\underset{\sim}{w}(s)) = 0, \tag{7.26}$$

which is instrumental in ultimately showing that (7.4b) holds. If (7.20b) is positive-definite, then the operator appearing in the weak equations is pseudo-monotone (cf. [41]), and an existence theory is obtained without severe difficulty. If (7.21) is positive-definite, this operator is strictly monotone and we readily get both existence and uniqueness of the solution. The regularity theory away from $s = 0$ follows the lines of [6]. (The full power of [9] is not required here; much of the effort of [9] was devoted to showing that minimizers of certain variational problems satisfy the weak Euler equations.) One can then use the regularity of solutions on $[0,1]$ to show that the solutions are in fact regular on $[0,1]$ by a careful use of differential inequalities along the lines of the proof of Nagumo's uniqueness theorem [30].

Thus we can reduce our full system to a single second-order, ordinary-functional differential equation for θ when (7.21) is positive-definite and to a family of such equations when (7.20b) is positive-definite. The resulting equations are susceptible of a full analysis along the lines indicated in Section 2. The linearization of the final equation is a Bessel equation. A uniqueness theorem of Nagumo type ensures that the nodal structure that a

branch inherits from the linearization is preserved along the entire branch. The full analysis shows that the family of solutions of the nonlinear equations exhibit a budget of pathologies just like the problem of Section 2.

Bibliographic Remarks. The material of this section is based on [10]. The problem was inspired by Wolkowisky's [62] beautiful analysis of v. Kármán's equations. (Also, see [19-21].)

8. Other Problems

Preliminary evidence indicates that the axisymmetric buckling of nonlinearly elastic spherical shells and caps under hydrostatic pressure can be given a global analysis that modifies the treatment of Section 7 in much the same way that the treatment of Section 5 modifies that of Section 2. Some variational aspects of this problem were briefly noted in [2]. See [18] for a fascinating and complete numerical study of this problem in the context of v. Kármán theory.

Among the most interesting remaining buckling problems for nonlinearly elastic structures are three which are governed by quasilinear systems of partial differential equations: (i) the buckling of a plate of any shape (e.g., rectangular) under the action of in-plane edge loads, (ii) the non-axisymmetric buckling of spherical shells under hydrostatic pressure, and (iii) the buckling of cylindrical shells under edge loads acting parallel to the generators. (Within the context of the v. Kármán theory, Problem (i) has been thoroughly analyzed in [19,21] by means of Lyusternik-Shnirel'man category theory, and Problem (iii) has been briefly analyzed in [20].)

Heretofore, one of the main difficulties obstructing the analysis of geometrically exact theories of nonlinearly elastic shells has been the choice of suitable constitutive

restrictions: The strong ellipticity condition, despite a number of attractions, has proved too weak for analysis, the analytic virtues of Coleman-Noll conditions were difficult to perceive, while the monotonicity condition, which can support a relatively simple analysis, also ensures the physically unsuitable uniqueness of solutions to problems in which the loading is dead. (In three-dimensional theories, the principle of frame-indifference implies that semi- or pseudo-monotone constitutive functions must be monotone. Similar results apply to shell theories.) A felicitous resolution of these difficulties has just been provided by Ball [17], who proposed conditions, somewhat more restricted than strong ellipticity, which are both convenient for analysis and physically reasonable. With such conditions, the Problems (i)-(iii) seem to be tractable either by the methods of Rabinowitz [50-52] or by Lyusternik-Shnirel'man category theory.

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Bifurcation Theory in Non-Linear Hydrodynamic Stability

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1. Introduction

Nonlinear hydrodynamic stability theory is really concerned, ultimately, with phenomena such as transition to turbulence. In practice, however, that phenomenon is so complex as to defy rational understanding at the present time. A more limited objective is that of gaining some understanding of nonlinear processes in fluid mechanics, perhaps with reference to the early, relatively-simple stages of the evolution of laminar flow to turbulence. Even then, the mathematical problems posed are challenging enough.

Attention has therefore been focussed in the literature on a number of prototype problems, namely, those of Taylor vortices between concentric rotating cylinders, Bénard cells in a convecting fluid layer heated from below, Poiseuille flow between parallel planes and, quite recently, the oscillatory Stokes' layer. In this lecture an attempt will be made to describe the experimental observations, the mathematical attempts to explain them, the results obtained and important questions which remain open.

It is appropriate in this Introduction, and may be helpful, to say something of the observations, so as to give some justification for the mathematical structures which are assumed or sought.

(a) Some 90-100 years ago, Lord Rayleigh and others were interested in the problem of the instability of a jet in a liquid. In experiment a sinuous

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wave can develop, which evolves in the direction of flow and takes on a highly contorted form. The theory of Rayleigh considered the development of a sinuous normal mode, developing from and propagating in a "parallel" flow appropriate to a local region; both viscosity and nonlinearity were neglected; but, more especially from the present viewpoint, the non-parallel, or developing aspect of the flow was neglected also.

(b) During the present century a good deal of interest has focussed on the aerodynamical boundary layer, through the theoretical work in Germany of Prandtl, Heisenberg, Tollmien and Schlichting in the 192Os and 193Os, followed by experimental observations and further theoretical developments in the United States, Europe, Japan and elsewhere. Again, as in Rayleigh's work on instability of a jet, attention has been directed towards sinuous propagating waves, of the form which were first observed by Schubauer and Skramstad at the National Bureau of Standards in Washington in 1943. Theoretical developments have been based on the "parallel-flow" approximation, but with viscosity, weak nonlinearity (for small amplitude of oscillation) and flow development taken into account. Three-dimensional boundary-layers have also received attention, especially in connexion with swept-wings and with geophysical Ekman layers. One example is that of flow due to a rotating disk, where a "logarithmic-spiral" vortex pattern appears in the boundary layer (see frontispiece to Rosenhead (1963)). In this case theoretical work has been based on linearized equations in a "parallel-flow" approximation.

(c) Taylor vortices occur in curved flows, and especially between concentric rotating cylinders when the outer is at rest but the inner one is rotating. The cylinders are often, but not always, long compared with the gap between them. A parameter, the Taylor number, which depends upon angular speed, radius of curvature, gap width and kinematic viscosity, determines whether the toroidal Taylor vortices, spaced regularly along the axis, occur or not. They do occur if a critical Taylor number is exceeded, and they have a

definite wavelength. At a higher Taylor number still the vortices become time-dependent and wavy and, in fact, travel around the annulus. These first and second transitions are followed by others at still higher Taylor numbers.

Variations on the Taylor-vortex theme include effects of non-concentric axes of the cylinders, when the critical Taylor number is known to depend on the eccentricity, and the effects of time-dependent angular velocity, when the critical Taylor number depends on the modulating amplitude and frequency. Both these features have been the subject of theoretical enquiry, with some degree of success.

One aspect of the experimental observations, which has not yet been treated adequately by theorists, is that concerned with end effects. Many experiments have cylinders whose length is several hundred times the annular gap, but several recent observations have been made with cylinders only some 20 or 30 gap-widths in length. It is clear from very recent work of Cole (1974a,b) that the effects of the ends can be considerable and dramatic, especially with reference to the development of wavy vortices. Both the critical Taylor number and the critical axial wave number for the second transition are strongly influenced by the end boundary conditions.

(d) Bénard cells can occur in very simple experimental configurations, such as a horizontal layer of liquid. The lower, rigid boundary is heated, while the upper boundary, which is also horizontal, can be rigid or free and is cooled relative to the lower boundary. A cellular convection pattern is possible due to gravity acting on the density variations, and is characterized by the Rayleigh number, a parameter which depends on expansion coefficient, temperature gradient, acceleration due to gravity, distance between the planes, kinematic viscosity and thermometric conductivity. Alternatively we note that the cellular motion can also be driven from a free surface by the dependence of surface tension on viscosity.

In many of the problems outlined above, the occurrence of a critical parameter (for which we use the generic term "Reynolds" number) for the linearized perturbation problem implies a bifurcation for the nonlinear equations. Such bifurcations have frequently been calculated for idealized cases in which the geometry extends to infinity in one or more coordinates. The amplitude of the perturbation is zero up to the critical Reynolds number, but thereafter is zero; in the notation of Benjamin (1976a) the supercritical, subcritical and transcritical cases of bifurcation are illustrated schematically in Fig.1(a),(b),(c).

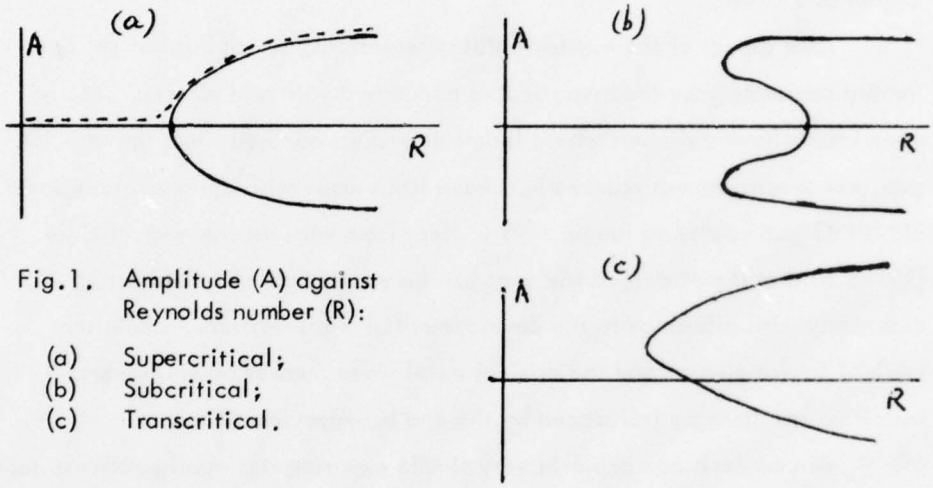


Fig.1. Amplitude (A) against Reynolds number (R):

(a) Supercritical;
(b) Subcritical;
(c) Transcritical.

Such problems, which lead to bifurcations of the type illustrated in Fig.1 are sometimes known as "perfect", a term prevalent in the literature of the theory of elastic stability. If small "imperfections" are added, such as that of allowing for a large but finite container, the problem is sometimes known as "imperfect". In many cases there is no strict bifurcation: rather, the amplitude follows a smooth path, shown dotted, close to one of the bifurcating amplitude curves of Fig.1. The work of Matkowsky and Reiss (1976), as explained by Professor Reiss at this meeting, deals with such problems. On the other hand, there are known examples in fluid mechanics where an "imperfection" (i) allows but shifts the bifurcation and (ii) possibly changes

the structure between Fig.1(a), (b) or (c); examples include eccentricity of the cylinders in the Taylor-vortex problem (DiPrima and Stuart 1975) and viscosity variations with temperature in the Bénard problem (Segel and Stuart 1962).

2. Equations of Motion

These are

$$\frac{\partial \underline{v}}{\partial t} - \underline{v} \wedge \underline{\zeta} = -\nabla\left(\frac{p}{\rho} + \frac{1}{2}\underline{v}^2\right) - \nu \operatorname{curl} \underline{\zeta}, \tag{2.1}$$

$$\underline{\zeta} = \operatorname{curl} \underline{v}, \tag{2.2}$$

$$\operatorname{div} \underline{v} = 0, \tag{2.3}$$

representing the equations of momentum, vorticity ($\underline{\zeta}$) and continuity for an incompressible fluid of constant kinematic viscosity (ν), where $\underline{v}$ is the velocity, p the pressure, t the time and ρ the density. For discussions of the Bénard and related problems, it is necessary also to consider the "energy" equation for the temperature distribution and also an equation of state. The set of equations then needs to be solved with appropriate boundary and initial conditions.

3. Prototype problems

In order to explain in a relatively simple way the progress which has been made, we shall treat four prototype problems first and then allow for recent and important variations in Section 4. These four problems are (a) plane Poiseuille flow, (b) circular Couette flow between rotating concentric cylinders, (c) Bénard (gravitational) convection and (d) the Stokes layer.

(a) Plane Poiseuille flow is driven between two parallel planes by a uniform pressure gradient acting in the x direction. The flow is unidirectional, with a velocity distribution $\bar{u}(y) = 1-y^2$, which is zero at each of the planes $y = \pm 1$. The Reynolds number is $R = U_o h/\nu$, where U_o is the maximum velocity and 2h the distance between the planes. Perturbations proportional to

exp $i(\alpha x + \gamma z - \alpha c t)$, where z is the transverse or spanwise coordinate, must satisfy the classical Orr-Sommerfeld equation. If, in accordance with one convention, we insist on α and γ being real, then the imaginary part of c $(=c_r + i c_i)$ yields instability if $c_i > 0$. The neutral curve defined by $c_i = 0$ is illustrated in Figure 2, of which further theoretical details can be found in the article by Stuart (1963). Within the curve, $c_i > 0$, and there is a temporal instability. The critical Reynolds number (R_c) is about 5772 with a critical wave number $\alpha = \alpha_c = 1.02$ at $\gamma = 0$.

More recently Nishioka, Iida and Ichikawa (1975) have given substantial experimental confirmation of Figure 2 in the Reynolds number range 6500-7500. Furthermore the work of Reynolds and Potter (1967), Pekeris and Shkoller (1967), Chen and Joseph (1973) and Itoh (1974), based on the earlier work of Stuart (1960) and Watson (1960), indicates the bifurcation at $R = R_c$, $\alpha = \alpha_c$ to be of the subcritical form (Fig.1(b)). Consequently there is instability possible at $R < R_c$ if a threshold amplitude is exceeded. An encouraging comparison between theory and experiment for $R = 5000$ is shown in Fig.3, where the threshold amplitude of Itoh's version of the theory is compared with the observations of Nishioka et al. This experimental work is of considerable importance in giving confidence in the basic theoretical ideas.

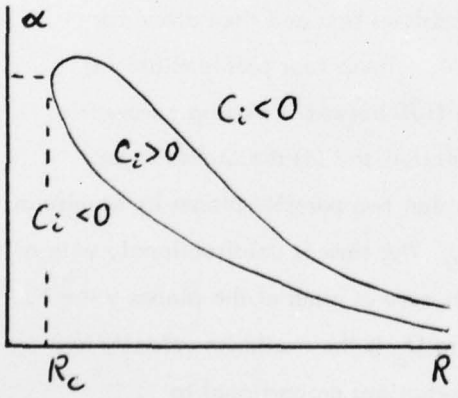


Fig.2. Neutral curve for plane Poiseuille flow.

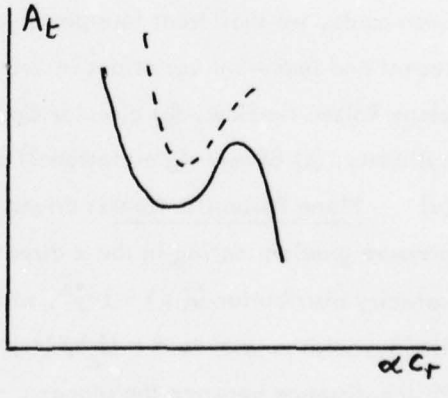


Fig.3. Threshold amplitude (A_t) against frequency (αc_r).

(b) Circular Couette flow occurs between two concentric cylinders of infinite length in differential rotation. If the outer cylinder is at rest, but the inner one rotates, perturbation modes of the form $\exp(\sigma t) \cos \lambda z$ occur if the Taylor number, T, exceeds a critical value T_c, where z is the axial coordinate; here λ, the wave number, and σ, the growth rate, are known from computations to be real. From linearized theory it is known that $\sigma = 0$ on a neutral curve shown schematically in Fig.4, with instability ($\sigma > 0$) above the curve. The critical Taylor number T_c is about 1695 with the critical wave number λ_c about 3.13. Many calculations, by Stuart (1958), Davey (1962), Kirchgässner and Sorger (1969) show that the bifurcation at T_c, λ_c is of supercritical type (Fig.1(a)), and there is ample confirmation by experiment within the limits of experimental error.

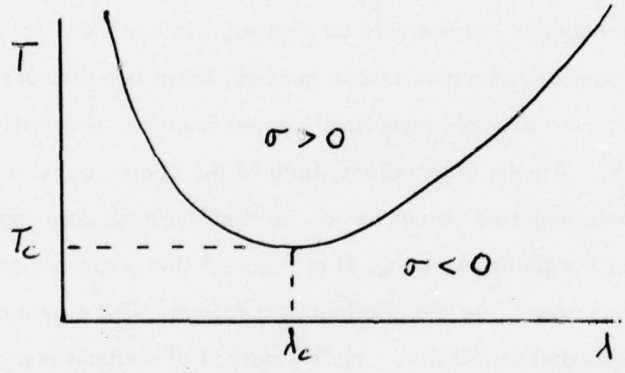


Fig.4. Neutral curve for circular Couette flow.

(c) Bénard (gravitational) convection occurs between two horizontal planes, when the lower one is heated, provided that the Rayleigh number, Ra, exceeds a critical value Ra_c. A variety of plan forms is possible. The neutral curve $\sigma = 0$ for perturbation cellular modes of the form $\exp(\sigma t)\, h(x,y)$, where h is periodic in x and y, is of the form of Fig.4, with Ra replacing T and λ

being a representative wave number within $h(x,y)$. This "perfect" problem, with constant viscosity, is of supercritical type (Fig.1(a)) in its bifurcation structure.

(d) The Stokes layer occurs on an infinite flat wall which oscillates in its own plane with a frequency ω rad/sec. The thickness of the layer is of order $(\nu/\omega)^{\frac{1}{2}}$ and the velocity parallel to the wall ($y=0$) is of the form $\exp(-y)\cos(\tau - y)$, where $\tau = \omega t$. Perturbations of the form $\psi(y,\tau)\exp ia(x-ct)$, with a real and c complex, satisfy a partial differential equation closely related to the Orr-Sommerfeld ordinary equation. An appropriate Reynolds number defined by Hall (1976) is $R = U_o(2/\nu\omega)^{\frac{1}{2}}$. Since there is a semi-infinite domain of flow $0 \leq y < \infty$, there is a continuous spectrum of eigenvalues c (with $c_r \equiv 0$), for which the eigenfunctions $\psi(y, \tau)$ are bounded but not zero as $y \to \infty$. In addition, for certain ranges of Reynolds number, at a given wave number, there is a discrete spectrum, with $c_r \neq 0$. An example from Hall's paper is given, illustratively in Fig.5 for $a = 0.15$. All the eigenvalues, both of the continuous and discrete varieties, have negative values for c_i, so that the modes are damped. Moreover, at the points A, B, C, D of Figure 5 the discrete eigenvalues and eigenfunctions merge into the continuous spectrum, with eigenfunctions which are merely bounded at infinity. Hall's method of analysis is a combination of Floquet theory with numerical analysis. (An earlier analysis of Kerczek and Davis(1974)for a Stokes' layer between two planes gave results which do not reduce to the discrete modes mentioned above, when the stationary plane recedes to infinity.)

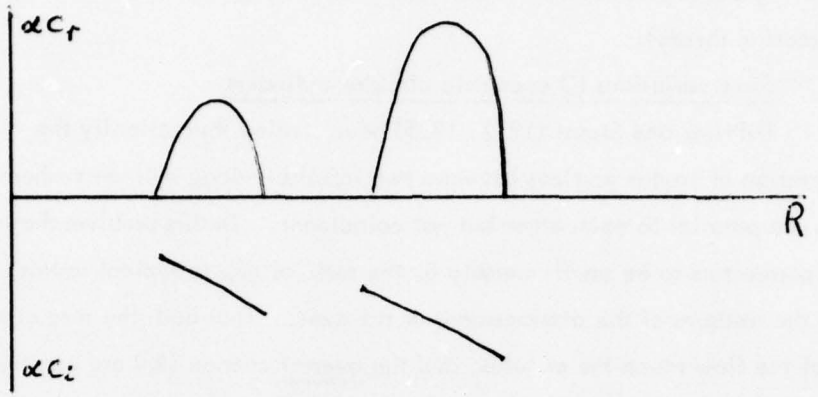


Fig.5. Discrete spectrum for Stokes' layer.

A curved Stokes layer is developed on a circular cylinder of infinite length, which oscillates torsionally about its axis with frequency ω. The Stokes layer is thin compared with the radius a if $\omega\ a^2/\nu \gg 1$, as is assumed. This flow is centrifugally unstable, in a way similar to circular Couette flow, and Taylor-type vortices are developed provided the Taylor number exceeds a critical value (plate 2 of Seminara and Hall 1976). Theoretical work by the same authors shows that the Taylor number $2\,\Delta^2 a^{-1}(\omega/\nu)^{\frac{1}{2}}$, where Δ is the tangential displacement of the cylinder, varies with wave number for a neutral mode in the way illustrated by Fig.4. Here, however, T_c is 164, while λ_c is O.86; experimental observation gave 150 and O.88 for these parameters. In a later paper Seminara and Hall (1977) show that the nonlinear bifurcation is of supercritical type Fig.1(a).

4. Variations on the prototype problems

In this section we consider a number of modifications to the prototype problems, in order to illustrate other aspects of the bifurcation problem. These are (a) slow variations or small changes leading to a shift of the bifurcation Reynolds number, (b) the spectrum of modes following a bifurcation, (c) secondary bifurcations and (d) "end" and other changes which prevent bifurcation

occurring and replace it by a smooth amplitude change (the so-called "perturbed" bifurcation theory).

(a) Slow variations (i) eccentric circular cylinders

DiPrima and Stuart (1972, 1975) have studied theoretically the generation of Taylor vortices between two infinitely-long cylinders when the axes are parallel to each other but not coincident. In this problem they assume two parameters to be small, namely δ, the ratio of gap to typical radius, and ε, the measure of the displacement of the axes. Thus both the rate of change (δ) of the flow round the annulus, and the overall change (ε) are assumed to be small. By assuming that δ and ε are related by $\delta^{\frac{1}{2}} = 2K\varepsilon$ for the case of the inner cylinder rotating and the outer one at rest, with $\varepsilon \to 0$ and K fixed, the authors were able to show in their 1972 paper that the critical Taylor number, T_c, varies with ε in the way shown by Fig.6. For $T > T_c$ Taylor vortices occur, and experimental observations are in substantial agreement with theory for $\varepsilon < 0.4$, with reference to the values of T_c predicted by Fig.6. In the nonlinear calculation, DiPrima and Stuart (1975) showed that the bifurcation is supercritical (Fig.1(a)) but shifted in T according to Fig.6. The Taylor-vortex strength is predicted to have its maximum value at an azimuthal angle of about 50° from the place of maximum gap, a result in accordance with experiment.

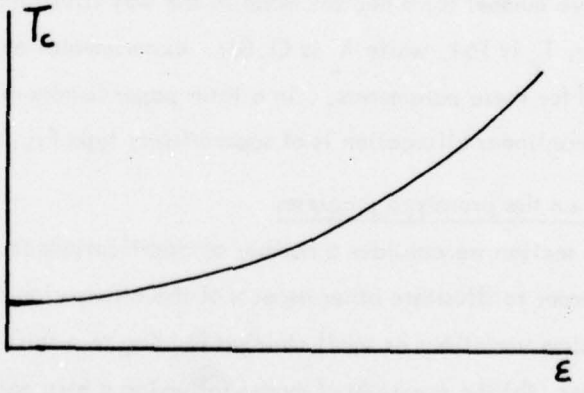


Fig.6. Critical Taylor number (T_c) against eccentricity (ε).

(a) Slow variations (ii) Blasius flow

In contrast to the scheme of approximation mentioned above, slow variations in boundary layers are usually associated with an O(1) change in the flow itself. Thus if in (i) it were decided to study the problem with ε of order 1, a different scheme of approximation would be needed. Such a method is the W.K.B.J. scheme, which has been applied recently by Bouthier (1973), Nayfeh (1974), Gaster (1974) and Eagles and Weissman (1975) to the problem of the stability of the developing Blasius and other boundary layers, with linearized perturbations. Comparison between theory and experiment in the sense of a diagram like Fig.2 is improved by including flow divergence, and a bifurcation still arises (Gaster 1974). However, as Gaster has rightly pointed out, the amplication rate in the x coordinate depends on the property being considered. This is important for the comparison with experiment.

(a) Slow variations (iii) Convection

Many authors (see, for example, Segel and Stuart 1962) have shown that the variation of viscosity with temperature can be important near to the critical Rayleigh number for convection in a layer of fluid heated from below. For example, in the particular case of convection in hexagonal cells a bifurcation point exists, but is changed from the supercritical case (Fig.1(a)) to the Transcritical Form (Fig.1(c)). Thus instability is possible at finite amplitude for Rayleigh numbers below Ra_c.

(b) The Spectrum of modes following bifurcation

As can be seen from Figs.2 and 4 for the particular examples of plane Poiseuille flow and circular Couette flow, once the critical point is exceeded (R_c or T_c) a band of wave numbers is possible for which exponential growth takes place at small amplitudes. At larger amplitudes, for which the nonlinear terms are important, there is a competition between modes of different wave numbers. This competition has been described in various ways, but perhaps

most successfully in terms of the amplitude or evolution equation introduced in a general way by DiPrima, Eckhaus and Segel (1971), Newell and Whitehead (1971) and Stewartson and Stuart (1971).

This topic is perhaps best discussed in terms of the case of plane Poiseuille flow. Suppose we consider a perturbation wave of the form $\exp i\alpha_c(x-c_{cr}t)\ A(\xi,\tau)$, where the suffix c denotes the critical point (α_c, R_c) of Fig.2, and the suffix r denotes the real part. Moreover $\tau = \varepsilon^2 t$ and $\xi = \varepsilon(x-c_g t)$, where $\varepsilon^2 = \alpha_c c_i$ is the growth rate at a Reynolds number slightly above R_c, with wave number α_c and group velocity c_g. If ε is sufficiently small it can be shown that the Amplitude function $A(\xi,\tau)$ satisfies the equation

$$\frac{\partial A}{\partial \tau} - a_2 \frac{\partial^2 A}{\partial \xi^2} = \sigma_c A - kA|A|^2 , \tag{4.1}$$

where σ_c is a real number and a_2 and k are in general complex numbers. Several phenomena of interest can be discussed in the context of this parabolic differential equation: (i) if $k_r > 0$, $\sigma_c > 0$ then an equilibrium solution of (4.1) exists with $|A|^2 = \sigma_c k_r^{-1}$. Whether it is stable or not depends on the Eckhaus-Benjamin-Feir instability mechanism, according to the value of α_c and the complex numbers a_2 and k (Newell and Whitehead 1969, Stuart and DiPrima 1976). (ii) If $k_r < 0$ and for certain ranges of the parameters a_2 and k, a focussed bursting phenomenon can occur, in which the amplitude grows to large values, outside the range of validity of (4.1), but in a finite time (Hocking and Stewartson 1972).

In the case of circular Couette flow and Bénard convection, equation (4.1) is valid but with a_2 and k both real. Work on the question of the stability, and of the preferred mode, has been done by Kogelman and DiPrima (1970) and by Büsse (see, e.g., Clever and Büsse 1974), though the work of the latter is not based on (4.1).

(c) Secondary bifurcations

One of the most interesting examples of such phenomena in fluid flow is that of the development of wavy vortices between rotating circular cylinders. It is known from observations (Coles 1965) that the Taylor-vortex flow is replaced as a stable form of flow by a doubly-periodic flow, periodic in both the azimuthal and axial coordinates and in time. Calculations of instability by Davey, DiPrima and Stuart (1968) and by Eagles (1971) have given a value for the Taylor number, at the secondary bifurcation point, in reasonable agreement with experiment. In a later calculation, of the torque on the inner cylinder, Eagles (1974) showed that this is lower than it would be if the flow had remained of Taylor-vortex form, as shown in Figure 7. Additional work on this problem has been done by Nakaya (1975) and Weinstein (1977).

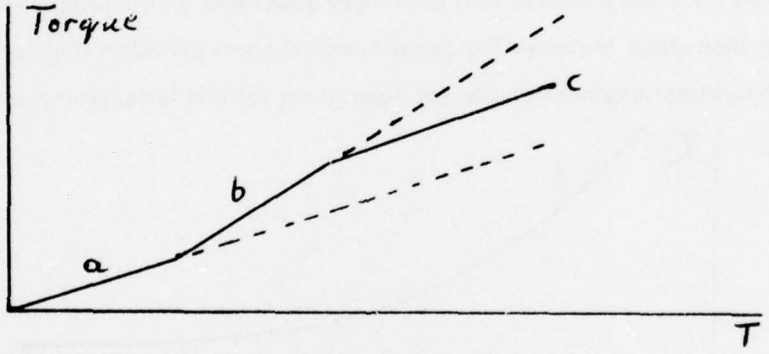


Fig.7. Torque against Taylor number (T)
(a) Laminar flow, (b) Taylor-vortex flow,
(c) wavy-vortex flow; dotted lines unstable.

Further work is needed on this problem in the neighbourhood of the secondary bifurcation point: Eagles (1974) found that the best fit of torque with experiment was obtained by assuming 4 waves in the azimuth, as in Coles observations. No calculation of nonlinear wave interactions has been made to justify this assumption.

(d) Effects of finite geometry

It has been known for some time that, in experiments on instabilities of fluid flows, the lateral boundaries (which, in theoretical work, are assumed to be at infinity) can have a considerable influence on the development of transitions, or bifurcations, from one state of flow to another. For example, several authors have commented on the occurrence of "ghostly" Taylor vortices between concentric cylinders of finite length at speeds well below the critical Taylor number calculated from linearized theory for cylinders of infinite length (Cole 1974a,b, Jackson, Robati and Mobbs 1975, Benjamin 1976b). In addition, Cole (1976) has shown that, whereas the critical Taylor number for Taylor vortices (first bifurcation) is little affected by the length of the cylinders, the critical Taylor number for the occurrence of wavy vortices (secondary bifurcation) is strongly dependent on the length when this is less than about ten times the gap between the two cylinders (Figure 8). No theoretical explanation has yet been given for this latter phenomenon.

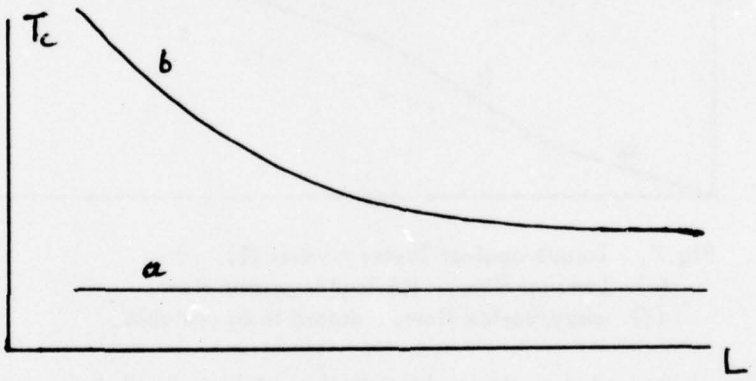


Fig. 8. Critical Taylor number against length (L) of cylinders for (a) Taylor vortices and (b) wavy vortices.

We return now to the matter of the "ghostly" incipient vortices, which are known to occur at low speeds. In various, though limited, ways this phenomenon has been the subject of theoretical investigations by Joseph (1971),

Benjamin (1976a,b), Kelly and Pal (1976), Daniels (1976) and Hall and Walton (1976). A summarized view of the present position can best be given by reference to Fig.1(a) and to Figure 9 which is given below. In the case of cylinders of infinite length a bifurcation of supercritical form occurs (Fig.1(a)), from which stable Taylor vortices occur. When the cylinders are of finite length, however, it is believed (though is in no sense yet proven) that a smooth transition takes place, as shown in the amplitude diagram of Figure 9. This view is in accordance with long established work in elastic stability theory, dating from W.T. Koiter's thesis at Delft in 1945 (see also E.L. Riess's contribution in this volume, and Matkowsky and Riess 1976).

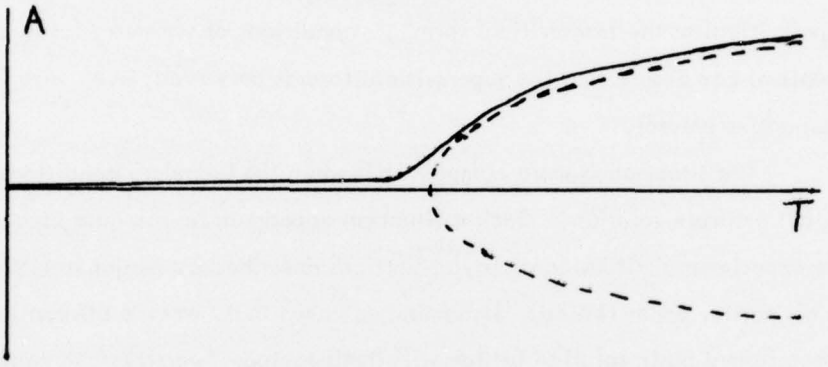


Fig.9. Amplitude (A) against Taylor number (T): full curve, smooth transition; dotted curve, supercritical case.

Thus, the first (Taylor-vortex) bifurcation is replaced by a smooth change, so that Taylor vortices are possible at all speeds, albeit only weakly at low speeds. We note, furthermore, the observation of Cole (1974a,b) that the Taylor vortices occur first at the ends and then spread inwards towards the middle region. In Benjamin's apparatus (1976b) only two vortices occur at lower speeds, starting from rest, and this continues for a range of speeds until a true (secondary) bifurcation occurs to a flow with four vortices. Further bifurcations occur at still higher speeds, and are described by Benjamin in some detail.

From a theoretical point of view not a great deal can yet be said with certainty about instability of flows in a bounded domain. In 1971 Joseph discussed buoyancy-driven convection in a container with some arbitrariness in geometry, but with the crucial condition that the boundary conditions on the container permit a simple solution for the temperature, with no convection. Joseph then shows, by "formal" arguments, that the bifurcation is of the "transcritical" type (Fig.1c) with effects such as (a) viscosity variation with temperature and (b) internal heating. However, when the two latter effects are zero the bifurcation is supercritical (Fig.1a). Benjamin (1976a,b) seems to have misread Joseph's paper when he says (1976a, p.386) that "Joseph has shown that the effect of lateral boundaries is to change the bifurcation from the supercritical to the transcritical form"; conditions of symmetry (as in Joseph's problem) can ensure that the supercritical form is preserved, even with lateral boundaries present.

The situation is more complex still when the boundary conditions do not permit a simple solution. Such a situation appears to be the case especially in the experimental situation of Taylor vortices described by Benjamin (1976b). In his earlier paper (1976a), Benjamin has shown that, when a bifurcation takes place from a basic solution (which will itself include "vortices" in general), the bifurcation is transcritical. Schauder-Leray degree theory is applied with two assumptions: (a) that the eigenvalue is simple; (b) that a certain non-linear expression ($\underline{a}$) is non zero. The latter assumption is tantamount to assuming that the bifurcation is transcritical rather than supercritical. When symmetry dictates that $\underline{a} = 0$, however, the bifurcation will be supercritical, as in Joseph's example quoted above. It seems that Benjamin's view, that bifurcations in closed domains are normally transcritical, overstates the case; supercritical bifurcations can occur.

We now turn to the matter of the calculation (by so-called "formal" means) of the basic solution when a simple solution (without vortices) cannot be obtained. This has not been done for the Taylor-vortex problem. Calculations have, however, been made by Daniels (1976) and by Hall and Walton (1976) for convection in a two-dimensional box with the Rayleigh (free surface and given temperature) conditions on the top and bottom horizontal surfaces of the box, but with zero velocity and a specified normal temperature gradient on each of the vertical end walls. These latter conditions simulate, in a mathematical way, the ends of the cylinders in a Taylor vortex experiment. Daniels' work, which considers the case when the distance between the end walls, L, is very large compared with the depth, d, of the layer, stems from a paper of Segel (1969) on the related problem with homogeneous conditions on the side walls. In Segel's problem a simple solution (with no convection vortices) is possible, and is shown by Segel to lead to a bifurcation at a Rayleigh number of order $(\frac{d}{L})^2$ higher than that for infinite planes (no end walls). Segel's method was that of the amplitude equation and this is followed by Daniels. The non-homogeneous conditions at the end walls force the development of a basic solution with convection cells for the whole range of Rayleigh numbers from zero. The amplitude of the convection velocity then follows the form shown in Fig.9, smoothly by-passing the bifurcating solution for non-homogeneous end conditions. In Daniels' work the deviation of the Rayleigh number from the critical value is limited to be of order $(d/L)^2$, while the normal temperature gradient on the end walls is of order $\delta = \lambda dL^{-1}$, with λ constant.

In the companion, and independent, paper by Hall and Walton (1976), the ratio L/d is not necessarily large, and the authors study the problem defined in the previous paragraph, by reference to the work of Drazin (1975) on the linear eigenvalue problem for homogeneous end-wall conditions. When the normal temperature gradient on the end walls is small and of order δ, the

solution in the interior is forced by those conditions, and convection develops even at a low Rayleigh number. More particularly if the normal temperature gradient on each of the end walls, as a function of height, is of the form of an eigenfunction of the linear problem, the solution will become unbounded if the Rayleigh number is at its critical, or bifurcating, value. In this neighbourhood it is necessary to introduce the nonlinear terms in the differential equations, and to effect this the authors follow an argument introduced by Kelly and Pal (1976) in an important pioneering contribution on a related problem, that of convection in an infinite layer, heated from below with a spatially-varying temperature gradient. This reasoning suggests that, in the neighbourhood of the critical Rayleigh number (Ra_c) of the Hall-Walton problem, the convection amplitude A, the Rayleigh number Ra and the magnitude δ of the normal temperature gradient at the end walls are related by

$$A^3 = A(Ra - Ra_c) + \delta \tag{4.2}$$

A multiple-scaling argument confirms this suggestion. We note, moreover, that if $\delta = 0$, the amplitude bifurcates at $Ra = Ra_c$ from $A = 0$ to $A^2 = (Ra - Ra_c)$, as in Fig.9. On the other hand, if A^3 is ignored we have $A = -\delta/(Ra - Ra_c)$, which is singular at $(Ra - Ra_c)$, due to resonance in the linearized solution. Although Hall and Walton's analysis has no restriction on L/d, a more stringent restriction is necessary for the Ra as $d/L \to \infty$ than in Daniels's work, namely that $Ra - Ra_c \ll O(d/L)^2$. Hall and Walton also analyse cases when the normal temperature gradient on the side walls is not of the form of an eigenfunction; in such cases a true bifurcation can occur, of supercritical or transcritical type (Figures 1a, 1c).

It is clear that much work remains to be done on the calculation of Taylor-vortex and other convective flows, especially to bridge the gap between abstract theory on the one hand and experimental observation on the

other. The work of Kelly and Pal, of Daniels and Hall and Walton constitutes a series of important first attempts in this area. Extensions to convection with more realistic boundary conditions, and to the Taylor-vortex problem, are awaited with interest.

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Preference in Pattern and Cellular Bifurcation in Fluid Dynamics

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1. Introduction

The question, why certain motion patterns are selected in some models of fluid dynamics which exhibit cellular instability, has been in the background of every bifurcation analysis in this field. For twenty years numerous theorectical and experimental contributions with varying physical insight or mathematical rigour have appeared, yet our knowledge is still fragmentary. This survey is devoted explicitly to this problem. However, since the excellent monograph of Joseph [14] which covers the state of affairs completely has appeared recently, we shall raise the question from a new point of view.

The celebrated Taylor - resp. Benard models in fluid dynamics are described by boundary value problems which are invariant under axial translations resp. the rigid motions of the plane.[+)] The invariance property causes a highly degenerate situation and solutions of different invariance structure are possible. Earlier theoretical work in this area has exclusively considered motions which are periodic or doubly periodic with given and fixed periods. An exception are the important contributions of Busse, but there other difficulties arise (see section 4). Recently, in a series of papers, Sattinger has given a systematic treatment of

[+)] Naturally, for this to be true, effects of finite boundaries have to be neglected, an assumption which is not undisputed.

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bifurcation and pattern selection among doubly periodic solutions using group theoretical methods.

In this contribution we shall raise the question whether the assumption of periodicity is reasonable or not, give some answers, substantiate some conjectures, state some open problems, and survey occasionally related results. To avoid technical difficulties and for the sake of a better insight we have chosen a sequence of model equations of increasing degree of complexity. Ad hoc methods which by their very nature are confined to the model equations - such as maximum principle, sub- and supersolutions - are sometimes used but their limited range of applicability is emphasized.

First we treat an easy one-dimensional problem where the reader can check the results by using phase plane methods. However, we phrase our results in a way which opens itself to generalisations. It is shown that almost all "small" solutions are periodic, and that, if a certain geometric condition is met, singular solutions (constant or non-periodic) exist as limits of periodic solutions with large periods. For every positive value of the parameter the set of solutions form a full neighborhood of 0. Nevertheless, also in this case, the selection of a definit solution depends on the geometry of the bifurcation. At the end of section 2 we shall explain the phenomenon for some special nonlinearities.

Only the two-dimensional models in the sections 3 and 4 provide a realistic analogy to the Taylor- and the Bénard problem. Since for these problems, to every supercritical λ the linearisation has solutions with at least two different periods [17], we expect quasiperiodic solutions to occur. In section 3 we show that the λ-range where all solutions are periodic is confined to the interval $(\pi^2, 4\pi^2)$. If $\lambda > 4\pi^2$ holds then, if we accept formal power series as solutions, all small solutions are quasiperiodic. Again

stability is studied for special nonlinearities and conclusions are drawn for the Taylor problem.

In section 4 we briefly discuss the model equation in the plane. The uniqueness result which is the key to the characterisation of all solutions in the earlier sections, is not known in this case. Therefore, we can only describe a formal method consistent with the equations yielding quasiperiodic solutions. However, as is seen by comparison with the linearized equation, the method is far from giving all possible solutions.

Whereas our method consists in a series expansion of the solution and its basic frequencies in terms of the amplitudes of different modes for fixed λ, Busse's method [2] works with fixed frequencies and variable parameter λ and thus is confined to socalled semiregular solutions. The methods are compared and stability is studied. In particular it is shown that, for the model equation, a unique asymptotically stable solution is selected for a special nonlinearity.

2. The model equation in dimension one

In this section we briefly discuss the simplest possible model in one dimension, where the method of phase plane provides a full picture of all solutions. However, we leave this instrument to the use of the reader to check the validity of the results, and we present a procedure which is applicable in higher dimensions as well. The main conclusion is that, under certain symmetry conditions for the nonlinearity, essentially all small solutions are periodic. Although obtained by general methods, its significance for higher dimensions - and especially for the hydrodynamical case - is rather limited, as the comparison with later sections will show. A generally valid feature is the existence of "singular" solutions, if a certain geometrical bifurcation property is met.

As for simple bifurcation problems stability and instability is intimately connected with the geometry of the bifurcation picture [7]; this will be shown for some special cases using the method of super- and subsolutions, which, however, is confined to second order problems exclusively.

Define the differential expressions

$$L(\lambda)u \equiv u_{yy} + \lambda u$$

$$N_j(u,\lambda) \equiv L(\lambda)\, u - f(u,u_y) - ju_t, \quad j = 0,1$$

where subscripts indicate partial differentiation, and where λ is a real parameter. The function $f \in C^2(\mathbb{R}^2,\mathbb{R})$ is supposed to be $o(u)$, i.e. $f(\underline{0}) = f_u(\underline{0}) = f_p(\underline{0}) = 0$. We study the solutions $u \in C^2(\mathbb{R},\mathbb{R})$ of the equation

$$N_0(u,\lambda) = 0 \quad , \tag{2.1}$$

in particular the qualitative characterisation of all small solutions.

If $f = 0$ holds then, for $\lambda < 0$, all nontrivial solutions grow exponentially, for $\lambda = 0$ they grow linearly, whereas for $\lambda > 0$ all solutions are periodic. Restricting attention to solutions of polynomial growth the question arises whether, for $\lambda > 0$, the qualitative picture is maintained under nonlinear perturbations. This is not true in general, as $f(u,p) = p^3$ shows. Before stating additional conditions on f, we formulate a uniqueness result which is trivial for this simple case.

Define

$$\tilde{P}u \equiv \begin{cases} u(0)\cos\sqrt{\lambda}\, y + \frac{1}{\sqrt{\lambda}}\, u_y(0)\, \sin\sqrt{\lambda}\, y, & \text{if } \lambda > 0 \\ u(0) + y u_y(0), & \text{if } \lambda = 0 \end{cases}$$

then, for every $\tilde{u} \in \ker L(\lambda)$ there is at most one solution of (2.1) with $\tilde{P}u = \tilde{u}$.

Now we introduce the following conditions:

(2.2)
$$\text{(a)} \quad f(u,-p) = f(u,p)$$
$$\text{(b)} \quad f(u,-p) = -f(u,p),\ f(-u,-p) = f(u,p)$$

Proposition 2.1. If the function f satisfies the conditions (2.2) a) or b), then, to every $\lambda > 0$, there is a constant $c(\lambda) > 0$, such that all solutions u of (2.1) with $|\tilde{P}u| \leq c(\lambda)$ are periodic.

The proof will be given in the next section. It is of interest to notice that the existence of "singular" solutions is tied to the geometry of the bifurcation picture; e.g. if $f(u,p) = -u^3$ all solutions are periodic, no singular solution exists. Every point $\lambda > 0$, $u = 0$ is a bifurcation point of periodic solutions and, if attention is confined to a fixed period, we have bifurcation to the left. It

turns out that the essential feature is the fact that at $\lambda = 0$ small nontrivial solutions exist.

To be specific: $H^k(\Omega)$ denotes the space of functions having distributional, square integrable derivatives up to the order k in Ω, $H^k_{loc} = \{u \in H^k(K) \mid K \text{ compact in } \Omega\}$. Set $u^\sigma(y) \equiv u(y+\sigma)$ and

$$E(u) \equiv \sup_{\sigma \in \mathbb{R}} \| u^\sigma \|_{H^2(-1,1)} .$$

For fixed λ, the component of connectedness in $H^2_{loc}(R)$ whose closure contains $u = 0$ is denoted by Γ_λ.

Proposition 2.2. Suppose that f satisfies (2.2) a) or b). Assume that for every $\varepsilon > 0$ there is a positive δ such that for all $\lambda \in (0, \delta)$

$$\sup_{u \in \Gamma_\lambda} E(u) < \varepsilon$$

holds. Then, for sufficiently small $\lambda > 0$, there is an open and bounded neighborhood $U(\lambda)$ of 0 in $\ker L(\lambda)$ such that, for every $\tilde{u} \in$ closure $U(\lambda)$, a solution u of (2.1) exists with $\tilde{P}u = \tilde{u}$. For $\tilde{u} \in U(\lambda)$ u is periodic, if $\tilde{u} \in \partial U(\lambda)$ this u is singular (either constant or nonperiodic). In the latter case u is the $H^2_{loc}(R)$-limit of periodic solutions having arbitrarily large irreducible periods.

Typical examples are u^2, u^3, with λ resp. $\pm\sqrt{\lambda}$ and exactly one resp. two nonconstant singular solutions (up to translations). The case $f(u,p) = up$ provides an example where the condition for $E(u)$ is violated and where no bounded singular solution exists. For the proof of proposition 2.2 see [18]. Similar arguments for the existence of solitary waves have been used in [1].

Stability is understood in the sense of Liapunov with respect to solutions of the parabolic equation

$$N_1(u,\lambda) = 0. \quad u\big|_{t=0} = u_0 \tag{2.3}$$

As measure of closeness we use the norm $\|u\| = \sup|u(y)|$. Accordingly, u is called stable if, for every $\varepsilon > 0$ there is a positive δ such that $\|u(\cdot,t)\| < \varepsilon$ if $\|u(\cdot,0)\| < \delta$ for all solutions u of (2.3); furthermore, u is called asymptotically stable, if $\lim_{t\to\infty} \|u(\cdot,t)\| = 0$. A solution u of (2.3) is called unstable if it is not stable. Note, that the proof of instability requires the existence of a solution of (2.3) of arbitrary small norm at $t = 0$ leaving eventually a certain neighborhood of $\tilde{u}$. We shall not dwell on this point since, by the use of the estimates in [20], p. 273, existence can be shown easily.

The special structure of (2.3) can be used to give detailed information about the stability behavior.

However, since the methods (maximum principle, differential inequalities) restrict the range of possible applications to second order problems, we confine ourselves to some special situations exhibiting typical phenomena. We make use of the following

Proposition 2.3. Consider sufficiently smooth functions $\hat{u}$, $\check{u}$ which, together with their derivatives are bounded in $\mathbb{R} \times [0,T]$. Suppose

$$N_1(\hat{u},\lambda) \leqq 0 \quad \text{and} \quad N_2(\check{u},\lambda) \geqq 0$$

$$\hat{u}\big|_{t=0} \geqq \check{u}\big|_{t=0}$$

then $\hat{u} \geqq \check{u}$ in $\mathbb{R} \times [0,T]$.

The proof is a consequence of Theorem 10, [22], p. 183. Obviously

$$f(\hat{u},\hat{u}_y) - f(\check{u},\check{u}_y) = c(\hat{u} - \check{u}) + d(\hat{u} - \check{u})_y$$

with bounded coefficients c and d. The growth condition is fulfilled in view of the boundedness of $\hat{u}$, $\check{u}$ and their derivatives.

We formulate the stability results for nonlinearities f depending on u only. They remain true locally near 0 for more general f, if the leading term is independent of p.

Proposition 2.4

a) Let u_0, u_1 be solutions of (2.1) such that $u_0(y) > u_1(y)$ for at least one y. Assume f to be convex near u_1. Then u_1 is not asymptotically stable.

b) If, in addition, $(u_0-u_1)(y) \geq a > 0$ for all $y \in \mathbb{R}$ and if f is strictly convex, then, u_1 is unstable.

c) Let be $uf'(u) - f(u) \geq 0$ for $u > 0$. Then, every positive solutions of (2.1) is stable. If the same inequality holds for $u < 0$, every negative solutions is stable.

d) If, in addition, $u_0 f'(u_0) - f(u_0) \geq \gamma > 0$ holds for some positive (negative) solution u_0 of (2.1) then, u_0 is asymptotically stable.

For the proof of a) use proposition 2.3 with $\check{u} \equiv \varepsilon u_0 + (1-\varepsilon)u_1$, $\varepsilon \in (0,1)$, and u solution of (2.2) with $u_{|t=0} = \hat{u}$. Assertion b) follows via the same ansatz, if we set

$$\varepsilon(t) = \frac{\varepsilon_0 e^{\frac{a\beta}{2}t}}{1 + \varepsilon_0(e^{\frac{a\beta}{2}t} - 1)}, \quad 0 < \beta \leq f''(u) .$$

To show c), use $\hat{u} = c_1 u_0$, $\check{u} = c_2 u_0$, $0 < c_2 < 1 < c_1$ and remark that $f(u)/u$ is increasing; a similar argument for u_0 negative. For the proof of d) use the same ansatz and set $(a \leq u_0 \leq M)$

$$c_1(t) = \frac{c_1^0}{c_1^0 - (c_1^0 - 1)\exp(-a\gamma t/M^2)} ,$$

$$c_2(t) = \frac{c_2^0 \, e^{\frac{a\gamma}{M^2}t}}{1 - c_2^0 + c_2^0 \exp(a\gamma t/M^2)}, \quad 0 < c_2^0 < 1 < c_1^0$$

We conclude that, if $\lambda < 0$ and $f(u) = u^k$, k even, all periodic solutions are not asymptotically stable, since they are bounded above by $u_0 = \lambda^{1/(k-1)}$. Moreover, for $\lambda > 0$ and $f(u) = u^2$ all periodic solutions are unstable (according to b)), the nonconstant singular solution is not asymptotically stable (see a)) and the constant solution $u_0 = \lambda$ is asymptotically stable (see d)). For $f(u) = u^k$, k odd, and $\lambda > 0$, the constant solutions $\pm\lambda^{1/(k-1)}$ are asymptotically stable.

To prove that all periodic solutions are not asymptotically stable in this case one can use a local version of proposition 2.1 (it is a special form of a theorem which I owe to P. Fife): Let u_1 resp. u_2 be bounded solutions of (2.1) in the open regions Ω_1 resp. Ω_2, where Ω_2 is bounded, $\mathbb{R} = \Omega_1 \cup \Omega_2$ and let $u_{2|\partial\Omega_2} \leq u_{1|\partial\Omega_2}$. Assume u to be a solution of (2.3) in $\mathbb{R} \times [0,T]$ with $u_{|t=0} \geq \max(u_1, u_2)$, then $u \geq \max(u_1, u_2)$ in $\mathbb{R} \times [0,T]$. The proof proceeds along the same line as that of proposition 2.1.

Now, take any periodic solution u_1 and another solution u_2 near u_1 which intersects u_1 in y_0 and y_1, such that $u_2 > u_1$ holds in (y_0, y_1) and

$u_2 < u_1$ outside $[y_0,y_1]$ then, $\max(u_1,u_2)$ is a lower bound for all solutions of (2.3) starting with some $u_0 \geq \max(u_1,u_2)$. Hence, u_1 is not asymptotically stable.

Let us remark, that, if $f(u) = -u^k$ there are only periodic solutions for $\lambda > 0$ and, according to the preceding argument, they are all asymptotically unstable. A usual bifurcation analysis for solutions of (2.1) with a fixed period shows that all branches bifurcate to the left, whereas for $f(u) = u^k$, where stable solutions exist, they bifurcate to the right. All results remain true locally near $u = 0$ if f is of the form $f(u) = u^k(\pm 1 + g(u,u_y))$ where $g(u,u_y) = o(|u| + |u_y|)$.

3. The model equation for a two-dimensional strip

This case already contains all characteristics of some celebrated hydrodynamical models, so that a well founded conjecture about the set of all small solutions is possible. We shall show that their character changes dramatically while λ passes through certain critical points. Quasi-periodic solutions appear. Stability properties are summarized using the "ad hoc" method of proposition 2.1. Less strong results are derived using a general method often used for stability problems in hydrodynamics. At the end we survey possible extensions.

Define the differential expressions

$$L(\lambda) \equiv \Delta + \lambda, \quad \Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$N_j(u,\lambda) \equiv (\Delta + \lambda)u - f(u,u_x,u_y) - ju_t, \qquad j = 0,1$$

where $f \in C^2(\mathbb{R}^3,\mathbb{R})$ and $f(u,p,q) = o(|u| + |p| + |q|)$ holds.

Consider the following boundary value problem

$$N_0(u,\lambda) = 0, \quad u(0,y) = u(1,y) = 0 \tag{3.1}$$

where $(x,y) \in [0,1] \times \mathbb{R} \equiv \Omega$. A solution which is periodic or quasiperiodic in y is simply called periodic resp. quasi-periodic. (3.1) always possesses the trivial solution $u = 0$. We are interested in characterizing all small non-trivial solutions.

Consider the linearized equation $(\Delta + \lambda)u = 0$. For $\lambda < \pi^2$ all nonzero solutions grow exponentially, for $\lambda \in (\pi^2, 4\pi^2)$, $\ker(\Delta + \lambda)$ is 2-dimensional and all solutions are periodic, for $\lambda \in (n^2\pi^2, (n+1)^2\pi^2)$, $n \geq 2$, $\ker(\Delta + \lambda)$ is 2n-dimensional and all solutions are quasiperiodic, i.e. they are of the form $u(x,y) = v(x,\omega_1^0 y,\ldots,\omega_n^0 y)$, $\omega_j^0 = \{\lambda - j^2\pi^2\}^{1/2}$, and $v(x,z_1,\ldots,z_n)$ is 2π-periodic in every z_j. (We say: v is 2π-periodic in z). For the exceptional points $\lambda = n^2\pi^2$ the 2n-dimensional space $\ker(\Delta + \lambda)$ consists of solutions having polynomial growth.

Hence, the spectrum of $\Delta + \lambda$ consists of the semiinfinite interval $[\pi^2, \infty)$ and every point is a nonisolated eigenvalue of finite multiplicity. To be specific: Define $g_k(y) = \max(1, |y|^k)$, $k \in \mathbb{N}_0$, and the Banach-spaces

$$X_k^s = \{u \in H^s_{loc}(\Omega) \mid \|u\|_{s,k} \equiv \sup_{|\alpha| \leq s} \{\int_\Omega g_k^{-2} |D^\alpha u|^2 d\mu\}^{1/2} < \infty\}$$

$$\overset{\circ}{X}{}_k^s = \{u \in X_k^s \mid u(0,\cdot) = u(1,\cdot) = 0\}, \qquad s > 1/2$$

and consider the inductive limits X^s resp. $\overset{\circ}{X}{}^s$:

$$X^s \equiv \bigcup_{k=0}^{\infty} X_k^s, \qquad \overset{\circ}{X}{}^s \equiv \bigcup_{k=0}^{\infty} \overset{\circ}{X}{}_k^s$$

then, $\Delta + \lambda: \overset{\circ}{X}{}^2 \to X^0$ is a topological isomorphism for all $\lambda < \pi^2$ whereas, for $\lambda \in [n^2\pi^2, (n+1)^2\pi^2)$, $\dim \ker(\Delta + \lambda) = 2n$ and $\Delta + \lambda$ is onto. (see [18]).

The question arises under which conditions the qualitative picture is maintained for small nonlinear perturbations. As a first step we formulate a uniqueness result. Define

$$u_\nu(y) \equiv \sqrt{2} \int_0^1 u(x,y) \sin \nu\pi x \, dx$$

$$\tilde{P}_n u = \sum_{\nu=1}^{n} \sin \nu\pi x \{u_\nu(0)\cos \omega_\nu^0 y + \frac{1}{\omega_\nu^0} u_{\nu,y}(0)\sin \omega_\nu^0 y\}$$

if $\lambda \in (n^2\pi^2, (n+1)^2\pi^2)$ and

$$\tilde{P}_n u = \tilde{P}_{n-1} u + \sin n\pi x \, (u_n(0) + u_{n,y}(0)y)$$

if $\lambda = n^2\pi^2$. Set $u^\sigma(x,y) \equiv u(x,y+\sigma)$ and $K_1 \equiv [0,1] \times [-1,1]$.

Proposition 3.1. Let $\lambda \in [n^2\pi^2, (n+1)^2\pi^2)$, $n \in \mathbb{N}$. Then, there exists a number $\varepsilon > 0$, such that, if u and v are solutions of (3.1) satisfying $\tilde{P}_n u = \tilde{P}_n v$ and

$$\sup_{\sigma\in\mathbb{R}} \| u^\sigma \|_{H^2(K_1)} < \varepsilon, \quad \sup_{\sigma\in\mathbb{R}} \| v^\sigma \|_{H^2(K_1)} < \varepsilon ,$$

it follows that $u = v$.

Moreover, for $\lambda < \pi^2$, $u = 0$ is an isolated solution of (3.1) in $\overset{\circ}{X}^2$

In contrast to the one-dimensional case it is here by no means trivial that the behavior at infinity can be controlled by the trace of u in $\ker(\Delta + \lambda)$ since many solutions of exponential growth are present. The proof goes far beyond the frame of this survey; we refer the interested reader to [18].

As the preceding section suggests, existence cannot be proved without further symmetry properties of f. Hence we make one of the following two assumptions

(3.2)

a) $f(u,p,-q) = f(u,p,q)$

b) $f(u,p,-q) = -f(u,p,q)$, $f(-u,-p,-q) = f(u,p,q)$

Proposition 3.2. Let be $\lambda \in (\pi^2, 4\pi^2)$ and assume that f satisfies (3.2) a) or b). Then, there is a constant $c(\lambda)$ such that, for every $\tilde{u} \in \ker(\Delta + \lambda)$ with $\sup_{\Omega} |\tilde{u}| < c(\lambda)$ there exists a periodic solution of (3.1) with $\tilde{P}_1 u = \tilde{u}$.

Together with the preceding proposition we conclude that, for $\lambda \in (\pi^2, 4\pi^2)$ all small solutions (small in the sense of prop. 3.1) are periodic. Actually, having constructed the solutions u_e which are even in y, every other solution can be obtained by a simple translation $u^\sigma = u_e(\cdot, \cdot + \sigma)$.

We indicate the proof. Set $u(x,y) = v(x,\omega y)$, and consider the auxiliary problem

$$(D(\omega) + \lambda)v - F(\omega, v) = 0$$

(3.3)

$$v(0,z) = v(1,z) = 0, \quad v \ 2\pi\text{-periodic in } z$$

where $D(\omega) \equiv \partial^2/\partial x^2 + \omega^2 \partial^2/\partial y^2$, $F(\omega, v) \equiv f(v, v_x, \omega v_z)$. Solve (3.3) by the method of Liapunov-Schmidt. Set

$$\varphi^j(x,z) = \frac{1}{\sqrt{\pi}} \sin|j|\pi x \cdot e^{iz_j}, \quad j = \pm 1$$

$z_1 = -z_{-1} = z$ and define

$$v = v_1 + v_2, \quad v_1 = \sum_{|j|=1} c_j \varphi^j$$

where v_2 is in the orthogonal complement of v_1 then, v^2 is a function of the c_j's . Using property (3.2) one is able to prove that $N_j(\tau, \underline{c}) \equiv (F(\tau, v_1+v_2), \varphi^j)/c^j$, $\tau = \omega^2 - \omega_0^2$, is a C^{p-1}-function near $\tau = 0$, $\underline{c} = \underline{0}$ and that $N_j(\underline{0}, \tau) = 0$, $N_1(\tau, \underline{c}) = N_{-1}(\tau, \underline{c})$. Therefore, the bifurcation equations read for $\underline{c} \neq \underline{0}$

$$- N_j(\underline{c}, \tau) = 0, \quad |j| = 1$$

and thus have locally a C^{p-1}-solution $\tau(\underline{c})$.

Together with Proposition 3.1 we have thus a complete description of all small solutions in the parameter interval $(\pi^2, 4\pi^2)$. When λ approaches π^2 from the right, singular solutions exist if a certain geometric property is met. The result is literally the same as that given in Proposition 2.2, if $H^2(-1,1)$ resp. $H^2_{loc}(R)$ is replaced by $H^2(K_1)$ resp. $H^2_{loc}(\Omega)$. When λ passes through $4\pi^2$ a new situation arises. For fixed $\lambda \in (4\pi^2, 9\pi^2)$ we have two frequencies $\omega_1^0 = (\lambda-\pi^2)^{1/2}$, $\omega_2^0 = (\lambda-4\pi^2)^{1/2}$ which, in general, are rationally independent, i.e. $n_1\omega_1^0 + n_2\omega_2^0 = 0, n_1, n_2 \in \mathbb{Z}$, implies $n_1 = n_2 = 0$. However, since $n_1\omega_1 + n_2\omega_2$ can be made arbitrarily small, small denominators appear when differential operators are inverted. Formally the constructive method of Liapunov and Schmidt used above for the periodic case can be carried through if f is analytic.

Hence, let us assume that f is analytic in its variables and suppose that one of the conditions (3.2) a) or b) holds. Set $u(x,y) = v(x,\omega_1 y, \omega_2 y)$, where $v(x,z_1,z_2)$ is 2π-periodic in $\underline{z}$. Then, problem (3.1) is transformed into

$$
\begin{aligned}
&(D(\underline{\omega}) + \lambda) - F(\underline{\omega}, v) = 0 \\
&v(0,\underline{z}) = v(1,\underline{z}) = 0, \quad v \ 2\pi\text{-periodic in } \underline{z}
\end{aligned}
\tag{3.4}
$$

where $D(\underline{\omega}) \equiv \partial^2/\partial x^2 + \sum_{j,k=1}^{2} \omega_j\omega_k \partial^2/\partial z_j \partial z_j$,

$$
F(\underline{\omega}, v) = f(v, v_x, \sum_{j=1}^{2} \omega_j v_{z_j}) \ .
$$

Suppose that $\lambda \in (4\pi^2, 9\pi^2)$ is such that ω_1^0 and ω_2^0 are rationally independent. Set $T_2 \equiv (0,1) \times (0,2\pi) \times (0,2\pi)$, $\tau \equiv \omega_j^2 - \omega_j^{02}$ and define the spaces

$$
\begin{aligned}
H^0_{\#} &\equiv \{v \in L^2(\Omega) \mid v \ 2\pi\text{-periodic in } \underline{z}\} \\
H^2_{\#} &\equiv \{v \in H^2(\Omega) \cap H^0_{\#} \mid v(0,\cdot) = v(1,\cdot) = 0\}
\end{aligned}
$$

Decompose v as follows: $v = v_1 + v_2$, where $v_1 \in \ker(D(\underline{\omega}^0) + \lambda)$ and $v_2 \in C^0 \subset H^0_{\#}$, $C^0 \equiv \ker(D(\underline{\omega}^0)+\lambda)^{\perp}$. Then, $D(\underline{\omega}^0) + \lambda: \overset{\circ}{H}{}^2_{\#}/\ker(D(\underline{\omega}^0) + \lambda) \to C^0$ is one to one, has dense range, but is not continuously invertible. Hence, the method of Liapunov-Schmidt fails. Yet, there is a formal power series solution of the form

$$v_2 = \sum_{\substack{|\alpha|\geq 0 \\ |\beta|\geq 2}} c_1^{\beta_1} c_2^{\beta_2} \, {}_1^{\alpha_1} \, {}_2^{\alpha_2} w_{\alpha,\beta}$$

with

$$v_1 = \sum_{|j|=1}^{2} c_j \varphi^j, \quad \varphi_j(x,\underline{z}) = \frac{\sqrt{2}}{\pi} \sin|j|\, \pi x \, e^{iz_j}$$

$j = \pm 1, \pm 2$, $z_{-j} = -z_j$, $c_{-j} = \overline{c_j}$. It is easily seen that there is a unique solution $w_{\alpha,\beta}$ for every α,β. However, the proof of convergence seems to be quite hard since small denominators occur. Apart from the convergence, the system of equations determining v_1 can be solved quite similarly as in the periodic case. Using property (3.2) one is able to prove that $N_j(\underline{\tau},\underline{c}) \equiv (F(\underline{\tau},v_1+v_2),\varphi^j)/c_j$, $|j| = 1,2$, is an analytic function near $\underline{\tau} = 0$, $\underline{c} = 0$, satisfying $N_{-j}(\underline{\tau},\underline{c}) = N_j(\underline{\tau},\underline{c})$. Then, if $\underline{c} \neq \underline{0}$, the bifurcation equations, determining v_1, read:

$$\tau_j - N_j(\underline{\tau},\underline{c}) = 0, \quad j = 1,2$$

with $N_j(\underline{0},\underline{0}) = D_\tau N_j(\underline{0},\underline{0}) = 0$. Hence, a unique analytic solution $\underline{\tau}(\underline{c})$ exists near $\underline{c} = 0$ (c.f. [18]). A similar analysis holds for $\lambda \in (n^2\pi^2,(n+1)^2\pi^2)$, $n \geq 2$. Together with Proposition 3.1 it supports the following

Conjecture 3.3. Let be $\lambda \in (n^2\pi^2,(n+1)^2\pi^2)$, $n \in \mathbb{N}$, $n \geq 2$. Set $\omega_j^0 \equiv (\lambda - j^2\pi^2)^{1/2}$ and suppose that $\omega_1^0,\ldots,\omega_n^0$ are rationally independent. Assume that (3.2) a) or b) holds for f which is real analytic near 0. Then, there is a positive constant $c(\lambda)$ such that for every $\tilde{u} \in \ker(\Delta+\lambda)$ with $\sup_\Omega|\tilde{u}| < c(\lambda)$ there exists a quasiperiodic solution of (3.1) with $\tilde{P}_n u = \tilde{u}$.

Very likely, the assumption of rational independence is not sufficient and stronger non-resonance conditions have to be imposed (c.f. [21]). It is of interest that - at least formally - this method gives a systematic approach for the determination of all solutions.

Stability and instability can be studied with the same method as in the preceding section. Proposition 2.3 is still valid and hence, we conclude from Proposition 2.4, that periodic and quasiperiodic solutions u_1 are not asymptotically stable if a solution u_0 exists satisfying $u_0(x,y) > u_1(x,y)$ for $(x,y) \in (0,1) \times \mathbb{R}$. A bifurcation analysis near $\lambda = \pi^2$ shows that, if the leading terms of f are $au^2 + bu^3$, $ab > 0$, $a + b \geq 0$, the even solution, starting in $u = 0$, $\lambda = \pi^2$, majorizes all small periodic solutions.

But for the proof of this local result one does not really need the special structure of our model equation. A well known argument from hydrodynamical stability theory shows that a periodic solution bifurcating in $(0,\lambda(\omega^0))$ cannot be stable if $\lambda(\omega^0)$ is not a local minimum of $\lambda(\omega)$ (c.f. [27], [16], [19]). We shall briefly indicate this argument for the present case.

Let ω_0 satisfy $\pi^2 < \omega_0^2 < 4\pi^2$ and consider $\omega_1 = \omega_0/n$ for some large integer n. Then $\lambda_1 = \pi^2 + \omega_1^2 < \pi^2 + \omega_0^2 = \lambda_0$ holds. Fix ω_1, let λ vary and confine the bifurcation analysis of (3.1) to solutions which are $2\pi/\omega_1$-periodic in y. The solution $(\tilde{u},\tilde{\lambda})$ emanating from $(0,\lambda_0)$, which has $2\pi/\omega_0$ as irreducible period, lies in this larger class too. The principle of linearized stability is well established for the problem under consideration (c.f. [15]). Hence, the spectrum of $\mathcal{L} \equiv (\Delta + \lambda) - f'(u)$ determines stability and instability of $\tilde{u}$. However, for $\lambda = \lambda_0$, $\tilde{u} = 0$, $\mathcal{L}$ has the positive eigenvalue $(n^2-1)\omega_0^2/n^2$. Since the eigenvalues vary

continuously with $\tilde{u}$ and $\tilde{\lambda}$, there is a full neighborhood $\tilde{u} = 0$, $\tilde{\lambda} = \tilde{\lambda}_0$ with a positive eigenvalue of $\mathcal{L}$. Hence, all solutions bifurcating in $(0,\lambda(\omega^0))$ where $\lambda(\omega^0)$ is not a minimum of $\lambda(\omega)$ are unstable near the bifurcation point.

At the end of this section we transfer our results to the Taylor problem of hydrodynamical stability. The viscous flow between two rotating coaxial circular cylinders of infinite length bifurcates into a space-periodic stationary motion (the Taylor vortices) if the Reynolds-number surmounts a certain critical value λ_{cr}. There are three coupled equations for the three velocity components describing this phenomenon. They are of the type (3.1) if rotationally symmetric solutions are considered (f is purely quadratic). The parameter ω denotes the wavenumber of the vortices ([6]). Ample numerical evidence shows (no proof is known) that the neutral curve $\lambda(\omega)$ is strictly convex with $\lambda \to \infty$ for $\omega \to 0$ and for $\omega \to \infty$. There is a unique wavenumber ω_{cr} determined by $\lambda(\omega_{cr}) = \lambda_{cr}$. Quite a number of experiments indicate that, for $\lambda > \lambda_{cr}$, wavenumbers ranging in a "band" around ω_{cr} can be realised. The wavenumber actually observed depends e.g. on the way the stationary state is reached. ([28]).

For values of λ slightly beyond λ_{cr} there are exactly two wavenumbers ω_1, ω_2 with $\lambda = \lambda(\omega_j)$, which, for almost all λ, are rationally independent ([13]). Hence we may conjecture that in this region, <u>all small rotationally symmetric solutions of the Taylor problem are quasiperiodic</u>.

Some indications in this direction are found [9] and in DiPrima, Eckhaus and Segel [8]. They show that, for λ slightly larger than λ_{cr}, the full system of the Navier-Stokes equations can be formally solved by vortex flows having arbitrarily large periods. If the amplitudes of the harmonic modes are chosen carefully enough one can construct solutions by superimposing periodic flows with wavenumbers

regularly distributed in the amplified interval of the linearisation. Although this interval is bounded below by a positive number the resulting solution has an irreducible period which corresponds to the difference of two neighboring modes and thus is arbitrarily large.

The role of the singular solution in our models is played by the vortex flow with wavenumber ω_{cr} for the Taylor problem. According to the preceding argument it is the only possible stable solution near λ_{cr}. We have already mentioned that - as experiments show - for larger values of λ a band of wavenumber seems to be stable. Our preceding stability argument implies the existence of a strip of unstable wavenumbers inside the neutral curve and tangent to it at $\omega = \omega_{cr}$. Kogelman and DiPrima [19] determined this region quantitatively. However, the theoretical and experimental data's are still far apart. Moreover, nothing is known about regions of attractions for stable solutions.

In the light of the analysis presented here another problem arises. Up to now the theoretical and experimental interest was concentrated on space-periodic solutions of the Taylor problem. However, they very likely form a small subset of all solutions which are quasiperiodic. Naturally it will be almost impossible to decide experimentally whether a given solution is periodic or quasiperiodic. Most of these methods determine phenomenologically the periodicity of the flow by counting the observed number of vortex cells. However, an experimental decision whether a purely periodic motion occurs or not must include an exact determination of the stagnation planes. It seems that for this point numerical experiments simulating the selection properties of nature could be of great value.

4. The model equation in the plane

In this section we discuss equation (3.1) when Ω is the whole of $\mathbb{R}^2$. A full understanding of the qualitative structure of solutions and of their stability would shed new light on the famous Bénard-problem which describes the self excited convective motion in a viscous fluid layer heated from below (c.f. [6]). Although a classical problem of fluid dynamics, it has recently attracted interest of various other fields of science such as geophysics [5], of Laser-theory [11]; for other applications see [24]. Despite the long history and the strong efforts to explain the appearance of a definite cellular instability and despite the beautiful results of Busse [2] and Iudovich [12], our systematic knowledge of it is still fragmentary, to say the least. In an important series of papers D. Sattinger has recently shown that by group theoretic methods the bifurcation equations can be classified according to the invariance properties of cell patterns considered [25] and that these classifications can be used to describe pattern preference by stability arguments among the set of doubly periodic solutions [26].

Let us return to our model equation

$$N_0(u,\lambda) = 0 \tag{4.1}$$

which now is considered in $\Omega = \mathbb{R}^2$. The linearized equation (f=0) has only solutions of exponential growth if $\lambda < 0$; for $\lambda = \lambda_0 > 0$ one concludes, using Fourier transform, that u - as a tempered distribution - has support on the circle $|\underline{x}| = \lambda_0$ and, according to the Paley-Wiener theorem can be extended to an entire function ([29],p.305). Hence u can be represented as follows

$$u(\underline{x}) = \int_{|\underline{\omega}|^2 = \lambda_0} c(\underline{\omega})\, e^{i\underline{\omega}\cdot\underline{x}}\, d\sigma \tag{4.2}$$

where $c(\underline{\omega})$ is an integrable function and $d\sigma$ denotes the line element of the unit circle in $\mathbb{R}^2$.

The first step of a systematic analysis should prove that every small solution has a unique trace in $\ker(\Delta + \lambda)$ determined by (4.2). We are not yet able to give such a proof. Up to now there is also little hope to construct sufficiently many solutions of (4.1) - even formally - "covering" the whole of $\ker(\Delta + \lambda)$. However, confining attention to finite Riemann-sums approximation of (4.2) one can proceed as in the previous sections. Set $\underline{\omega}_j \equiv (\alpha_j, \beta_j)$, $|\underline{\omega}_j|^2 = \alpha_j^2 + \beta_j^2 = \lambda_0$, $\lambda_0 > 0$ fixed, $\underline{\omega} = (\underline{\omega}_1, \ldots, \underline{\omega}_n)$, and let

$$u(x,y) = (\alpha_1 x, \ldots, \alpha_n x, \beta_1 y, \ldots, \beta_n y)$$

where $v(z_1, \ldots, z_n, \zeta_1, \ldots, \zeta_n)$ is 2π-periodic in every z_j and ζ_k (2π-periodic in $\underline{z}$ and $\underline{\zeta}$). Then (4.1) is transformed in the following boundary-value-problem for v :

$$(D(\underline{\omega}) + \lambda)v - F(\underline{\omega}, v) = 0 \tag{4.3}$$

where

$$D(\underline{\omega}) \equiv \sum_{j,k=1}^{n} (\alpha_j \beta_k \, \partial^2/\partial z_j \partial z_k + \beta_j \beta_k \partial^2/\partial \zeta_j \partial \zeta_k)$$

and

$$F(\underline{\omega}, v) \equiv f(v, \sum_{j=1}^{n} \alpha_j v_{zj}, \sum_{j=1}^{n} \beta_j v_{\zeta j}) \, .$$

We treat (4.3) as a bifurcation problem with the 2n-dimensional vector $\underline{\omega}$ as the bifurcation parameter varying near $\underline{\omega}^0$, $|\underline{\omega}_j^0| = \lambda_0$, $j = 1, \ldots, n$. The method of Liapunov-Schmidt is formally applicable if $\underline{\omega}^0$ and f satisfy certain conditions. The condition imposed on $\underline{\omega}^0$ states that

$$(\sum_{j=1}^{n} n_j \alpha_j^0)^2 + (\sum_{j=1}^{n} m_j \beta_j^0)^2 = \lambda_0, \quad n_j, m_k \in \mathbb{Z} \tag{4.4}$$

implies $(n_j, m_j) = \underline{0}$ with the exception of exactly one index. It corresponds to the rational independence of the ω_j^0's of the last section.

The condition imposed on f requires that one of the following alternatives should hold

$$\text{a)} \quad f(u,-p,-q) = f(u,p,q)$$

$$\text{b)} \quad f(u,-p,-q) = -f(u,p,q) \tag{4.5}$$

$$f(-u,-p,-q) = f(u,p,q)$$

Given (4.4), $(\Delta + \lambda)v = 0$ implies

$$v = \sum_{|j|=1}^{n} c_j \, e^{i(z_j+\zeta_j)}$$

where $c_{-j} = \overline{c_j}$, $z_{-j} = -z_j$, $\zeta_{-j} = -\zeta_j$. The formal analysis of the previous section can be carried out for this quasi-periodic case too and we obtain: *If the conditions* (4.4) *and* (4.5) *are satisfied then* (4.3) *has a formal power series solution* $(v,\tau_1,\ldots,\tau_n)(c_1,\ldots,c_n)$ *where* τ_j *is defined by* $\alpha_j^2 - \alpha_j^{0\,2} = \beta_j^2 - \beta_j^{0\,2}$.

We are not yet able to prove this assertion since obviously problems of small denominators arise. In addition, we are not sure whether this approach provides all small solutions.

For comparison let us shortly review the methods of constructing solutions of the Bénard-problem. The most general is the one of Schlüter-Lortz-Busse [27] first applied in Busse's dissertation [2] where u and λ are expanded in a power-series of ε, a formal parameter measuring the amplitude of u. While λ varies, $|\underline{\omega}_j|$ is fixed at the point ω_{cr} where the neutral curve $\lambda(|\underline{\omega}|)$ attains its unique minimum (the neutral curve $\lambda(|\underline{\omega}|)$ being the graph of the smallest eigenvalue of $-\Delta$ for doubly periodic solutions $e^{i(\alpha x+\beta y)}$, with $\underline{\omega} = (\alpha,\beta)$; as for the Taylor problem it

is numerically known to be strictly convex).

Busse shows that his approach is consistent with the so-called Boussinesq-approximation if the lattice of the $\underline{\omega}_j$ is semiregular, i.e. the angle between two neighboring $\underline{\omega}_j$ alternates between two fixed values. In general these solutions are not doubly periodic and again, as Joseph [14] pointed out already, small denominators have to be handled if the method should be justified. Evidently, all regular cell patterns such as rolls, squares and hexagons are contained in Busse's approach. For these cases however, proofs of existence are known, analytical as well as topological ([12], [23], [10]). Our approach however seems to be more general as the previous ones, and the results of the preceding sections even nourish the hope that they eventually yield all solutions.

Stability and instability for the model equation is understood again with respect to solutions of the Cauchy problem $N_1(u,\lambda) = 0$, $u_{|t=0} = u^0$. Observe that, for $f(u,p,q) = u^2$ + higher order terms, there is a constant positive solution bifurcating in $u = 0$, $\lambda = 0$ to the right. Hence, invoking Proposition 2.4 we conclude that, for $\lambda > 0$, among all bounded solutions in some neighborhood of 0 (depending on λ), the constant solution is the only asymptotically stable one.

For the Bénard problem the role of this distinguished solution is played by the motions with $|\underline{\omega}| = \omega_{cr}$. As for the Taylor case one concludes that the graph of $\lambda(|\underline{\omega}|)$ bounds a region of unstable pairs $(|\underline{\omega}|, \lambda)$ below. Among the semiregular solutions of Busse, the rolls are the only stable ones if the Boussinesq approximation is considered. In [27] an argument is given that, if the first order term in the expansion of u contains more than one harmonic mode ($n > 1$) then the linearisation of (4.1) about u contains a positive eigenvalue in its spectrum. The situation for generalisations of the Boussinesq-model is somewhat more complicated.

The interested reader is referred to [3].

If one accepts the principle of linearized stability (no proof is known in the present case) then the results of Busse suggest a strong preference for roll patterns. According to [27] only those rolls can be stable for which $|\underline{\omega}| \geq \omega_{cr}$. The stability regions for rolls in a wide range of Rayleigh-numbers has been determined numerically in [4].

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Mathematical Problems of Dynamo Theory

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1. Introduction

Dynamo theory is concerned with those solutions of the magnetohydrodynamic equations with finite magnetic fields which bifurcate from solutions without a magnetic field. Expressed in physical terms this bifurcatation phenomenon may be described in the following way: Velocity fields without an electric current can be realized in an electrically conducting fluid as in any other fluid. Electrically conducting fluids, however, offer an additional degree of freedom in that they can support electric currents. Dynamo theory deals with those instabilities in which velocity fields with electric currents arise spontaneously from velocity fields without electric currents. The instability mechanism is referred to as the dynamo process.

Since magnetohydrodynamics can be considered as an extension of hydrodynamics to the case of electrically conducting fluids, dynamo theory may be regarded as an extension of the theory of hydrodynamic instability and turbulence. Indeed, dynamo theory parallels the latter in many respects and the same mathematical methods are used in both subjects. The dynamo problem differs, however, from the problem of hydrodynamic stability in that the bifurcating solution introduces a new physical quantity, the magnetic field, which is foreign to ordinary hydrodynamics. It is this phenomenon which has made dynamo theory a particularly fascinating subject of continuum

physics.

Dynamo theory has developed fairly recently. It was not until 1958 that Backus [2] and Herzenberg [22] showed convincingly that the dynamo process of generation of magnetic fields in a homogeneous electrically conducting fluid is possible. Traditionally there has been little interest in the subject among fluid dynamicists. Since it is difficult to achieve magnetic Reynolds numbers in the laboratory which are sufficiently high to provide dynamo action and since it is unlikely that the homogeneous dynamo will ever have technical applications, there has been little reason for engineers to get involved in the subject.

On the other hand the dynamo process plays a prominent role in geophysical and astrophysical applications. The origin of geomagnetism and the mechanism of the solar magnetic cycle have long been puzzling problems to natural philosophers. In 1919 Larmor [25] was the first to propose the dynamo process as the cause for the generation of solar magnetic fields. Cowling [17] made an important contribution in 1934 by demonstrating that axisymmetric and two-dimensional magnetic fields cannot be generated by the dynamo process. Cowling's theorem cast severe doubts on the dynamo hypothesis of the origin of geomagnetism since the observed magnetic field of the earth is nearly axisymmetric. Only the absence of any viable alternative led Elsasser [18] and Bullard [6] to pursue the dynamo hypothesis further. Today we know that nearly all velocity fields give rise to dynamo action if their amplitudes are sufficiently high and that the generation of magnetic fields with minimal deviations from axisymmetry is possible. Since the doubts of whether the dynamo process in the liquid outer core of the earth is feasible have disappeared, the attention of theoretical geophysicists has shifted towards the construction of more or less realistic models of the geodynamo. Many basic

problems of dynamo theory, however, remain unsolved, and it will be an objective of this paper to point out those of particular mathematical interest.

Our review of basic aspects of dynamo theory starts with the derivation of the dynamo equation in section 2. Some insight into the dynamo process can be gained by considering the homopolar or disk dynamo, which is described in section 3. In hydrodynamics it is often important to avoid instability and a number of sufficient conditions for the stability of steady fluid flow have been derived. In dynamo theory the bifurcation of a solution with a magnetic field is considered a positive event which is also referred to as dynamo action. For this reason criteria similar to those in hydrodynamics assume the form of necessary conditions for bifurcation. The simplest necessary conditions for dynamo action result from energy integrals, which are discussed in section 4. The close analogy to energy bounds in the theory of hydrodynamic stability becomes apparent at this point. The anti-dynamo theorems in section 5 represent a more unique aspect of dynamo theory. The theorems have played an important role in the historical development of the field and some of the unproven conjectures continue to pose challenging mathematical problems. Because of the extensive literature it is not possible to give a survey of all solutions of the kinematic dynamo problem. The discussion in section 6 is therefore restricted to general aspects and a particularly simple example. In recent years the focus of theoretical efforts has moved towards the magneto-hydrodynamic dynamo problem, which is considered in section 7. The complexities of this problem have so far prevented the discovery of any general result, such as a general relationship determining the equilibrium amplitude of the magnetic field. The variety of phenomena exhibited by the few existing solutions for magnetohydrodynamic dynamos indicates that much more work will be required before a

comprehensive understanding of this problem is achieved. The paper closes with some remarks on the applications of dynamo theory.

2. The Dynamo Equation

The mathematical description of the dynamo process is based on Ohm's law and on Maxwell's equations in the magnetohydrodynamic approximation in which the displacement current is neglected. This approximation is well satisfied in all problems of interest since in general the fluid velocities measured relative to an inertial system are small compared to the velocity of light. It is convenient to eliminate from the basic equations all variables but the magnetic induction $\underset{\sim}{B}$,

$$\nabla \cdot \underset{\sim}{B} = 0, \quad \frac{\partial}{\partial t}\underset{\sim}{B} = -\nabla \times \underset{\sim}{E}, \quad \nabla \times \underset{\sim}{B} = \mu \underset{\sim}{j}, \quad \underset{\sim}{j} = \sigma(\underset{\sim}{v} \times \underset{\sim}{B} + \underset{\sim}{E})$$

$$\frac{1}{\mu\sigma}\nabla \times \underset{\sim}{B} = \underset{\sim}{v} \times \underset{\sim}{B} + \underset{\sim}{E}$$

$$(1) \qquad \nabla \times \lambda(\nabla \times \underset{\sim}{B}) = \nabla \times (\underset{\sim}{v} \times \underset{\sim}{B}) - \frac{\partial}{\partial t}\underset{\sim}{B}$$

$$(2) \qquad \left[\frac{\partial}{\partial t} + \underset{\sim}{v} \cdot \nabla - \lambda\nabla^2\right]\underset{\sim}{B} = \underset{\sim}{B} \cdot \nabla \underset{\sim}{v}$$

In the last step the assumption has been made that the fluid is incompressible, with the consequence that a solenoidal velocity field can be assumed. The dynamo problem has been almost exclusively considered under this restriction. The inverse of the product of conductivity σ and permeability μ is called the magnetic diffusivity λ. In deriving the second form (2) of the dynamo equation it has been assumed that λ is constant. It is evident from the first form (1) of the dynamo equation that $\underset{\sim}{B}$ remains solenoidal if its initial conditions satisfy this property. For this reason Maxwell's equation $\nabla \cdot \underset{\sim}{B} = 0$ is required only for the initial conditions. For geophysical and astrophysical applications it is of interest to note that the dynamo equation is invariant with respect to

transformations to rotating systems.

In the context of the dynamo problem it is important to emphasize the boundary condition for the magnetic field $\underset{\sim}{B}$ at infinity

$$|\underset{\sim}{B}| = o(|\underset{\sim}{x}|^{-3}) \text{ for } |\underset{\sim}{x}| \to \infty \tag{3}$$

where $\underset{\sim}{x}$ is the position vector. Any field with a "source" at infinity cannot be considered a solution of the dynamo problem. A brief discussion of the boundary conditions at surfaces bounding the electrically conducting fluid is given in the Appendix.

The dynamo problem has traditionally been considered at two different levels. The kinematic dynamo problem addresses itself to the question of under which conditions equations (1) or (2) allow solutions $\underset{\sim}{B}$ that are growing in time when an arbitrarily chosen solenoidal velocity field $\underset{\sim}{v}$ is given. Since the dynamo equation is linearly homogeneous, the amplitude of $\underset{\sim}{B}$ cannot be determined. When a steady velocity field is assumed the growth of the magnetic field is exponential, which is clearly unphysical beyond a finite range of time. In reality the velocity cannot be prescribed independently of the magnetic field, since it must satisfy the equations of motion in the presence of the Lorentz force. Accordingly, the full magnetohydrodynamic dynamo problem is described by the nonlinear system of the equations of motion and the dynamo equation. The nonlinear coupling of the equations reduces the growth of the magnetic field and leads to an equilibration of the magnetic energy. Since the Lorentz force is proportional to $|\underset{\sim}{B}|^2$, however, there is always a finite range in which the kinematic dynamo problem correctly describes the growth of the magnetic field.

Restricting our attention to the stationary aspects of the problem and using the amplitude A of the velocity field as a parameter we may use the language of bifurcation theory to describe the two parts of the dynamo

problem. The kinematic dynamo problem determines the points of bifurcation A_n at which magnetic solutions bifurcate from the purely hydrodynamic basic state. The nonlinear post-bifurcation behavior of the magnetic solution can be described only by the coupled system of the equations of motion and the dynamo equation.

3. The Disk Dynamo

In spite of the simplicity of the dynamo equation (2) it was not known until about 20 years ago whether non-decaying solutions did exist. This may appear surprising in view of the fact that the dynamo principle has long been used for the generation of electromagnetic energy from mechanical energy. The main difference between the dynamo process in a contained homogeneous fluid and in technical dynamos is that the latter depend on a multiply connected distribution of electrical conductivity. Since the self-excited technical dynamo is otherwise quite similar to the homogeneous dynamo it is of interest to consider the disk dynamo, which provides the simplest example of dynamo action.

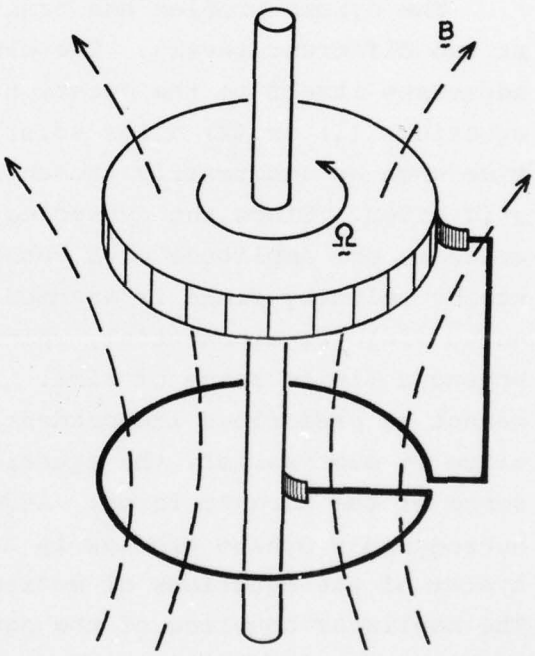


Fig. 1: Disk dynamo

As shown in figure 1, the disk dynamo consists of a metal disk connected to a shaft which is rotating with a angular velocity Ω. An initial weak magnetic field B_o

induces a radial electromagnetic force in the disk which leads to a potential difference U between the rim and axis of the disk. This force is used to drive an electric current I in a circuit which is closed in such a way that the magnetic field of the current has the same direction as the initial magnetic field. When the resistance R of the circuit is low enough, the possibility for a self-excited dynamo exists. Mathematically the process is described by

$$L\dot{I} + RI = U \tag{4a}$$

$$U = \frac{1}{2\pi} MI\Omega \tag{4b}$$

where L is the inductance of the circuit and M is the mutual inductance between the loop and the disk. Equations (4) yield for I and hence for the associated magnetic field a solution of the exponential form,

$$\underset{\sim}{B} = \underset{\sim}{B}_o \exp\{\frac{\Omega M}{2\pi} - R)t/L\}$$

Growing solutions exist when the condition for self-excitation

$$M\Omega/R > 2\pi \tag{5}$$

is satisfied. The complex distribution of conductivity of the disk dynamo permits a simple rotational motion. In the case of the homogeneous dynamo the opposite extreme is realized. A uniform distribution of conductivity requires a complex velocity field for dynamo action. In both cases, however, the condition for self-excitation or "bifurcation" yields expressions of the form (5).

The hydromagnetic problem for the disk dynamo is obtained when the angular velocity is no longer regarded as constant and the torque exerted by the Lorentz force on the disk is taken into account. The earlier stability analysis of this problem by Bullard [7] has recently been extended into the nonlinear regime by Robbins [42] using a modified dynamo suggested by Malkus [28]. In this case the disk dynamo exhibits nonlinear oscillations and aperiodic

reversals similar to those found in previous work [26, 2, 36] on systems of two or more coupled disk dynamos. The time-dependent properties of disk dynamos resemble phenomenologically at least, the excursions and reversals of the earth's magnetic field indicated by the paleomagnetic records.

4. Energy Bounds

In order to obtain necessary conditions for the existence of growing solutions of the dynamo equation (2) we consider a finite domain V of conducting fluid. For simplicity we assume that the magnetic diffusivity λ of the fluid is constant and that the outside space V' is insulating. Multiplication of equation (1) or (2) by $\underset{\sim}{B}$ and integration over V + V' yields

$$\frac{1}{2}\frac{d}{dt}\int_{V+V'}|\underset{\sim}{B}|^2 = -\lambda\int_V|\nabla\times\underset{\sim}{B}|^2+\int_V\underset{\sim}{u}\cdot(\underset{\sim}{B}\times(\nabla\times\underset{\sim}{B})) \tag{6}$$

In deriving this equation it has been assumed that the normal component of $\underset{\sim}{u}$ vanishes on the surface of V and that $\nabla\times\underset{\sim}{B}=0$ holds in V'. For details we refer to Backus [2]. Using the maximum velocity u_m of $|\underset{\sim}{u}|$ in V the inequality

$$\frac{1}{2}\frac{d}{dt}\int_{V+V'}|\underset{\sim}{B}|^2 \le -\lambda\int_V|\nabla\times\underset{\sim}{B}|^2 + u_m\Big\{\int_{V+V'}|\underset{\sim}{B}|^2\int_V|\nabla\times\underset{\sim}{B}|^2\Big\}^{1/2} \tag{7}$$

is obtained from (6). Let k^2 be the solution of the variational problem

$$k^2 = L^2\min\Big\{\int_V|\nabla\times\underset{\sim}{B}|^2\Big/\int_{V+V'}|\underset{\sim}{B}|^2\Big\} \tag{8}$$

where L is the diameter of V and where the variation is extended over all solenoidal fields $\underset{\sim}{B}$ that decay towards infinity. Since (8) always has a finite solution we can use the positive number k to rewrite inequality (7)

$$\frac{1}{2}\frac{d}{dt}\int_{V+V'}|\underset{\sim}{B}|^2 \le -\int_V|\nabla\times\underset{\sim}{B}|^2\{\lambda-u_mL/k\} \tag{9}$$

Thus we obtain as a necessary condition for dynamo action

that the magnetic Reynolds number R_m exceeds the value k (Childress [15]),

$$R_m \equiv u_m L/\lambda > k \tag{10}$$

Backus [2] has calculated the value

$$k_s = 2\pi$$

for a spherical volume of diameter L which can serve as a lower bound for all possible values of k.

The derivation of the necessary condition for dynamo action, $R_m > k$, is essentially the same as for the energy stability criterion for viscous flow [43]. As in the latter case the inequality (9) can be replaced by

$$\frac{1}{2}\frac{d}{dt}\int_{V+V'} |B|^2 \leq -\int_{V+V'} |B|^2\{\lambda k^2/L^2 - m\} \tag{11}$$

where m is the maximum of the largest principal value of the rate of strain tensor $\varepsilon_{ij} \equiv \frac{1}{2}(\partial_j u_i + \partial_i u_j)$ in V. The criterion

$$mL^2/\lambda > k^2 \tag{12}$$

for dynamo action was first derived by Backus [2]. The advantage of inequality (11) is that it remains invariant with respect to a superimposed rotation.

The fact that the dynamo equation does not contain a pressure term as do the Navier-Stokes equations for an incompressible fluid allows the derivation of more specific criteria. The following criterion turns out to be useful in geophysical applications

$$\frac{1}{2}\frac{d}{dt}\int_V |\underset{\sim}{B}\cdot\underset{\sim}{r}|^2 \leq \{-\lambda + D \max\ (\underset{\sim}{u}\cdot\underset{\sim}{r})\}\int_{V+V'} |\nabla\underset{\sim}{r}\cdot\underset{\sim}{B}|^2 \tag{13}$$

where $\underset{\sim}{r}$ is the position vector and D is defined by

$$D \equiv \{\int |\underset{\sim}{B}|^2 / \int_{V+V'} |\nabla\underset{\sim}{r}\cdot\underset{\sim}{B}|^2\}^{\frac{1}{2}}$$

Although the criterion for dynamo action implied by inequality (13) depends on the magnetic field, it is applicable to the earth's core since upper bounds on D are available independently [10].

Criteria of the form (10) or (12) are not really

independent of the magnetic field either, since the Lorentz force may change the maximum velocity considerably. It is desirable therefore to derive criteria which are given in terms of forces acting on the fluid or velocities prescribed at the boundaries. Such criteria have been derived in the case of hydrodynamic stability theory by Payne [33, 34]. The extension to magnetohydrodynamics appears to require new techniques and represents a challenging mathematical problem.

5. Anti-Dynamo Theorems

The most famous theorem of dynamo theory is Cowling's, which states that axisymmetric and two-dimensional magnetic fields cannot be steady solutions of the dynamo equation (2). The original proof by Cowling [17] has been extended by Backus and Chandrasekhar [3] to include toroidal fields and by Lortz [27] to allow for variations of magnetic diffusivity.

Because of Cowling's theorem it was doubtful for a long time whether the dynamo hypothesis of the origin of the earth's magnetic field is feasible. Later work, in particular the work of Braginskiy [4, 5], has demonstrated that magnetic fields with arbitrarily small deviations from axisymmetry can be generated by the dynamo process if the magnetic Reynolds number is high enough. Since Cowling's theorem is often misquoted in the literature it should be emphasized that it does not state that axisymmetric and two-dimensional velocity fields cannot generate magnetic fields. In fact, these velocity fields provide the most convenient way to obtain solutions for the dynamo problem.

A theorem which is probably more important than Cowling's theorem from the practical point of view is the toroidal theorem. It is implicitly contained in an early paper by Elsasser [18] and was proven rigorously by Bullard and Gellman [8]. This theorem is based on the

property that any solenoidal vector field $\underset{\sim}{v}$ can be written in the form

$$\underset{\sim}{v} = \nabla \times (\nabla \times \underset{\sim}{r}\phi) + \nabla \times \underset{\sim}{r}\psi \tag{14}$$

with arbitrary scalar functions ϕ and ψ. The first term on the right hand side of (14) is the poloidal component of the velocity field, the second term is the toroidal component. The toroidal theorem states that a velocity field with a vanishing poloidal component cannot give rise to steady or growing solutions of the dynamo equation (1) if the magnetic diffusivity λ is a function of $|\underset{\sim}{r}|$ only. Using the center of the earth as the origin for the position vector the theorem can be directly applied to the earth's core. It implies that for the geodynamo a non-vanishing radial velocity is required since the latter is associated only with the poloidal field of motion.

The toroidal theorem can also be formulated for a plane geometry using the general representation

$$\underset{\sim}{v} = \nabla \times (\nabla \times \underset{\sim}{k}\phi) + \nabla \times \underset{\sim}{k}\psi \tag{15}$$

for a solenoidal velocity field where $\underset{\sim}{k}$ denotes a constant unit vector. The theorem states that a velocity field of the form

$$\underset{\sim}{v} = \nabla \times \underset{\sim}{k}\psi \tag{16}$$

cannot give rise to dynamo action if λ is a function of $\underset{\sim}{k} \cdot \underset{\sim}{r}$ only. The proof is sufficiently simple that it can be described in a few lines. Using the representation

$$\underset{\sim}{B} = \nabla \times (\nabla \times \underset{\sim}{k}h) + \nabla \times \underset{\sim}{k}g$$

for the magnetic field we obtain by multiplying equation (1) by $\underset{\sim}{k}$

$$(\frac{\partial}{\partial t} - \lambda\nabla^2)\Delta_2 h = \underset{\sim}{v} \cdot \nabla\Delta_2 h \tag{17}$$

where Δ_2 is defined by

$$\Delta_2 \equiv \nabla^2 - (\underset{\sim}{k} \cdot \nabla)^2$$

Equation (17) represents the heat equation with $\Delta_2 h$ replacing the temperature. The variation of λ corresponds to variations of the heat capacity rather than the heat

conductivity. Since there are no "heat sources", $\Delta_2 h$ and consequently the poloidal component of $\underset{\sim}{B}$ must decay. Assuming that this decay has taken place we obtain an equation for g by taking the $\underset{\sim}{k}$-component of the curl of equation (1)

$$\Delta_2 \left(\frac{\partial}{\partial t} + \underset{\sim}{v} \cdot \nabla + \lambda \nabla^2\right) g = 0 \tag{18}$$

where terms involving h have been neglected. Using the requirement that g must be bounded at infinity we obtain by integration

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{v} \cdot \nabla + \lambda \nabla^2\right) g = f(\underset{\sim}{k} \cdot \underset{\sim}{r}) \tag{19}$$

Without affecting the magnetic field $\underset{\sim}{B}$ we can assume

$$\bar{g} = 0$$

where the bar indicates the average over any plane normal to $\underset{\sim}{k}$. By taking this average over equation (19) and using $\underset{\sim}{v} \cdot \underset{\sim}{k} = 0$ we find $f(\underset{\sim}{k} \cdot \underset{\sim}{r}) \equiv 0$. Thus (19) becomes identical to the homogeneous heat equation, which yields only decaying solutions for g.

The proof depends critically on the separation of the equations for the poloidal and toroidal parts of the magnetic field. Since this property cannot be carried over to the case of a cylindrical geometry, it has not yet been possible to prove the theorem in the cylindrical case. In fact, a dynamo has recently been proposed [48] which would violate such a theorem. However it is believed that the respective calculations are in error [41] and that there does not exist a dynamo for a cylindrically symmetric function λ if the radial component of the velocity field vanishes.

The literature of dynamo theory contains a number of conjectures of anti-dynamo theorems which have remained unproven. The most interesting conjecture, first mentioned by Bullard and Gellman [8], concerns the intriguing possibility of an "invisible" dynamo, i.e. a dynamo which generates a purely toroidal magnetic field in

a planetary core. Since the toroidal field in a conducting sphere does not continue into the insulating space outside, in contrast to the poloidal component, it would be difficult to infer the existence of such a dynamo. It is generally believed that the dynamo equation neither admits a purely toroidal magnetic field as a solution nor the twin case of a purely poloidal magnetic field. (The reference for a proof mentioned in [15] does not contain such a proof, unfortunately.) Another unproven conjecture of geophysical interest is that a velocity field confined to meridional planes cannot generate a magnetic field. It appears desirable to obtain a generalization of the toroidal theorem which would include this and other cases and thereby clarify the topological nature of velocity fields required for dynamo action.

6. Kinematic Dynamos

It is not possible in this article to review the numerous solutions of the dynamo equation (2) that have been obtained during the past twenty years. The most extensive and up to 1970 nearly complete review has been given by P.H. Roberts [39] to which we refer the interested reader. Here we shall restrict our attention to some general features and open questions of kinematic dynamos.

A common feature of nearly all homogeneous dynamos is that a large scale magnetic field is generated by a small scale velocity field. Even numerical models that do not require a difference in scales as part of the method of solution exhibit the tendency that dynamo action disappears when the scale of the motion becomes comparable to that of the large scale magnetic field (see, for instance, Gubbins [21]). Although the reduction of Ohmic dissipation because of the large scale of the main component of the magnetic field enhances the chances for dynamo action, the primary reason for the separation of scales is the efficient two-step interaction with the velocity field: the

advection of the large scale magnetic field by the small scale velocity field creates a small scale magnetic field. Since this latter field and the velocity field have approximately the same scale, their interaction can create a mean electromotive force which is required to produce the large scale field.

A second feature of efficient dynamo action is the property of helicity of the velocity field. Helicity is defined by

$$\overline{\underset{\sim}{v} \cdot \nabla \times \underset{\sim}{v}}$$

where the spatial average may be extended over two- or three-dimensional domains. Helicity is neither sufficient nor necessary for dynamo action. A velocity field of the form (16) may have finite helicity, even though it cannot provide dynamo action according to the toroidal theorem. On the other hand there are dynamo solutions [38, 9] based on velocity fields with vanishing helicity. In these cases, however, the interaction of the fluctuating fields which generates the mean magnetic field is of higher order.

To illustrate some typical features of dynamo solutions we describe briefly what is likely to be the simplest example of a spatially periodic dynamo. We assume a two-dimensional velocity field of the form

(20a) $$\underset{\sim}{v} = \nabla \times \underset{\sim}{k}\psi + \underset{\sim}{k}w$$

where $\underset{\sim}{k}$ is the unit vector in the z-direction and ψ and w are given by

(20b) $$\psi = A \sin \alpha x \sin \alpha y; \quad w = C\psi/A$$

Since neither z nor t appear explicitly in the dynamo equation

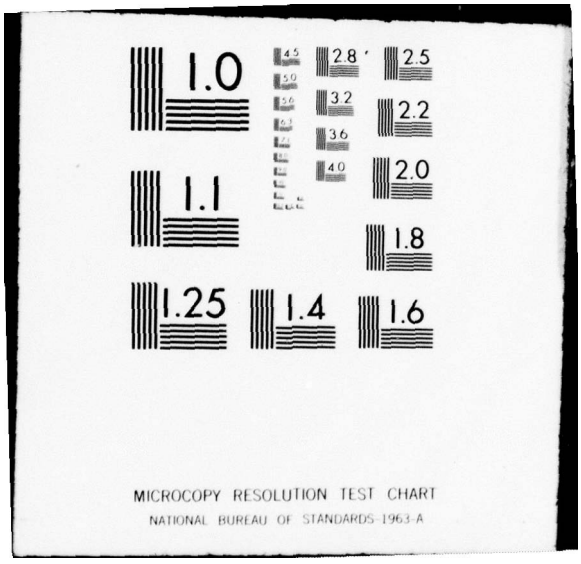
$$\underset{\sim}{B} \propto \exp\{pt + i\gamma z\}$$

can be assumed. Denoting the x, y-average by a bar we obtain separate equations for the mean $\overline{\underset{\sim}{B}}$ and the fluctuating field $\underset{\sim}{b} \equiv \underset{\sim}{B} - \overline{\underset{\sim}{B}}$ by first taking the x, y-average of equation (2) and then subtracting it from equation (2),

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$$(21a) \qquad (p + \lambda\gamma^2)\bar{\underset{\sim}{B}} = i\gamma \underset{\sim}{k} \times (\overline{\underset{\sim}{v} \times \underset{\sim}{B}})$$

$$(21b) \qquad (p - \lambda\nabla^2)\underset{\sim}{B} = -i\gamma w \bar{\underset{\sim}{B}} + \bar{\underset{\sim}{B}} \cdot \nabla \underset{\sim}{v} + \ldots$$

On the right hand side of (21b) we have neglected terms involving $\underset{\sim}{B}$ since we shall assume

$$|\underset{\sim}{B}| << |\bar{\underset{\sim}{B}}|$$

This assumption is justified in the limit $\gamma^2 << \alpha^2$ if we restrict p to the order of magnitude of $\lambda\gamma^2$ as is suggested by equation (21a). The solution for $\underset{\sim}{B}$ is readily obtained,

$$\underset{\sim}{B} = \frac{1}{2\lambda\alpha^2} \bar{\underset{\sim}{B}} \cdot \nabla \underset{\sim}{v} + \ldots$$

and its use in equation (21a) yields

$$(22) \qquad (p + \lambda\gamma^2)\bar{\underset{\sim}{B}} = -i\gamma \frac{AC}{4\lambda} \underset{\sim}{k} \times \bar{\underset{\sim}{B}}$$

This equation is solved by

$$(23) \qquad \bar{B}_x + i\bar{B}_y = B_1 \exp\{pt + i\gamma z\},$$

$$(24) \qquad p = -\lambda\gamma^2 + \gamma AC/4\lambda$$

where B_1 is a constant depending only on the initial conditions. Obviously, steady or growing solutions exist for

$$(25) \qquad \lambda\sqrt{\alpha AC/\alpha\gamma} \geq 2$$

This dynamo condition is written in such a form that the left hand side represents the harmonic mean of the magnetic Reynolds number with respect to the small length scale α^{-1} and the large scale γ^{-1}. The lines of the magnetic field (23) resemble the steps of a right hand staircase for $AC > 0$. The same sense of turning is exhibited by the velocity field corresponding to a positive helicity. An illustration is given in figure 2. The above dynamo represents one of G.O. Roberts' [38] dynamos in the limit of small γ and we refer to his paper for a more exact treatment. Periodic kinematic dynamos based on three-dimensional velocity fields have been considered by Childress [16] and G.O. Roberts [37].

The property exhibited by equation (22) that the interaction of small scale velocity and magnetic fields

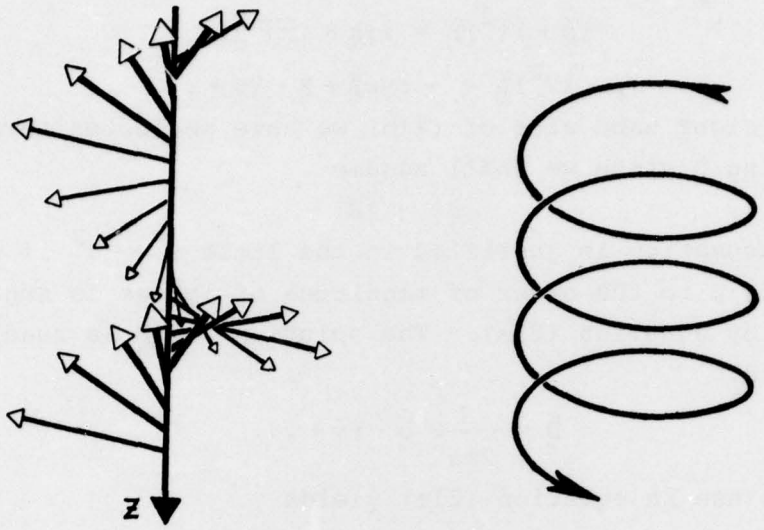


Fig. 2: a) Turning sense of velocity field
b) Mean magnetic field B as given by (23)

provides a mean electromotive force has led Steenbeck, Krause, and Rädler [47] to propose the concept of mean field electrodynamics in which the dynamo equation (2) is replaced by

$$\frac{\partial}{\partial t}\bar{B} + \lambda \nabla \times (\nabla \times \bar{B}) = \alpha \nabla \times \bar{B} \tag{26}$$

in the simplest case. The quantity α is proportional to the helicity of the small scale turbulent velocity field. A finite value of α is commensurate with the assumption of isotropy as long as the turbulence is not mirror symmetric. In general α must be replaced by a tensor in order to take into account the non-isotropy of the small scale velocity field.

Because of their simplicity equation (26) and its extensions which include the effects of large scale motions have been favored in applications of dynamo theory to problems of geophysical and astrophysical interest. Starting with the early work of Parker [32] who used an equation similar to (26) the applications have demonstrated

that turbulent dynamo theory is capable of modeling the generation of observed magnetic fields. Because of the simplifying assumptions and because of the unknown properties of the turbulence, however, the models are not unique and do not offer opportunities for detailed comparison with observational data.

7. The magnetohydrodynamic dynamo problem

The nonlinear magnetohydrodynamic dynamo problem is an unusual bifurcation problem of intrinsic mathematical and physical interest besides its important applications. Because the bifurcating solution is characterized by a new physical quantity it is governed by a significantly different system of equations than the hydrodynamic solution from which it bifurcates. Unfortunately, the same properties which make the problem particularly interesting also contribute to its difficulties and at the present time the theoreticians of the field are far from comprehending the variety of phenomena which are described by the full set of coupled equations.

Before we enter the discussion we wish to mention the two most urgent reasons for an understanding of the magnetohydrodynamic problem from the point of view of applications. A striking property of stellar and planetary magnetic fields is the large variation of magnetic field strength. In many cases this observational quantity may well contain the key for the understanding of the underlying dynamo mechanism. While kinematic dynamo theory can predict the form of the magnetic field if the velocity field is known, it cannot provide an estimate for the amplitude. Only the magnetohydrodynamic dynamo problem incorporates the Lorentz force, the balance of which with the other forces in the equation of motion determines the equilibrium value of the magnetic energy.

An even more important limitation of kinematic dynamo models is their lack of uniqueness. In contrast to

earlier views which associated dynamo action with special velocity fields only, it has become evident as the number of solutions increased during the past decade that nearly all velocity fields possess dynamo action if the magnetic Reynolds number is high enough. Hence little can be concluded from the form of large scale magnetic fields about the velocity fields that provide the dynamo action. Since direct observations of the relevant velocity fields are usually scarce the degeneracy of kinematic models can only be resolved if the equations of motion are solved. Even though the forces are not known either in many cases, the choice of physically reasonable forces is much more restricted and solutions of the full magnetohydrodynamic dynamo problem are likely to provide additional constraints such that a unique determination of the dynamo process becomes possible.

A typical difficulty in solving the magnetohydrodynamic dynamo problem is encountered already at the level of the purely hydrodynamic solution. There does not seem to exist an exact analytical solution of the Navier-Stokes equations for physically realizable forces which provides dynamo action. Indeed, nearly all known exact solutions can be written in the form (16) which is barred from dynamo action by the toroidal theorem. Thus the solutions of the magnetohydrodynamic dynamo problem depend on hydrodynamic approximations in addition to those connected with the magnetic part.

Thermal convection in rotating or non-rotating layers heated from below has been used in all magnetohydrodynamic models since it offers convenient small scale motions in addition to its relevance to geophysical and astrophysical applications. The fact that convection itself represents a bifurcating solution from a basic static state facilitates the mathematical analysis. The equations of motion for the velocity vector χ, and the heat equation for the deviation θ of the temperature from

the static solution T_o are given by

$$\frac{\partial}{\partial t}\underset{\sim}{v} + \underset{\sim}{v} \cdot \nabla \underset{\sim}{v} = -\nabla\pi - \beta \underset{\sim}{g}\theta + \nu\nabla^2 \underset{\sim}{v} + \frac{1}{\mu}(\nabla \times \underset{\sim}{B}) \times \underset{\sim}{B} \tag{27a}$$

$$\nabla \cdot \underset{\sim}{v} = 0 \tag{27b}$$

$$\frac{\partial}{\partial t}\theta + \underset{\sim}{v} \cdot \nabla\theta = -\underset{\sim}{v} \cdot \nabla T_o + \kappa\nabla^2\theta \tag{27c}$$

where β, ν and κ denote the coefficient of expansion, the kinematic viscosity, and the thermal diffusivity, respectively. $\underset{\sim}{g}$ is the gravity vector. The Boussinesq approximation has been used which takes into account the temperature dependence of the density in the gravity term only. All terms that can be written as gradients are included in $\nabla\pi$ since they do not affect the dynamics of the system. In the case of a rotating system the Coriolis force term $-2\underset{\sim}{\Omega} \times \underset{\sim}{v}$ must be added on the right hand side of equation (27a).

The only systematic analytical method for the investigation of the post-bifurcation behavior of dynamo solutions is the expansion in powers of the amplitude of the magnetic field. Since the magnetic field enters the equations of motion (27a) quadratically, the velocity field can be expressed in an expansion of even powers while the magnetic field is given by an expansion in odd powers. By the same argument $(\underset{\sim}{v}, -\underset{\sim}{B})$ is a solution of the magnetohydrodynamic dynamo problem for any solution $(\underset{\sim}{v}, \underset{\sim}{B})$, which is the basic reason for the fact that both polarities of the earth's magnetic field have occurred about equally in the geologic history of the earth.

The analysis starts with the hydrodynamic problem described by equations (27) with $\underset{\sim}{B} \equiv 0$ and appropriate boundary conditions. After the velocity field has been determined the kinematic dynamo problem (2) is solved in the second step. Among the solutions for the magnetic field the fastest growing mode is assumed to be the physically realized one. The Lorentz force exerted by this mode

is introduced as a perturbation into equation (27a) and the resulting modification of the velocity field is calculated in the third step. In the fourth step the modification of the velocity field is taken into account in the dynamo equation (2). According to Lenz' rule it can be expected that the modification leads to a reduction of the growth rate of the magnetic field. Since the velocity modification grows proportional to $|\underset{\sim}{B}|^2$ the growth rate decreases monotonically until it vanishes, at which point the equilibrium amplitude of the magnetic field is reached. This behavior characterizes steady dynamos as well as those producing an oscillatory magnetic field, in which case the above conclusions apply to the mean amplitude.

While the four-step procedure is generally applicable, it has been applied only to the case of convection. In accomplishing the first step it is not a trivial matter to avoid motions of the form (16) since in general the most simple kind of convection in the form of rolls is physically realized [43, 24]. For this reason convection rolls with a superimposed mean shear flow were used in an early magnetohydrodynamic dynamo [9]. The equilibrium value of the magnetic energy in this case is of the order of the change of the gravitational energy of the layer owing to convection multiplied by the ratio of magnetic to viscous diffusivity. Soward's dynamo model [45] based on convection in a rotating layer requires a three-dimensional convection pattern. Although steady three-dimensional convection solutions are unstable [24], Soward finds that there exist time-dependent three-dimensional magnetic solutions which appear to be stable. While Soward investigated the case of asymptotic high Taylor numbers for which the horizontal scale of motion becomes small compared to the vertical scale, Iooss [23] has more recently analyzed the case of moderate Taylor numbers. Using functional

analytic methods he gives a comprehensive discussion of the various possibilities of bifurcation. Of particular geophysical relevance is the case of convection with gravity and rotation vectors at right angles. Convection columns aligned with the axis of rotation are realized in this case. They are representative of motions in the earth's core if the Lorentz force is small compared to the Coriolis force. The dynamo model [11] based on this type of convection exhibits a steady magnetic field in the form of a dipole aligned with the axis of rotation, which agrees in first approximation with the earth's magnetic field. Since the model shares with the other models the restriction to weak magnetic fields, a prediction for the amplitude of the earth's magnetic field cannot be made.

Because Lenz' rule does not have the character of a rigorous law in magnetohydrodynamics it is not surprising that the equilibrium amplitude of the magnetic field is not always reached monotonously. Soward [45] finds that for sufficiently vigorous convection no equilibrium value can be obtained within the framework of his model. The property that the critical value for dynamo action decreases with increasing magnetic field strength corresponding to unstable bifurcation must be regarded as the rule rather than the exception in rotating systems since the Lorentz force tends to counteract the stabilizing influence of the Coriolis force. An example of this phenomenon has been studied recently [13] in an extension of the model [11]. In a particular case the nonlinear dynamo equation for the dimensionless magnetic field $\bar{\underset{\sim}{B}}$ can be expressed in a simple form in analogy to the kinematic equation (22),

$$\text{(28a)}\quad \frac{d}{dz}\bar{\underset{\sim}{B}} = \underset{\sim}{k}\times\bar{\underset{\sim}{B}}\left[\int_0^z |\bar{\underset{\sim}{B}}|^2 dz - z\left(\delta + \int_0^1 |\bar{\underset{\sim}{B}}|^2 dz\right)\right] + d\underset{\sim}{j}$$

where $\underset{\sim}{j}$ represents the unit vector in the y-direction and d is the constant of integration. The boundary conditions

are

(28b) $$\underset{\sim}{B}(0) = 0; \quad \underset{\sim}{j} \cdot \underset{\sim}{B}(1) = 0$$

Solutions of this nonlinear boundary value problem exist at much lower values of δ when $\int_0^1 |\underset{\sim}{B}|^2 dz$ is finite than when it vanishes as in the linear limit of problem (28).

Various aspects of magnetohydrodynamic dynamo theory have been considered on the basis of turbulent dynamo action. Moffatt [30, 31] and Soward [46] considered dynamo action by random inertial waves in a rotating system. Malkus and Proctor [29] investigated the effect of a large scale velocity field induced by the Lorentz forces of a turbulent dynamo. Frisch et al. [20], and Pouquet et al. [35] have emphasized the interaction between the helicities of magnetic and velocity fields as a dominant feature of turbulent dynamo action. The main difficulty of these theories is that the concept of an isotropic turbulence with finite helicity appears to be too idealized to be directly applicable to physically realized dynamos. The variety of effects exhibited in the idealized case of isotropic magnetic turbulence gives some indication, however, of the much larger variety of phenomena to be expected in more realistic cases of the magnetohydrodynamic problem.

8. Concluding remarks

It is evident from the discussion in the preceding section that answers to some of the key questions of nonlinear dynamo theory such as the question of the equilibrium amplitude of the magnetic field are still elusive except in the small amplitude limit. This is not surprising in view of the fact that similar questions in the case of hydrodynamic turbulence cannot be answered on the basis of present theories. There is hope, however, that in special cases, as for example, rapidly rotating systems, the fully nonlinear dynamo may be more readily accessible to theoretical analysis.

From geophysical and astrophysical observations it is evident that there is a close relationship between rotation and the dynamo process. It is generally believed that a dynamic reason for the generation of a magnetic field is that it tends to alleviate the constraint of rotation on fluid motion. Indeed, even the simple example of convection in a layer heated from below shows that while magnetic field and rotation inhibit convection when occurring separately their inhibitory effect is strongly reduced when they occur together [14]. Roberts and Stewartson [40] have recently studied nonlinear aspects of this interaction and derived arguments for a balance between Coriolis force and Lorentz force in the earth's core. It can be argued that only the ageostrophic component of the Coriolis force requires balancing by the Lorentz force and that for this reason the strength of the magnetic field in the earth's core is much lower than commonly assumed [11]. The balance of the ageostrophic component of the Coriolis force in the case of convection appears to yield a relationship for the amplitude of the magnetic field which compares well with the values of the three planetary magnetic fields for which a dynamo origin can be assumed and detailed measurements are available [12]. Hopefully, these speculations can be replaced by rigorous theoretical arguments in the not too distant future.

Appendix: Boundary conditions

The boundary conditions at a stationary surface which separates regions of different magnetic diffusivity λ are usually given in terms of the magnetic field $\underset{\sim}{B}$ and the electric field $\underset{\sim}{E}$

(A1) $$[\underset{\sim}{B}] = 0$$

(A2) $$[\underset{\sim}{n} \times \underset{\sim}{E}] = 0$$

where $\underset{\sim}{n}$ is the normal unit vector of the surface and the brackets indicate the difference between the values of the

enclosed quantity on both sides of the interface. It has been assumed that magnetic permeability does not change across the interface.

The condition (A2) has two disadvantages. First, it is not independent of condition (A1); secondly, it is formulated in terms of the electric field $\underset{\sim}{E}$ which has been eliminated from the dynamo equation (1). In order to obtain a more suitable boundary condition we assume for simplicity the case of a plane interface. Replacing $\underset{\sim}{n}$ by the constant normal vector $\underset{\sim}{k}$ of the plane we find that (A2) is equivalent to

$$\text{(A3)} \qquad [\nabla \cdot \underset{\sim}{k} \times \underset{\sim}{E}] = 0, \quad [\underset{\sim}{k} \cdot \nabla \times (\underset{\sim}{k} \times \underset{\sim}{E})] = 0$$

except in the case of a homogeneous field $\underset{\sim}{E}$, which is not of much interest in dynamo theory and can be considered separately if necessary. Since $\nabla \cdot \underset{\sim}{k} \times \underset{\sim}{E} = -\underset{\sim}{k} \cdot \nabla \times \underset{\sim}{E}$ the first of conditions (A3) is included in condition (A1) because of Maxwell's equation $\partial/\partial t \, \underset{\sim}{B} = -\nabla \times \underset{\sim}{E}$. The second condition (A3) can be expressed in terms of $\underset{\sim}{B}$ by using Ohm's law,

$$\text{(A4)} \qquad [\lambda \underset{\sim}{k} \cdot \nabla \, \underset{\sim}{k} \cdot \nabla \times \underset{\sim}{B} + \underset{\sim}{k} \cdot \nabla \times (\underset{\sim}{\gamma} \, \underset{\sim}{k} \cdot \underset{\sim}{B})] = 0$$

In the derivation of (A4) it has been used that $\underset{\sim}{\gamma} \cdot \underset{\sim}{k}$ vanishes at the interface. Conditions (A1) and (A4) represent necessary and sufficient conditions at a plane interface. For the spherical surface a condition similar to (A4) can easily be derived. In special cases where the outside medium is insulating ($\lambda_o/\lambda_i \to \infty$) or infinitely conducting ($\lambda_o/\lambda_i \to 0$) simplifications of the conditions can be achieved.

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Bifurcation Theory and Permanent Waves

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The purpose of this paper is to give an overview of some aspects of existence proofs for permanent waves that have recently been published by the author using a uniform method (cf. [34]). The subjects treated here include:

- relationships between sinusoidal capillary-gravity waves and cnoidal capillary-gravity waves;
- capillary-gravity waves of two superposed fluids;
- capillary-gravity waves with general vorticity distributions;
- tidal waves and capillary-gravity waves in circular channels;
- flow around a gas bubble with a prescribed parallel and circulatory flow at infinity, the force of gravity being taken into consideration.

The last section of this article deals with a number of open problems.

The theory of permanent waves has a long history. Airy, Boussinesq, Kelvin, Korteweg, de Vries, Rayleigh, and Stokes were among the nineteenth-century investigators who concerned themselves with this theory. The first rigorous proofs of the existence of a permanent wave motion (infinite -depth gravity waves) were independently given by Nekrassov (1921) and Levi-Civita (1924). Investigators who thereafter occupied themselves with this subject include Struik, Kotschin, Lichtenstein, Dubreil-Jacotin, Lavrentyev, Friedrichs, Hyers, and Krasnoselski. In the past twenty years, numerous other mathematicians also concerned themselves with the subject (cf. [31], [34]).

Our own studies followed the lines set out by the work of H. Beckert ([6, 7, 8, 9]). The objective of these studies was to

- simplify the constructive existence proofs given by Levi-Civita, Struik, Kotschin, and Dubreil-Jacotin;
- replace topological existence proofs by constructive existence proofs;
- produce existence proofs for problems that have not yet been dealt with;
- study correlations between the various problems.

Our treatment of the problems is by the steps summarized below:

- The plane hydrodynamic problem is expressed as a nonlinear free boundary value problem for the Laplace equation or more general nonlinear elliptic differential equations (in the case of vorticity distributions).
- A conformal or quasiconformal mapping gives a nonlinear boundary value problem with an eigenvalue parameter for the Cauchy-Riemann differential equations or for more general integro-differential equations (in the case of vorticity distributions). The equations are defined in the unit circle for infinite depth channels, in an annulus for the finite depth case, or in nonlinearly coupled unit circles or annuli for superposed fluids.
- These problems may be studied with the aid of bifurcation theory.

Only a few basic principles will be outlined below. Information about further contributions to these problems can be found in the references (see especially [31], [34]) at the end of this paper. In particular see [34] for more details on the mathematical formulation of the problems studied below.

§ 1. Capillary-Gravity Waves

The waves considered here are plane waves of wavelength λ, under the action of a gravitational accelleration g and surface tension β which propagate with a constant velocity c in a channel of depth h. The shape of the surface $y = y_0(x)$ is fixed in an x, y coordinate system moving with the wave.

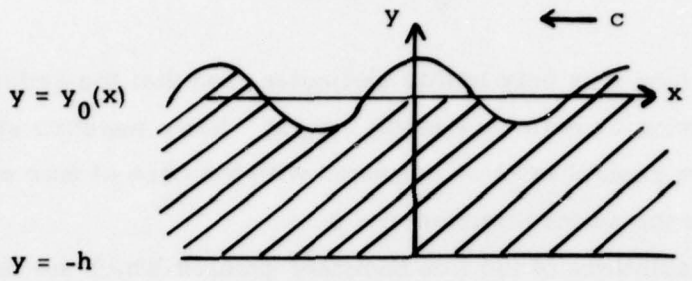


Figure 1

To solve this free boundary value problem, we seek $y = y_0(x)$ and $c = c(\lambda)$. We assume that the fluid is of density μ and is inviscid, homogeneous, and incompressible, and that the motion is irrotational. Standard approximations [34] to the partial differential equations which describe this problem show that there are two types of waves for $\lambda < \infty$, namely

i) sinusoidal waves (Kelvin, 1871)

$$(1) \qquad y_0(x) = \frac{\lambda}{2\pi}\, s \cdot \sin\frac{2\pi x}{\lambda}, \quad c^2 = \left(\frac{g\lambda}{2\pi} + \frac{2\pi\beta}{\lambda\mu}\right)\tanh\frac{2\pi h}{\lambda}$$

ii) cnoidal waves (Korteweg, de Vries, 1895)

$$(2) \qquad y_0(x) = 3a^2h\left(1 - \frac{3\beta}{g\mu h^2}\right)k^2 \cdot \mathrm{cn}^2\left(\frac{3ax}{2h}, k\right)$$

$$c^2 = gh, \quad \lambda = 4K(k)\, h/3a, \quad 0 < k < 1,$$

where $\mathrm{cn}(.)$ is Jacobi's elliptic function and $K(k)$ is the complete elliptic integral of the first kind.

In (1) and (2) s and a are small parameters so $h \ll \lambda$ always holds in ii). This is called shallow water theory. What is also important is the existence in ii) of a critical depth

(3) $$h_0 = \sqrt{3\beta/g\mu} \quad .$$

For $h < h_0$ (and it is only in this particular case that the author succeeded in giving an existence proof), $y_0(x)$ has a negative sign giving us a negative cnoidal wave as compared with the case of pure gravity waves where the surface tension $\beta = 0$.

Exact solutions of the free boundary problem which are power series expansions with respect to the small parameters s and a can be determined. We pose the following questions:

- How can these exact solutions be found and their Taylor coefficients be calculated ?
- How are i) and ii) related?
- Can one let $h \to \infty$ and $\beta \to 0$, respectively, in the exact solutions in order to obtain, in i), infinite depth capillary-gravity waves ($h = \infty$) and finite depth gravity waves ($\beta = 0$)?

It will be seen that the answer to these questions involves an interesting bifurcation effect occurring because our problem leads to a nonlinear boundary value problem for the annulus $\{\zeta: 0 < q < |\zeta| < 1\}$. Here q is a parameter in terms of which we will determine the allowable range of values of h.

If $0 < q < 1$, we get sinusoidal capillary-gravity waves resulting from bifurcation from simple eigenvalues;

as $q \to 0$, we get sinusoidal infinite depth capillary-gravity waves;

and as $q \to 1$, cnoidal waves are obtained in an interesting case of asymptotic bifurcation.

Mathematically, the problem is to find an analytic function $W = \varphi + i\psi$ in the region of flow ratifying the boundary conditions:

$$(4)\qquad \left.\begin{cases} g\mu y_0(x) + \frac{1}{2}\mu |W'(z)|^2 + p_0 - \beta \frac{d\vartheta}{dt} = \text{constant} \\ \psi = 0 \end{cases}\right\} \text{at } y = y_0(x)$$

$$\psi = -\psi_0 = \text{constant at } y = -h .$$

(Here p_0 = external pressure, $\vartheta = \arctan y_0'(x)$, t = arclength along $y = y_0(x)$, and $z = x + iy$). Making the ansatz

$$W' = \bar{c}\, e^{-i\omega}, \quad \omega = \vartheta + i\tau ,$$

after a conformal mapping [34], the following boundary value problem results for the annulus

$$K_q \equiv \{\zeta = \rho e^{i\sigma} : 0 < q < \rho < 1\}$$

(where ρ, σ are polar coordinates).

Required are constants $b, p > 0$, and an analytic function in K_q, $\omega = \vartheta + i\tau$, satisfying the following boundary conditions

$$(5)\qquad -b \frac{d\vartheta}{d\sigma} = \frac{1}{2}(e^{\tau} + d(\vartheta, \tau) e^{-\tau}) \frac{\bar{c}}{c} - p \frac{c^2}{\bar{c}^2} \int_0^{\sigma} e^{-\tau} \sin\vartheta d\sigma$$

$$\equiv F(\vartheta, \tau) \quad \text{at } \rho = 1$$

$$\vartheta = 0 \quad \text{at } \rho = q$$

and symmetry conditions

$$\vartheta(\rho, -\sigma) = -\vartheta(\rho, \sigma), \quad \tau(\rho, -\sigma) = \tau(\rho, \sigma)$$

$$\int_{-\pi}^{\pi} \tau(q, \sigma) d\sigma = 0 ,$$

where

$$b = \frac{2\pi\beta}{\mu\lambda c^2}, \quad p = \frac{g\lambda}{2\pi c^2}, \quad \frac{\tilde{c}}{c} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\tau(q,\sigma)}\, d\sigma$$

$$q = \exp(-\frac{2\pi\bar{h}}{\lambda}), \quad \bar{h} = \frac{\psi_0}{c} = \text{mean depth}.$$

The functional $d(\vartheta, \tau)$ in (5) is chosen so that

$$\int_{-\pi}^{\pi} F(\vartheta, \tau)d\sigma = 0 .$$

In order to be able to study the relationship between sinusoidal and cnoidal waves, this problem is transformed into an operator equation in the Banach space C_α^-, $0 < \alpha < 1$, of Hölder continuous functions on the boundary of the unit circle $\rho = 1$ with the symmetry condition $\vartheta(1, -\sigma) = -\vartheta(1, \sigma)$.

For each function $\vartheta(1, \sigma) \in C_\alpha^-$ there is exactly one analytic function $\omega = \vartheta + i\tau$, in K_q, with

$$\vartheta(q, \sigma) = 0 , \qquad \int_{-\pi}^{\pi} \tau(q, \sigma)d\sigma = 0 .$$

We set

$$\tau(1, \sigma) \equiv -X_q\, \vartheta(1, \sigma)$$

and

$$Tf(\sigma) \equiv f(\sigma) - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\sigma)d\sigma, \quad Jf(\sigma) \equiv T\int_0^{\sigma} Tf(\sigma)d\sigma .$$

From (5) one gets the operator equation

$$-b\,\vartheta = JF(\vartheta, \tau = -X_q\vartheta), \quad \vartheta \in C_\alpha^- . \tag{6}$$

This operator equation has the following form

$$bx = \sum_{k=1}^{\infty}{}' (N_k(q)x^k + p\, M_k(q)x^k), \quad x \in Y , \tag{7}$$

where $b, p > 0$, $C \leq q \leq q_1 < 1$ are real parameters; $N_1(q)$, $M_1(q)$ are linear, completely continuous operators in the Banach space Y, continuous functions of $q \in [0, q_1]$ in the norm topology of Y; the maps $q \to N_k(q)x^k$, $q \to M_k(q)x^k$ are continuous for all $q \in [0, q_1] = I$, for $\|x\| \leq r$, $k = 2, 3, \ldots$; $N_k(q)x^k$, $M_k(q)x^k$ are k-linear with respect to x; and

$$(7') \qquad \sum_{k=1}^{\infty} \sup_{q \in I} \|N_k(q)\| r^k, \quad \sum_{k=1}^{\infty} \sup_{q \in I} \|M_k(q)\| r^k < \infty .$$

Theorem 1. Suppose

(8) For all $q \in I$, the linearized problem

$$bx = N_1(q)x + pM_1(q)x$$

has a simple eigensolution $x = x_q$, $p = p_q$ which is continuously dependent upon $q \in I$;

(9) There is a linear, continuous functional $y \in Y^*$, so that the following holds for all $q \in I$:

$$bx = N_1(q)x + p_q M_1(q)x + P_q x$$

implies $x = 0$ where $P_q x \equiv y(x)x_q$, $y(x_q) = 1$;

$$(10) \qquad y(R_q M_1(q)x_q) \neq 0, \qquad \forall\, q \in I .$$

(Here $R_q \equiv (b \cdot \mathrm{id} - N_1(q) - p_q M_1(q) - P_q)^{-1}$ exists according to the Riesz-Schauder theory. For $N_1(q) \equiv 0$, (10) follows from (8), (9).)

Then there are numbers r_0, ε_0, $s_0 > 0$, so that for given s with $0 < |s| \leq s_0$ (7) has exactly one solution (x, p) with

$$\|x\| \leq r_0, \quad y(x) = s, \quad |p - p_q| \leq \varepsilon_0 .$$

for each $q \in I$. This solution is analytic with respect to the small

parameter s:

$$(11) \qquad x = sx_q + \sum_{k=2}^{\infty} s^k \bar{x}_k, \quad p = p_q + \sum_{k=1}^{\infty} \varepsilon_k s^k, \quad y(\bar{x}_k) = 0, \quad \forall k \geq 2 ,$$

where x, p, x_k, and ε_k are continuous functions of q.

The proof, which uses the method of Lyapunov-Schmidt is given in [34]. Theorem 1 is closely related to a general theorem by Crandall and Rabinowitz [11]. The relationship to simple algebraic eigenvalues is discussed in [38].

Introducing the Fourier coefficients

$$a_k(\tau) \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} \tau(l,\sigma) \cos k\sigma \, d\sigma ,$$

$$b_k(\vartheta) \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} \vartheta(l,\sigma) \sin k\sigma \, d\sigma ,$$

it is easy to verify the assumptions (8), (9) and (10) for a variety of classes of permanent waves because the effect of all of the above operators on the Fourier coefficients is explicitly known. One then obtains existence proofs for sinusoidal waves ([34]).

The following is a description of a method for the explicit construction of (11). Given that $p = p_q + \varepsilon$, (7) is equivalent to

$$(12) \qquad x = sx_q + \varepsilon R_q M_1 x + R_q \sum_{k=2}^{\infty} N_k(q)x^k + (p_q + \varepsilon)M_k(q)x^k ,$$

and (11), (12) imply that

$$s^2 \bar{x}_2 = s^2(\varepsilon_1 R_q M_1 x_q + H(x_q)) .$$

Then $y(\bar{x}_2) = 0$ gives

$$\varepsilon_1 = -y(H(x_q)) / y(R_q M_1 x_q) .$$

This permits $\bar{x}_2$ to be determined. The coefficients $\bar{x}_3$, ε_2, etc. are obtained in the same manner.

Application of this method to our problem (6) yields

$$\vartheta(1,\sigma) = s \cdot \sin n\sigma + \sum_{k=2}^{\infty} \vartheta_k(\sigma) s^k \tag{13}$$

where ϑ_k are trigonometric polynomials, $p = p_q + \sum_{k=1}^{\infty} \varepsilon_k s^k$, and p_q, ϑ_k, ε_k may be calculated explicitly.

In [38] a number of other methods of calculating (11) without solving the bifurcation equation are given.

The case of infinite depth capillary-gravity waves gives a problem which is analogous to (5) for which K_q becomes the unit circle, i.e. $0 \leq z < 1$. Using the continuous dependence of Theorem 1 in q, it is possible to show

$$\vartheta \text{ (capillary-gravity waves of depth } h < \infty)$$
$$\to \vartheta \text{ (capillary-gravity waves of depth } h = \infty)$$

as $h \to \infty$ in the norm of C_α^-.

The study of the limit process $b \to 0$ (surface tension $\beta \to 0$) is far more difficult because the structure of the operator equations (6), (7) then changes. However, it is possible to show

$$\vartheta_k,\ \varepsilon_k \text{ (capillary-gravity waves } \beta \neq 0)$$
$$\to \vartheta_k,\ \varepsilon_k \text{ (gravity waves } \beta = 0)$$

as $\beta \to 0$ in the norm of C_α^-. Only the coefficients in (13) converge in this case.

The existence proof for cnoidal waves ($q \to 1$) has the following underlying idea. In

$$bx = \sum_{k=1}^{\infty} (N_k(q)x^k + pM_k(q)x^k) , \tag{7}$$

for large p we set

$$q = \exp(-1/p), \quad b = \gamma/p, \quad p \to \infty ,$$

with γ = constant and

$$a = K(k)/\pi p .$$

It is possible to show that

$$S_p \equiv (b \cdot id - N_1(q) - pM_1(q))^{-1}$$

exists for $\gamma \gg 1/3$. Then (7) implies

$$x = S_p\Big(\sum_{k=2}^{\infty} N_k(q)x^k + p\,M_k(q)x^k\Big) \tag{14}$$

with the asymptotic structure

$$x = p^2 \sum_{k\geq 1,\, m\geq 2} p^{-k} S_{km} x^m . \tag{15}$$

Therefore making the ansatz

$$x = p^{-1}u + p^{-2}v ,$$

one obtains the following nonlinear equation for u:

$$u = S_{12}\, u^2 . \tag{16}$$

If u is known, then the equation obtained for v is

$$v = A(p^{-1}, v) .$$

This equation may be solved using Banach's fixed-point theorem.

For this procedure to succeed it is essential that the nonlinear equation (16) may be solved and that asymptotic expansions do exist. Demonstration of the existence of such asymptotic expansions is rather difficult in the case of cnoidal waves, especially if consideration is given to the surface tension. This is due chiefly to the fact that the operators involved are closely related to singular integral operators.

Moreover an existence proof is complicated by (14) having a much more complex structure than (15) (cf. [35]).

Littman's proof of the existence of cnoidal waves ([20]), which does not take the surface tension into account results in a boundary value problem for the strip $0 \leq \psi \leq 1$. The small parameter a in (2) results from the use of Friedrichs' "stretching of the independent variable" method ([13]).

§ 2. Capillary-Gravity Waves of Two Superposed Fluids

Two fluids are considered here, the conditions being identical to those referred to in §1 above. Propagating at the interface is a permanent wave, propagation being influenced by both the gravitational force and the surface tension. After conformal mapping two boundary value problems (which are analogous to (5)) are obtained for two required analytic functions

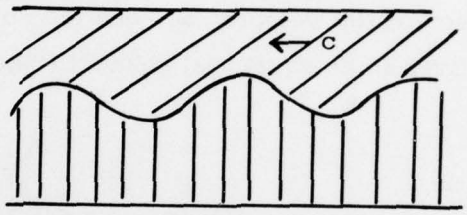


Figure 2

$$\omega_j(\zeta_j) = \vartheta_j(\zeta_j) + i\tau_j(\zeta_j), \quad \zeta_j = \rho_j e^{i\sigma_j}, \quad j = 1, 2$$

in two annuli in the ζ_1-plane and in the ζ_2-plane. What is peculiar about this problem is that it is necessary to determine, not only $\omega_j(\zeta_j)$, but also the unknown relationship between the angular coordinates, namely, $\sigma_2 = g(\sigma_1)$.

The method of proof consists of reducing the problem to an operator equation along $\rho_1 = 1$ for $(\vartheta_1(1, \sigma_1), g(\sigma_1))$ and involving Theorem 1. To verify (7') it is necessary to use sharp estimates for singular integral operators, which are given in [34]. Also, the tools provided in [34] enable other such coupling problems to be dealt with effectively.

In 1928, Kotschin ([16]), with the aid of a complicated method of

majorants was able to demonstrate the existence of gravity waves of the form shown in Figure 2 above. The method of proving the existence of gravity and capillary-gravity waves given in [34] is much simpler.

The existence of capillary-gravity waves is quite interesting from a physical point of view. Whereas in the case of gravity waves the upper fluid must always be lighter than the lower fluid, the reverse is also possible in the case of capillary-gravity waves.

§ 3. Tidal Waves

We will consider here a plane dynamic tidal model in an $x-y$ coordinate system in which the Moon is nonrevolving, the Earth is rotating at an angular velocity ω, and the curve $r = r(\alpha)$ of the ocean surface is fixed in an equatorial channel (Figure 3; r, α are polar coordinates).

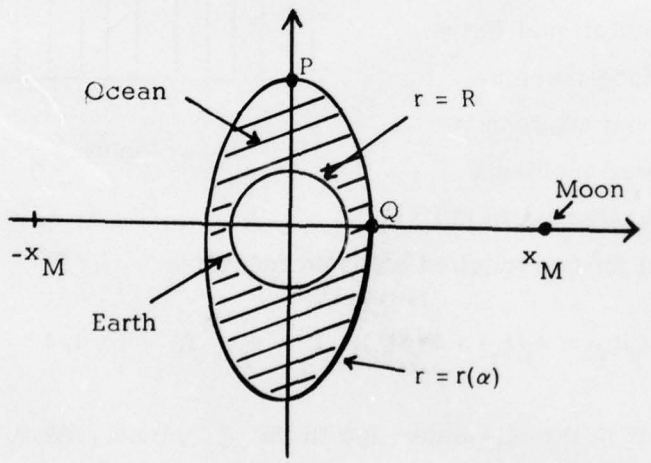


Figure 3

In order to obtain a stable model, it is necessary to use the square approximation of the Moon's gravitational potential. This involves using an anti-Moon at $x = -x_M$ and halving the effect produced by the two moons. Also, assume the ocean fluid be inviscid, incompressible, and homogeneous, and let the motion be an irrotational motion or have a general vorticity distribution.

Of greatest interest is the surface curve $r = r(\alpha)$. The exact proof of existence shows that $r(\alpha)$ has a maximum at P and a minimum at Q (Figure 3). This is in agreement with what is actually observed in nature where the tidal wave does not arrive until the Moon has passed through the zenith. While this effect is qualitatively correctly obtained in our model, the amplitudes of our tidal waves are too small as compared with actual measurements. However, this is not surprising since our tidal model (which was originally proposed by H. Beckert [9]) represents an exact formulation of Airy's channel theory of tides in straight-line channels, where, to a first approximation, the effects referred to above can also be seen to occur.

H. Beckert ([9]) was able to give a topological existence proof for irrotational tidal waves. A constructive existence proof will be outlined here for general vorticity distributions ([3], [5]). In this context we will also study the much more complex case of natural oscillations occurring without a perturbing body (i.e. a moon) being present (i.e. tidal waves on planets having no moons revolving around them). This involves the following remarkable alternative: There are critical angular velocities ω_j such that:

- for $\omega \neq \omega_j$ there exist tidal waves in the presence of perturbing bodies;
- in the neighborhood of ω_j there exist natural oscillations in the absence of perturbing bodies.

To explain the existence of spring and neap tides, it is necessary to also consider the effect of the Sun located at the point $(x_S, 0)$ for a spring tide or $(0, y_S)$ for a neap tide.

For the velocity vector (u, v) we have the differential equations

$$u_x + v_y = 0, \quad u_y - v_x = cf(\psi)/R$$

for the stream function ψ with $u = \psi_y$, $v = -\psi_x$ and the boundary conditions

$$\begin{cases} \mu k \frac{r}{R} + \mu U_M + \mu U_S + p_0 + \frac{\mu}{2}(u^2 + v^2) = \text{const.} \\ \psi = 0 \qquad\qquad\qquad\qquad\qquad \text{at } r = r(\alpha) \end{cases}$$

$$\psi = -\psi_0 = \text{const. at } r = R$$

where U_M and U_S are respectively the Moon's and Sun's potentials, p_0 is the pressure, μ is the density, R is the radius of the Earth, f is the given vorticity distribution, and k is a constant.

After a quasiconformal mapping the following problem for the annulus

$$K_q \equiv \{\zeta = \rho e^{i\sigma} : 0 < q < \rho < 1\} .$$

is obtained.

For a given vorticity distribution $f \in C_\gamma[q, 1]$, $0 < \gamma < 1$, we find two functions with Hölder continuous derivatives

$$\vartheta, \tau \in C_\gamma^{(1)}(\bar{K}_q) ,$$

which satisfy the integro-differential equations

$$\vartheta_\rho - \frac{\tau_\sigma}{\rho} = 0_2(\vartheta, \tau, \tau_\sigma)$$

$$\tau_\rho + \frac{\vartheta_\sigma}{\rho} + 2\tau_0'(\tau + \int_q^\rho \frac{\tau d\rho}{\rho}) = 0_2(\vartheta, \tau, \vartheta_\sigma)$$

$$\exp \tau_0(\rho) = \int_q^\rho \rho(f(\rho)/q^2)\, d\rho + 1 > 0 ,$$

the boundary conditions

$$T\tau - \lambda T \int_0^{\sigma} \vartheta \, d\sigma = (a_M \pm a_S)(\cos 2\sigma + 0_1(\vartheta, \tau)) + (2 + \lambda)\, 0_2(\vartheta, \tau) \qquad \text{at } \rho = 1$$

$$\vartheta = 0 \ , \qquad \text{at } \rho = q$$

as well as the symmetry conditions

$$\vartheta(\rho, -\sigma) = -\vartheta(\rho, \sigma), \quad \tau(\rho, -\sigma) = \tau(\rho, \sigma), \quad \int_{-\pi}^{\pi} \tau(q, \sigma) d\sigma = 0 \ .$$

Here, $0_1, 0_2$ denote first - and second order terms, respectively,

$$Tg(\sigma) \equiv g(\sigma) - \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\sigma) d\sigma \ ,$$

a_M, a_S are small parameters describing the small perturbations due to the Moon and Sun, respectively, and λ is an eigenvalue parameter which is dependent upon the Earth's angular velocity.

The structure of the problem is as follows

$$F(x, \lambda) = 0, \quad x = (\vartheta, \tau) \in C_{\gamma}^{(1)}(\overline{K}_q) \times C_{\gamma}^{(1)}(\overline{K}_q) \ . \tag{17}$$

The existence of natural oscillations where $a_M = a_S = 0$, i.e. in the absence of the Moon and Sun leads to a bifurcation problem and may be treated using Theorem 1. What is rather difficult here is to verify the simplicity of the eigenvalues. This is carried out by showing that the eigenfunctions have the following structure

$$\vartheta(\rho, \sigma) = b(\rho) \sin n\sigma, \quad \lambda = \lambda_n, \quad n = 1, 2, \ldots$$

and that $(b(\rho), \lambda_n)$ satisfies a second-order differential equation that is dependent on n. $\lambda_n \neq \lambda_m$ for $n \neq m$ then follows from nodal properties.

The existence of tidal waves in the presence of both the Moon and Sun may be proved by means of the implicit function theorem because for

$\lambda \neq \lambda_n$, $n = 1, 2, \ldots$ the inverse operator $F_x^{-1}(0, \lambda)$ exists in (17).

The solutions have the following structure

$$\vartheta = \sum_{k, m, p} a_M^k a_S^m s^p \vartheta_{kmp}; \quad \tau = \sum_{k, m, p} a_M^k a_S^m s^p \tau_{kmp}$$

($p \equiv 0$ when perturbing bodies are present).

For irrotational waves ($f \equiv 0$), ϑ_{kmp}, τ_{kmp} may be constructed explicitly (see [34]). In the general case ($f \not\equiv 0$), ϑ_{kmp}, τ_{kmp} are obtained through the solution of ordinary differential equations by means of an iterative method (see [5]).

The method outlined above was used in [39] to prove the existence of sinusoidal capillary-gravity waves of the type referred to in §1 and characterized by general vorticity distributions. However, it should be noted that the method described in [5] has, in a number of points, several improvements over that given in [39].

Further information about capillary-gravity waves in circular channels is given in [34].

§ 4. Flow Around a Gas Bubble

We are interested here in the existence of a plane gas bubble in a steady-state flow of an inviscid, incompressible, and homogeneous fluid, which at infinity, may be seen to be superimposed on a parallel and circulatory flow. Also acting is the force of gravity. What is required is the shape of the gas bubble. (Figure 4).

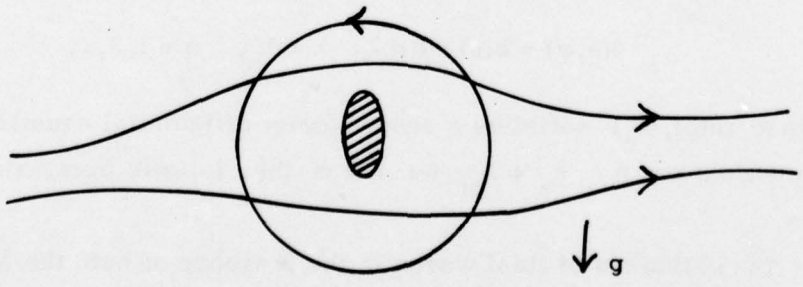


Figure 4

After a conformal mapping one has the following problem: Find an analytic function $\omega = \vartheta + i\tau$ in the unit circle $K \equiv \{\zeta = \rho e^{i\sigma},\ 0 \leq \rho < 1\}$ with the boundary condition at $\rho = 1$

$$\frac{d\tau}{d\sigma} = -1 + Pe^{-\vartheta} + (Z + C\cos\sigma)^2 e^{\vartheta} +$$

$$+ G\, e^{-\vartheta} \int_0^{\sigma} e^{-\vartheta} \sin(\tau(\sigma) + \sigma) d\sigma$$

and the symmetry conditions

$$\vartheta(\rho, -\sigma) = \vartheta(\rho, \sigma),\ \tau(\rho, -\sigma) = -\tau(\rho, \sigma),\ \int_{-\pi}^{\pi} \vartheta(1, \sigma) d\sigma = 0\ .$$

P, Z, C, G are constants.

This problem may be treated using the method outlined in §1 above (Theorem 1) (cf. [37][34]). For previous studies of special cases see [21], [3].

§ 5. Open Problems

Problems which, in the author's opinion, have not yet been studied include the following ones:

- Treatment of the problem in §4 as a three-dimensional problem;
- studies of exact three-dimensional tidal models;
- demonstration of the existence of solitary waves with the surface tension taken into consideration (Figure 5, cf. [13])

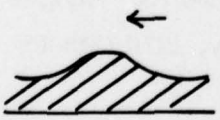


Figure 5

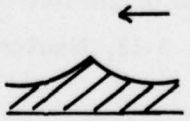


Figure 6

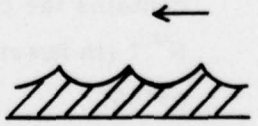


Figure 7

- demonstration of the existence or nonexistence of cnoidal waves with due consideration of the surface tension for $h > h_0$ (cf. (3)) (proof of nonexistence of infinite depth solitary waves has been produced by Krasovski [18]);
- demonstration of the existence of highest waves of permanent type (Figure 6, Figure 7; the wave crest is not smooth and has an angle of magnitude $2\pi/3$, cf. Yamada [32, 33];
- Krasovski [17] showed that there is a permanent gravity wave of infinite depth for every α: $0 < \alpha < \pi/6$, with the angle of the tangent to $y_0(x)$ being equal to α in one point. Does this result also hold for capillary-gravity waves?
- It is recommended that a study should be made to investigate the global structure of branches of solutions for permanent waves in the sense of the basic alternative of Rabinowitz [23].*
- What is the range of values of the parameter s for which the solution expansion (12) converges (cf. [25])?
- In the case of capillary-gravity waves there are critical wavelengths to which double eigenvalues are known to correspond. It is recommended that a study should be made to prove the existence of waves in this case (cf. [34]).

In conclusion, the author wishes to thank the organizing committee and its chairman, Professor Paul H. Rabinowitz, for the invitation to this most interesting symposium.

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* We have learned from J. Norbury at this conference that he and J. Toland have obtained some joint results in this direction.

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The Finite Amplitude Dynamics of Baroclinic Waves

Joseph Pedlosky

1. Introduction.

Solar heating is the ultimate energy source for the motion of both the atmosphere and the oceans; lunar tidal oscillations (relatively weak) excepted. The radiant energy emitted by the sun may vary slightly over very long periods, but a sensible meteorological idealization consists of holding the solar source strength fixed. The rotation of the earth therefore produces a diurnal variation of heating and the orbital progress of the earth produces seasonal change. Nevertheless the movement of the atmosphere as evidenced by the phenomenon of weather reveals the existence of motions at spatial and temporal scales not directly imposed by these astronomically fixed variations of solar heating. A similar examination of oceanic motions has shown the presence of very energetic fluctuations† whose horizontal spatial and temporal scales (of order 300 km and weeks to months, respectively) are not directly imposed by either solar heating or the atmospheric wind forcing which combine to produce the general oceanic circulation.

It is, however, easy (in principle) to imagine a dynamical state of the atmosphere and ocean, consistent in every respect with basic physical principles in which all change is predictable with the appropriate astronomical period, in which each season is identical to its

†The so-called "meso-scale" eddies.

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predecessor, and such that an almanac of the past might serve as a predictor of the future. Although such a dynamical state might be consistent with physical principle, it is not the state which is realized in nature. Instead we see an ocean and atmosphere with a rich spectrum of motions churning and seething, all to the delight of poets and the near despair of those responsible for prediction.

It is natural to take the view that the simpler, periodically repetitive state of the atmosphere and ocean is hydrodynamically unstable with respect to the realized state and that the observed fluctuations are manifestations of the instability of simpler, more serene states of motion. It is not the purpose of this lecture to pursue the full implications of this view (the reader is referred to the excellent monograph of Lorenz for an interesting discussion). Instead, I shall assume its validity and here discuss some questions related to the theory of stability of atmospheric and oceanic currents.

The pioneering studies of Charney (1947) and Eady (1949) showed that large scale atmospheric (and oceanic) flows with realistic distributions of velocity would become unstable with respect to disturbances which release the available potential energy associated with the nearly ubiquitous presence of horizontal density gradients in such flows. The basic explanation for the instability may be easily described.

Consider a fluid under the influence of gravity ($\vec{g}$) which possesses both vertical (parallel to $\vec{g}$) and horizontal density variations. For example,

$$\rho = \rho_0(y,z). \tag{1.1}$$

For the moment suppose $\partial\rho_0/\partial z$ and $\partial\rho_0/\partial y$ are constant. Then the lines of constant density will be inclined to the horizontal as shown in Figure 1. Normally such a state is not in equilibrium. Such inclined density surfaces would tend to rotate into coincidence with horizon-

tal planes (equi-potential gravitational surfaces) giving rise to motion in the y-z plane. If, however, the entire fluid is rotating about the z axis with angular velocity Ω, this density state is a possible equilibrium state in the presence of a fluid flow in the x direction (into the paper) which leaves the density surfaces undisturbed.

Such a rectilinear flow in the x direction gives rise to a Coriolis acceleration in the positive y direction, which in equilibrium is balanced by a pressure variation with y, viz:

$$\rho(2\Omega)u = -\frac{\partial p}{\partial y} \tag{1.2}$$

while in the vertical direction the pressure gradient is balanced by buoyancy,

$$\rho g = -\frac{\partial p}{\partial z} \tag{1.3}$$

Consistency of (1.2) and (1.3) requires that

$$2\Omega \frac{\partial u}{\partial z} = +\frac{g}{\rho}\frac{\partial \rho}{\partial y}\left[\,1 - \frac{(\partial z/\partial y)_\rho}{(\partial z/\partial y)_p}\,\right] \tag{1.4a}$$

or

$$2\Omega \frac{\partial u}{\partial z} = +\frac{g}{\rho}\left(\frac{\partial \rho}{\partial y}\right)_p \tag{1.4b}$$

Hence, when density surfaces and pressure surfaces do not coincide the vertically varying Coriolis force/unit mass, $2\Omega\,\partial u/\partial z$, can balance the pressure forces that in a non-rotating system would give rise to circulations which tend to align the surfaces of density, pressure and geopotential. Since the velocity required to produce the Coriolis force is perpendicular to the density gradient, this density state is a dynamic equilibrium state. Such states are called baroclinic and they occur frequently in the atmosphere, where, for example, the increase of the westerly winds with height balances the poleward increase of density

and in the ocean where the surface intensification of the Gulf Stream is observed to occur in regions of sharp horizontal density variations.

Consider now the vertical displacement of a fluid element from position A in Figure 1 to another position, say B. For simplicity suppose that the fluid element preserves its density during the displacement. When the fluid element arrives at B it will have a density differing from its surroundings by an amount

$$\delta\rho = -\left(\eta \frac{\partial \rho_o}{\partial y} + \zeta \frac{\partial \rho_o}{\partial z}\right) \tag{1.5}$$

where η and ζ are the elemental displacements in the y and z directions, respectively. The Archimedean buoyancy force in the vertical direction per unit mass is $-g\,\delta\rho/\rho$ which when projected on the virtual trajectory gives rise to a force in the direction of motion equal to

$$-g\frac{\delta\rho_o}{\rho_o}\sin\theta = \frac{1}{\rho_o}\frac{\partial \rho_o}{\partial z}\sin^2\theta\left\{1 - \frac{\partial z}{\partial y}\Big)_\rho / \tan\theta\right\}\ell \tag{1.6}$$

where θ is the angle the trajectory makes with the horizontal and ℓ is the length of the virtual displacement. In almost all natural situations of interest the fluid is gravitationally <u>stable</u> to vertical displacements, i.e. $\partial\rho_o/\partial z < 0$, i.e. if the fluid element is displaced vertically to point C in Figure 1, it experiences a <u>restoring</u> force proportional to displacement and returns to its equilibrium position. However, (1.6) reveals that if the element is displaced so that its trajectory satisfies

$$0 < \tan\theta < (\partial z/\partial y)_\rho \tag{1.7}$$

the resulting force in the direction of motion is positive and the fluid element will further accelerate away from equilibrium. Referring to Figure 1, it is evident that this requires fluid trajectories to lie within the wedge defined by the horizontal geopotential surface and the

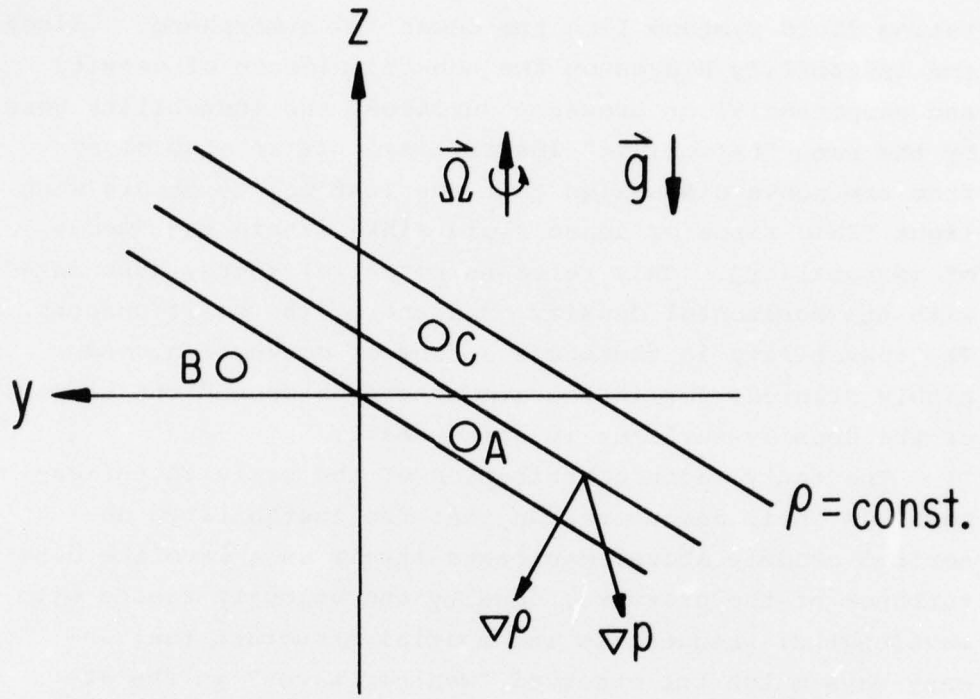


Fig. 1. The "wedge of instability" is determined by the lines ρ = constant and the geopotential surface z = constant. Displacements which interchange fluid elements A and C are stable. Those which interchange A and B are unstable.

surface of constant density. Such a wedge opening exists only due to the role of rotation in the first place and consequently the resulting instability is peculiar to rotating fluid systems like the ocean and atmosphere. Since the instability hinges on the non-coincidence of density and geopotential or pressure surfaces, the instability goes by the name "baroclinic" instability. It is also clear from the above discussion that the instability occurs when light fluid rises or dense fluid sinks within the "wedge of instability". This releases potential energy associated with the horizontal density gradient of the basic current. The instability is therefore a form of convection, and highly slanted, for in the atmosphere or oceans the slope of the density surfaces is quite small.

The truly major contribution of the early investigators was their demonstration that the instability, described crudely above, manifests itself as a wavelike disturbance of the pressure, density and velocity fields with wavelengths, frequencies and spatial structure that in many ways match the observed "weather waves" in the atmosphere. The theory of baroclinic instability then became a central concept in understanding the nature of the atmospheric general circulation and recent and future oceanic field experiments will determine whether, as suspected, it is of similar importance in the ocean.

An implicit philosophical assumption of the early instability theories was that although the dynamical state without the waves is unstable, the instability is sufficiently weak so that the deduced linear instabilities would bear a recognizable structural connection to the observed fluctuations, rather than leading to a catastrophic transition to a fully turbulent state of motion. In many ways, i.e. wave-length, frequency and geographical distribution, such expectations have been realized. This fact leads naturally to the next stage of theoretical development. That is, the description of this important instability phenomena

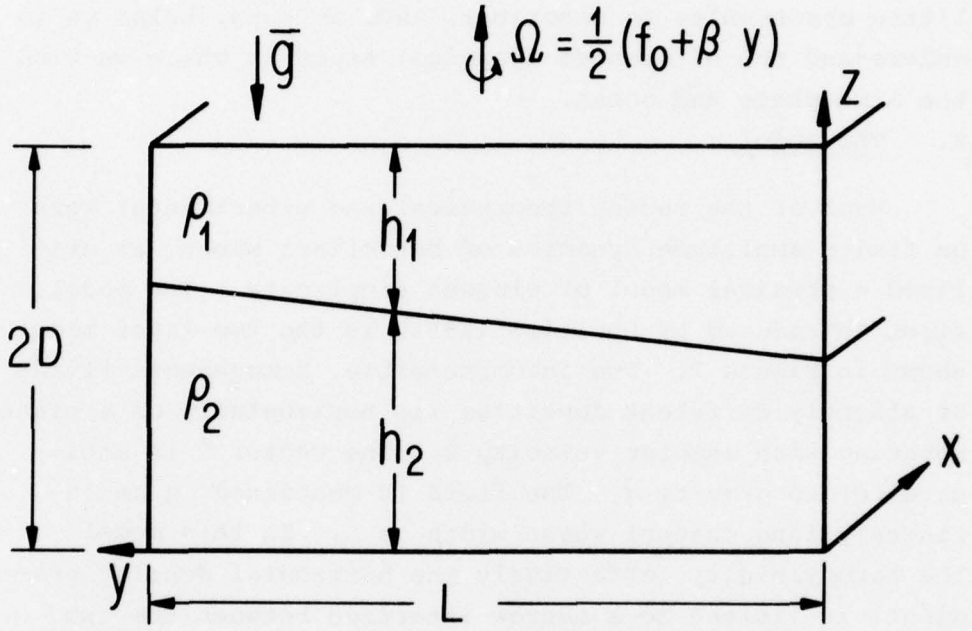


Fig. 2. Schematic rendering of the two-layer model model for baroclinic waves.

in finite-amplitude is needed so that the amplitude limiting mechanism, the effect of the instability on the general circulation, and the interaction with other modes of motion can be described. The goal of such description is the development of a conceptual foundation which helps understand the results of geophysical observation, numerical and laboratory experiments (which this lecture gives little opportunity to describe), and, in turn, helps us to understand the bifurcated dynamical state in which we find the atmosphere and ocean.

2. The model.

Much of the recent theoretical and experimental work on finite-amplitude dynamics of baroclinic waves has utilized a physical model of elegant simplicity. The model, first introduced by Phillips (1951) is the two-layer model shown in Figure 2. Two incompressible, homogeneous fluids of slightly different densities lie horizontally on a plane rotating with angular velocity $\vec{\Omega}$. The vector $\vec{\Omega}$ is anti-parallel to gravity $\vec{g}$. The fluid is contained in an infinitely long channel whose width is L. In this model the baroclinicity (effectively the horizontal density gradient) is limited to a narrow interface between the two layers. The fluid is considered to have a small enough kinematic viscosity ν, so that the effects of viscosity are limited to thin viscous boundary layers at the upper and lower rigid lids and, in some studies (Hart 1973, 1976) also at the interface.

The depth of each layer in the absence of motion is D and it is assumed that $\Omega^2 L^2/gD$ is sufficiently small so that the centrifugal curvature of the interface may be neglected.

Scales U, L, L/U, D are used to non-dimensionalize the horizontal velocity, horizontal lengths, time and depth respectively. If the parameters $\varepsilon = U/2\Omega L$ (Rossby number) and $E = \nu/\Omega D^2$ (Ekman number) are both small, it can be shown

that to lowest order the motion is quasi-geostrophic (Phillips 1963). That is, to order ϵ and E the Coriolis force due to the horizontal velocity is balanced by the horizontal pressure gradient while the vertical pressure gradient balances buoyancy forces. Under these circumstances it can be shown (Pedlosky (1970)) that the governing equations of motion are the equations for the 0(1) vorticity in each layer viz:

$$(2.1)\quad \frac{d}{dt}\,(\nabla^2\psi_n + \beta y) = (-1)^n F \frac{d}{dt}\,(\psi_2 - \psi_1) - r\,\nabla^2\psi_n \qquad n = 1,2$$

where the subscripts 1, 2 refer to variables defined in the upper and lower layers, respectively. In (2.1) $\psi_n(x,y,t)$ is the non-dimensional dynamic pressure field, which due to the quasi-geostrophic balance between Coriolis and pressure forces, acts as a stream function for the horizontal velocity seen relative to the rotating frame. That is

$$(2.2)\qquad u_n = -\,\partial\psi_n/\partial y,\quad v_n = \frac{\partial\psi_n}{\partial x}$$

where u_n and v_n are the velocities down the channel (x direction) and across the channel (y direction). The vertical component of vorticity is

$$(2.3)\qquad \zeta_n = \frac{\partial v_n}{\partial x} - \frac{\partial u_n}{\partial y} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi_n$$

It is important to note that within each homogeneous layer, the horizontal velocities are independent of height (1.4a). In deriving (2.1) it has been assumed that the vertical vorticity of the rotating frame, $2\vec{\Omega}$, is linearly increasing in the y direction. This approximation, first introduced by Rossby (1939), can be shown to contain the essential dynamical effects of the earth's sphericity if y is associated with the northward directed co-ordinate. Thus (2.1) states that in each layer the rate of change of the <u>total</u>

vorticity (ζ_n which is the vorticity relative to the earth plus βy, the variable frame or planetary vorticity) is due to two effects. The first term on the right hand side of (2.1) is the production of vorticity by the stretching of fluid columns that is produced by the motion of the interface. Its effect is equal and opposite in the two layers. The second term represents the destruction of vorticity by viscous dissipation in the viscous boundary layers.

The non-dimensional parameters appearing in (2.1) are

$\beta = L^2\left(\frac{df}{dy}\right)/U$, sphericity factor

$F = 4\Omega^2L^2/g \ \frac{(\rho_2-\rho_1)}{\rho_2}$, stratification parameter

$r = E^{\frac{1}{2}}/\varepsilon$, viscosity measure.

For a spherical earth $df/dy = 2\Omega\cos\theta/a$ where θ is latitude and a, the earth's radius. β, F and r are constants in the geostrophic limit, and the nature of the stability problem and the associated finite amplitude dynamics depend on their relative magnitudes. In (2.1) they are assumed to be 0(1) or smaller.

The appropriate boundary conditions for (2.1) are (Pedlosky 1970)

$$\text{(2.4a,b)} \qquad \frac{\partial\psi_n}{\partial x} = 0 \qquad y = 0,\ 1$$

$$\lim_{X\to\infty} \int_{-X}^{X} \frac{\partial^2\psi_n}{\partial t\partial y}\,dx = 0 \qquad y = 0,\ 1$$

The fundamental non-linearity in (2.1) is provided by the fact that in each layer

$$\text{(2.5)} \quad \frac{d}{dt} = \frac{\partial}{\partial t} + u_n\frac{\partial}{\partial x} + v_n\frac{\partial}{\partial y} = \frac{\partial}{\partial t} - \frac{\partial\psi_n}{\partial y}\frac{\partial}{\partial x} + \frac{\partial\psi_n}{\partial x}\frac{\partial}{\partial y}\,.$$

So far, almost all the finite amplitude studies have considered the instability and subsequent bifurcated states associated with the mean flow

$$\psi_n^{(0)} = - U_n y \tag{2.6}$$

where U_n represents a y independent flow in the x direction which is different in each layer. The constant slope of the interface in the basic state is proportional to $U_1 - U_2$ and provides the available potential energy for the instability.

The total stream function is given by

$$\psi_n = - U_n y + \phi_n(x,y,t) \tag{2.7}$$

from which it follows that the ϕ_n satisfy

$$\left(\frac{\partial}{\partial t} + U_n \frac{\partial}{\partial x}\right)\left(\nabla^2\phi_n + (-1)^n(\phi_1-\phi_2)\right) + \left(\beta - (-1)^n F(U_1-U_2)\right)\frac{\partial \phi_n}{\partial x} \tag{2.8}$$

$$= - r \nabla^2\phi_n - J(\phi_n, \nabla^2\phi_n + (-1)^n(\phi_1-\phi_2))$$

where the Jacobian operator is defined

$$J(f,g) \equiv \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} \tag{2.9}$$

for any two functions (f,g). In addition

$$\left.\begin{aligned} &\frac{\partial \phi_n}{\partial x} = 0, \qquad \lim_{X\to\infty} \int_{-X}^{X} \frac{\partial^2 \phi_n}{\partial y \partial t}\, dx = 0 \end{aligned}\right\} \quad y = 0,1 \tag{2.10}$$

With only minor alterations the same model is appropriate for the study of laboratory flows (Hart 1972, 1973, 1974) in cylindrical geometries and to the investigation of continuous velocity profiles in the case of constant shear and zero sphericity (Drazin 1970, 1972).

3. Linear instability.

If the disturbance has a small amplitude, at least initially a linear stability analysis will be valid. If

the last term in (2.8) is then ignored, wave solutions of (2.8) which satisfy (2.10) can be found in the form

$$\phi_n = R_e\, A_n\, e^{ik(x-ct)} \sin m\pi y \tag{3.1}$$

The eigenvalue c is given by the relation

$$c = \frac{U_1+U_2}{2} - \frac{(a^2+F)}{a^2(a^2+2F)}\{\beta + ir\, a^2/k\} \tag{3.2}$$

$$\pm \{4F^2(\beta+ir\, a^2/k)^2 - (U_1-U_2)^2(4F^2-a^4)a^4\}^{1/2}/2a^2(a^2+2F)$$

where

$$A_2/A_1 = \frac{a^2+F}{F} - \frac{(\beta+F(U_1-U_2))}{(U_1-c)} - \frac{ia^2 r}{kF(U_1-c)} = \gamma \text{ and} \tag{3.3}$$

$$a^2 = k^2 + m^2\pi^2. \tag{3.4}$$

Instability occurs where c_i, the imaginary part of c, is positive.

The theoretical development of the non-linear problem depends crucially on examining cases in which the amplitude of the wave, although finite, remains small. This in turn requires that the analysis be limited to that part of parameter space where the linear growth rate (kc_i), is small compared to the advective time scale kU. An important mathematical question then arises. As the neutral boundary (i.e. the surface in parameter space on which c_i is zero) is approached, do both roots (3.2) have vanishing imaginary parts or only one? That is, is the neutral curve defined as a coalescence of two roots (3.2)? Consider the case r = 0 (inviscid instability). The shear flow will be unstable when $a^2 < 2F$ and

$$(U_1-U_2)^2 > 4F^2\beta^2/(4F^2-a^4)a^4 \equiv U_c^2(a^2) \tag{3.5}$$

For

$$U_1 - U_2 = U_c(1+\Delta) \tag{3.6}$$

where $\Delta << 1$

$$c_i = \pm \beta F(2)^{\frac{1}{2}} \Delta^{\frac{1}{2}}/a^2(a^2 + 2F) \tag{3.7}$$

Hence in the inviscid limit the neutral curve is defined by a coalescence of two roots; c_i has a branch point at $\Delta = 0$ and the growth rate is proportional to $\Delta^{\frac{1}{2}}$. On the other hand, when $\beta = 0$, $r \neq 0$, instability requires $a^2 < 2F$ and

$$(U_1 - U_2)^2 > \frac{4r^2a^2/k^2}{(2F-a^2)} = U_c^2 \ . \tag{3.8}$$

When $U_1 - U_2 = U_c(1+\Delta), \Delta << 1$

either

$$c_i = \frac{U_c a (2F-a^2)^{\frac{1}{2}}}{2(a^2+F)} \ \Delta \tag{3.9a}$$

or

$$c_i = - \frac{2(a^2+F)}{(a^2+2F)} \ r/k. \tag{3.9b}$$

The stability curve, $U_1 - U_2 = U_c$, is therefore a transition curve for one root of (3.2) in this case (3.9a), while the second root (3.9b) remains 0(1) and negative. Hence, one root (3.9b) represents rapid decay on an 0(1) time scale while only the root (3.9a) yields slow amplitude evolution at a rate proportional to Δ. This distinction between the inviscid case (coalescing roots) and the viscous case (distinct roots) is crucial to the nonlinear theory.

4. Finite amplitude theory.

The finite amplitude theory of baroclinic waves pivots about the analysis of the dynamics for those supercriticalities Δ, which are small. This restriction is required so that the amplitude of the wave at equilibration remains

sufficiently small, though finite, that a perturbation theory appropriate for weak non-linearity is valid. Small Δ also implies long development times compared to the advective time scale. It is useful to recognize this explicitly by introducing a new, "slow" time variable

$$T = \sigma(\Delta)t \tag{4.1}$$

where

$$\sigma(\Delta) = |\Delta|^j \tag{4.2}$$

The exponent j depends on whether the marginal curve is a transition point for one or two roots of the dispersion relation as discussed in section (3). If the marginal curve is a coalescence point $j \approx \frac{1}{2}$ (3.7), if not $j = 1$ (3.9a). The absolute value is used in (4.2) to allow an examination of sub-critical states. The disturbance stream function ϕ_n is treated as a function of the two time variables t and T so that in (2.8)

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \sigma(\Delta)\frac{\partial}{\partial T} \tag{4.3}$$

In each of the cases ϕ_n is developed in the asymptotic series

$$\phi_n = |\Delta|^{\frac{1}{2}}\phi_n^{(1)} + |\Delta|\phi_n^{(2)} + |\Delta|^{3/2}\phi_n^{(3)} + \ldots \tag{4.4}$$

where $\phi_n^{(1)}$ is written in the form

$$\phi_n^{(1)} = A_n(T)e^{ik(x-ct)}\sin m\pi y \tag{4.5}$$

where c(k,m) is the <u>real</u> phase speed on the neutral curve. $U_1 - U_2 = U_c(k,m)$. The substitution of (4.3), (4.4) into (2.8) leads to a sequence of linear problems, the first of which yields the marginal wave, i.e. c(k,m) and $\gamma(k,m)$. The suppression of secular terms, which would otherwise invalidate (4.4) for times $T = 0(1)$ in the subsequent higher

order problems determines both the equation for the amplitude evolution and the non-linear correction to the mean field. The reader is referred to Pedlosky (1970) for details of the calculations.

4.a. Viscous equilibration

When $\beta = 0$ and $r = 0(1)$ so that (3.9a) and (3.9b) yield the slightly supercritical growth rates, the exponent, j, in (4.2) is 1. The amplitude equation is

$$\frac{dA}{dT} = \{kc_{oi} - ik\frac{\Delta}{|\Delta|}\}A - k^2 NA|A|^2 \tag{4.6}$$

where

$$c_{oi} = \frac{\Delta}{|\Delta|}\frac{a(2F-a^2)^{1/2}}{2(a^2+F)} \tag{4.7a,b}$$

$$N = \frac{F}{r8(a^2+F)}\{4m^2\pi^2(a^2-F) + 3a^2(2F-a^2)\} > 0.$$

whose solution is

$$A = |A(T)|e^{ik\frac{\Delta}{|\Delta|}T} \tag{4.8a,b}$$

where

$$|A(T)|^2 = |A(0)|^2 e^{2kc_{oi}T}/(1 + \frac{|A(0)|^2}{|A_\infty|^2}(e^{2kc_{oi}T}-1))$$

The amplitude monotonically rises from its initial value A(0) and monotonically attains its final, steady equilibrated limit

$$|A_\infty|^2 = (\frac{c_{oi}}{kN}) \tag{4.9}$$

There are several important features of the solution. First, as mentioned above, the amplitude evolution is monotonic. The amplitude rise is inexorable and the rise time is determined by kc_{oi}, (4.7a), and this growth rate is

identical to that given by linear theory (3.9a). The sole new element of the finite amplitude theory in (4.6) is the introduction of an amplitude dependence to the effective growth rate, viz: $(kc_{oi} - k^2N|A|^2)$. The Landau constant, N, is the essential fruit of the non-linear calculation. For the baroclinic problem N is positive which implies i) only supercritical instability and ii) a steady final amplitude given by (4.9). A major conceptual weakness of the theory is its inability in particular cases to determine a`priori the sign of N. At the present time this requires tedious and lengthy calculation.

Drazin (1972) considered the finite-amplitude, viscous (r = 0(1)), extension of Eady's problem. In that problem both the density and velocity fields are continuous functions of height. The shear is constant as is the vertical density gradient. It was already known that the linear theory appropriate to this problem gave results qualitatively similar to the two-layer problem and Drazin showed that the amplitude equation for the continuous problem was identical to (4.6), with only quantitatively different results for c_{oi} and N. The Landau constant, N, was also positive.

4.b. Inviscid equilibration.

The strictly inviscid problem (r = 0) possesses a critical shear $U_c(a)$, given by (3.5). In this case the stabilizing dynamical constraint that is overpowered at the instability threshold is an inviscid vorticity constraint due to the sphericity (β) effect. For small supercriticality Δ, c_i is given by (3.7) and the index j in (4.2) is ½. The resulting amplitude equation, Pedlosky (1970) is

$$\frac{d^2A}{dT^2} = k^2c_{oi}^2 A - k^2NA[|A|^2 - |A(0)|^2] \tag{4.10}$$

where now

$$c_{oi}^2 = \frac{2\beta^2F^2}{a^4(a^2+2F)^2}\frac{\Delta}{|\Delta|} \tag{4.11a}$$

$$(4.11b) \qquad N = (2m^2\pi^2F)^{-1}\left[\frac{(\beta+FU_c)m^2\pi^2U_c}{8(a^2+2F)(U_1-c)^2}\right]$$

$$\left[a^2(2F-a^2)+4m^2\pi^2(a^2-F) + (2F-a^4)\frac{(4\tanh\sqrt{F/2}}{\sqrt{F/2}}\left(\frac{2m^2\pi^2}{2m^2\pi^2+F}\right)\right]$$

$$(4.11c) \qquad c = U_2 + \frac{U_c}{2} - \beta(a^2+F)/a^2(a^2+2F)$$

and where A(0) is the initial value of A(T).

In this problem as in the viscous case the mechanism for equilibration is the same, i.e. the production of a correction of the mean shear flow, to reduce its shear as the wave grows and depletes the energy source for the instability. An examination of (4.10) shows, however, that the inviscid amplitude behavior is distinctly different. Instead of monotonic equilibration to a steady state (4.10) yields an oscillation. Equation (4.10) can be thought of as an equation for a mass-spring oscillator whose potential is repelling at small amplitudes† and restoring (N>0) for large amplitudes.

In the case when A is real and $(1/A\ dA/dT)_{T=0} = c_{oi}$, the maximum amplitude during the oscillations is given by

$$(4.12) \qquad A^2_{max} = A^2(0) + \frac{c^2_{0i}}{N} + \frac{c^2_{0i}}{N}\left(1 + \frac{2N}{c^2_{0i}}A^2(0)\right)^{\frac{1}{2}}$$

which occurs at an amplitude in excess of the amplitude

$$(4.13) \qquad A^2_* = A^2(0) + c^2_{0i}/N$$

at which the instantaneous mean shear flow appears stable.

† In reality, for amplitudes close to the initial amplitude.

Thus because of the "inertia" associated with the inviscid dynamics, the finite amplitude wave experiences its largest amplitudes during that part of its cycle of perpetual growth and decay when the instantaneous mean flow would be stable to small disturbances. The dnoidal oscillation is described by the elliptic function, viz:

$$A(T) = A_{max}\,dn\{[\frac{kA_{max}N^{\frac{1}{2}}}{\sqrt{2}}(T-\tau)],\kappa\} \tag{4.14a}$$

where

$$\tau = \sqrt{2}[dn^{-1}\{\frac{A(0)}{A_{max}};\kappa\}]/kN^{\frac{1}{2}}A_{max} \tag{4.14b}$$

$$\kappa = (A_{max}^2 - A_{min}^2)^{\frac{1}{2}}/A_{max} \tag{4.14c}$$

$$A_{min}^2 = A^2(0) + \frac{c_{oi}^2}{N} + \frac{c_{oi}^2}{N}\left(1 + \frac{2N}{c_{oi}^2}A^2(0)\right)^{\frac{1}{2}}$$

A crucial step in the inviscid problem occurs in the linear problem for $\phi_n^{(2)}$. A solvability condition arises at that order in the form

$$\frac{dA}{dT}\left[\frac{\beta+FU_c}{(U_1-c)^2} + \gamma^2\frac{\beta-FU_c}{(U_2-c)^2}\right] = 0 \tag{4.15}$$

For A to indeed vary on the $\Delta^{-\frac{1}{2}}$ time scale, the bracket in (4.15) must vanish, as it does in the inviscid case. Indeed (4.15) is reminiscent of Rayleigh's flex point theorem in ordinary shear flows. For r = 0(1), the equivalent bracket does not vanish which requires T to be scaled for temporal variations on the Δ^{-1} scale. In fact, (4.15) is merely a condition for the coalescence of the two roots for c on the linear, inviscid marginal curve.

4.c. Slightly viscous equilibration.

It is natural to investigate what the fate of the oscillation described in section (4.b) would be in the

presence of a small amount of dissipation. To investigate this, it is useful to recast the stability problem such that for a given, k, (U_1-U_2) and r, the parameter F is considered the critical one. In the case $\beta = 0$, the threshold value for F for instability is

$$F_c = \frac{a^2}{2} + \frac{2r^2a^2}{k^2(U_1-U_2)^2} \tag{4.16}$$

F is then augmented by an amount Δ and r is chosen to be $0(\Delta^{\frac{1}{2}})$. This last setting is guided by the results of linear theory so that the viscous decay rate (r) will be of the same order as the inviscid growth rate $(\Delta^{\frac{1}{2}})$.

It then follows that A satisfies (Pedlosky (1972))

$$\frac{d^2A}{dT^2} + \frac{3}{2}\frac{r}{|\Delta|^{\frac{1}{2}}}\frac{dA}{dT} + \frac{\Delta}{|\Delta|}\frac{k^2(U_1-U_2)^2}{4a^2}A + \frac{Ak^2(U_1-U_2)}{2a^2}m\pi\int_0^1 dy\ \sin 2m\pi y\ \frac{\partial^2\phi}{\partial y^2} = 0 \tag{4.17a}$$

$$\frac{\partial}{\partial T}\left[\frac{\partial^2\phi}{\partial y^2} - a^2\phi\right] + \frac{r}{|\Delta|^{\frac{1}{2}}}\frac{\partial^2\phi}{\partial y^2} = \frac{a^2m\pi}{2(U_1-U_2)} \times \left[\frac{d|A|^2}{dT} + \frac{2r}{|\Delta|^{\frac{1}{2}}}|A|^2\right]\sin 2m\pi y. \tag{4.17b}$$

In (4.17a), $T = |\Delta|^{\frac{1}{2}}t$, while the function ϕ in (4.17a,b) is the correction to the x-independent mean flow shear. When the flow is inviscid, r = 0, (4.17b) can be integrated immediately. The resulting equation for A is identical in form to (4.10) although the particular parameters have different values. The inviscid problem was also investigated in the case of continuous shear by Drazin (1970), who derived a similar amplitude equilibration equation.

When $r/|\Delta|^{\frac{1}{2}}$ is very large (i.e. the dissipation time is short compared to the linear e-folding instability time)

(4.17b) becomes an algebraic equation linking $\partial^2\Phi/\partial y^2$ with $|A|^2$. On the other hand (4.17a) is approximated by the singular perturbation resulting from dropping the second time derivative in (4.17a) when $r/|\Delta|^{\frac{1}{2}}$ becomes large. The resulting first order equation, identical in form with (4.6), matches asymptotically the equation derived under the assumption $r = 0(1)$ in the limit $r \to 0$. This coincidence of limits shows that the Landau equation (4.6) is a singular perturbation of the more general problem (4.17a,b) valid for large dissipation ($r = 0(1)$). As the dissipation decreases the equation takes on more the character of the inviscid problem in section (4.b).

Nevertheless, for all $r \neq 0$, no matter how small, a steady solution exists for A, namely

$$A^{(s)\,2} = \frac{\Delta}{|\Delta|}\,\frac{(U_1-U_2)^2}{a^2m^2\pi^2} \tag{4.18}$$

The question naturally arises as to the relationship of the inviscid problem and its oscillatory behavior and the steady solution (4.18), whose wave amplitude is independent of r (for small r).

To answer this question I (Pedlosky, (1971), (1972)) relaxed the boundary conditions on Φ from the vanishing of $\partial\Phi/\partial y$ on $y = 0,1$ to the vanishing of Φ on those boundary points. By both analogue computer calculation and analytic asymptotic analysis, the following picture emerged. For $r = 0$ the solution executes a simple closed trajectory in the phase plane (Figure 3). For $r/|\Delta|^{\frac{1}{2}}$ $0(1)$, the solution oscillates several times but eventually settles down to the steady solution (4.18) (Figure 4). For $r/|\Delta|^{\frac{1}{2}} \ll 1$ however, the solution seldom becomes static. Instead the solution continuously oscillates. The steady wave solutions become linearly unstable for small enough r and for $k^2 > 3m^2\pi^2$. Unsteady solutions, therefore, are expected there. For $k^2 < 3m^2\pi^2$, the steady wave amplitude solutions are <u>linearly</u> stable, but calculation shows that they are unstable to

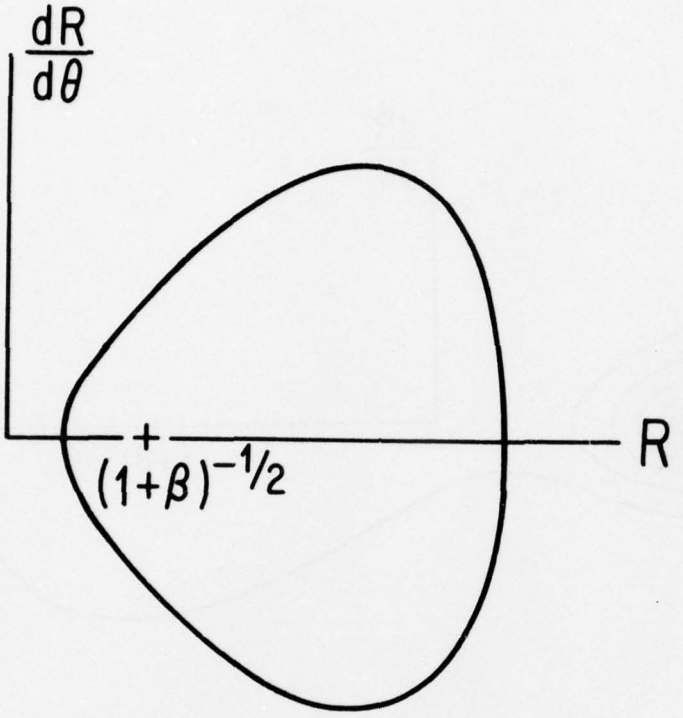


Fig. 3. The phase space trajectory for inviscid finite amplitude oscillations of the wave amplitude.

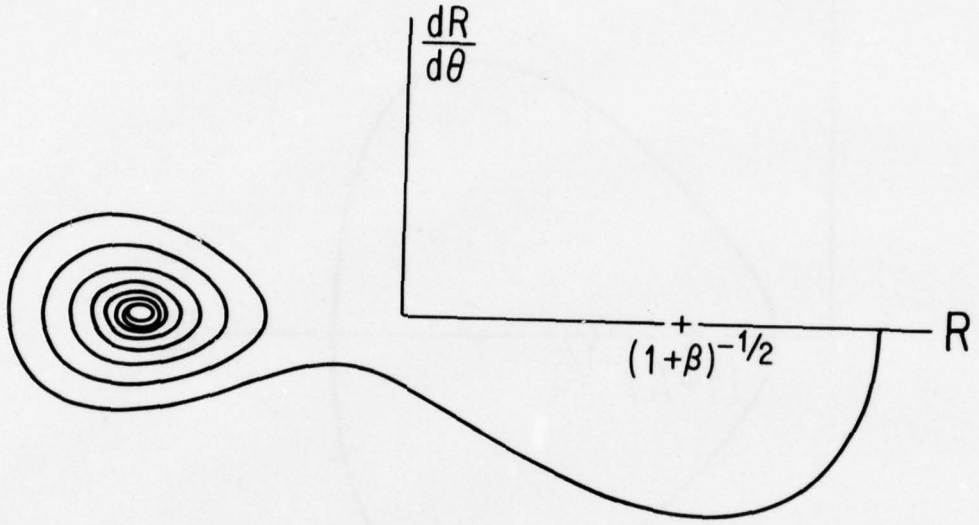


Fig. 4. The phase space trajectory for slightly viscous waves, $r/\Delta^{\frac{1}{2}} = 0(1)$. After several oscillations the wave amplitude attains its steady equilibrium value.

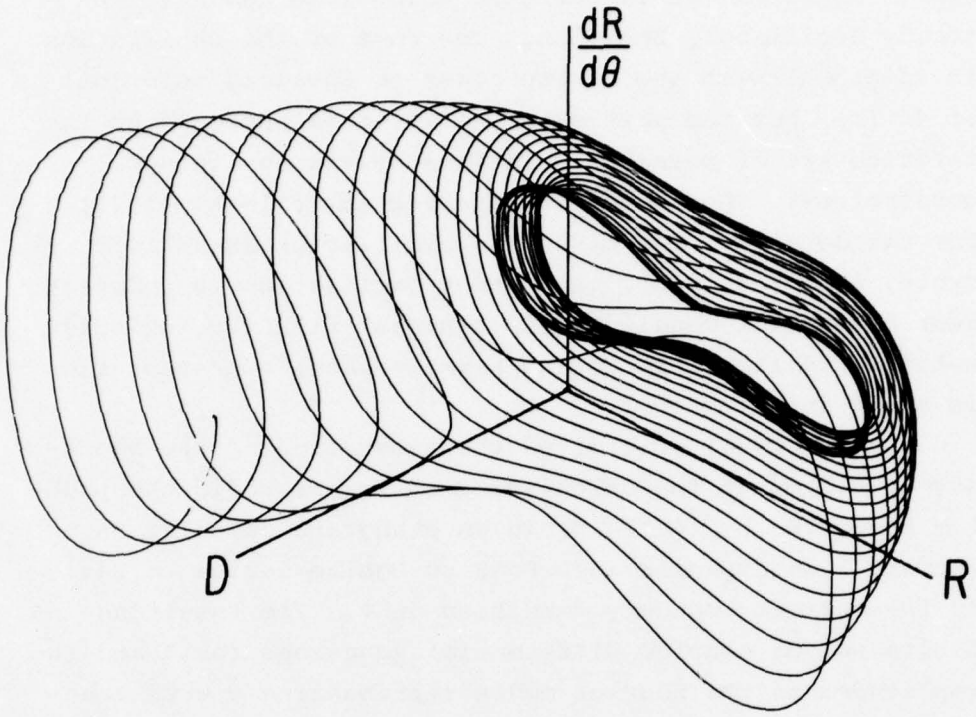


Fig. 5. The phase space trajectory (in perspective) for $r/\Delta^{\frac{1}{2}} \ll 1$. The wave amplitude perpetually oscillates in a limit cycle.

finite amplitude perturbations, which lead again to unsteady oscillatory behavior. The form of the oscillation is identical with one of the class of inviscid solutions of (4.17), but the particular solution is chosen from the infinite set of possible solutions by viscous integral constraints. The reader is referred to Pedlosky (1971) for the details. The resulting oscillation is a limit cycle, i.e. the final, persistent oscillation is independent of initial conditions in contrast with the inviscid solution (4.14a). The phase plane (space) representation is shown in (Figure 5).

Smith (1974) criticized the alteration of the boundary condition in the calculation of Φ. He redid the problem presented by (4.17a,b) in an elaborate calculation wherein Φ is expanded in a Fourier cosine series to satisfy the correct boundary condition on Φ. The resulting infinite set of coupled differential equations for A and the amplitudes of the Fourier modes representing Φ were truncated and integrated numerically. Smith's results (1976) reproduce, essentially, the results of the simpler theory outlined above. This, no doubt, occurs because of the fact that the steady wave solutions have their instability preferred for large a^2 and small $r/|\Delta|^{\frac{1}{2}}$, for which (4.17b) becomes in the "interior approximation", an ordinary differential equation in T for Φ with y carried parametrically.

Hart (1973, 1974, 1976) has carried out an important and interesting series of laboratory experiments. His experiment consisted of a rotating cylinder containing two immiscible fluid layers. He redid the caculations leading to (4.17a,b) taking into account the effects of the cylindrical geometry, interface friction and the centrifugal slope of the interface. These alterations do not affect the structure of the amplitude equation which he derived (Hart 1974) for small supercriticalities in the parameter F. Hart found that for small supercriticalities

and fairly small r (~.05) a steady wave regime occurred in the experiment. The amplitude of the observed steady wave was in good agreement with the theoretical prediction which is basically a refinement of (4.18) suitable for the experimental situation. Hart was unable to reduce the experimental setting for r to small enough values to examine the theory near the neutral curve (for such small r the viscous boundary layers on the horizontal surfaces become turbulent). Significantly, if he kept r constant but instead increased Δ sufficiently ($\Delta = 0(1)$), a limit cycle oscillation qualitatively similar to that predicted for small Δ and smaller was found. Apparently, larger supercriticalities, beyond the limits of validity of the present theory of (4.17a,b), qualitatively reproduce the limit $r \to 0$. Hart (1976) has also produced a challenging diagnostic numerical model in which certain non-linear interactions are arbitrarily suppressed but which nevertheless reproduces substantially his experimental observations.

5. Finite Amplitude Dynamics as a function of supercriticality.

One of the features of the finite amplitude dynamics that is usually inaccessible to weakly non-linear theory is the alteration of the amplitude dynamics as the supercriticality increases to such a degree that higher thresholds of instability are crossed. These linear thresholds are usually inaccessible since they generally correspond to order one supercriticalities above the lowest threshold. In an effort to examine this question, I studied the case $\beta = 0$, $r = 0(1)$ (for which (4.6) is relevant for small supercriticalities) when $F \gg 1$ (Pedlosky (1976)). The virtue of this parameter setting is that for the k corresponding to the minimum critical shear, the stability threshold for the m^{th} cross stream mode is given by

$$(5.1) \qquad \Delta = \frac{(U_1 - U_2 - U_c)}{U_c} = \Delta_m = \frac{(m^2-1)\pi}{2\sqrt{2}} / F^{\frac{1}{2}}$$

Therefore for large F, such that $F\Delta^{\frac{1}{2}} = O(1)$, small supercriticalities will penetrate several modal thresholds for instability. This is significant as a device for examining higher supercriticality dynamics within the systematic framework of the asymptotic theory. The method of analysis is similar to that given in Section 4 with one crucial difference. The modal structure in y is not determined a`priori by the linear marginal stability problem. Rather, the modal structure is itself linked to the nonlinear dynamics.

With

$$\phi_n^{(1)} \sim A(y,T)e^{ikx} \tag{5.2}$$

it can be shown that A satisfies the partial differential equation ($\tau = \Delta F^{\frac{1}{2}}\, r\epsilon/2$)

$$\frac{\partial A}{\partial \tau} - \frac{\partial^2 A}{\partial y^2} = s^2 A(1+u) \tag{5.3a,b}$$

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2}{\partial y^2}|A|^2$$

where $u(y,\tau)$ is the correction to the mean zonal shear. Both A and u vanish on y = 0,1. Linear instability is assured whenever the supercriticality s, related to Δ by

$$s^2 = 2\frac{k^2}{F^{\frac{1}{2}}}\left(\Delta F^{\frac{1}{2}} - \frac{k^2}{4F^{\frac{1}{2}}}\right);\ \frac{k}{F^{\frac{1}{2}}} = O(1); \tag{5.4}$$

exceeds $m\pi$. For small supercriticality,

$$s = \pi + \omega, \quad \omega << \pi \tag{5.5}$$

only the first stability threshold is crossed. In this case it can be shown (Pedlosky (1976)) $A(y,\tau) \sim A_o(\tau)\sin\pi y$ and the equation for A_o is given by the Landau equation

$$\frac{dA_o}{d\tau} = \omega A_o - \frac{3\pi^2}{4} A_o|A_o|^2 \tag{5.6}$$

as in (4.6). The problem of higher supercriticality

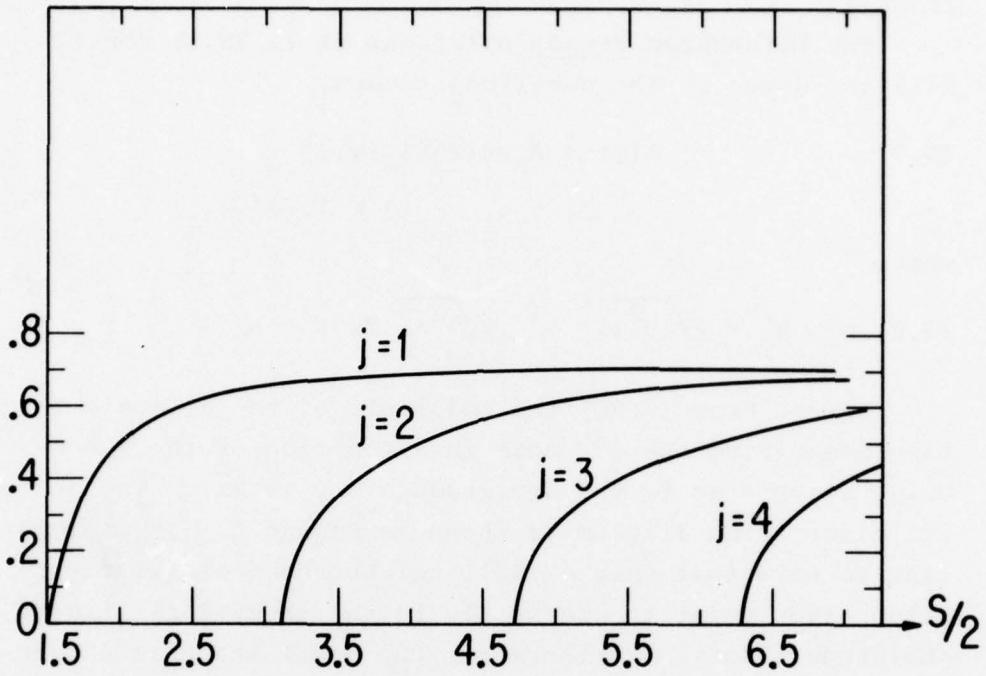


Fig. 6. Bifurcation diagram for steady wave solutions $r = 0(1)$, $(U-U_c)U_c F^{\frac{1}{2}} = 0(1)$, as a function of shear supercriticality s.

corresponds to s = 0(1) so that several thresholds are crossed.

The bifurcated steady solutions of (5.3a,b) for s = 0(1) are given by the non-linear modes.

$$A(y) = A_o \, sn(2jK(\nu)y, \nu) \qquad j = 1,2,3... \tag{5.7}$$

where

$$A_o = \nu/\sqrt{\nu^2+1} \qquad 2\sqrt{\nu^2+1}\, K(\nu) = s/j \tag{5.8a,b}$$

Hence, from (5.8b) the modulus ν of the elliptic function describing the j^{th} mode is a function of the supercriticality s as is the amplitude A_o by (5.8a). The analytic bifurcation diagram is shown in Figure 6. It is important to note that only a small neighborhood of the first bifurcation point is accessible to the traditional finite amplitude theory. Furthermore, the modal shape is a function of s. Rather than being dominated by a single Fourier harmonic each non-linear mode contains a mixture of many Fourier modes. For any $s>2\pi$ more than one steady mode is possible, but of course not simultaneously, since the superposition principle for non-linear modes is not valid (perhaps regrettably). Numerical methods are required to integrate the initial value problem. Figure 7 shows the result of such a calculation. The function ϕ_1 would be the first Fourier component of the first non-linear steady mode while ϕ_2 would be the first Fourier component of the second non-linear mode. In general for the numerical calculation the truncated representation

$$\begin{pmatrix} A \\ u \end{pmatrix} \sum_{k=1}^{N} \begin{pmatrix} \phi_k(\tau) \\ u_k(\tau) \end{pmatrix} \sin k\pi y \tag{5.9}$$

was used to integrate (5.3a,b). What Figure 7 shows is that after a non-monotonic oscillatory period the solution

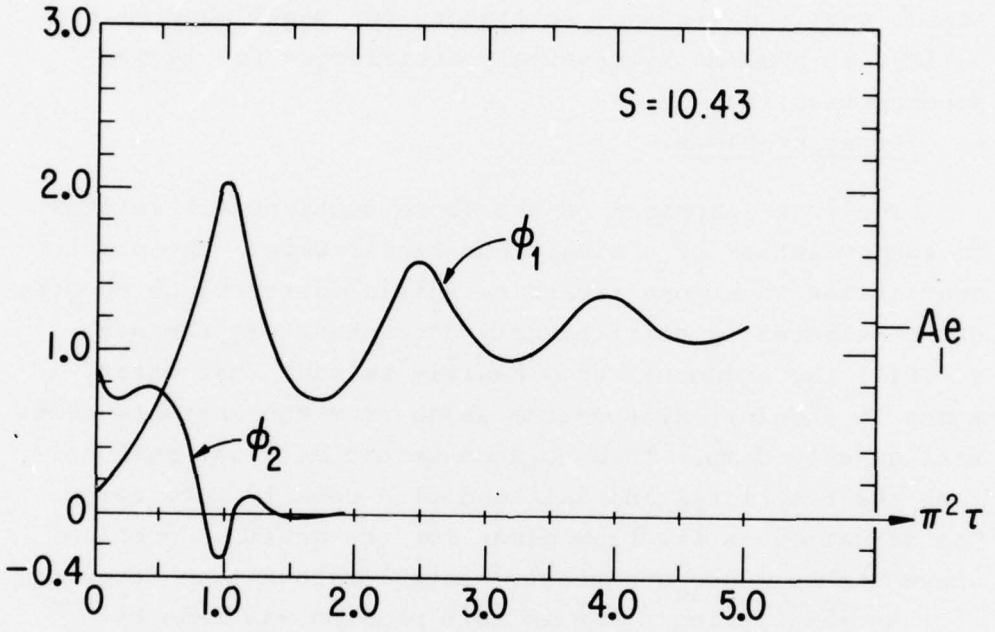


Fig. 7. The amplitude evolution of the dominant Fourier harmonics of the first two non-linear modes for large supercriticality. Note the non-monotonic equilibration though $r = 0(1)$.

asymptotes to the first non-linear steady mode, even though the initial spectrum ($\phi_1 << \phi_2$) lies close to the second non-linear mode. The results of several calculations demonstrated the model's preference for that non-linear mode (j=1) that has the largest steady state amplitude. It is furthermore significant that although the approach to the steady amplitude is non-oscillatory for small supercriticality, it becomes increasingly oscillatory for larger supercriticality.

6. Other problems.

The work described in the above sections all relates to the evolution of a single non-linear wave. The problem that arises when more arbitrary initial disturbance spectra are considered is still largely untouched. In the case $r = 0(1)$ the argument can plausibly be made that other modes in the initial spectrum aside from the unstable modes will quickly damp. This to some extent at least rationalizes the restricted initial condition usually considered. The situation is far from clear for the inviscid problem where stable modes are merely neutral rather than decaying.

An interesting start on this problem was made by Loesch (1974). He considered the problem with $r=0$, $\beta \neq 0$ for which a minimum critical shear occurs at $a^2 = \sqrt{2}F$, $U_1 - U_2 = \beta/F$. He considered an initial spectrum composed of three waves when the mean flow shear slightly exceeds its minimum critical value. The three waves formed a resonant triad wherein one wave was the slightly unstable wave while the other two were waves which are linearly stable on the mean flow but resonantly interact with the unstable wave.

If ($k_1, m_1, \omega_1; A_1(T)$) are the x wave number, the y modal number, the (real) frequency and the amplitude respectively of the unstable wave, while (k_i, m_i, ω_i, A_i), $i=2,3$, are those for the neutral waves, then to form the resonant triad

(6.1)
$$k_1 \pm k_2 \pm k_3 = 0$$

$$\omega_1 \pm \omega_2 \pm \omega_3 = 0$$

$$m_1 \pm m_2 \pm m_3 = 0$$

where, recall, each wave is of the form $A_j e^{i(k_j x - \omega_j t)} \sin m_j \pi y$. Loesch showed that the amplitudes satisfied the equations

(6.2a,b,c)
$$\frac{d}{dT}\left\{\frac{dA_1}{dT} - iN_1 A_2^* A_3^*\right\}$$

$$= k_1^2 C_i^2 A_1 - k_1^2 N A_1 (|A_1|^2 - |A(0)|^2)$$

$$+ \frac{k_1^2 N N_1}{2} A_1 \left\{\frac{|A_2|^2 - |A_2(0)|^2}{N_2} + \frac{|A_3|^2 - |A_3(0)|^2}{N_3}\right\}$$

$$\frac{dA_2}{dT} = iN_2 A_3^* A_1^*$$

$$\frac{dA_3}{dT} = iN_3 A_1^* A_2^*$$

where N, N_2, N_3 are numbers calculated from the usual suppression of secularity condition. The problem posed by (6.2a) combines elements of the finite amplitude theory and resonant interaction theory and allows the efficient transfer of energy from an unstable wave to waves that are stable on the mean flow by linear theory. Loesch found numerically for F>10.5, regardless of the initial amplitude ratios in the triad, that the "neutral" waves reached an amplitude as large as the "unstable" wave by the non-linear transfer of energy between triad members.

Pedlosky (1975) examined a similar case for very large F where the only equilibration mechanism for the primary unstable wave was its resonant interaction with neutral waves and suggested this mechanism as a possible explanation

for meso-scale oceanic eddies.

This review would certainly be incomplete without mention of the pioneering work of Lorenz (1963). The work is extensive and a complete review is beyond the scope of this paper. Lorenz, fundamentally, considered a truncated Fourier representation of each dependent variable in a dynamical model similar to (2.1) but one which allows, in a basically ad hoc (but energy preserving) way for time variations in F (the static stability). The truncation level for each variable is drastic and it is surprising, in a sense, how effective such severe truncations can still be in reproducing qualitatively the vacillation, steady wave and no-wave regimes found experimentally. Since the truncation is not related to any formal parametric ordering it is difficult to evaluate the theoretical to a`priori validity of the theory. As in Hart's numerical work, the striking qualitative success of Lorenz's model serves as a challenge to more formal theories to achieve as much.

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Bifurcation and Stability Problems in Astrophysics

Norman R. Lebovitz

1. Introduction

Astrophysics is a rich source of stability problems, and this is especially true of that branch of the subject dealing with stellar structure and evolution. This is not surprising, since a star is a fluid mass, and therefore subject to the classical hydrodynamical instabilities. Moreover, the mechanical and thermal balances peculiar to stars create further instabilities in certain parameter ranges, and these can interact with hydrodynamical effects. The net result is a somewhat bewildering variety of instability mechanisms. Good reviews of the theory of stellar stability are available [18, 19], however.

Two kinds of stellar instability problems are discussed in this article. They were selected because the corresponding bifurcation problems have particularly important implications for astrophysics. Despite their particular importance, their choice for discussion here is inevitably somewhat personal, and there are many other important bifurcation problems in the theory of stellar structure and evolution, not to mention in other areas of astrophysics [10].

The first problem discussed is intimately linked with the general problem of turbulence in fluid dynamics, and may therefore be extremely formidable. Nevertheless, it seems worthwhile to state it clearly as a bifurcation problem because (a) it arises naturally in this way, yet (b) does not seem to have been expressed as such in the astro-

physical literature. The second problem (or class of problems) addresses the following fundamental question in astronomy: what happens to a self-gravitating mass endowed with angular momentum when it undergoes (as it must) an enormous contraction?

2. The Equations of Stellar Structure

A star is a gaseous mass and is therefore described by the equations of fluid dynamics. The form that these equations take must reflect certain characteristic physical assumptions peculiar to stars: (1) The gravitational field acting on the star is due to matter in the star, (2) Viscous effects are normally assumed neglibile, (3) Energy is dissipated through radiation, and replenished by nuclear reactions, or by gravitational contraction. The equations of conservation of momentum, mass, and energy may be written

$$\frac{D\underset{\sim}{v}}{Dt} = -\frac{1}{\rho}\,\text{grad}\,p + \text{grad}\,V, \tag{1}$$

$$\frac{D\rho}{Dt} = -\rho\,\text{div}\,\underset{\sim}{v}, \tag{2}$$

and

$$T\frac{Ds}{Dt} = -\frac{1}{\rho}\,\text{div}\,\underset{\sim}{q} + \varepsilon, \tag{3}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + \underset{\sim}{v}\cdot\text{grad}$ and

$$V(\underset{\sim}{x},t) = G\int_R \frac{\rho(\underset{\sim}{x}',t)}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\,d\underset{\sim}{x}' \,. \tag{4}$$

In these equations, $\underset{\sim}{v}$ is the fluid velocity, ρ the density, p the pressure, V the gravitational potential (expressed by the equation (4), where G is the gravitational constant and R is the region occupied by the fluid), s the entropy per unit mass, $\underset{\sim}{q}$ the radiative flux vector, and ε the rate of nuclear energy production. For most purposes it is sufficient to write [33]

$$\underset{\sim}{q} = -Q\,\text{grad}\,T, \tag{5}$$

for a certain function $Q(\rho,T)$, so the radiative transport of heat has the same local properties as material conductivity.

The expression of Q as a function of (say) ρ and T is only one of several constitutive relations needed to make equations 1 - 3 determinate. One also needs ε expressed in terms of ρ and T, and an equation of state similarly expressing p and s. All these quantities depend as well on the chemical composition of the matter. We shall assume for the present that the latter is constant. The correct expression of these constitutive relations, expecially those for Q and ε, is a complicated physical problem which we shall assume solved.

What further conditions are needed in specifying a solution of equations (1)-(3)? Initial conditions on $\underset{\sim}{v}$, ρ, and s (or T) are needed, of course, but these are not enough. The specification of a star is normally supposed to require giving its mass M and chemical composition. The latter having been assumed fixed, we still have for the former the integral constraint

$$\int_R \rho(\underset{\sim}{x},t)\,d\underset{\sim}{x} = M. \tag{6}$$

A second integral constraint that may be imposed without loss of generality is that the center of mass remain at the origin of coordinates:

$$\int_R \rho(\underset{\sim}{x},t)\,x_k\,d\underset{\sim}{x} = 0 \quad (k = 1,2,3). \tag{7}$$

Finally, there are boundary conditions to be satisfied on the surface S bounding R (the latter defined as the set of points where $\rho > 0$). Let S have the equation

$$F(\underset{\sim}{x},t) = 0 \quad \text{(defines } S\text{)}.$$

Then we may take as boundary conditions:

$$\frac{DF}{Dt} = 0, \quad p = 0, \quad T = 0 \quad \text{on} \quad S. \tag{8}$$

Whether the initial-boundary value problem expressed by equations (1)-(8) is mathematically well-set is not known, at least in the generality with which the problem has been stated. It is a free-surface problem, i.e., the function F defining S is not given *a priori* but is to be discovered; this only increases the complexity of an already complex problem. Other difficulties of the formulation given above could be pointed out, but let us pass instead to the simpler problems that have in fact been successfully treated by astrophysicists: the construction of spherically symmetric equilibrium configurations.

Most of the effort in the theory of stellar structure has been directed toward constructing models for which $\underset{\sim}{v} = \underset{\sim}{0}$ and p, ρ, etc. depend only on r, the distance from the origin. If we define the luminosity $\ell(r)$ by the formula

$$\ell(r) = 4\pi r^2 q_r(r), \tag{9}$$

then we find in place of equations (1), (3), and (5)

$$p'(r) = \rho(r)V'(r), \tag{10}$$

$$\ell'(r) = 4\pi r^2 \rho(r)\varepsilon, \tag{11}$$

and

$$T'(r) = -[4\pi r^2 Q]^{-1}\ell(r), \tag{12}$$

with boundary conditions $\ell = 0$ at $r = 0$, $p = 0$ and $T = 0$ at $r = R$. Here R is not given *a priori*, but is to be discovered along with p, ℓ, and T. In place of equation (4) one has, in virtue of Newton's theorem on spherical distributions of mass,

$$V'(r) = -\frac{4\pi G}{r^2}\int_0^r s^2\rho(s)ds, \tag{13}$$

which can be substituted for $V'(r)$ in equation (10). A more convenient system of equations is obtained by replacing the independent variable r by m, where

$$m = 4\pi \int_0^r s^2 \rho(s)\,ds. \tag{14}$$

Then one finds the system of four equations

$$\frac{dr}{dm} = [4\pi r^2 \rho]^{-1}, \tag{15}$$

$$\frac{dp}{dm} = -[4\pi r^4]^{-1} Gm, \tag{16}$$

$$\frac{d\ell}{dm} = \varepsilon, \tag{17}$$

$$\frac{dT}{dm} = -[(4\pi r^2)^2 \rho Q]^{-1} \ell. \tag{18}$$

With ρ, ε, and Q regarded as functions of p and T, these form a nonlinear system of four ordinary differential equations for (r,p,ℓ,T), with the two-point boundary conditions

$$r = 0,\ \ell = 0 \text{ at } m = 0;\ p = 0,\ T = 0 \text{ at } m = M. \tag{19}$$

Hidden in these equations is a dependence on chemical composition. For given mass and chemical composition, one finds (in practice, numerically) a solution of the problem (15)-(19). To construct an evolutionary sequence, we now recognize that the chemical composition must change in virtue of the nuclear reactions. Since it changes on a timescale longer than other natural timescales appearing in the problem, it is reasonable to suppose that the equilibrium equations continue to apply (cf. Sec. 8). The procedure used by astrophysicists to calculate the "next" equilibrium model in an evolutionary sequence is to use the present model to estimate (according to known rules) the change in chemical composition during an appropriate time interval, and then use the altered chemical composition

in equations (15)-(19).

The physical considerations of this section may be found in fuller form in a number of books, e.g., references [3], [6], and [33].

3. Convective Instability and Convective Equilibrium

As one follows an evolutionary family of equilibrium configurations, one may find that, at some point, the equilibrium changes from stable to unstable. One may anticipate that nonuniqueness of the equilibrium solutions occurs at such a point. This possibility is now fairly widely recognized among astrophysicists in virtue of the actual numerical construction of multiple solutions in some cases [27]. This contrasts with a long-held view that equations (15)-(19) must have a unique solution (The Vogt-Russell Theorem; see, e.g., [3]).

Even at points where equations (15)-(19) have a unique solution, the more general equations (1)-(8) may allow solutions beyond the spherically symmetric, equilibrium solutions. If one linearizes the equations (15)-(18), the perturbations of the known solution are required also to have spherical symmetry. Consequently, if one wants to test the stability of a solution of (15)-(19) to non-spherical disturbances, it is necessary to return to the more general equations (1)-(8), and linearize these about the state of spherical equilibrium. In order to study stability against convection, the purpose of this section, this more general procedure is essential, because the convective motions are not radially symmetric.

In linearizing the equations (1)-(8) for the purpose of determining the onset of convective instability, the assumption is usually made that equation (3) can be replaced by

$$\frac{Ds}{Dt} = 0. \tag{20}$$

This says that the motion is adiabatic, i.e., that a parcel of fluid, as it moves along, does not exchange heat with its surroundings. The physical basis for this argument is that the convective motions, stable or unstable, are on the very short timescale T_1 ("dynamical" timescale),

$$T_1 = (G\bar{\rho})^{-1/2}, \tag{21}$$

where $\bar{\rho}$ is the mean density; whereas the righthand side of equation (3) only changes the entropy very slowly, on the so-called contraction timescale T_2:

$$T_2 = \frac{GM^2}{RL}, \tag{22}$$

where L is the luminosity. Hence the change in entropy should be negligible during the convective motions.

Accepting this reasoning, one finds that the stability problem thus obtained leads to a very simple criterion, called the Schwarzschild criterion in the astrophysical literature [33]. Let

$$A(r) = p'(r) - \frac{\gamma p(r)}{\rho(r)} \rho'(r) \tag{23}$$

where γ is the ratio of specific heats. Then the star is stable against convection if $A > 0$ for $0 \leq r \leq R$, unstable if $A < 0$ in any r-interval. An alternative formulation in terms of the temperature gradient is more familiar in meteorology. Let the "adiabatic gradient" T'_{ad} be defined as follows:

$$T'_{ad} = \frac{(\gamma-1)}{\rho(r)} T(r)\rho'(r). \tag{24}$$

Then one concludes stability if $|T'| < |T'_{ad}|$, $0 \leq r \leq R$, and instability otherwise.

The spectrum of eigenvalues associated with this stability problem is peculiar, and must affect the character of the solution (if there is one) bifurcating at the onset

of instability. Typically [7, 28, 13] the spectrum divides into two classes: (1) that for which the eigenvalues and eigenvectors of the linear stability problem represent sound waves, (2) that for which they represent the much slower, convective motions. The modes of class (1) are safely stable. Those of class (2) are all stable if $A > 0$ everywhere, but infinitely many of them become unstable when $A < 0$ somewhere. Hence if the chemical composition of the star changes slowly and in such a way that A changes from positive to negative in some r-interval (no matter how small this interval or how small $|A|$ is in it), infinitely many eigenvalues of the linear problem pass from stable to unstable type.

The principal gap in the theory of stellar structure and evolution appears at this point. What is the character of the steady-state solution (if there is one) past the point where the static, spherically symmetric solutions become unstable? Even if there is a solution, it is likely to be so complicated that calculating it is a practical impossibility. Astrophysicists would be happy to have an answer to the following more limited question. How much heat is transported by the convection?

The circumstance that infinitely many eigenvalues of the linear problem change from stable to unstable type when A changes sign would mean that the familiar existence theorems of bifurcation theory fail to apply, if the stability problem corresponded to the steady-state equations in the usual way. But it does not, because we have replaced equation (3) with equation (20). What would happen if the stability theory based on equation (3) were carried out? This is not known. Even if it made things better from the viewpoint of existence theory, it might not help from the viewpoint of finding the fluid motions or estimating the heat transport, since many, many of the linear problem's eigenfunctions would probably be impli-

cated in any realistic description of the flow or the heat transport.

Let us conclude this section by outlining the procedure actually followed by astrophysicists. Suppose a sequence of static models is constructed along the lines described in the preceding section, and suppose that, at some point in this sequence, A becomes negative in some layer $r_1 < r < r_2$, so that convection sets in. Crude estimates valid for many stars show that convection is far more efficient than radiation in transporting heat. To maintain the star's luminosity would require, in a convective region, a temperature gradient so minutely in excess of T_{ad} that the difference would be negligible. One therefore arrives at the following procedure. Continue to assume that the star is static and spherically symmetric, but replace equation (18) by

$$\frac{dT}{dm} = \begin{cases} -[(4\pi r^2)^2 \rho Q]^{-1} \ell & \text{if, as a result, } A > 0 \\ \frac{(\gamma-1)T}{\rho} \frac{d\rho}{dm} & \text{otherwise ,} \end{cases} \tag{25}$$

Hence one constructs two families of equilibrium configurations, one in radiative equilibrium, one in convective equilibrium. The latter family is stable past the point where the former is not.

The prescription (25) is adequate for many stars, but fails in some cases, where the balance between radiative and convective transport is more delicate. In these cases one tries to estimate the heat transport via the so-called mixing-length theory. These estimates are rather crude. Substantial efforts have been made toward improving these estimates, but they have not yet succeeded in providing a reliable means for doing so.

4. Rapid Rotation: Riemann's Idealized Models

Rotation is a common feature of the universe: stars rotate more or less rapidly (the sun less rapidly), binary

stars rotating about a common center of mass make up about half the stars in the sky, galaxies tend to be highly flattened by rotation. Accordingly, it is of interest to study the evolution and dynamics of rotating, self-gravitating masses. There are numerous obstacles in the path of such a study, however. For example, the gravitational force is much more difficult to treat when the figure is distorted from the spherical by rotation or other fluid motions, and the free-surface aspect of the boundary conditions becomes a severe hindrance as well. These difficulties are manageable when the rotation is small enough that the distortion from the spherical is small (cf. [2]), but for the case when the rotation is sufficiently large to have a decisive influence on the dynamics or evolution, the distortion is likely to be large.

Various approaches to the problem of finding steady-state figures having large amounts of angular momentum have been used. Numerical models have been constructed [26] and existence theorems have been proved [1, 22]. These approaches have not yielded enough information to enable one to make far-reaching inferences concerning the effect of angular momentum on the evolution of a star. There remain two approaches: Physical reasoning and idealized models. We shall concentrate on the latter, though the former is inevitably involved in interpreting the results. The idealized models are figures of uniform density whose bounding surfaces are ellipsoids. The restriction to uniform density makes these figures poor candidates for stars, which are centrally condensed, but one can hope that their behavior is qualitatively similar to that of more realistic models.

Seeking solutions bounded by an ellipsoidal surface enormously reduces the free-surface aspect of the problem. If the semiaxes are a_1, a_2, a_3, one must determine these three parameters, rather than the infinitely many needed to characterize a general surface. The further assumption that the density ρ is independent of position gives for

the gravitational potential the well-known [4] expression

$$(26) \qquad V(x) = \pi G\rho \{I - \sum_{k=1}^{3} A_k x_k^2\}, \quad \sum_{k=1}^{3} \frac{x_k^2}{a_k^2} \leq 1,$$

where the axes of x_1, x_2, x_3 are chosen along the principal axes of the ellipsoid. The coefficients I and $A_k (k = 1,2,3)$ are known functions of the semiaxes $a_k (k = 1,2,3)$. In 1742, Maclaurin showed that an oblate spheroid $(a_2 = a_1, a_3 \leq a_1)$ could be a figure of relative equilibrium in a frame of reference rotating with angular velocity ω about the x_3-axis provided

$$(27) \qquad \omega^2 = 2\pi G\rho (A_1 - \frac{a_3^2}{a_1^2} A_3), \quad 0 \leq \frac{a_3}{a_1} \leq 1.$$

These Maclaurin spheroids were thought to be the only figures of equilibrium of ellipsoidal shape until 1834, when Jacobi discovered that ellipsoids with three unequal axes are possible if

$$(28) \qquad a_3^2 A_3 = \frac{a_1^2 a_2^2}{a_1^2 - a_2^2} (A_2 - A_1), \quad 0 \leq \frac{a_2}{a_1} \leq 1,$$

and

$$(29) \qquad \omega^2 = 2\pi G\rho \frac{a_1^2 A_1 - a_2^2 A_2}{a_1^2 - a_2^2} .$$

Equation (28) gives a_3/a_1 as a function of a_2/a_1, hence specifying the shape of the ellipsoid. In the limit as $a_2 \to a_1$, one finds $a_3/a_1 = 0.5827\ldots$. For this value of a_3/a_1, there also exists a Maclaurin spheroid. One can show that the limiting value of the angular velocity (29) agrees with that given by equation (27). Hence the limiting Jacobi figure is a Maclaurin spheroid: The Jacobi ellipsoids bifurcate from the Maclaurin spheroids at this point.

In a paper published in 1861, Riemann substantially generalized these results, still in the context of ellipsoidal figures of uniform density. He sought ellipsoidal figures satisfying equations (1) and (2) (equation (3) gets decoupled and can be ignored if ρ is constant in space and time) which had principal axes aligned along coordinate axes in a frame of reference rotating with angular velocity $\underset{\sim}{\omega}$; relative to this frame, there is in general a velocity field whose components are linear functions of position. He found that this velocity field could be characterized by a vector $\underset{\sim}{\lambda}$. These assumptions lead to a solution of equations (1) and (2) provided the time-dependent parameters $a_k (k = 1,2,3)$, $\underset{\sim}{\omega}$, and $\underset{\sim}{\lambda}$ satisfy a certain nonlinear system of ordinary differential equations.

The discussion here will be limited to those Riemann ellipsoids for which $\underset{\sim}{\omega} = (0,0,\omega)$ and $\underset{\sim}{\lambda} = (0,0,\lambda)$. The velocity field relative to the rotating frame is given by

$$(30) \qquad u_1 = \frac{a_1}{a_2}\lambda x_2 + \frac{\dot{a}_1}{a_1}x_1, \quad u_2 = -\frac{a_2}{a_1}\lambda x_1 + \frac{\dot{a}_2}{a_2}x_2, \quad u_3 = \frac{\dot{a}_3}{a_3}x_3,$$

where the dot represents time derivative. Referring to equation (1), where the velocity field $\underset{\sim}{v}$ is relative to inertial space, we have

$$(31) \qquad \underset{\sim}{v} = \underset{\sim}{\omega} \times \underset{\sim}{x} + \underset{\sim}{u},$$

and

$$(32) \qquad p(\underset{\sim}{x},t) = p_c(t)\left(1 - \sum_{k=1}^{3} \frac{x_k^2}{a_k^2}\right).$$

The differential equations are:

$$(33) \qquad \ddot{a}_1 = a_1(\lambda^2+\omega^2) - 2a_2\lambda\omega - 2\pi G\rho a_1 A_1 + 2p_c/\rho a_1,$$

$$(34) \qquad \ddot{a}_2 = a_2(\lambda^2+\omega^2) - 2a_1\lambda\omega - 2\pi G\rho a_2 A_2 + 2p_c/\rho a_2,$$

$$(35) \qquad \ddot{a}_3 = -2\pi G\rho a_3 A_3 + 2p_c/\rho a_3,$$

$$a_1\dot{\lambda} - a_2\dot{\omega} + 2\dot{a}_1\lambda - 2\dot{a}_2\omega = 0, \tag{36}$$

$$a_1\dot{\omega} - a_2\dot{\lambda} + 2\dot{a}_1\omega - 2\dot{a}_2\lambda = 0. \tag{37}$$

Equations (36) and (37) can be integrated, giving

$$(a_1-a_2)^2(\omega+\lambda) = \frac{5}{M}J - \frac{1}{\pi}C \equiv 2K_1, \tag{36'}$$

$$(a_1+a_2)^2(\omega-\lambda) = \frac{5}{M}J + \frac{1}{\pi}C \equiv 2K_2, \tag{37'}$$

where J is the angular momentum and C the equatorial circulation, both conserved under Riemann's assumptions. Equation (2) above expressing mass conservation is replaced, under the present assumptions, by

$$\frac{4}{3}\pi\rho a_1a_2a_3 = M. \tag{38}$$

Of course, if the fluid is incompressible, then ρ is constant and $a_1a_2a_3$ is constant. These equations, as well as various modifications of them, have been integrated numerically [30, 32].

Consider the equilibrium solutions of equations (33)-(37). One finds without difficulty that they are satisfied if and only if two relations of the following form hold:

$$(\omega+\lambda)^2 = \pi G\rho f\left(\frac{a_2}{a_1},\frac{a_3}{a_1}\right), \tag{39}$$

$$(\omega-\lambda)^2 = \pi G\rho g\left(\frac{a_2}{a_1},\frac{a_3}{a_1}\right), \tag{40}$$

where f and g are certain combinations of a_k and A_k $(k = 1,2,3)$ which are homogeneous of degree zero in a_1, a_2, a_3. One can assume without loss of generality that $a_2 \leq a_1$. One can then infer that $a_3 \leq a_1$, and the domain $(\frac{a_2}{a_1})(\frac{a_3}{a_1})$-plane where equilibrium solutions exist is inside

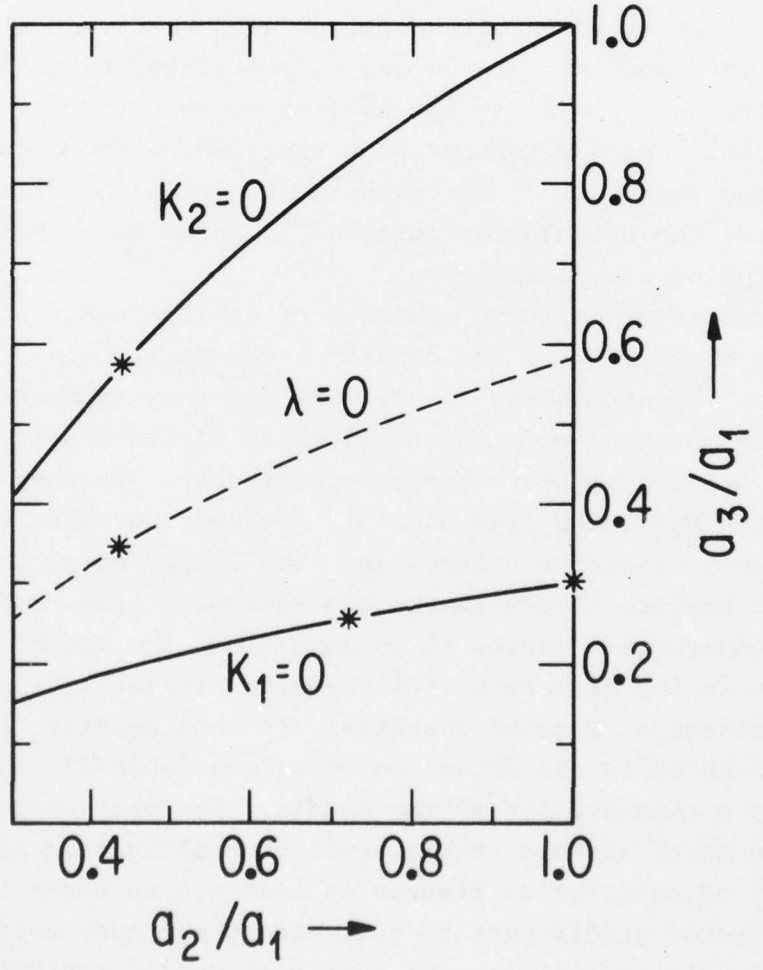
the horn-shaped region of the Figure . To each point in this domain there correspond two essentially distinct Riemann ellipsoids. This can be seen easily from equations (39) and (40), which are unchanged if ω and λ are interchanged. Two solutions of the equations obtained from one another in this way are called "adjoint". Along the boundaries, the figures are self-adjoint. This notion of adjointness extends to the time-dependent equations (33)-(37), and even to the more general time-dependent equations where no assumption is made regarding the directions of $\underset{\sim}{\omega}$ and $\underset{\sim}{\lambda}$ (cf. [4], where a full discussion of the Riemann ellipsoids is given).

5. The Fission Theory: $J = \text{const.}, \lambda = 0$.

The influence of angular momentum on a star's evolution is likely to be particularly pronounced in those stages of the star's existence when it undergoes a large contraction. One of the persistent ideas in astronomy is that a binary system might be the outcome of the evolution of a rotating, self-gravitating mass undergoing contraction. Since about half the stars in the sky are binary stars, and the mechanism whereby they are formed is not at all agreed upon, this idea, that a single star might undergo fission to create a pair, has received considerable attention.

The notion was first broached in 1883 in the second edition of the famous physics text by Thomson and Tait [35]. They noted that, for the same mass, density, and angular momentum, the Jacobi ellipsoid has lower energy than the Maclaurin spheroid. They reasoned that, in the presence of viscosity to dissipate the energy difference, the Maclaurin spheroids in question should be unstable to a disturbance deforming them in the direction of the Jacobi ellipsoids. This conjecture has only been verified rather recently [31]. They further noted that the total energy of a binary system having the same mass, density, and angular momentum as a sufficiently elongated Jacobi ellipsoid,

The boundaries of region where Riemann ellipsoids exist are indicated by $K_1=0$ and $K_2=0$. The Maclaurin sequence is at the right ($a_2/a_1=1$). The Jacobi sequence is dashed and indicated by $\lambda=0$.



has lower energy. They concluded that, far enough along the Jacobi sequence, that figure should be unstable to a deformation tending to thin it in the middle and thicken it at the ends.

This leads to a picture of evolution along equilibrium configurations, first described by Poincaré [29]. In the following description, it is necessary to note that viscosity plays a major role, whereas the Riemann ellipsoids of the preceding section were assumed inviscid. However, if $\lambda = 0$, we can show that the corresponding figures, which are precisely the Maclaurin and Jacobi figures, continue to be steady-state solutions even when viscosity is taken into account. Hence one is restricted to the family of solutions $\lambda = 0$ by the assumption that viscosity is important. We may further note that, while the angular momentum integral J survives the inclusion of viscous effects, the circulation integral C does not. Poincaré's description was as *follows*.

Suppose a Maclaurin spheroid of small eccentricity contracts slowly, i.e., its density increases slowly, its angular momentum being fixed. One can show that it then becomes more and more oblate (just as if its density were fixed and its angular momentum increased). It then moves down the Maclaurin line until it reaches the point where the Jacobi sequence bifurcates. The latter being the stable branch, it now leaves the Maclaurin line, and further contraction causes it to move along the Jacobi sequence in the direction of increasing elongation, until that sequence becomes unstable. It then departs from the Jacobi shape in the direction of a "pear-shaped" figure, having a constriction in the middle. The pear-shaped figure being assumed stable, evolution along this new family of equilibrium figures is imagined to cause the constricted middle part to grow slowly narrower until it finally disappears, leaving a pair of masses orbiting one another.

This picture was based partly on established results, partly on conjecture. In his memoir of 1885, Poincaré established many of the ideas and much of the terminology of bifurcation theory in the context of systems having finitely many degrees of freedom, and applied them to studying bifurcation along the Jacobi sequence. He found that it becomes unstable to a particular disturbance belonging to the third ellipsoidal harmonics. He drew a diagram showing what the perturbed figure would look like in a first approximation, and it indeed contains a constriction, and looks rather like a pear. However, the diagram is inaccurate: careful calculations show no such constriction, and the figure looks more like an egg than a pear. This is not so serious in the theory as some other defects, however.

Poincaré's analysis was linear and it consequently left open the question whether the new series bifrucates "forward" (i.e., is stable) or "backward" (i.e., is unstable). It is essential for the picture above that the bifurcating figures be stable. Laborious computations are needed for deciding this. They were carried out first by Sir George Darwin [9] who concluded stability, next by Lyapunov [23] who concluded instability, and (presumably) finally by Jeans [12], who not only concluded instability in agreement with Lyapunov, but also tracked down an error in Darwin's work which, when corrected, brought it into agreement with the others. This was a heavy blow to the fission theory, at least as pictured by Poincaré. Others, notably Jeans himself [12], have argued in its favor despite this and other setbacks, but they have not been very convincing.

6. The Evolution of Gravitationally Radiating Stars: $C = \text{const.}$, $\omega = 0$.

There are two stages in a star's existence when it can undergo a large contraction. An early stage, before the

temperature near the center is high enough to cause nuclear reactions to take place, and a late stage, after the nuclear fuel has been consumed. In this late stage, bizarre things seem to happen. If the mass of the star is less than a critical mass M_1 (about $1.4M_\odot$, where $M_\odot$ is the mass of the sun), it collapses to a state of high density known as a white dwarf. If its mass exceeds M_1 it cannot become a white dwarf but becomes a neutron star at still higher density, provided its mass is less than a second critical mass M_2 (about $3M_\odot$). If its mass exceeds M_2 it collapses out of sight and becomes a black hole. At some point, at about the density of a compact neutron star, the gravitational field becomes large enough for effects of general relativity to be important.

One of these effects is gravitational radiation. This is produced by the rapid motions of massive bodies. For example, the rotation of a superdense star about an axis with respect to which it is asymmetric would generate such waves. Astrophysicists have used the Riemann ellipsoids in an attempt to follow the evolution of a contracting, gravitationally radiating mass, in the following way [11, 25].

The Newtonian equations (1) and (2) may be regarded as the first terms in a formal power series expansion of the general-relativistic equations (cf. [5]; the parameter of expansion is v/c, where v is a characteristic velocity and c is the speed of light). At a certain stage in this expansion, a term appears on the right-hand side of (1) that is identified as the radiation reaction: it represents the effect on the motions of the radiation of gravitational energy. To find the effect of gravitational radiation, one neglects all relativistic contributions but this one. Hence the equations are as before, with the exception of the forcing term on the right-hand side. One can ask whether ellipsoidal solutions of the kind envisaged

by Riemann are still possible. Now, the radiation-reaction term is the gradient of a scalar V_R, where

$$V_R = -\frac{G}{15c^5}\left(\frac{d^5 D_{jk}}{dt^5}\right)x_j x_k, \tag{41}$$

and

$$D_{jk} = 3\int_R \rho x_j x_k dx - \delta_{jk}\int_R \rho |x|^2 dx \quad (j,k = 1,2,3);$$

summation on j,k is implied in equation (41). Since V_R is quadratic in the coordinates, one indeed finds that ellipsoidal solutions are possible. Equations (33)-(38) above are suitably modified [11]. The circulation integral C survives the introduction of radiation reaction because it is conservative, but angular momentum is no longer a conserved quantity. What about steady-state solutions?

The Maclaurin spheroids, being symmetric about the rotation axis, do not radiate and continue to be steady-state solutions of the relativistically modified equations. Any asymmetric ellipsoid would have to be at rest in inertial space in order not to radiate, i.e., in order to be in a steady-state. Such peculiar objects indeed exist. This can be seen as a consequence of the existence of adjoint configurations (Section 4). The configuration adjoint to a Jacobi ellipsoid ($\omega = \omega_o, \lambda = 0$) is a figure with the same shape, but $\omega = 0$, $\lambda = \omega_o$: its surface appears at rest in inertial space, and it is supported by internal motions. Hence, in the presence of gravitational radiation, the sequence of steady-state figures along which evolution may proceed is characterized by $C = \text{const.}$, $\omega = 0$, in remarkable reciprocity to the classical fission theory.

The evolutionary picture is now: contraction makes the Maclaurin spheroids progressively more oblate; at the same point as that where the Jacobi sequence branches off, the Maclaurin spheroids become unstable; further contraction

takes place along the adjoint to the Jacobi sequence (called the Dedekind sequence); the latter also becomes unstable to disturbances described by third ellipsoidal harmonics, at a different point along the sequence from the point where the "pear-shaped" figures branch off from the Jacobi sequence. Nothing further is known about the bifurcating figures at this point, and only the wildest speculations can be made regarding the creation of superdense binary stars.

7. The Fission Theory Reformulated: J = const. and C = const.

In each of the preceding sections, dissipative mechanisms were invoked to infer that the Maclaurin sequence becomes unstable past the point where the Jacobi-Dedekind sequence intersects it. Neither of these dissipative mechanisms is contemplated in the equations of Section 4. If they are indeed absent, then J and C are both constant (eqs. (36') and (37')). The constancy of J and C delineates a family of steady-state figures (just as the constancy of J and the vanishing of λ, or the constancy of C and the vanishing of ω). In particular, a configuration starting on the Maclaurin line would have $K_1 = 0$ initially, hence always. The unique asymmetric family of ellipsoids having $K_1 = 0$ is so labeled in the Figure. It intersects the Maclaurin sequence at the point where the latter becomes unstable in the absence of dissipation. This point where instability sets in was discovered by Riemann.

A reformulation of the fission theory along the lines described above has been initiated in [14]. Neglecting dissipation seems reasonable in view of estimates as to its effectiveness (cf. [24]), but a potentially more important feature has been incorporated into this reformulation: compressibility. In the theories outlined in Sections 5

and 6, it is assumed that the density increases along the sequences considered, but the stability results are obtained under the assumption of incompressibility. This not only seems inconsistent, but also suppresses changes in the internal energy, which, in a star, are comparable in size with the changes in kinetic and gravitational energies. It turns out that Riemann's equations (Section 4) remain valid if ρ is a function of t (and t only). Then one must adjoin the energy equation (3) and an equation of state. This is done in [14], and indeed leads to evolution along the branch of ellipsoids marked $K_1 = 0$. These become unstable at the point indicated in the Figure [15]. Whether the bifurcating figures are stable or not is not yet known, but the stability analysis is now sufficiently different (because the effect of compressibility is included), that one may seriously entertain the possibility that, in contrast to the classical result, they are stable.

Whatever the outcome of the fission theory may turn out to be, the problems outlined in this and the preceding sections contribute to answering the fundamental question expressed in the last sentence of the Introduction.

8. Singularly Perturbed Initial-Value Problems

The evolutionary problems discussed here, in common with a variety of analogous problems from scientific fields other than astrophysics, have a certain paradoxical element: on the one hand, the figures are in a steady state whereas, on the other hand, they evolve. Physically there is no paradox. One notes that (at least) two timescales are relevant to the problem, a short timescale T_1 and a long timescale T_2 (cf., e.g., equations (21) and (22) above). During evolution on the long timescale the system may be, to a good approximation, in a steady-state with respect to motions on the short timescale. To the extent that there is a mathematical paradox, it too can be resolved, as follows.

Consider a system of ordinary differential equations of the form (x and y may be vectors)

$$\frac{dx}{dt} = f(x,y), \quad \varepsilon\frac{dy}{dt} = g(x,y), \tag{42}$$

where ε is a small parameter (the ratio of the timescales T_1 and T_2, say). These equations are vastly simpler than those of Section 2 governing stellar evolution, but certain analogies can be made. The chemical composition, whose evolution equation was not written down in Section 2, is represented by the vector x, and the remaining variables (v,ρ,s) by y. Then (x,y) indeed satisfy a system of equations analogous to (42), provided that time t is measured in the unit T_2 (i.e., on the long or evolutionary, timescale). The "reduced" system of equations is (for $\varepsilon = 0$)

$$\frac{dx}{dt} = f(x,y), \quad 0 = g(x,y). \tag{43}$$

In this system, the second equation is the equation for the steady state. Hence, if for small ε the solutions of (43) approximate the solutions of (42) accurately enough, one can indeed speak of evolution along steady-state solutions.

The system (42) is a "singular" perturbation of (43) in the sense that it can accomodate initial data for x and y, whereas only intial data for x can be specified in connection with (43). This has the consequence that solutions of (42) can, at best, approach those of (43) nonuniformly in t as $\varepsilon \to 0+$. Moreover, solutions of (42) need not approach those of (43) at all, unless a certain condition is satisfied. This condition may be stated in the following way [20]: the Jacobian matrix $g_y(x,y)$ should be stable along the solution of (43), i.e., all its eigenvalues should have negative real parts there. This condition can be related to the stability, on the short timescale, of a steady-state configuration with x viewed

as a parameter (exactly as one does on physical grounds). If the reduced path satisfying (43) encounters a point of nonuniqueness (a bifurcation point), this condition fails, for g_y must be singular there. For the most familiar kinds of bifurcation points one can nevertheless analyze the relations of (42) to (43) [16, 17, 21], with the results that the solution of (42) "usually" stays close to stable branches of the steady-state solutions (although this can fail and the solution of (42) become rapidly variable in time). Note that the nonuniqueness associated with the bifurcation problem and hence also with the reduced problem (43) in no way contradicts the usual uniqueness conclusion for the "real" problem (42).

We note finally that the standard form of bifurcation problems with t as a parameter is [8]

$$G(t,y) = 0. \tag{44}$$

Equation (43) would reduce to this if f were independent of y. In general this is not true, however, and bifurcation in evolving systems is more accurately described by (43).

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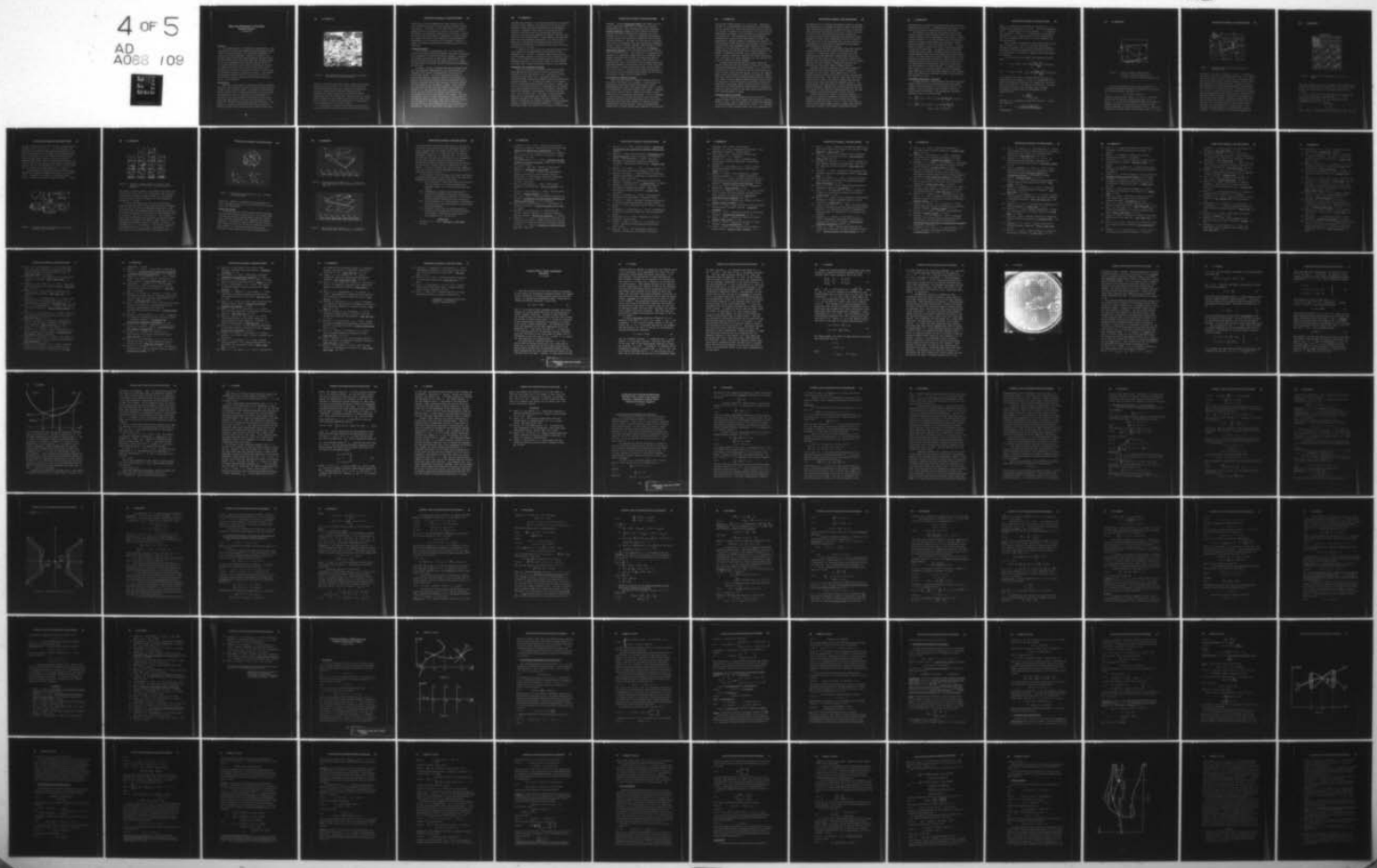
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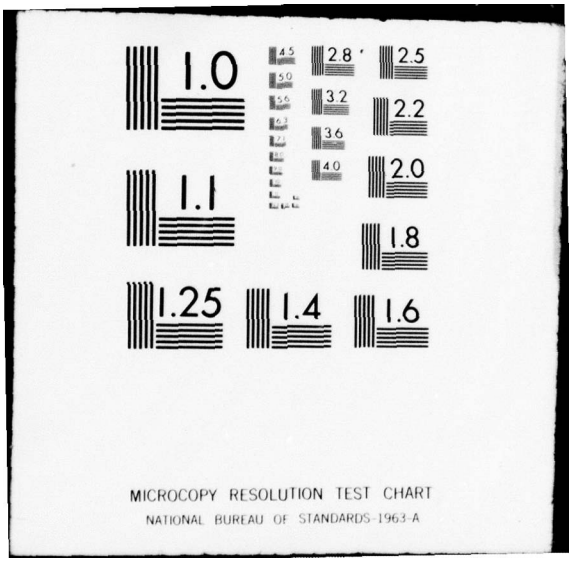
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Bifurcation Phenomena in Chemically Reacting Systems

W. Harmon Ray

ABSTRACT

An enormous amount of theoretical, computational, and experimental work has been reported showing multiple steady states, pattern formation, and oscillatory phenomena in chemically reacting systems. Although a number of the systems studied have been analyzed through the application of bifurcation theory, many more important problems remain virgin and untouched by this powerful tool of analysis. This paper discusses a wide range of problems of interest to engineers and scientists involving heat and/or mass transfer with chemical reaction and which have been or can be profitably treated by bifurcation theory. Case studies are presented showing some of the results which may be obtained for chemical reactors and how these results are useful to the engineer.

INTRODUCTION

There are a wide range of processes involving chemical reaction which have been observed to exhibit bifurcation to steady or time periodic states. Although the first experimental observations of these phenomena were certainly made by prehistoric cave men carrying out the oxidation of cellulose (i.e. wood burning), they could not have understood the complexities of ignition and extinction of the flame and the flame oscillations they were observing. (cf Figure 1). Some of the earliest scientific reports of multiple steady states and oscillatory behavior were with



Figure 1. Early Observations of Bifurcation Phenomena in Chemically Reacting Systems.

regard to catalytic and non-catalytic oxidation and oxidation-reduction reactions [1-4]. Since this early work, similar observations have been reported for biological reactors, polymerization process, homogeneous and heterogeneous tubular reactors, both open and closed homogeneous reaction systems with complex kinetics, single catalyst particles, and catalytic wires just to name a few. Thus the scope for potential applications of bifurcation theory appears to be very large indeed.

In this paper, the goal is to provide an overview of the many chemically reacting systems where multiple steady

states and periodic phenomena have been observed and to show how some of these have been analyzed through bifurcation theory. Therefore, in the next section, a survey is presented which is designed to provide the reader with a guidebook to the enormous literature in the field. Following this a case study shall be described in which bifurcation theory has led to the successful and comprehensive elucidation of the structure of the problem. Finally, a number of promising problem areas needing analysis will be described.

LITERATURE SURVEY

It would be an overwhelming task to survey exhaustively the extensive and wide ranging literature on bifurcation phenomena in chemically reacting systems. Therefore, the present survey will be selective, emphasizing those areas of greatest interest and familiarity to the author, and shall rely on existing reviews of particular subdomains of the field.

To begin, we shall consider branching phenomena which have been observed experimentally and through numerical simulation and without considerations of analysis by bifurcation theory. In a very interesting review of oscillating reaction systems, Higgins [5] discusses the types of oscillations observed and the particular kinds of kinetic mechanisms which might simulate the oscillations. His main emphasis was biological systems. More recently, two additional reviews over the dynamics of biological systems [6,7] also provide many useful references. A rather general overview of chemical oscillations was contributed by Nicolis and Portnow [8]; however, there are fundamental errors in their discussion of thermochemical oscillations and one should treat this review with caution. The literature on the dynamic behavior of reactors of interest to the chemical industry has been reviewed rather extensively. Ray [9], Sheintuch and Schmitz [10], and

Hlavaček and Vortruba [11] discuss the behavior of catalytic reactors, while Schmitz [12] and Gilles [13] review the reported results for both homogeneous and heterogeneous reactors. Recently a great deal of interest has been devoted to reactions such as the Belousov-Zhabotinsky reaction which show both temporal and spatial periodicities even in a semi-closed system. These systems and their wave patterns are extensively discussed in the fine review by Marek [14], who himself has reported some of the most interesting comparisons of experiments and theory with this reaction (e.g., [15-19]).

In addition to the review articles noted above, there are a number of monographs which are devoted to these problems (e.g., [20-27]). Although these review articles and monographs provide an entree into the literature in each area, it is instructive to consider some of the principal phenomena observed in the various fields and what efforts have been made to apply bifurcation theory to the problem.

Biological and Biochemical Systems

There is at present an enormous amount of interest among mathematicians as well as natural scientists and engineers in the oscillations (e.g., biological clocks), multiple steady states (e.g., biological switches), travelling waves and pattern formation found in biologically reacting systems. It is beyond the scope of the present effort to comprehensively discuss this field. However, one can note that major efforts are being made both in biological living systems as well as in biochemical reactors for wastewater treatment, fermentation, and food production (e.g., [6,7,22,25,27-66]). The major thrust at present seems to be to determine the richness of bifurcation phenomena possible from simple reaction networks in an effort to establish the simplest possible biological models capable of predicting observed phenomena. A number of directions seem to be evolving in the effort to solve these

problems. Thom's Catastrophe Theory [28] makes use of a visual topological approach in attempting to explain morphogenesis and other observed bifurcation phenomena. Pattern formation, waves and other spatial structures arising from partial differential equations are being analyzed [67-78] in an effort to understand experimental results arising from biological systems as well as model reaction networks. The goal of this work is to explain nerve signal transmission and other biochemical phenomena through a study of simple chemical model systems [79-110]. The most heavily studied model system is the Belousov-Zhabotinski Reaction [93] involving malonic acid, cerium ions, and bromate ions. This system is an interesting one because it is relatively simple and yet it displays both temporal periodicities as well as interesting dynamic pattern formation. The literature on this reaction is astonishingly large; thus only a few key papers describing observed phenomena [14-18,93-100] will be noted along with those efforts to apply bifurcation theory to the system [14,19,78,101-110]. Two particularly good discussions of the current work on analyzing this network may be found in the surveys of Marek [14] and Othmer [78].

Non-catalytic Chemical Reactors

There are many types of chemical reactors which do not involve the action of a catalyst. These include the homogeneous continuous stirred tank reactor (CSTR), heterogeneous gas-liquid and liquid-liquid CSTR's, as well as non-catalytic reactions carried out in tubes. A special case of the latter class of reactors are those in which combustion of a hydrocarbon takes place. It is of interest to those working in bifurcation theory that for certain reactions and under certain conditions all of the reactor types mentioned have been observed to exhibit multiple steady states and oscillations. Even at thermodynamic equilibrium in closed systems there appears to be branching

to multiple steady states (e.g., [111,112]). Excellent surveys of the experimental observations and their models for the reactions of short chain molecules may be found in the reviews of Schmitz [12] and Gilles [13]; thus the only augmentation necessary is in the areas of polymerization and combustion. Both homogeneous and heterogeneous polymerization reactors have been observed (either from experiment or through analysis of mathematical models) to show multiple steady states and oscillations [113-124] and in some cases these phenomena cause enormous control problems in the industrial reactor. Similarly, multiple steady states and oscillations are serious problems in the design of combustion reactors (e.g., heating devices, rocket engines, industrial reactors for olefin production, etc.) and there is a great literature (cf [13,125-130] for a brief overview) on these problems.

Although branching to multiple steady states and periodic solutions has been known for more than 20 years in continuous homogeneous stirred reactors (e.g., [131-134]), the structure of the problem was greatly clarified by the work of Poore [135] and Uppal et al. [136,137] who applied Hopf-Friedrichs bifurcation theory to the problem. There have been several further reported applications of bifurcation theory to more complex reaction systems [122,139-141]. The analysis of the bifurcation of periodic solutions has been made a great deal easier through the work of Poore [138] who has reformulated the Hopf-Friedrichs bifurcation theory in order to make it both more widely applicable and more straightforward to apply.

Catalytic Reaction Systems

Catalytic reaction systems most often involve a catalytic agent on a solid support and housed in a stirred vessel or tube. Thus the analysis of bifurcation phenomena in these systems must be made on two levels: (i) consideration of the solid catalyst alone, and (ii) examination of

the behavior of the reactor and catalyst material together. This separation is necessary because multiple steady states (both uniform and nonuniform in space) as well as oscillations and waves have been found at both levels of analysis. As noted in the introduction, there are a number of excellent surveys of the experimental phenomena and their analysis for this class of problems [9-13,26] so that we shall only note a few key papers here and then proceed to a discussion of those very few problems having already been analyzed by bifurcation theory.

On catalytic surfaces, experiments show that multiple steady states and various types of oscillations occur [9-13,26]. In addition, model calculations show that standing waves and asymmetric steady states are to be expected in some cases (e.g., [26,142-147]). However, it appears that the standing waves are expected to be unstable and only a limited number of asymmetric profiles are possible for the simple kinetic schemes studied.

Although there are hundreds of published studies of the branching of multiple steady states on catalytic surfaces (cf [9-13,26]), there appear only to be two papers employing bifurcation theory to study time periodic solutions [148,149]. The first of these [148] uses bifurcation theory to show that simple thermokinetic oscillations are not likely to arise on catalyst surfaces. The second paper [149] demonstrates a useful lumping procedure and the subsequent application of Hopf-Friedrichs bifurcation theory to the resulting lumped equations for the case of diffusion and reaction inside a porous catalyst.

Before leaving the subject of catalyst dynamics, it is useful to note that relaxation oscillations seem to be playing an important role in the observed oscillatory behavior of many reactions [10,13]. Since these oscillations involve sudden jumps between multiple quasi-stationary states, there is an intimate relationship between

bifurcation to relaxation oscillations and bifurcation to multiple steady states. A comprehensive theoretical treatment of this class of problems would be very useful.

The behavior of catalytic well mixed reactors may exhibit multiplicities and oscillations both due to the reactor itself or due to phenomena on the catalyst surface. In some instances, it is even possible for the reactor to behave in a stable fashion when the catalyst by itself would be unstable at the same conditions [150]. For the tubular catalytic reactor, the range of possible behavior is even more interesting. In addition to the existence of multiple steady states (as many as three stable non-uniform steady states have been observed in some instances [151]), one sees travelling waves of various types [9,13]. Although the branching to multiple steady states has been studied by many workers [9-13], there do not seem to be any applications of bifurcation theory to the periodic and wave behavior of these reactors. It appears that at least some of the travelling waves are of a relaxation type controlled by the local heat transfer at the reaction front, and these can be analyzed through a quasi-stationary analysis [13,152].

AN EXAMPLE CASE STUDY - THE CSTR

In this section we shall discuss the steady state and dynamic behavior of the Continuous Stirred Tank Reactor (CSTR) and the application of bifurcation theory to the problem. For the simplest type of reaction kinetics (first order irreversible), the mathematical modelling equations take the following form

$$(1) \qquad \frac{dx_1}{dt} = -x_1 + Da(1 - x_1) \exp\left\{\frac{x_2}{1 + x_{2/\gamma}}\right\} = f_1(x_1, x_2)$$

$$(2) \qquad \frac{dx_2}{dt} = -x_2 + B\,Da(1 - x_1) \exp\left\{\frac{x_2}{1 + x_{2/\gamma}}\right\}$$

$$-\beta(x_2 - x_{2c}) = f_2(x_1, x_2)$$

where x_1 denotes the conversion of reactant and x_2 a dimensionless temperature [136,137]. The system parameters are B, a dimensionless heat of reaction; γ a dimensionless activation energy; β a dimensionless heat transfer coefficient; x_{2c} a dimensionless coolant temperature, and Da a dimensionless ratio of reactor residence time to reaction time.

This system of equations has been under study for more than 20 years (e.g., [131-134]), but the structure of the periodic behavior was not clear until Hopf-Friedrichs bifurcation theory was applied to the problem [135-137]. In this section, the results of applying this theory shall be discussed.

If one considers the steady state behavior arising from

$$0 = -x_{1s} + Da(1 - x_{1s}) \exp\left\{\frac{x_{2s}}{1 + \frac{x_{2s}}{\gamma}}\right\} = f_1(x_{1s}, x_{2s}) \tag{3}$$

$$0 = -x_{2s} + B\,Da(1 - x_{1s}) \exp\left\{\frac{x_{2s}}{1 + \frac{x_{2s}}{\gamma}}\right\} = f_2(x_{1s}, x_{2s}) \tag{4}$$

it is easy to show that it has one of the two types of behavior shown in Figure 2. Curve (1) corresponds to the case where branching occurs as Da varies while curve (2) shows the behavior in the absence of branching. As noted, the branching points $x_{1s} = m_1(Da_1)$, $x_{2s} = m_2(Da_2)$ correspond to points where the determinant of the Jacobian matrix

$$\underset{\sim}{A} = \left\{\frac{\partial f_i}{\partial x_j}\right\}_{x_{1s}, x_{2s}} \tag{5}$$

vanishes. It is possible to show that $\det \underset{\sim}{A} > 0$ unless the condition

$$B > \frac{4[1 + \beta + (\beta x_{2c}/\gamma)]^2}{1 + \beta - (4/\gamma)[1 + \beta + \beta x_{2c}/\gamma]} \tag{6}$$

is satisfied.

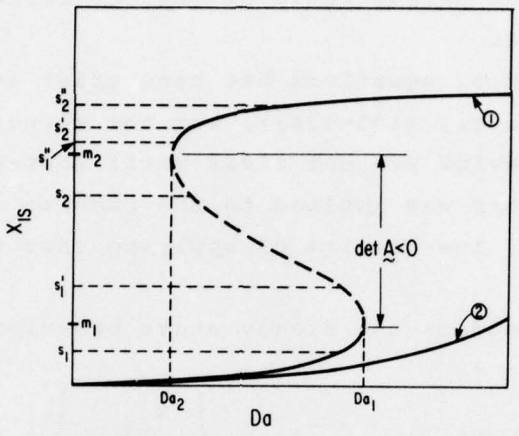


Figure 2. Locus of Reactor Steady States:
(1) when B satisfies Equation (6),
(2) when B fails to satisfy Equation (6).

The local stability as well as the existence of bifurcating periodic solutions is determined by the roots of the characteristic equation of $\underset{\sim}{A}$

$$\lambda^2 - (\text{tr } \underset{\sim}{A})\lambda + \det \underset{\sim}{A} = 0 \tag{7}$$

Stability requires that $\det \underset{\sim}{A} > 0$, $\text{tr } \underset{\sim}{A} < 0$ while the bifurcation of periodic orbits occurs when $\text{tr } \underset{\sim}{A} = 0$, $\det \underset{\sim}{A} > 0$. All of this information allows one to plot the dynamic character of the system as a function of the system parameters as shown in Figures 3 and 4. Line M is condition (6), while line S is the lowest value of B (as a

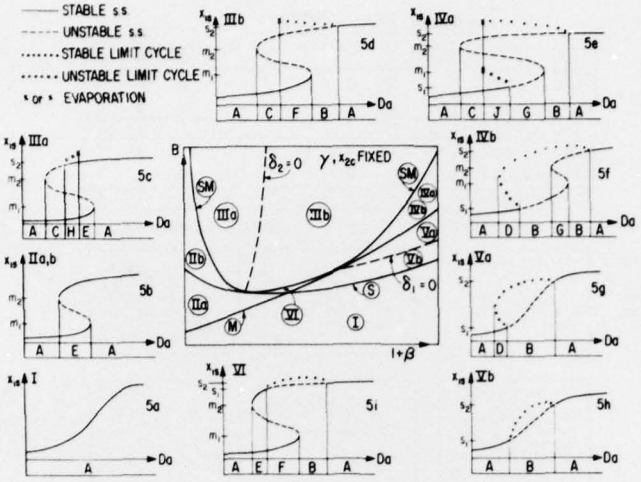


Figure 3. CSTR Behaviour for Different Regions in Parameter Space

function of β, γ, x_{2c}) for which Tr $\underset{\sim}{A} = 0$ is possible. Line SM is the locus of parameters for which tr $\underset{\sim}{A}$ = det $\underset{\sim}{A}$ = 0. Thus lines M,S,SM define six regions in parameter space which have qualitatively different dynamic behavior due to the placement of bifurcation points. Regions III and V have two distinct subregions determined by the direction of bifurcation of the limit cycles. The dividing lines $\delta_1 = 0$, $\delta_2 = 0$ were determined by a perturbation analysis from the bifurcation points [135,136,138]. Thus for this problem, almost the entire structure may be determined purely from local analysis. Numerical computations were used to trace the periodic orbits and this led us to the discovery of the "evaporation" of limit cycles as they encounter the separatrices in the phase space. This coincides with the period of the oscillation becoming infinite. An additional bit of structure uncovered is

TYPICAL PHASE PLOTS

CASE		A	B	C	D	E	F	G	H	J
NODE or FOCUS	STABLE	1	0	1	1	2	1	0	2	1
	UNSTABLE	0	1	1	0	0	1	2	0	1
SADDLE POINT		0	0	1	0	1	1	1	1	1
LIMIT CYCLES	STABLE	0	1	0	1	0	1	1	0	1
	UNSTABLE	0	0	0	1	0	0	0	1	1
TOTAL INVARIANTS		1	2	3	3	3	4	4	4	5

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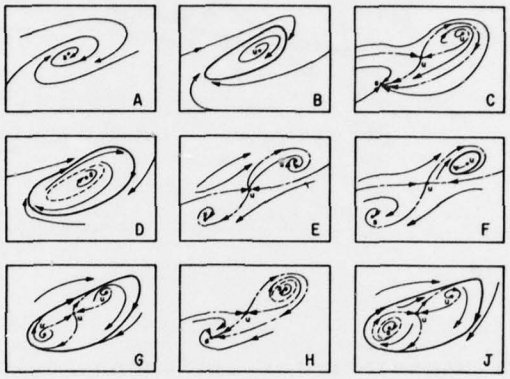


Figure 4. Classes of Possible Phase Plots for the CSTR.

that there appear to be only 9 distinct types of phase plots possible for this problem (denoted A,B--J in Figures 3 and 4), and their evolution, as Da is varied, may be seen clearly.

In an effort to aid the experimentalist who may most easily vary the reactor residence time, τ, in the laboratory (rather than Da), the system (1-2) was reparameterized [137] by letting

$$Da = Da_0 \tau \qquad \beta = \beta_0 \tau \tag{8}$$

where Da_0, β_0 become reference constants (in fact β_0 can

be made a unique function of B) so that the parameters of interest are B, Da_0, γ, x_{2c}, τ. To our surprise, the structure of the problem was greatly complicated by this simple transformation. The steady state bifurcation was found to have one of the five types of branching shown around the perimeter of Figure 5 and the regions in parameter space where each of these types of behavior are found is shown in the center of the figure. Of particular interest are the isolated branches (isola) and the existence of as many as 4 branch points as τ varies.

The bifurcation to periodic solutions was even more complicated, showing enormously fine detail of structure [137]. A summary of the types of steady state and periodic solution branching is shown in Figure 6. There

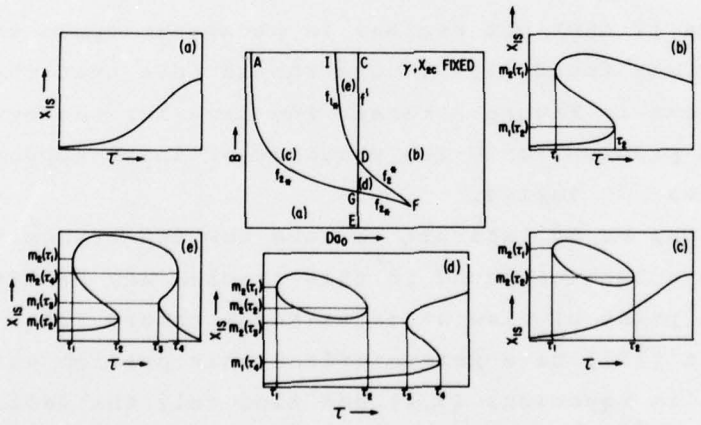


Figure 5. Multiple Steady State Branching as Reactor Residence Time is Varied.

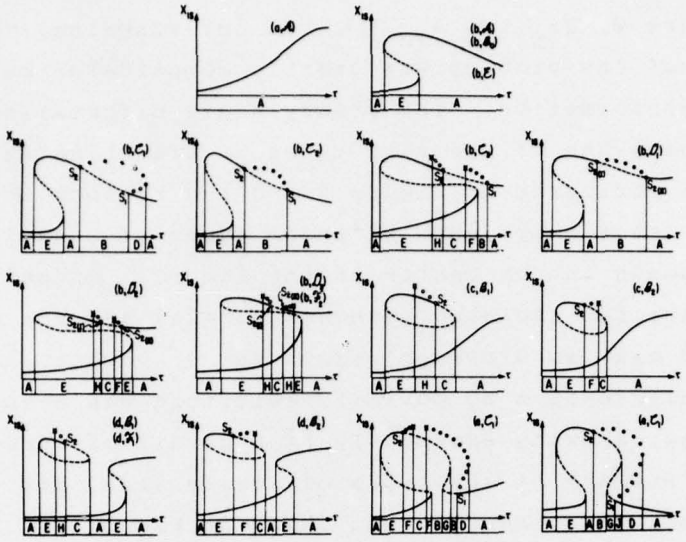


Figure 6. Summary of Steady State and Periodic Orbit Branching for the CSTR as Residence Time is Varied.

were some 17 distinct regions in parameter space where this behavior was found [137]. One should note that the phase plots shown in Figure 3 remain the same for the reparameterized problem, only the sequence of their appearance changes as τ varies.

It may be of interest to note how the branching to multiple solutions found in this problem may be interpreted from the point of view of catastrophe theory [28]. Blum and Mehra [153] have parameterized this problem essentially as shown in Equations (3,4) and find only the familiar cusp in a space of x_{2s}, Da, β (cf Figure 7). More recently Aris [154] has analyzed the bifurcation to multiple steady states shown in Figure 5 and seems to have found a very interesting catastrophe surface. As shown in Figure 8, for B between pt. G & D (in Figure 5), one obtains a cusp with an attached wing in an x_{1s}, Da_0, τ space while for B above point D, one seems to have an overhanging wing (cf

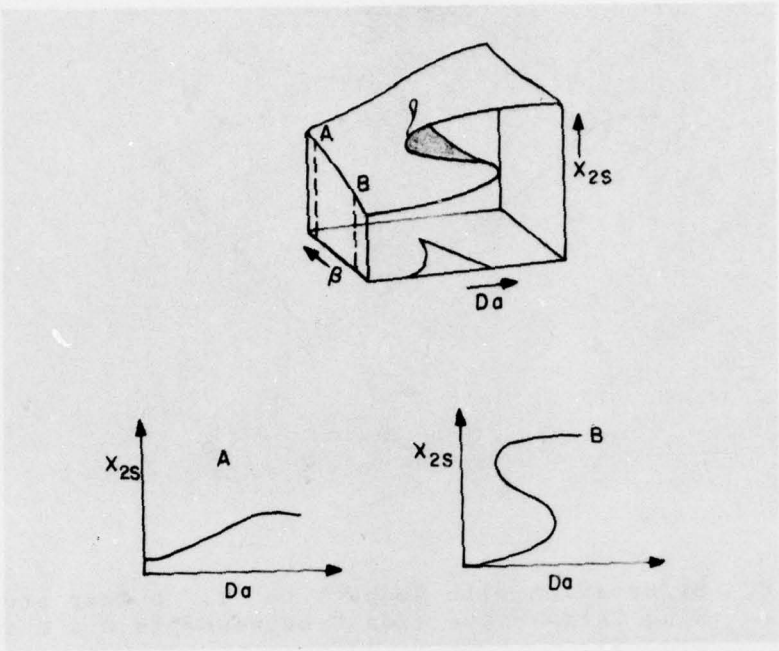


Figure 7. Bifurcation with Respect to Da, a Simple Cusp Catastrophe.

Figure 9). However, it appears that this surface is only a reparameterization of the cusp and does not represent a new catastrophe.

CONCLUDING REMARKS

It should be clear from the scope of the literature cited and the problems discussed that a very large number of chemically reacting systems exhibit multiple steady states, periodic solutions, wave phenomena, and pattern formation. In addition, many of these reaction systems have great scientific and practical importance and are of keen interest to natural scientists and engineers. Although it seems clear that a proper understanding of

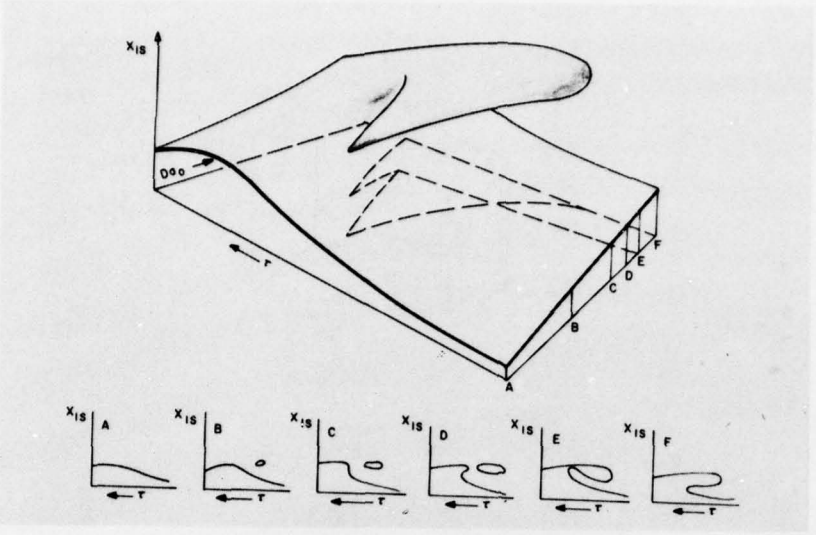


Figure 8. Bifurcation with Respect to τ, a Cusp and Wing Catastrophe (for B between pts G & D in Fig. 5).

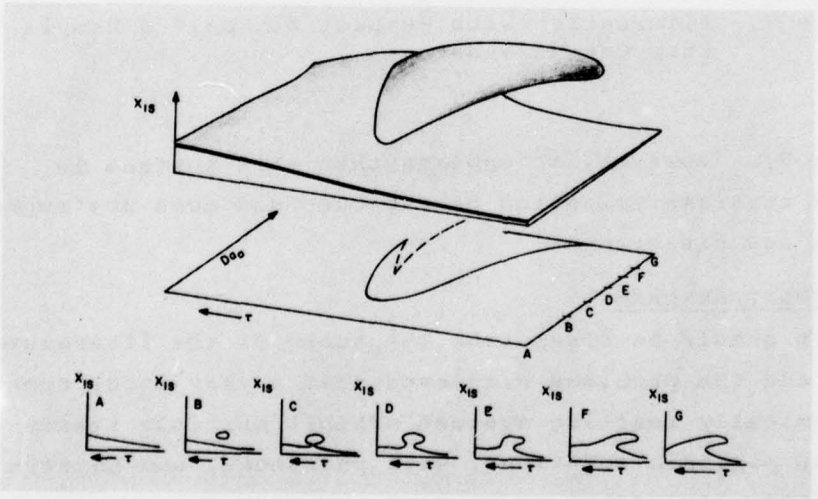


Figure 9. Bifurcation with Respect to τ, a Cusp and Wing Catastrophe (for B above pt D in Fig. 5).

the tools arising from bifurcation theory is the key to unravelling the structure of these important problems, a major responsibility for communicating these ideas in a understandable form to the user lies with the applied mathematics community. At the same time, the scientist and engineer must provide stimulation to the development of the theory through reporting observed phenomena, developing mathematical models as candidates for analysis and providing a testing ground for application of the theory. It is only through this type of mutual communication that the most fruitful development and application of bifurcation theory can occur.

In the spirit of this exchange, the present discussion of potential applications is offered along with some naive suggestions for future research directions:

(i) The development of some easily applied conditions for the existence of isolated steady state branches (isola) [137] and of isolated periodic orbits.

(ii) Contributions towards a more comprehensive distributed parameter bifurcation theory which allows a straightforward treatment of bifurcation from nonuniform steady states and wave motion.

(iii) The development of a unified theory of relaxation oscillations and the relationship to steady state branching behavior.

(iv) Studies of conditions for synchronization of oscillations when separate oscillating systems are allowed to communicate through mass or heat exchange (e.g., [14,16,17]).

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Chemical Wave–Trains and Related Structures

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1. The kind of waves with which this lecture is concerned do not involve any macroscopic mechanical motion, but depend on the interaction of certain chemical reactions with diffusion. An appropriate mathematical model is provided by "reaction-diffusion equations" of the form:

$$C_t = F(C) + K\nabla^2 C \quad . \qquad (1)$$

Here C is a vector whose components represent the concentrations of various substances which are participating in certain chemical reactions. Because of these reactions, the concentrations are changing with time--the rate at which this occurs is given by the vector F, which is a function of the concentrations themselves, and possibly of certain other things, notably temperature. (If temperature variations are important one may make temperature one of the components of C and retain the form (1).) In addition, if the concentrations are not spatially uniform, diffusion will alsc contribute to the local rates of change; this is represented by the last term of (1), in which K is a positive definite matrix of diffusivities.

The equations of "chemical kinetics", $C_t = F(C)$, (which might be obtained explicitly, for a given set of reactions, on the basis of the law of mass action for instance) would normally have an absolutely stable critical point representing the final equilibrium state after all the reactions have "run down". However, in some cases--and this

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includes those of interest in connection with chemical waves --this final state is not approached for a long time, and interesting things happen on shorter time-scales. It is then possible (and appropriate) to make approximations which produce a simpler set of equations, of the same general form as (1), usually for a subset of the original set of concentrations. (The simplest example of such approximations, appropriate for instance if some of the substances are initially distributed fairly homogeneously and in concentrations large compared to the amount by which they change over the short time-scales otherwise of interest, is that these particular concentrations may be taken as approximately constant in the equations for the rates of change of the others.) The approximate chemical kinetics equations, even though they describe accurately the course of the reactions on the short time-scale, need not then have an absolutely stable critical point, and might for instance have a stable limit-cycle solution. (They also need not particularly resemble the polynomial form given by the law of mass action.)

A (plane) wave-train solution of Equation (1) is a solution of the form $C = y(\sigma t - \alpha z)$ where σ and α are constants, z is a spatial variable (normal to the wave crests) and y is a 2π-periodic function of its argument. Such solutions to (1) exist if and only if the system of ordinary differential equations

$$y' = F(y) + \alpha^2 K y'' \qquad (2)$$

has, for suitable constants σ (frequency) and α (wave number), a 2π-periodic solution. Note that the order of (2) is twice that of the (approximate) equations of chemical kinetics, $C_t = F(C)$. In general, (2) probably has no 2π-periodic solutions, no matter what σ and α are. But with certain hypotheses about F it can be shown that it does, and in fact then has typically a one-parameter family

of them: for each α in a certain range there is a σ, say $\sigma = H(\alpha^2)$, such that (2) has a 2π-periodic solution y_α. This was shown by Kopell and me in [1], under the hypothesis that the equations $C_t = F(C)$ of the kinetics have an unstable critical point at which the linearization has two conjugate eigenvalues in the right half plane. It was also assumed that K was a scalar matrix, or nearly so. The proof made use of the Hopf bifurcation theorem, so this result should perhaps be mentioned at a conference on applications of bifurcation theory. Unfortunately, the wave trains given by this turn out to be unstable as solutions of (1), so probably the result should be mentioned only at such conferences. However, we were also able in that paper to show the existence of a one parameter family of plane wave solutions for all sufficiently small α, under the hypothesis that the kinetics equations have a stable limit-cycle solution. It appears to be the case that these waves are stable, as solutions of (1), for a certain range $0 \leq \alpha \leq \alpha_s$ of wave numbers (at least if the diffusivity matrix is sufficiently close to being scalar). This was not obtained by a bifurcation method, but by exploiting the singular perturbation character of (2) for small α; this also provides the basis of an effective numerical method for calculating such solutions of (2). Topological methods have also been used to show the existence of plane wave solutions in some circumstances. Roughly speaking, it seems that wave-train solutions are to be expected when the chemical kinetics is "oscillatory", as in the cases just mentioned, and also in some other cases, often described as "excitable". Here, although the chemical kinetics equations have an absolutely stable critical point, a suitable small (but finite) perturbation will send the phase-point off on a long excursion before it once again approaches the critical point.

2. Perhaps the simplest examples illustrating wave train solutions to reaction-diffusion equations are the "λ - ω systems" whose kinetic equations are of the form

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}_t = \begin{bmatrix} \lambda & -\omega \\ \omega & \lambda \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \tag{3}$$

where λ and ω are functions of $r = \sqrt{C_1^2 + C_2^2}$. (Here C_1 and C_2 are to be thought of as deviations of the concentrations from a critical point, so that they need not be positive.) If $\lambda(r)$ has a zero at some radius, say $r = 1$, and is decreasing there, then (3) has the circle $r = 1$ as the orbit of a stable limit cycle, assuming $\omega(1) \neq 0$. Such systems are a favorite example for illustrating limit cycles in elementary differential equations since the introduction of polar coordinates in the phase plane makes them particularly transparent. Polar coordinates ($C_1 + iC_2 = re^{i\theta}$) are also helpful for reaction-diffusion equations with $F(C)$ given by the right hand side of (3). Taking K to be the identity matrix, one finds, in the case of a single space coordinate z :

$$\begin{aligned} r_t &= r(\lambda(r) - \theta_z^2) + r_{zz} \\ \theta_t &= \omega(r) + \frac{1}{r^2}(r^2\theta_z)_z \end{aligned} \tag{4}$$

For these systems, the family of plane waves can be written down explicitly, and is:

$$r = r_o$$

$$\theta = \sigma t - \alpha z$$

where

$$\sigma = \omega(r_o) \quad , \quad \alpha^2 = \omega(r_o) \quad .$$

For those values of the "amplitude parameter" r_o for which $\lambda > 0$, we get plane wave solutions; for instance with $\lambda = 1 - r^2$, $\omega = 2 - r^2$, there are plane waves for all α on $(0,1)$, the wave of wave number α having frequency $\sigma = 1 + \alpha^2$ ($\equiv H(\alpha^2)$). In [1], the (linear) stability of these waves, as solutions of the reaction diffusion equations, was investigated. The principal conclusion was that they are not always stable, but those close enough to the stable limit cycle of the kinetics (i.e., those with small enough α) are stable.

3. The Belousov reaction is probably the most thoroughly studied oscillatory reaction, though there is still some uncertainty about details of the basic chemical mechanisms. Under appropriate conditions, it exhibits self-excited nearly periodic oscillations, in which the concentrations of some of the reactants change by several orders of magnitude, with a period of the order of 30 seconds (the "short time-scale"). The period and wave-forms gradually change with time as the major ingredients are used up, but the oscillation persists for an hour or two (the "long time-scale"). For references, one may consult [2]. Under other conditions, the same system is excitable, but not self-excited. It is also with this system that real chemical waves have been most clearly demonstrated, first by Zaiken and Zhabotinskii [3]. If the reacting mixture is placed in a shallow layer in a covered flat dish, one sees after a while an organized pattern of sets of concentric rings--or sometimes spirals--made visible by color changes in an indicator. Figure 1 is a photograph of such a pattern. Careful observation shows that the rings are propagating slowly outward from the centers of the "targets". Where adjacent targets meet, the meeting waves seem simply to annihilate one another, indicating their highly non-linear character. The spacing of the rings and the period of oscillation are quite uniform over a given target, but are not usually the same in different targets, nor is the

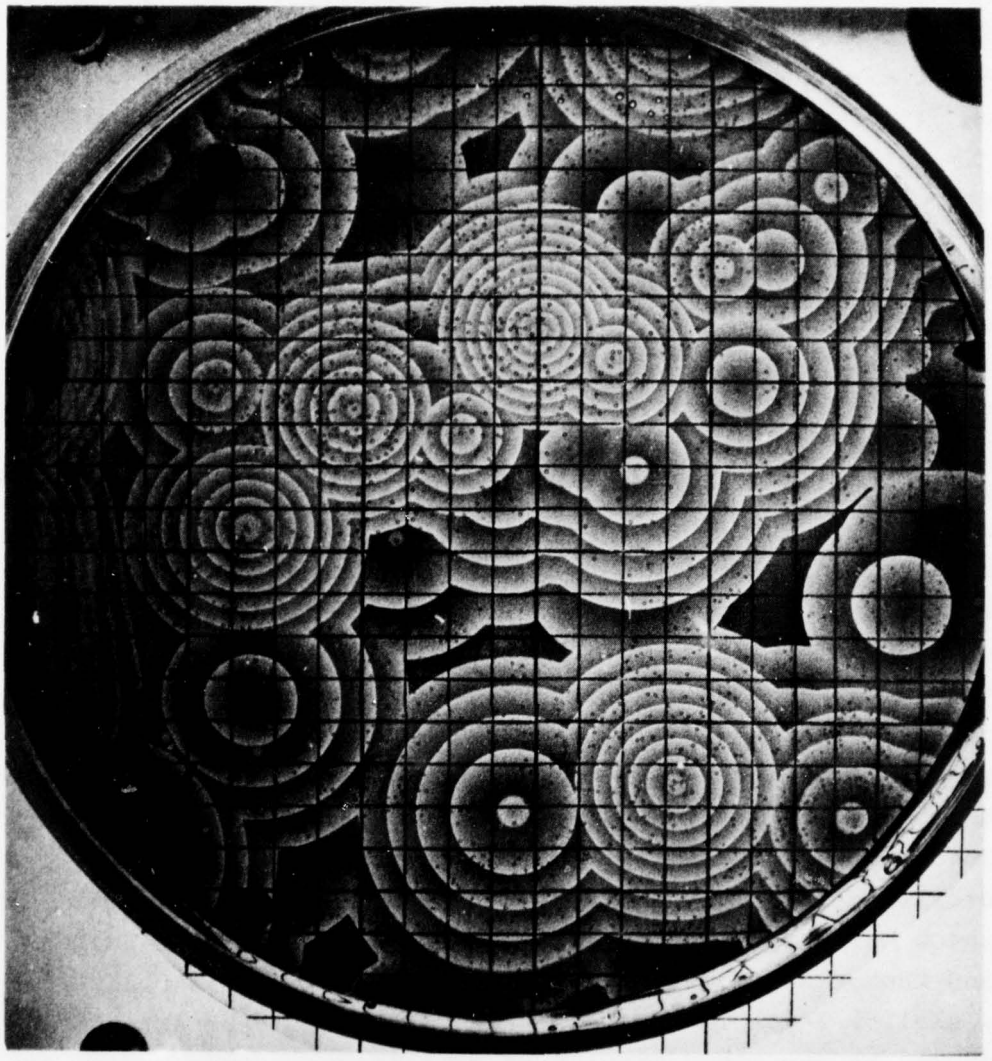


Figure 1

propagation speed. Besides the waves themselves, two other striking features of pictures like Figure 1 are the centers and the boundaries between different targets, the rather thin transition zones between regions of essentially uniform period and wave length. It is in trying to make a theoretical model of such transition zones that Kopell and I have recently found some useful applications of bifurcation methods. The centers appear to us to be somewhat different, and other methods seem more appropriate. I shall now describe these transition zone models and some mathematical issues connected with them. A detailed account will appear in [4].

4. The observed boundaries separating different targets are not usually fixed in position, but propagate slowly from a target with higher frequency into a neighboring one of lower frequency--indeed, a target will sometimes completely devour its neighbor or neighbors. But during this process, the transition zone remains thin and well-defined. This suggests that an appropriate one-dimensional idealization of the transition zone is a solution which provides a connection between two different plane waves and which, in a suitably moving coordinate system, retains a permanent structure. A plane wave is a "permanent structure" in that at each point, the concentrations are periodic functions of time, and in fact, in a coordinate system moving with the wave propagation speed σ/α it appears as a steady structure. We cannot expect to find a moving system in which the transition zone solution appears steady, but we shall try to find one in which the solution is everywhere periodic with the same period. Thus, we shall look for a solution C to the reaction-diffusion equations expressible as $C = X(\bar{\sigma}t, z - ct)$, where X is a 2π-periodic function of its first argument with the asymptotic behavior $X(\bar{\sigma}t, z - ct) \sim y_{\alpha_{\pm}}(\sigma_{\pm}t - \alpha_{\pm}z)$ for $z \to \pm\infty$. Setting

$\zeta = z - ct$ for the position coordinate in the moving system this means that

$$X(\bar{\sigma}t,\zeta) \sim y_{\alpha_{\pm}}((\sigma_{\pm} - \alpha_{\pm}c)t - \alpha_{\pm}\zeta)$$

as $\zeta \to \pm\infty$. Thus, for the above to be possible, we must have the two equations

$$\bar{\sigma} = \sigma_{\pm} - \alpha_{\pm}c \tag{5}$$

from which the propagation speed c of the transition zone and the "Doppler shifted" common frequency $\bar{\sigma}$ can be obtained in terms of the frequencies and wave numbers of the two asymptotic plane waves. The result of solving for c is:

$$c = \Delta\sigma/\Delta\alpha \quad , \tag{6}$$

the finite-difference analogue of the group-speed $d\sigma/d\alpha$ of the family of plane waves. This rule (6) for the speed of a transition zone appears to be well verified experimentally. It was noticed already by Zaiken and Zhabotinskii.

Let us consider now the $\lambda - \omega$ case, where once again the polar coordinates in the concentration space prove to be helpful. Transformed to the moving system, the equations (4) become:

$$\left.\begin{aligned} r_t - cr_\zeta &= r(\lambda - \theta_\zeta^2) + r_{\zeta\zeta} \\ \theta_t - c\theta_\zeta &= \omega + \frac{1}{r^2}(r^2\theta_\zeta)_\zeta \end{aligned}\right\} \quad . \tag{7}$$

It is readily seen that these equations are consistent with a solution of the form $r = r(\zeta)$, $\theta = \bar{\sigma}t - \int_0^\zeta a(\zeta')d\zeta'$,

which will make the concentrations 2π-periodic functions of $\bar{\sigma}t$ for each ζ. In this way, our problem is transformed into the third order system of ordinary differential equations for r, $u \equiv r'/r$, and the "local wave-number" a:

$$\left.\begin{aligned} r' &= ru \\ u' &= a^2 - u^2 - \lambda(r) - cu \\ a' &= \omega(r) - \bar{\sigma} - 2au - ca \end{aligned}\right\} \quad . \tag{8}$$

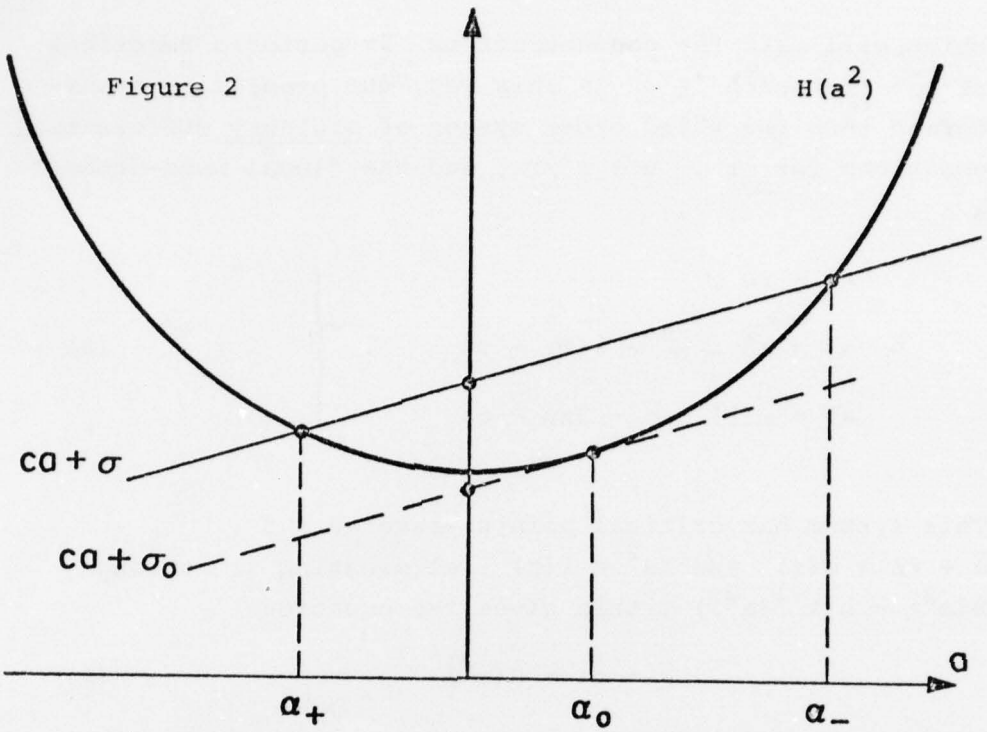
This system has critical points where $u = 0$, $\bar{\sigma} + ca = \omega(r)$ and $a^2 = \lambda(r)$; eliminating r through $H(a^2) = \omega(\lambda^{-1}(a^2))$, this gives the equation

$$\bar{\sigma} + ca = H(a^2) \tag{9}$$

which has the solutions $a = \alpha_+$ (cf. (5)). Thus, the two plane waves--which are indeed solutions of the special form under consideration which also have r and a constant--appear as critical points of the system (8). The transition zone solution which we are seeking is then a trajectory of (8) which goes from the critical point with $a = \alpha_-$ to that with $a = \alpha_+$. Let us assume now that

$$aH'(a^2) \geq 0 \quad . \tag{10}$$

This appears to be the case for the real waves in the Belousov reaction, where shorter waves seem to have higher frequency. For a $\lambda - \omega$ case, it corresponds to $\omega'(r) < 0$ if we also have $\lambda'(r) < 0$, as is necessary (over the relevant range of r) for stability of the plane waves. A sketch of the two sides of equation (9), plotted as functions of a, would then look like this:



the intersections of the line and the "parabola" determining the critical points. Instead of thinking of α_+ as the basic parameters, we now think of them as being c and $\bar{\sigma}$, and in fact consider a family of problems which c fixed but $\bar{\sigma}$ variable. By reducing $\bar{\sigma}$ from the value corresponding to the original α_+, we slide the line in Figure 2 downwards, thereby bringing the two critical points closer together until they finally coalesce (and disappear) when $\bar{\sigma} = \bar{\sigma}_0$ and the line becomes tangent to the "parabola". This indicates how bifurcation methods can be made use of in the critical-point connection problem to which finding the transition zone solution has been reduced. We start at $\bar{\sigma}_0$, where the critical points just come into existence, and push $\bar{\sigma}$ back up in the expectation that we shall at least be able to say something for cases where α_+ and α_- are not too different.

Critical-point connection problems occur in the problem of finding the structure of a shock layer in viscous, heat-

conducting, gas dynamics. There, the states being connected (at least in the idealized case) are physically constant in time, and the whole shock-structure is too, if examined in a coordinate system moving with the shock speed determined by the Rankine-Hugoniot conditions--of which Equation (6) is here the analogue. For the chemical waves, the states being connected are not really constant in time, but periodic; they have been made to look like constant states only after some mathematical transformations, notably the use of the polar coordinates in concentration space, a device which seems restricted to the very special (and quite non-chemical) $\lambda - \omega$ models. Still, the analogy seems to be quite close, and we have come to think of the transition zones as "shocks".

5. Here is a bifurcation theorem which enables one to show the existence of a connecting trajectory for Equations (8) when α_+ and α_- are sufficiently close to one another--the "weak shock" case:

Let $\dot{x} = F_\mu(x) = \mu g + Ax + Q(x,x) + E.E.$ be a family of autonomous systems of dimension n depending on the parameter μ, where E.E. represents "everything else"--terms of higher order in x and μ than those written explicitly. About the vector g, matrix A and vector-valued quadratic form Q we assume the following:

i) rank $A = n - 1$. Let e and ℓ be right and left eigenvectors of A corresponding to the eigenvalue 0, which has multiplicity k. (e and ℓ are unique in direction.)

ii) $\ell g \neq 0$.

(Under these hypotheses, there exists a locally unique curve of critical points in (x,μ) space passing through the origin.)

iii) $\ell Q(e,e) \neq 0$.

(With this additional hypothesis, there are exactly two critical points for all small positive values of $\hat{\mu} \equiv - \mu \ell g / \ell Q(e,e)$, and no others near the origin.)

iv) $k = 1$ or 3 .

Then there is a locally unique trajectory joining the critical points for all small positive values of $\hat{\mu}$.

(A proof of this and of related results for $k = 2$ can be found in [5].)

If one investigates the bifurcation point $\bar{\sigma} = \bar{\sigma}_o$ of Equations (8) one finds that all the hypotheses of this theorem are satisfied, with $k = 1$, and so we can conclude the existence of the idealized transition zone structure in the weak-shock case. In the paper [4], the existence of connecting trajectories for certain strong shock cases is also demonstrated, this time by an application of the above theorem in the $k = 3$ case. The connecting trajectory given by the bifurcation theorem goes in a definite direction; when $aH'(a^2) \gtrless 0$ this turns out to mean that the transition zones exist between wave trains which are approaching one another from opposite directions, or more generally, when the one on the left has a larger group velocity than the one on the right. In the opposite case, the shock structure does not exist (as it does not for "expansion shocks" in gas dynamics), at least between sufficiently close wave trains.

6. The success of the bifurcation method in the $\lambda - \omega$ case led us to attempt to treat the general case in a similar way. Formally, this can be done, but in order to make the plane waves look like critical points, it seems to be necessary to think of differential equations in an infinite dimensional space rather than just the 3 of (r,u,a). To do this, we consider the concentration vector in the moving system $X(t,\zeta)$ as being, for each ζ , a point in a space of vector-valued periodic functions of t . One can separate the ideas of "amplitude" and "phase" (paralleling the r and θ of the $\lambda - \omega$ case) in the following way. Since we are going to use a bifurcation method centered on a certain plane-wave y_{α_o} , we may suppose that all the

orbits (for various values of ζ) of the periodic functions $X(\cdot,\zeta)$ are near one another. We can then give a consistent rule for marking on any such orbit a standard point called the "phase-zero"--for instance, we may take a hyperplane in concentration space transverse to these orbits and call the phase-zero the point at which this hyperplane is crossed from one side to the other. (In [4] a somewhat different definition was found technically more convenient, but the idea is similar.) We then think of a space P_n of "normalized 2π-periodic functions" as vector functions $p(\theta)$ whose orbits are near that of y_{α_o} and which have their phase-zero at $\theta = 0$. We then try to solve the shock-structure problem by setting

$$X(t,\zeta) = U(\bar{\sigma}t - \int_0^\zeta a(\zeta')d\zeta',\zeta) \quad \text{where, for each } \zeta, \quad U(\theta,\zeta)$$

is in P_n. We may assume that the representation of the family of plane wave solutions has been adjusted so that for each α, $y_\alpha(\theta)$ is in P_n, so that in particular, the limiting plane waves $y_{\alpha_\pm}$ can be regarded as points in P_n, with constant values of a. Inserting the above form for $X(t,\zeta)$ into the reaction-diffusion equations in the moving system, one finds that the result can be put into the form of an autonomous system

$$\left.\begin{aligned} a_\zeta &= h(a,U,V) \\ U_\zeta &= V \\ V_\zeta &= G(a,U,V) \end{aligned}\right\} \qquad (11)$$

where U is in P_n, V is in a space P_o which is like P_n but with a slightly different normalization produced by the ζ-differentiation, and a is in R, the reals. h is a mapping from (part of) $RxP_nxP_o \to R$ and G is a similar one into P_o.

In this form, the problem looks formally just as before--we are seeking a trajectory in $RxP_n xP_o$ joining two critical points of the system (11). It is then possible to identify the analogues of the vector g, the linear operator A and the quadratic form Q mentioned in the bifurcation theorem. Under assumptions which are natural about the plane wave solutions (such as their stability), one then finds that all the hypotheses of the bifurcation theorem for $k = 1$--apart from the finite dimensionality--come out exactly right. It thus seems a very plausible conjecture that the idealized weak-shock structures exist in general.

However, there seems to be a serious difficulty in proving this along the lines sketched above. There are certainly many circumstances in which it makes sense to talk about differential equations in Banach spaces, and there is probably an analogue of the bifurcation theorem stated above in such a context. But it appears that the operators h and G in (11), which involve second derivatives with respect to θ, are too singular to really define a vector field on $RxP_n xP_o$. The underlying reason for this is that ζ is really a spatial variable in the reaction-diffusion equations, but in forcing the equations for the shock-structure into a form resembling the bifurcation theorem, we obtain equations of evolution in ζ. Everybody knows that it is at least half as dangerous to try to solve the heat equation sideways as it is to do it backwards! Nevertheless, we know that connecting trajectories exist for the $\lambda - \omega$ case, and have good physical reasons to expect that they do for general chemical kinetics which have plane wave solutions. This suggests then, that finite dimensional bifurcation theorems, like the one stated above, which assert the existence of connecting trajectories as well as of critical points, may have some other kind of infinite dimensional generalization which does not fit into the context of differential equations on a Banach

space. We suspect that Equations (11) do make sense on some very restricted part of the domain, a part which contains the connecting trajectory of interest but perhaps not a whole lot else. Possibly if these things were understood in a somewhat more general way, we should find a new kind of bifurcation theorem.

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A General Class of Nonlinear Bifurcation Problems from a Point in the Essential Spectrum. Application to Shock Wave Solutions of Kinetic Equations

B. Nicolaenko

1. Introduction and Background from Mechanics.

We investigate an abstract class of bifurcation problems from the essential spectrum of the associated Frechet derivative. This class is a very general framework for the theory of one-dimensional, steady profile traveling shock wave solutions to a wide family of kinetic integro-differential equations from non-equilibrium statistical mechanics [1,2]. Such integro-differential equations usually admit the Navier-Stokes system of compressible gas dynamics or the M.H.D. systems in plasma dynamics as a singular limit [3-5], and exhibit similar viscous shock layer solutions [6,7].

The mathematical methods associated to systems of Partial Differential Equations must however be replaced by the following considerably more complex Bifurcation Theory setting, first outlined in [8-10] for special cases. We actually consider a hierarchy of bifurcation problems, starting with a simple (solved) bifurcation problem from a simple eigenvalue.

Let $\mathcal{G}(\mu,f)$ be a nonlinear mapping from a Banach space X, into a Banach space Y, parametrized by μ:

$$\mathcal{G}(\mu,f) : R^1 \times X \to Y. \tag{1.1}$$

Consider

$$\mathcal{G}(\mu,f) = 0, \tag{1.2}$$

such that

$$\mathcal{G}(\mu,0) \equiv 0, \ \forall\ \mu \in R^1.$$

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We admit all the necessary hypotheses to insure bifurcation at $\mu = \mu^*$, from a simple isolated eigenvalue of the Frechet derivative

$$\mathcal{G}_f(\mu^*,0).$$

Classical theory [11] insures that, in some neighborhood of $(\mu^*,0)$ in $R^1 \times X$, there exists a second branch $\omega(\mu)$:

$$\begin{aligned} \mathcal{G}(\mu,\ \omega(\mu)) &= 0 \\ \omega(\mu^*) &= 0. \end{aligned} \tag{1.3}$$

Thus, the primary hypothesis is bifurcation from a simple eigenvalue for the operator $\mathcal{G}$. In concrete cases, the relative bifurcated and trivial branches correspond to different asymptotic steady states at the "tails" of the shock wave (space-independent subsonic and supersonic states related by Rankine-Hugoniot conditions; $\mu = \mu^*$ corresponds to the transonic regime).

We shall actually investigate the more involved extended operator equation, for $x \in R^1$, $-\infty \leqslant x \leqslant +\infty$:

$$\begin{aligned} A(\mu)\ \frac{\partial f}{\partial x} - \mathcal{G}(\mu,f) &= 0, \text{ or} \\ \mathcal{F}(\mu,f) &= 0, \text{ where} \end{aligned} \tag{1.4}$$

$$A(\mu) : R^1 \times X \to Y \tag{1.5}$$

is a linear operator from X into Y, parametrized by $\mu \in R^1$. f is now a vector valued function of $x \in R^1$, $-\infty \leqslant x \leqslant +\infty$, with values in the Banach space X. We may restrict ourselves to spaces of absolutely continuous functions. If $A(\mu) \equiv I$ and $x \equiv t$, (1.5) reduces to an evolution equation

$$\frac{\partial f}{\partial t} - \mathcal{G}(\mu,f) = 0, \tag{1.6}$$

and one looks for solutions which are trajectories between critical points of (1.1), i.e., the trivial solution and the bifurcated solution $\omega(\mu)$. Such a problem (1.6) of trajectories joining two steady asymptotic states, has first been considered by B. Matkowsky, using matched asymptotic expansions [12,13]; it has been investigated in depth by

G. Iooss [14] and K. Kirchgässner [15], within the Navier-Stokes context (see also [16]).

However, in (1.4-5), fundamental properties of the physical context impose somewhat pathological conditions on $A(\mu)$:

Hypothesis.

1) Zero belongs to the continuous spectrum of $A(\mu)$, $\forall\mu$, i.e.:

$$\overline{R(A(\mu))} = Y \text{ and } A(\mu)f = 0 \Leftrightarrow f = 0.$$

2) $A(\mu)$ is neigher positive nor negative semi-definite, nor is it accretive. As a corollary $A(\mu)^{-1}$ does not exist, $\forall\mu$. Recall that

$$\mathcal{G}_f(\mu^*, 0)^{-1}$$

does not exist either at $\mu = \mu^*$. In fact the properties of $A(\mu)$ are such that an initial value problem for (1.4) is ill-posed. Attempts to straightforwardly extend methods developed for (1.6) lead to erroneous results.

We still look for critical trajectories of (1.4), between the trivial solution and $\omega(\mu)$. We investigate the possible existence of a branch $\Omega(\mu, x)$, solution of (1.4) such that:

1) $\Omega(\mu^*, x) \equiv 0$, but $\Omega(\mu, x) \neq 0$, $\mu \neq \mu^*$;

2a) $\Omega(\mu, -\infty) = 0$, $\Omega(\mu, +\infty) = \omega(\mu)$; or

2b) $\Omega(\mu, +\infty) = 0$, $\Omega(\mu, -\infty) = \omega(\mu)$,

for μ close to μ^*. In an appropriate Banach space of absolutely continuous functions normalized at $\pm\infty$, the hypothetical non-trivial branch $\Omega(\mu, x)$ corresponds to bifurcation from the essential spectrum of:

$$\mathcal{J}_f(\mu,0) = A(\mu)\frac{\partial}{\partial x} - \mathcal{G}_f(\mu,0) \quad . \tag{1.7}$$

Specifically at $\mu = \mu^*$, zero is a limit point of the spectrum (a non-isolated eigenvalue in the essential spectrum). The kernel is non-trivial, as it includes that of $\mathcal{G}_f(\mu^*,0)$. The non-isolated character stems from the individual essential specra of $A(\mu)$ and $\frac{\partial}{\partial x}$. (1.4) must be considered as a bona-fide problem of bifurcation from the essential spec-

trum. Finally, we shall demonstrate the non-trivial result that $\Omega(\mu, +\infty) = \omega(\mu)$ or $\Omega(\mu, -\infty) = \omega(\mu)$ (critical trajectory). Since will will emphasize the mathematical techniques, we briefly review the relevance of (1.4) to fluid and statistical mechanics.

Steady profile shock waves in compressible fluid dynamics and magneto-hydrodynamics correspond to rather different mathematical theories according to the level and complexity of the fluid dynamical description. In order of increasing complexity, one has the well known hierarchy of equations, from the Euler level, to the compressible Navier-Stokes and the Magneto-hydrodynamic (M.H.D.) systems; and finally to the Boltzmann equation and the Kinetic integro-differential equations of collision-dominated plasmas. While viscosity terms are explicit in macroscopic Navier-Stokes equations, they are implicit in kinetic equations, where they result from explicit interparticle collision description on a microscopic scale. H. Grad [3-5] has carefully investigated the singular limit of the Boltzmann equation (for neutral gases) to the Navier-Stokes system when the mean free path between interparticle collisions (microscopic scale) becomes very small as compared to the macroscopic mean flow scale. His estimates do not cover, however, the shock case.

Hyperbolic systems are a standard tool for discontinuous shock solutions of Euler equations. Compressible Navier-Stokes systems exhibit viscous shock layers: in one dimension, Gilbard and Paolucci reduced them to a system of nonlinear autonomous Ordinary Differential Equations [6,7], and demonstrated that the shock layer is the unique trajectory between a node and a saddle point. For M.H.D. systems, such concepts have been extended by Conley and Smoller [17], using advanced tools of Topological Dynamics and Global Analysis. Yet none of the above mathematical methods apply to shock solutions of microscopic kinetic equations. Worse, it is well known that Partial Differential Equations

approximations of the "13 moments" type break down at a finite Mach number ≈ 2 (non-existence of trajectories between critical points) [18]. The major problem is whether one can still consider the latter kinetic equations within the framework of critical orbits between critical states.

Moreover, there is plenty of experimental and numerical evidence for important microscopically originating effects observed in shock layers ruled by integrodifferential kinetic equations. Even in neutral gases, at small Mach numbers of 1.2 (weak shocks), a 40% deviation has been observed for the local ratio of the heat diffusion to the viscosity coefficients in the shock, as compared to predictions from the Navier-Stokes equation with Transport Coefficients calculated by the time-honoured Chapman-Enskog expansion [19]. This deviation is especially marked in the "hot (subsonic) tail" of the shock [20]. Previous numerical and experimental results have missed these important distortions by focusing only on the geometry of the sharp transition profile [21]. In (collision-dominated) Plasmas, small-scale microinstabilities are as important as large scale (M.H.D.) macroinstabilities, and account for the difficulties in obtaining stable equilibria configurations.

The simplest kinetic equation - the Boltzmann operator for neutral rarefied gases, has been worked out in previous publications [8-10], and is summarized in the appendix. Technical estimates do, however, somewhat dissimulate the conceptual simplicity of the hierarchy of Bifurcation Problems.

Most of the pathology of the mathematical problem stems from the peculiar properties of the operator

$$A(\mu)\ \frac{\partial}{\partial x}\ .$$

This operator is nevertheless universally present in kinetic (statistical mechanics) equations. Usually called "the streaming operator", it represents transfer of very high velocity particles. The latter account for all deviations

observed from Navier-Stokes. Physically, by traveling almost instantaneously in opposite upstream and downstream directions, these very high velocity particles cause a strong coupling between the asymptotic "tails" of the shock.

2. The Mathematical Problem: Principal Results.

We first consider a classical bifurcation setting for Problem I:

$$\mathcal{G}(\mu, f) = 0 \tag{2.1}$$

where $\mathcal{G}$ is a bounded nonlinear mapping from a Banach space X into a Banach space Y (usually graph-norm spaces):

$$\mathcal{G}(\mu, f) : R^1 \times X \to Y ,$$

$$\mathcal{G}(\mu, 0) \equiv 0, \ \forall \mu . \tag{2.2}$$

$\mathcal{G}$ is assumed to be analytic, both in f and μ:

$$\mathcal{G}(\mu,f) = \sum_{n=1}^{\infty} \frac{1}{n!} \mathcal{G}^{(n)} (0; (f)^n) , \tag{2.3.a}$$

with the notations:

$$\mathcal{G}^{(1)} (0; f) = T_\mu , \tag{2.3.b}$$

$$\frac{1}{2!} \mathcal{G}^{(2)} (0; f, f) = \Gamma_\mu (f, f) , \tag{2.3.c}$$

$$\sum_{n=2}^{\infty} \frac{1}{n!} \mathcal{G}^{(n)} (0; (f)^{(n)}) = \mathcal{N}(\mu, f) . \tag{2.3.d}$$

Hypothesis 0.

a) $\forall \mu$, T_μ is a Fredholm operator of index zero.

b) at $\mu = \mu^*$, dim ker $\{T_{\mu^*}\} = 1$ (zero is an isolated eigenvalue of T_{μ^*}).

c) $\mathcal{G}_{f,\mu}(0;\mu^*)h \notin R (T_{\mu^*})$, $\forall h \in \ker \{T_{\mu^*}\}$.

Conclusion. In some neighborhood of $(\mu^*, 0)$ in $R^1 \times X$, there exists a second branch $\omega(\mu)$:

$$\mathcal{G}(\mu, \omega(\mu)) = 0 \text{ and } \omega(\mu^*) = 0 . \tag{2.4}$$

Hypothesis 0.d. The bifurcation is bilateral.

We now investigate Problem II. Let $x \in R^1$, $-\infty \leq x \leq +\infty$:

(2.5.a) $A(\mu)\ \frac{\partial f}{\partial x} - \mathcal{G}(\mu,f) = 0$, equivalently

(2.5.b) $\mathcal{F}(\mu,f) = 0$, where

(2.6) $A(\mu) : R^1 \times X \to Y$

is a bounded (in graph-norm) linear mapping from X into Y. $\mathcal{F}$ is a nonlinear mapping acting on spaces of vector-valued absolutely continuous functions:

(2.7) $\mathcal{F} : R^1 \times AC^I[R^1 \to X] \to R^1 \times AC[R^1 \to Y]$.

The space AC^I is such that $\frac{\partial f}{\partial x} \in AC$. Generally, the AC norms are defined by:

$$|||f||| = \int_{-\infty}^{+\infty} \left\| \frac{\partial f}{\partial x} \right\|_{X,Y} dx .$$

In fact, we restrict ourselves to spaces such that $\frac{\partial f}{\partial x}$ is continuous. The absolutely continuous functions are normalized:

$$f \to 0 \text{ as } x \to -\infty \text{ or } x \to +\infty .$$

Now, $\forall \mu$, $f \equiv 0$ is still a trivial solution. The question is whether there is still bifurcation for problem II, at $\mu = \mu^*$, such that:

$$\Omega(\mu,x) \in AC^I[R^1 \to X]$$

$$\Omega(\mu^*, x) = 0$$

$$\Omega(\mu,-\infty) = 0 \text{ or } \Omega(\mu,+\infty) = 0 .$$

In the affirmative, one might speculate that

$$\Omega(\mu,+\infty) \equiv \omega(\mu) \text{ or } \Omega(\mu,-\infty) \equiv \omega(\mu) .$$

This corresponds to bifurcation from the essential spectrum of:

(2.8) $$\mathcal{F}_f(\mu;0) = A(\mu)\ \frac{\partial}{\partial x} - T_\mu .$$

Specifically, at $\mu = \mu^*$, zero is a non-isolated point of the spectrum, with

$$\ker \{T_{\mu^*}\} \subset \ker \{\mathcal{F}_f(\mu^*;0)\} .$$

To insure the bifurcation, and as suggested by statistical mechanics, we need the further

Hypothesis 1. Zero belongs to the continuous spectrum of the linear operator $A(\mu)$, $\forall \mu$:

$$\overline{R(A(\mu))} = Y \text{ and } A(\mu)f = 0 \Leftrightarrow f = 0 .$$

Corollary. $(A(\mu))^{-1}$ is unbounded for every μ.

Remark that, for $\mu = \mu^*$, $T_{\mu^*}^{-1}$ does not exist either.

Hypothesis 1bis. $A(\mu)$ is neither positive nor negative definite, nor more generally accretive; moreover $(\lambda A(\mu) - T_\mu)^{-1}$ is not compact.

Thus one cannot construct any equivalent norm.

Hypothesis 2. The generalized spectrum of the operator

$$\lambda A(\mu) - T_\mu : X \to Y, \ \lambda \in C,$$

is included in two sectors, one in $\mathrm{Re}\ \lambda < 0$, the other in $\mathrm{Re}\ \lambda > 0$, uniformly in μ. (See Figure I). The generalized spectrum [27] is the set of λ such that $(\lambda A(\mu) - T_\mu)^{-1}$ does not exist, or is unbounded as a mapping from Y to X. Remark that $X \neq Y$, and $A(\mu) \neq I$.

From classical perturbation and invariance properties of Fredholm operators, we deduce from Hypotheses 0 and 1:

Theorem 2.1.

a) There exists a neighborhood of $\lambda = 0$ in C, where $\lambda A(\mu) - T_\mu$ is Fredholm of index zero, $\forall \mu$.

b) There exists a neighborhood of $(\mu^*, 0)$ in $R^1 \times C$, where $(\lambda A(\mu) - T_\mu)^{-1}$ has a simple pole in λ, corresponding to a simple generalized eigenvalue $\lambda_o(\mu)$:

$$\lambda_o(\mu)\, A(\mu)\, \hat{\varphi}_o(\mu) - T_\mu\, \hat{\varphi}_o(\mu) = 0 , \tag{2.9}$$

where $\hat{\varphi}_o(\mu) \in X$ is a generalized eigenfunction.

Hypothesis 3. $\lambda_o(\mu)$ is real, and

$$\lambda_o(\mu) \geq 0 \text{ for } \mu \geq \mu^*,$$
$$\lambda_o(\mu) \leq 0 \text{ for } \mu \leq \mu^*.$$

This last hypothesis implies that the linearization (2.8)

is unstable for $\mu > \mu^*$.

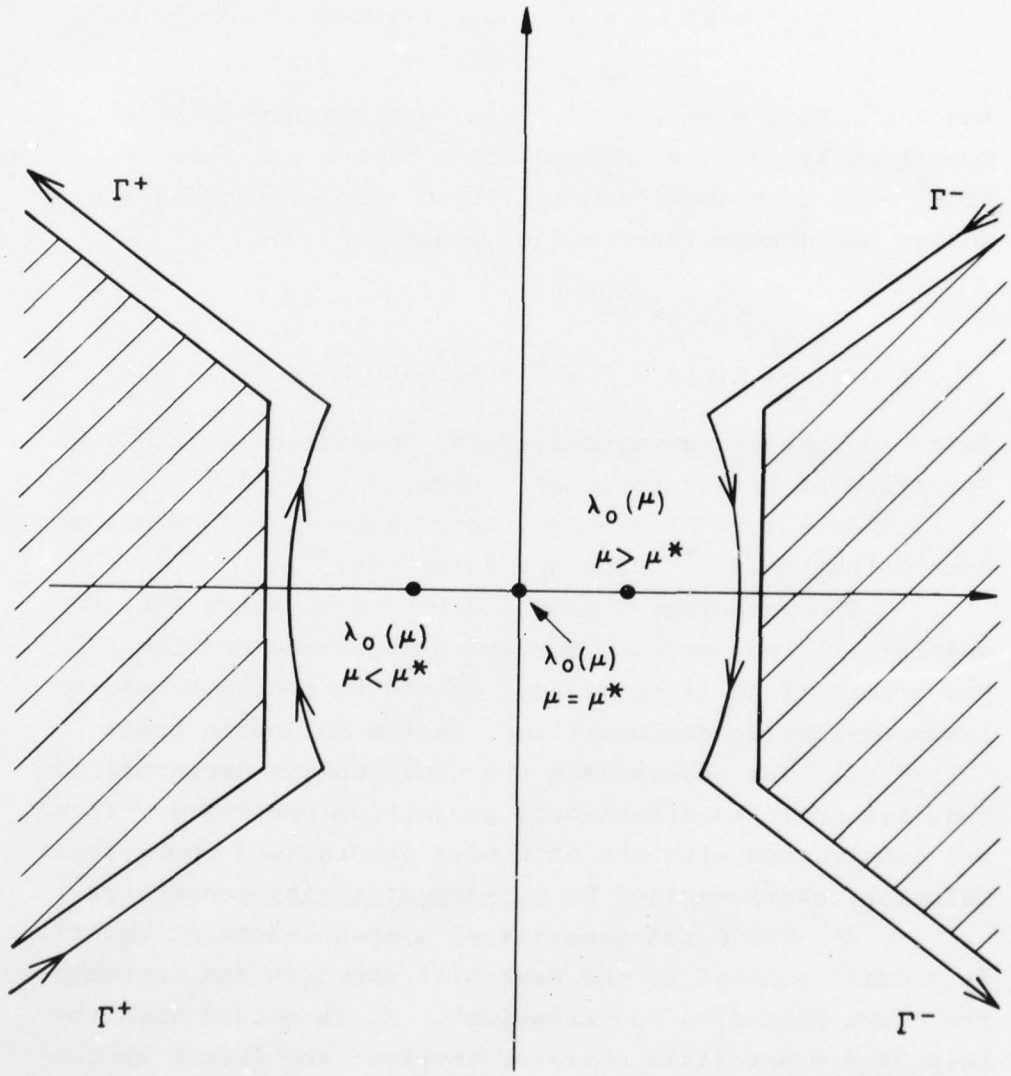


Figure I. The Spectrum of $(\lambda A(\mu) - T_\mu)$.

From Hypotheses 0-3, we demonstrate the fundamental Theorem 2.2. There exists a neighborhood $\mathcal{D}$ of $(\mu^*,0)$ in $R^1 \times AC[R^1 \to X]$, where there exists another branch solution of problem II, $\Omega(\mu,x)$, unique up to a translation, with:

$$\text{a)} \quad \mu \geqslant \mu^* : \Omega(\mu,-\infty) \equiv 0 \qquad \Omega(\mu,+\infty) \equiv \omega(\mu)$$

$$\text{b)} \quad \mu \leqslant \mu^* : \Omega(\mu,+\infty) \equiv 0 \qquad \Omega(\mu,-\infty) \equiv \omega(\mu) \ ,$$

and $\Omega(\mu^*, x) \equiv 0$ at $\mu = \mu^*$. $\omega(\mu)$ was defined in (2.4). Specifically, $\Omega(\mu, x)$ belongs to a closed subspace of $AC[R^1 \to X]$ such that, denoting by C^α the standard space of Hölder continuous functions of index α:

$$\frac{\partial \Omega}{\partial x} \in C^\alpha[R^1 \to X] \cap L^1[R^1 \to X]$$

$$\frac{\partial^2}{\partial x^2} A(\mu)\Omega \in C^\alpha[R^1 \to Y] \cap L^1[R^1 \to Y] \ ,$$

(with appropriate asymptotic decay conditions at $x = \pm \infty$, specified in later sections). Here, $0 < \alpha < 1$.

As a word of caution, note that $\frac{\partial^2}{\partial x^2} \Omega$ does not exist. Recall that $(A(\mu))^{-1}$ does not exist, $\forall \mu$.

The pathology introduced by the operator $A(\mu)$ re- specifically new methods for the bifurcation Problem II. The general line is to attempt to rescue the time-honored Lyapunov-Schmidt decomposition, at the following cost:

1) The generalized Lyapunov-Schmidt decomposition requires infinite dimensional projection operators. These are constructed with the help of a generalized Operational Calculus, characterized by non-commutativity properties.

2) The first generalized Lyapunov-Schmidt equation is closely related to the essential spectrum and represents the "fast particles contribution". It is solved with the help of a generalized operator inverse; the latter is constructed with generalized holomorphic semi-groups which do not admit any infinitesimal generator.

3) The second generalized Lyapunov-Schmidt equation

is not a mapping on finite-dimensional spaces. Rather it is a Functional-Differential equation in the sole variable $x \in R^1$, $-\infty \leq x \leq +\infty$, and global (non-local) in character: the initial value problem is ill-posed. Moreover, this equation in itself is again a bifurcation problem from a purely continuous spectrum.

We outline the mathematical techniques in the next sections. Full details will appear in [22] and elsewhere.

3. A Generalized Operational Calculus, and the Derivation of the Generalized Lyapunov-Schmidt Equations.

Let

$$R(\lambda,\mu) = (\lambda\, A(\mu) - T_\mu)^{-1} \tag{3.1}$$

where T_μ is defined as the Frechet derivative of $G(\mu,f)$ at $f = 0$ (2.3.b).

In order to construct appropriate projections associated to the isolated pole $\lambda_o(\mu)$, one cannot use the classical operational calculus based on Dunford Integrals of $R(\lambda,\mu)$, since the standard resolvent identity

$$R(\lambda) - R(\lambda') = (\lambda' - \lambda)\, R(\lambda)\, R(\lambda')$$

is false (non commutativity of $A(\mu)$ and T_μ). It must be replaced by the following correct identities:

$$A\,R(\lambda) - A\,R(\lambda') = (\lambda' - \lambda)\, A\,R(\lambda)\,A\,R(\lambda') \tag{3.2.a}$$

$$R(\lambda)\,A - R(\lambda')A = (\lambda' - \lambda)\, R(\lambda)A\,R(\lambda')A \tag{3.2.b}$$

Based on (3.2), a Generalized Operational Calculus is constructed, characterized by anticommutativity properties.

Proposition 3.1. There exists two families of projection operators:

$$E_{av}\,(\lambda_o(\mu)) : Y \to Y \quad \text{(range)}$$

$$E_{ap}\,(\lambda_o(\mu)) : X \to X \quad \text{(domain)}$$

associated to the generalized eigenvalue $\lambda_o(\mu)$, such that:

$$E_{av}(\lambda_o)\, A(\mu) = A(\mu)\, E_{ap}(\lambda_o)$$

$$E_{av}(\lambda_o) T_\mu = T_\mu\, E_{ap}(\lambda_o) = \lambda_o\, E_{av}\, A(\mu)\ ,$$

where

$$E_{av}\ (\lambda_o(\mu)) = \frac{1}{2\pi i}\int_{c(\lambda_o)} A(\mu)\ R(\lambda,\mu)\ d\lambda$$

$$E_{ap}\ (\lambda_o(\mu)) = \frac{1}{2\pi i}\int_{c(\lambda_o)} R(\lambda,\mu)\ A(\mu)\ d\lambda$$

and $c(\lambda_o)$ are closed contours around λ_o in the complex λ-plane.|

(Note: av ≈ avant, ap ≈ après).

Crandall and Rabinowitz [23] introduced a somewhat related notion of "K-eigenvalue": λ is a K-eigenvalue associated to $(T - \lambda K)$ if 1) dim $N(T - \lambda K)$ = codim $R(T - \lambda K)$ = 1, and 2) $Kx_o \notin R(T - \lambda K)$, where $N(T - \lambda K)$ = span x_o. This notion can be considerably strengthened by the

Theorem 3.2. The necessary and sufficient condition for λ to be "K-simple" is that λ is an isolated eigenvalue corresponding to a simple pole of $(T-\lambda K)^{-1}$. The generalized operational calculus always applies, with the stronger properties:

$$\text{a)}\quad R(T - \lambda K) \equiv (I - E_{av})Y$$

$$\text{b)}\quad \text{span}\ Kx_o \equiv E_{av}\ Y \qquad .|$$

Remark: Theorem 3.2 is easily generalized to the case of a simple pole, but with dim N > 1 (algebraically K-simple eigenvalue).

Concerning Problem II, we have the

Corollary 3.3. These projections $E_{ap}\ (\lambda_o(\mu))$ and $E_{av}(\lambda_o(\mu))$ respectively reduce the domain and the range of both $(\lambda A(\mu) - T_\mu)$ and $(A(\mu)\frac{\partial}{\partial x} - T_\mu)$, where the latter is the Frechet derivative of $\mathcal{J}(\mu,f)$ at $f = 0$, defined in (2.8).|

We have, indeed the following decomposition diagram, with the major problem of constructing a pseudo-inverse for $\mathcal{J}_f\ (\mu;0)$ from $(I - E_{av}\ (\mu))\ Y$ onto $(I - E_{ap}\ (\mu))\ X$:

$$\begin{array}{lccccc} Y = & (I - E_{av}\ (\mu))\ Y & \oplus & E_{av}\ (\mu)\ Y \\ (A(\mu)\ \frac{\partial}{\partial x} - T_\mu): & \uparrow \quad \downarrow\ (\ )^{-1}?? & & \uparrow \\ X = & (I - E_{ap}\ (\mu))\ X & \oplus & E_{ap}\ (\mu)\ X \end{array} \tag{3.3}$$

Following the general trend of the Lyapunov-Schmidt method, we decompose the solution f in the space X as:

$$f = E_{ap}(\mu)\, f + (1 - E_{ap}(\mu))\, f \tag{3.4.a}$$

$$= c(x)\, \hat{\varphi}_o(\mu) + w(\mu, x), \text{ where} \tag{3.4.b}$$

$$c(x)\, \hat{\varphi}_o(\mu) = E_{ap}(\mu)\, f, \tag{3.4.c}$$

$$w(\mu, x) = (1 - E_{ap}(\mu))\, f, \tag{3.4.d}$$

and $\hat{\varphi}_o(\mu) \in X$ is the generalized eigenfunction associated to $\lambda_o(\mu)$ and defined in (2.9). In fact:

$$c(x) \in A\, C^I\, [R^1]$$

$$w(\mu, x) \in A\, C\, [R^1 \to (I - E_{ap})X]$$

Applying the projection $(I - E_{av}(\mu))$ to Problem II (2.5), using the anticommutation properties of Proposition (3.1) and the reduction properties of Corollary (3.3), we obtain the following generalized Lyapunov-Schmidt equations, which we shall call L1 and L2:

(3.5) (L1)

$$A(\mu)\, \frac{\partial w}{\partial x} - T_\mu\, w = (I - E_{av}(\mu))\, \mathcal{N}(\mu;\, c(x)\hat{\varphi}_o + w)$$

where $\mathcal{N}$, defined in (2.3.d) is the remainder of the analytic expansion of $\mathcal{G}(\mu, f)$ at $f = 0$, excluding the first order Frechet derivative T_μ, but <u>including</u> the second order derivative $\Gamma_\mu(f, f)$.

Should the major problem of constructing a pseudo-inverse for the left hand side of (3.5) be solved, the implicit function theorem in Banach Spaces would yield w as a functional of c(x):

$$w\,\{c(x)\} : AC^I[R^1] \to AC\,[R^1 \to (I - E_{ap})X]\,. \tag{3.6}$$

To obtain the second Lyapunov-Schmidt equation, we apply the projection $E_{av}(\mu)$ to (2.5), use the anticommutation properties of Proposition (3.1) and assume the result (3.6). To simplify L2, we also need

<u>Theorem 3.4</u>. For a simple isolated eigenvalue $\lambda_o(\mu)$, there

exists $\eta(\mu) \in Y^*$ and $\psi(\mu) \in X^*$, such that:

$$A^*(\mu)\ \eta(\mu) = \psi(\mu)\ ,$$

$$E_{ap}(\mu)\ f = \hat{\varphi}_o(\mu)\ <\psi(\mu), f>/<\psi(\mu), \hat{\varphi}_o(\mu)>,$$

$$E_{av}(\mu)\ f = A(\mu)\ \hat{\varphi}_o(\mu)\ <\eta(\mu), f>/<\psi(\mu), \hat{\varphi}_o(\mu)>\ .|$$

With the previous result, L2 simplifies as:

$$\text{(L2)} \qquad \frac{dc}{dx} = \lambda_o(\mu)\ c(x) - k(\mu)\ (c(x))^2$$

$$\text{(3.7.a)} \qquad + \mathcal{H}\{c(x), w\},$$

where

$$\text{(3.7.b)} \qquad k(\mu) = - <\eta(\mu), \Gamma_\mu(\hat{\varphi}_o, \hat{\varphi}_o)>_{Y^*}$$

$$k(\mu) \neq 0,\ k(\mu) = O(1),$$

$$\text{(3.8.a)} \qquad \mathcal{H}\{c(x), w\} = <\eta(\mu), \mathcal{N}(\mu;\ c(x)\hat{\varphi}_o + w)>_{Y^*}$$

$$- <\eta(\mu), \Gamma_\mu\ (c(x)\hat{\varphi}_o, c(x)\hat{\varphi}_o)>_{Y^*}$$

and the normalization $<\psi(\mu), \hat{\varphi}_o(\mu)>_{X^*} = 1$. ($\mathcal{N}$ and Γ have been defined in (2.3.c-d)).

A priori $\mathcal{H}$ is a mapping on $c(x)$ and w:

$$\text{(3.8.b)} \qquad \mathcal{H}: AC^I\ [R^1] \times AC\ [R^1 \to X] \to AC\ [R^1]\ ;$$

but with the implicit functional $w\{c(x)\}$ in (3.6):

$$\mathcal{H}: AC^I\ [R^1] \to AC\ [R^1],$$

where $w\{c\}$ depends <u>globally</u> upon $c(x)$, $-\infty \leq x \leq +\infty$. So in fact, L2 (3.7) is a <u>functional differential equation</u> for $c(x)$, <u>global</u> in nature. The initial value problem is a nonsense, as initial data ought to be specified $\forall x$, $-\infty \leq x \leq +\infty$! We remark that the differential part (including the quadratic term) of the functional differential equation L2 is in fact Landau's Equation [24]. The exact corrective term to Landau's model is, interestingly enough, neither polynomial, nor differential, but a non-local mapping $\mathcal{H}$.

We now define exactly the functional subspaces $\mathcal{C}_x$, $\mathcal{S}_y$, and $\mathcal{S}_c$ appropriate for the investigation of L1-L2.

$$\mathcal{S}_x \subset AC[R^1 \to (1-E_{ap})X] \tag{3.9}$$
$$\mathcal{S}_y \subset AC[R^1 \to (1-E_{av})Y]$$

$f \in \mathcal{S}_x$ if:

a) $\frac{\partial f}{\partial x} \in C^\alpha[R^1 \to (I-E_{ap})X] \cap L^1[R^1 \to (1-E_{ap})X]$

b) $\frac{\partial^2}{\partial x^2} Af \in C^\alpha[R^1 \to (I-E_{ap})X] \cap L^1[R^1 \to (I-E_{ap})X]$

c) $\exp(-2\lambda_o x)\ \frac{\partial f}{\partial x}$ and $\exp(-2\lambda_o x)\ \frac{\partial^2}{\partial x^2} Af \in L^\infty[R^1]$ for $x<0$

(asymptotic decay at $x = -\infty$)

d) $\exp((\lambda_o-\varepsilon)x)\frac{\partial f}{\partial x}$ and $\exp((\lambda_o-\varepsilon)x)\frac{\partial^2}{\partial x^2} Af \in L^\infty[R^1]$ for $x>0$

(asymptotic decay at $x = +\infty$; $\varepsilon > 0$).

$f \in \mathcal{S}_y$ if the conditions a, c, d, (excluding b and any conditions for $\frac{\partial^2}{\partial x^2} Af$) are satisfied, with X replaced by Y, and E_{ap} by E_{av}. C^α is the usual Hölder space of index α, $0 < \alpha < 1$.

$c(x) \in \mathcal{S}_c$ if:

a) $\frac{dc}{dx} \in C^o[R^1] \cap L^1[R^1]$ and

b) $\exp(-\lambda_o x)\ \frac{dc}{dx} \in L^\infty[R^1]$, for $x < 0$,

c) $\exp((\lambda_o-\varepsilon)x)\ \frac{dc}{dx} \in L^\infty[R^1]$, for $x > 0$.

$c(x) \in \mathcal{S}_c^I$ if:

a) $c(x) \in \mathcal{S}_c$ and

b) $\frac{dc}{dx} \in \mathcal{S}_c$.

4. Methods of Solution for the Lyapunov-Schmidt and the Functional Differential Equations.

Consider the first Lyapunov equation (3.5) as a mapping $\mathcal{A}$:

$$\mathcal{A}_{(c,w)} : \mathcal{S}_c^I \times \mathcal{S}_x \to \mathcal{S}_y , \tag{4.1.a}$$

$$\mathcal{A}_{(0,0)} = 0, \tag{4.1.b}$$

$$\mathcal{A}_{(c,w)} = A(\mu)\,\frac{\partial w}{\partial x} - T_\mu w$$

$$-(I-E_{av})\ \mathcal{N}(\mu;\ c(x)\,\hat{\varphi}_o + w) = 0 . \tag{4.1.c}$$

Theorem 4.1. $\mathcal{A}_w(0,0)$ is an isomorphism of $\mathcal{S}_x$ onto $\mathcal{S}_y$.|

Corollary 4.2. In some neighborhood of (0,0) in $\mathcal{S}_c^I \times \mathcal{S}_x$, there exists a unique continuous mapping:

$$w\{c(w),\mu\} : \mathcal{S}_c^I \to \mathcal{S}_x$$

such that:

$$\mathcal{A}_{(c(x),\ w\{c(x),\mu\})} \equiv 0.|$$

Theorem 4.1 hinges upon the existence of $\mathcal{A}_w(0,0)^{-1}$; let:

$$\mathcal{A}_w(0,0)f = A(\mu)\,\frac{\partial f}{\partial x} - T_\mu f = S(x) \tag{4.2}$$

To solve for f in (4.2), we construct generalized holomorphic semi-groups. The mapping $\mathcal{A}_w(0,0)$ acts from $(I-E_{ap})X$ into $(I-E_{av})Y$, cf. the reduction diagram (3.3). In particular the reduced operator $\lambda A(\mu) - T_\mu$ is invertible in a neighborhood of $\lambda_o(\mu)$ (deletion of the eigenvalue $\lambda_o(\mu)$). The essential spectrum (cf. figure I) remains only, which allows for the definition of Dunford Path Integrals along it. Let Γ^+, Γ^- be such paths along respectively the left and right side essential spectra (cf. figure I).

Proposition 4.3. If $S(x) \in \mathcal{S}_y$ in (4.2), then the solution $f \in \mathcal{S}_x$ of (4.2) is given by:

$$f(x) = \int_{-\infty}^{x} U^+(x-y)[S(y)-S(x)]\,dy \tag{4.3}$$

$$+\int_{+\infty}^{x} U^-(x-y)[S(y)-S(x)]\,dy - T_\mu^{-1}\,S(x),$$

(T_μ^{-1} is the pseudo-inverse, which now does exist even at $\mu = \mu^*$);

$$U^\pm(x) = \frac{1}{2\pi i}\int_{\Gamma^+,\Gamma^-} e^{\lambda x}(\lambda A(\mu)-T_\mu)^{-1}\,d\lambda; \tag{4.4}$$

$\|U^\pm\| = O(\frac{1}{|x|}),\ |x| \to 0;$

although $U^\pm$ are <u>not</u> semi-groups, $A\,U^\pm$ and $U^\pm A$ are:

$$A\,U^\pm(x+y) = A\,U^\pm(x)\ A\,U^\pm(y), \tag{4.5}$$

for $x > 0,\ y > 0$ <u>or</u> $x < 0,\ y < 0$;

$$\frac{\partial A U^{\pm}}{\partial x} f = T_{\mu} U^{\pm} f , \tag{4.6}$$

$$\frac{\partial U^{\pm} A}{\partial x} f = U^{\pm} T_{\mu} f \ ; \ \text{but} \tag{4.7}$$

$$\| T U^{\pm} \| = O(\frac{1}{x}), \ |x| \to 0, \tag{4.8}$$

and the holomorphic semi-groups (4.5) have no infinitesimal generator.|

A technical hypothesis needed for Proposition (4.1), and suggested by statistical mechanics is:

Hypothesis 4. Let $R(\lambda,\mu) = (\lambda A(\mu) - T_{\mu})^{-1}$; then as $|\lambda| \to \infty$, within the resolvent set:

$$\| R(\lambda,\mu) \| = O(\frac{1}{|\lambda|^{\alpha}}), \ 0 < \alpha < 1 \tag{4.a}$$

or:

$$\| R(\lambda,\mu) \| = O(1). \tag{4.b}$$

Proof of Proposition (4.1) is more complicated with hypothesis 4.b. Specifically, $A\ U^{\pm}(0^{\pm})$ exist in the case of hypothesis 4.a (although limits are projections, but not the identity!), but are undefined under hypothesis 4.b.

We now investigate the functional differential equation L2 (3.7-8):

$$L2 : \mathcal{S}_c^I \to \mathcal{S}_c$$

$$\frac{dc}{dx} = \lambda_o(\mu)\ c(x) - k(\mu)\ (c(x))^2 + \mathcal{H}(\mu, c, w\{c\}). \tag{4.9}$$

$\forall \mu$, $c \equiv 0$ is a trivial solution (as $\mathcal{H}$ is multilinear in c and $w\{c\}$). (4.9) is again a full-sized bifurcation problem from a continuous spectrum, at $\mu = \mu^*$;

$$\lambda_o(\mu^*) = 0, \ k(\mu^*) = O(1), \ \mu = \mu^*.$$

At $\mu = \mu^*$, the Frechet derivative of (4.9) reduces to $\frac{dc}{dx}$. The latter's spectrum, in spaces of absolutely continuous functions, is a purely continuous spectrum containing the full left or right half complex plane, including the

imaginary axis (depending on normalization of the AC spaces).

New techniques are needed for (4.9). We first make the following remarks; in a neighborhood of $\mu = \mu^*$:

$$\lambda_o(\mu) \approx O(\mu-\mu^*) ,$$

$$||c(x)|| = O(\mu-\mu^*) ,$$

$$f = c(x)\,\hat{\varphi}_o(\mu) + O(\mu-\mu^*)^2 \tag{4.10}$$

$$= \frac{\lambda_o(\mu)}{k(\mu)} \frac{\exp(\lambda_o x)}{\exp(\lambda_o x)+1}\,\hat{\varphi}_o(\mu) + O(\mu-\mu^*)^2 ;$$

the lowest order Landau differential operator approximation (4.10) is accurate only to $O(\mu-\mu^*)$. The exact $\mathcal{H}(\mu,c,w)$ contribution appears at $O(\mu-\mu^*)^2$ and corresponds to deviations from the "Navier-Stokes" solution (so called since the Landau equation (4.9) <u>without</u> the functional $\mathcal{H}$ admits the universal hyperbolic tangent Taylor weak shock profile for one-dimensional Navier-Stokes systems).

The key concept is to consider (4.9) not as a bifurcation from $c(x) = 0$, but as a <u>branching from the Landau-Taylor profile</u>

$$f = \frac{\lambda_o(\mu)}{k(\mu)} \frac{\exp(\lambda_o x)}{1+\exp(\lambda_o x)}\,\hat{\varphi}_o(\mu) . \tag{4.11}$$

To do so, we introduce a change of function, a change of variable and a change of parameter in (4.9):

$$\tau = \lambda_o(\mu) \Leftrightarrow \mu = \mu(\tau), \tag{4.12.a}$$

$$y = \tau x = \lambda_o x , \tag{4.12.b}$$

$$c(x) = \tau \frac{1}{k(\mu)} \frac{e^y}{e^y+1} (1+\theta(y)) . \tag{4.12.c}$$

L2 becomes a functional-differential equation for $\theta(y)$, on $-\infty \leqslant y \leqslant +\infty$, parametrized by τ:

$$\frac{d\theta}{dy} + \frac{e^y}{e^y+1}\theta = -\frac{e^y}{e^y+1}\theta^2 + \tau\, k(\tau)(1+e^{-y})\,\tilde{\mathcal{H}}\{\theta\}, \tag{4.13.a}$$

$$\theta \equiv 0 \text{ at } \tau = 0 ; \tag{4.13.b}$$

let L3 be the operator defined by (4.13), then:

$$L3 : \mathcal{S}_c^I \to \mathcal{S}_c ; \tag{4.13.c}$$

$\tilde{\mathcal{H}}(\theta(y))$ is identical to $\mathcal{H}(c(x))$ (3.8), after substitution of (4.12.a-b-c).

Whereas we look for $\theta \to 0$ as $\tau \to 0^+$ (branching from Landau's solution (4.11)), the former trivial branch $c(x) \equiv 0$ now becomes $\theta(y) \equiv -1$, $\forall \tau$. We have effectively achieved separation of branches. This is confirmed by:

Theorem 4.4. Let $\mathcal{L}(\theta)$ be the Fréchet derivative of L3 at $\theta = 0$:

$$\mathcal{L}(\theta) = \frac{d\theta}{dy} + \frac{e^y}{e^y+1}\,\theta \; ; \tag{4.14}$$

then $\mathcal{L}^{-1}$ is a bounded mapping from $\mathcal{S}_c$ onto $\mathcal{S}_c^I$, $\forall \tau \geq 0$.|

Remark: $\mathcal{L}^{-1}$ is an integral operator on $-\infty \leq y \leq +\infty$, which is in general unbounded on spaces of integrable functions. This required a much more complicated theory in [10]. If we do take into account the asymptotic decay conditions included in $\mathcal{S}_c$, $\mathcal{S}_c^I$ (3.9):

$$\exp(-y)\,\frac{d\theta}{dy} \text{ and } \exp(-y)\,\frac{d^2\theta}{dy^2} \in L^\infty(R^1)$$

for $y \leq 0$, and

$$\exp((1-\varepsilon)y)\,\frac{d\theta}{dy} \text{ and } \exp((1-\varepsilon)y)\,\frac{d^2\theta}{dy^2} \in L^\infty[R^1]$$

for $y \geq 0$, $\varepsilon > 0$, then $\mathcal{L}^{-1}$ is bounded from $\mathcal{S}_c$ onto $\mathcal{S}_c^I$. These decay conditions are, of course, suggested by the behavior of the derivatives of Landau's solution (4.11) at $y = \pm\infty$. To conclude:

Corollary 4.5. In some neighborhood of $\tau = 0$ in R^1, there exists a unique mapping

$$\begin{array}{ccc} \tau & \to & \theta\{\tau\} \\ R^1 & \to & \mathcal{S}_c^I \end{array},$$

such that $\theta\{\tau\}$ is the unique solution of (4.13) with $\theta\{0\} \equiv 0$.|

To demonstrate the corollary, we use the implicit function theorem applied to (4.13.a) considered as a mapping from: $R^1 \times \mathcal{S}_c^I \to \mathcal{S}_c$.

Finally from $\theta\{\tau\}$, we reconstruct

$$c(\tau) = \frac{\tau}{k} \frac{\exp(\tau x)}{1+\exp(\tau x)} \; (1+\theta\{\tau\})$$

$$f = c(\tau)\, \hat{\varphi}_o(\tau) + w\{c(\tau)\} \; .$$

The solution is actually unique up to a translation, since we have chosen an arbitrary (normalized) origin is obtaining the Landau profile (4.11) solution of:

$$\frac{dc}{dx} = \lambda_o(\mu)\, c(x) - k(\mu)(c(x))^2 \quad . \tag{4.15}$$

The asymptotic behavior of f at $x = \pm\infty$ shows that deviations from the "Navier-Stokes" component $c(x)$, caused by $w\{c\}$, appear $O(\tau^2)$ in the "hot tail" of the shock. Roughly speaking, $c(y)$ decays $O(\exp(y))$ as $y \to -\infty$, whereas $w(y)$ decays $O(\exp(2y))$. As $y \to +\infty$, both $c(y)$ and $w(y)$ decay $O(\exp-(1-\varepsilon)y)$.

To conclude, we remark that the concept of modified Landau's equation has also been introduced by N. N. Janenko [25,26]: he has added higher order polynomial terms in $c(x)$ to (4.15), in order to study the transition to turbulence in incompressible Navier-Stokes flows. Here, at the kinetic level, we have a corrective global functional operator $\mathcal{H}$.

A natural extension of Problems I-II is:

$$\frac{\partial f}{\partial t} + A(\mu)\, \frac{\partial f}{\partial x} - \mathcal{G}(\mu, f) = 0 \; ; \tag{4.16}$$

in this respect, we have the

Conjecture. For $\mu < \mu^*$, $\tau < 0$, the second branch $\Omega(\mu,x)$ is unstable in time; it is stable for $\mu > \mu^*$, $\tau > 0$. (This corresponds to well-known Entropy Conditions across the shock for Navier-Stokes). Also more general wave solutions of (4.16) may be investigated, including Burgers-like waves. Work is in progress on these questions.

APPENDIX

We summarize technical results of [8-10]. The Boltzmann equation [27] rules the evolution of a local velocity particle distribution $F(\vec{c})$, with the local velocity

vector

$$\vec{c} = (c_1, c_2, c_3);\ c = |\vec{c}| \ . \tag{A.1}$$

The space-independent Boltzmann operator:

$$Q[F, F] = 0 \tag{A.2}$$

is a bilinear integral operator in $L^2(R^3)$. It acts only upon the velocity vector $\vec{c}$. Classically:

$$Q[F, F] = 0 \Leftrightarrow F \equiv \omega(\mu, \vec{c}) \ , \tag{A.3}$$

where ω is a maxwellian (gaussian) distribution:

$$\omega(\mu, \vec{c}) = \frac{\rho}{(2\pi RT)^{3/2}} \exp\left\{ - \frac{(c_1-\mu)^2+c_2^2+c_3^2}{2RT} \right\} \ ,$$

where ρ is the density, μ the mean velocity (directed along the x-axis) and T the temperature. These macroscopic quantities which appear in the Navier-Stokes equations, are simply related to weighted averages of $F(\vec{c}, x)$:

$$\rho = \int F(\vec{c},x)d\vec{c}, \quad \rho\mu = \int c_1\, F(\vec{c},x)d\vec{c} \ ,$$
$$3\ \rho\ R\ T = \int (\vec{c}-\vec{\mu})^2\ F(\vec{c},x)d\vec{c} \ ,$$

where R is the perfect gas constant.

In one space dimension, the space dependent Boltzmann equation for the velocity distribution

$$F(c_1, c, x, t),\ c = |\vec{c}| \ , \quad x \in R^1,\ -\infty \leqslant x \leqslant +\infty \ , \tag{A.4}$$

becomes:

$$\frac{\partial F}{\partial t} + c_1 \frac{\partial F}{\partial x} = Q[F,F] \tag{A.5}$$

The second term on the left side is the one dimensional version of the ubiquitous "streaming operator"

$$\vec{c}.\ \nabla_x F \ .$$

We look for traveling waves of the type

$$F(c_1, c, x + \mu t) \ .$$

A viscous shock is defined as a nonlinear transition profile between two asymptotic ($x = \pm\infty$) Maxwellians; one with mean velocity μ^+ subsonic; the other with μ^- supersonic. It must be noted that the same Rankine-Hugoniot conditions as for Navier-Stokes uniquely relate μ^+, ρ^+, T^+ and μ^-, ρ^-, T^-. After renormalization [9,10]:

$$(A.6) \qquad (\mu+c_1)\frac{\partial f}{\partial x} = L_\mu f + \Gamma_\mu[f,f] ,$$

where L_μ is the Fréchet derivative of Q, and Γ_μ, an appropriate second order derivative; together with the normalization:

$$(A.7) \qquad f(c_1, c, -\infty) = 0 \text{ or } f(c_1, c, +\infty) = 0 .$$

(A.6) is investigated in a space AC of absolutely continuous functions:

$$AC[R^1 \to X] \to L^1_{\ell oc}[R^1 \to Y]$$

(normalized at $\pm\infty$, cf. (A.7)) and X,Y are appropriate graph-norm Banach spaces defined uniquely on the velocity variable. The following is then demonstrated:

Proposition A.1. In appropriate spaces X,Y (implicitly incorporating the Rankine-Hugoniot conditions),

$$(A.2bis) \qquad Q[f,f] = L_\mu f + \Gamma_\mu[f,f]$$

is a bifurcation problem from a simple isolated eigenvalue of L_μ* at the critical sonic value of $\mu = \mu^*$. The two branches correspond to a subsonic and a supersonic Maxwellian, identical at $\mu = \mu^*$.|

Looking for a critical trajectory joining the two asymptotic bifurcated subsonic and supersonic maxwellians, we consider (A.6) as a bifurcation problem from the essential spectrum, superimposed upon the simple bifurcation problem (A.2bis). In (A.6), $f \equiv 0$ is indeed a trivial solution $\forall\mu$. The essential spectrum is evident from the identification:

$$(A.8) \qquad A(\mu) \equiv (\mu+c_1)I ,$$

which does not possess an inverse in $L^2(R^3)$, since

$$-\infty \leqslant c_1 \leqslant +\infty$$

(cf. $c_1 = -\mu$). The hypothesis required by the abstract setting are completed through the:

Proposition A.2. The generalized eigenvalue problem

$$\lambda(\mu+c_1)\varphi - L_\mu\varphi = 0$$

has a real, simple, isolated eigenvalue $\lambda_o(\mu)$, in the spaces X and Y:

$$\lambda_o(\mu) \leqslant 0, \ \mu \leqslant \mu^*$$
$$\lambda_o(\mu) \geqslant 0, \ \mu \geqslant \mu^*.$$

Similar results were obtained by H. Weyl in 1949 [28], for the Navier-Stokes equations linearized about sub- or supersonic equilibria. Finally, the "streaming operator" $A(\mu)$ defined in (A.8), though responsible for the pathology of the problem, is universally present in kinetic (statistical mechanics) equations. It represents transfer of very high velocity particles, and generates the essential spectrum of kinetic operators.

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Numerical Solution of Bifurcation and Nonlinear Eigenvalue Problems

Herbert B. Keller

1. Introduction.

We show in this paper how a large class of bifurcation and nonlinear eigenvalue problems can be solved and the numerical methods justified. Only equilibrium problems are treated here, say in the general form

$$G(u, \lambda) = 0 \tag{1.1}$$

where $G : \mathbb{B} \times \mathbb{R} \to \mathbb{B}$ for some Banach space, $\mathbb{B}$. We find it most instructive to refer occasionally to the cannonical example of the matrix eigenvalue problem:

$$Au - \lambda u = 0 \ , \tag{1.2}$$

where A is an $n \times n$ real matrix with real eigenvalues.

By a smooth branch or arc of solutions

$$\Gamma_{ab} : [\, u(s), \ \lambda(s)] \ , \quad s_a \leq s \leq s_b \tag{1.3}$$

we mean a one parameter family of solutions of (1.1), $u(s) \in \mathbb{B}$, $\lambda(s) \in \mathbb{R}$ depending twice continuously differentiably on some parameter $s \in [\, s_a, s_b]$. Of course the parameter, s, is quite arbitrary on each such branch and this fact is crucial in our study. In figures 1a and 1b we sketch, respectively, typical solution branches for problems (1.1) and (1.2). The "points" at λ_A and λ_B represent "simple" bifurcation while λ_D is a multiple bifurcation point. The "arcs" Γ_j in figure 1b for $j \neq 0$ represent the eigenspaces of A belonging to the eigenvalues λ_j while Γ_0 is the trivial solution. For simple eigenvalues the bifurcation is simple (i.e. a one parameter family of solutions branches off). At $\lambda = \lambda_C$ on branch Γ_1 in figure 1a we

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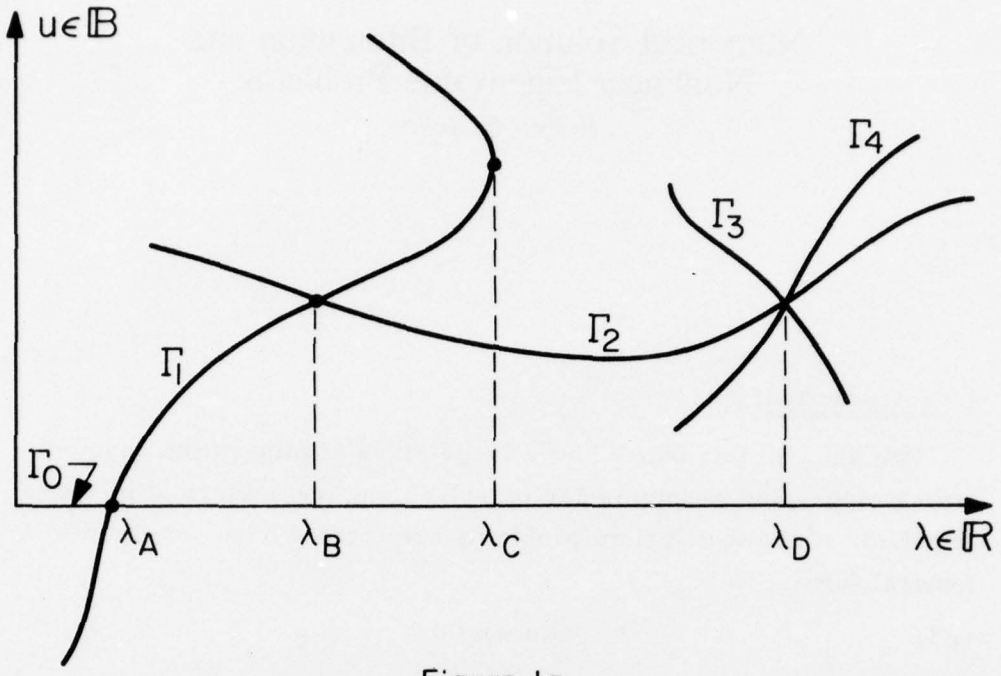


Figure Ia.

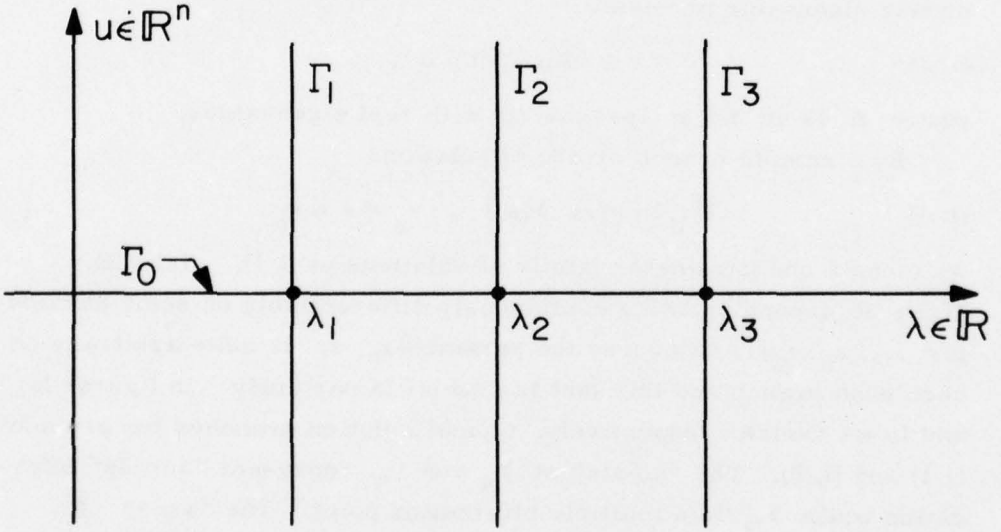


Figure Ib.

have what is called a "limit point" in the applied mechanics literature. These present no difficulties analytically or computationally in our current theory although they seem to have been quite troublesome in the past [1, 12, 20]. Note that all the branches Γ_1, Γ_2, ... in the eigenvalue problem are composed entirely of these limit points.

We assume that the basic problem is to compute large segments of solution branches of (1.1) including all branches bifurcating from each segment.

2. Parametrization and Continuation of Solution Arcs

The standard approach is almost invariably to use λ, one of the naturally occuring parameters of the problem, as the parameter defining solution arcs, $u(\lambda)$. Indeed if for some $\lambda = \lambda_0$ a solution $u = u_0$ of (1.1) is isolated, that is

$$G_u^0 \equiv G_u(u_0, \lambda_0) \tag{2.1}$$

is nonsingular, and if $G(u, \lambda)$ is C_1 in some ρ_0 - sphere about $[u_0, \lambda_0]$ then the implicit function theorem insures the existence of a unique smooth arc of solutions $u = u(\lambda)$ for $|\lambda - \lambda_0| < \rho_1$, say. Furthermore, with our assumed smoothness, it follows that $du(\lambda)/d\lambda$ exists and satisfies:

$$G_u(u(\lambda), \lambda) \frac{du}{d\lambda} = - G_\lambda(u(\lambda), \lambda) \tag{2.2}$$

Many procedures are now available for extending or approximating the solution branch through $[u_0, \lambda_0]$. The implicit function theorem suggests contraction mapping techniques while (2.2) suggests predictor-corrector continuation. A nice survey of the latter ideas (for finite dimensional problems) is given by Rheinboldt [18]. In particular we have frequently used [3] one step of Euler's method in (2.2) as a predictor:

$$u^0(\lambda + \delta\lambda) = u(\lambda) + \delta\lambda \frac{du(\lambda)}{d\lambda} \tag{2.3 a}$$

to supply the initial iterate for Newton's method to solve (1.1) at $\lambda + \delta\lambda$:

$$G_u^\nu \, \delta u^\nu(\lambda + \delta\lambda) = - G^\nu \quad , \quad \nu = 0, 1, \ldots \; ; \tag{2.3 b}$$

where:

$$(2.3)\ c)\quad \begin{cases} G_u^\nu \equiv G_u(u^\nu(\lambda+\delta\lambda),\ \lambda+\delta\lambda)\ ,\quad G^\nu \equiv G(u^\nu(\lambda+\delta\lambda),\ \lambda+\delta\lambda)\ , \\ u^{\nu+1}(\lambda+\delta\lambda) = u^\nu(\lambda+\delta\lambda) + \delta u^\nu(\lambda+\delta\lambda)\ . \end{cases}$$

It is not difficult to base existence proofs on such techniques provided the arc consists of isolated solutions.

All of the indicated continuation procedures may fail or encounter difficulties as a nonisolated solution is approached; that is a point $[u_0, \lambda_0]$ where G_u^0 is singular. We also call these singular points. As we shall see in §3 the above indicated continuation procedures could skip over some singular points but not over limit points as at $\lambda = \lambda_C$ in figure 1a. Also at bifurcation points some special procedures are required to switch from one branch to another. A simple analysis shows that the scheme (2.3) is incapable of tracing out any nontrivial solutions of the eigenvalue problem (1.2)!

To circumvent these difficulties we recall that the parametrization of solution arcs is at our disposal. Thus we are free to impose some additional constraint or normalization on the solution and we do this, quite generally, by replacing (1.1) by:

$$(2.4)\ a)\qquad G(u, \lambda) = 0\ ,\quad b)\ N(u, \lambda, s) = 0\ .$$

Here $N : \mathbb{B} \times \mathbb{R}^2 \to \mathbb{R}$ and $s \in \mathbb{R}$ is the independent parameter on the solution arc. We shall show several choices for N which make s an approximation to "arclength" on the solution branch. Then, as we shall see, limit points essentially disappear, it is easy to jump over singular points or to compute them, relatively large steps in s can be taken and it is easy to switch branches at bifurcation points. Note that we are not simply changing the parameter in the problem as was first done in [12] or in another way in [1].

By introducing $x \in \mathbb{X} \equiv \mathbb{B} \times \mathbb{R}$ and $P : \mathbb{X} \times \mathbb{R} \to \mathbb{X}$ as

$$(2.5)\ a)\qquad x \equiv [u, \lambda]\ ,\quad P(x, s) \equiv \begin{pmatrix} G(u, \lambda) \\ \\ N(u, \lambda, s) \end{pmatrix}$$

a solution arc of (1.1) or (2.4) is $x(s) \equiv [u(s), \lambda(s)]$ and it satisfies

$$(2.5)\ b)\qquad P(x(s), s) = 0\ .$$

For fixed s a solution $x(s)$ is isolated if

$$\mathcal{A}(s) \equiv P_x(x(s),\ s) = \begin{pmatrix} G_u(u(s),\ \lambda(s)) & G_\lambda(u(s),\ \lambda(s)) \\ N_u(u(s),\ \lambda(s),\ s) & N_\lambda(u(s),\ \lambda(s),\ s) \end{pmatrix} \tag{2.6}$$

is nonsingular. Furthermore on a smooth arc $\dot{x}(s) \equiv dx(s)/ds$ satisfies

$$\mathcal{A}(s)\,\dot{x}(s) = -\begin{pmatrix} 0 \\ \cdot \\ N_s(u(s),\ \lambda(s),\ s) \end{pmatrix} \tag{2.7}$$

Now continuation in s could proceed in exact analogy with our prior continuation in λ. However the possible advantages of our reformulations are clarified by the fact that P_x can be nonsingular while G_u is singular. Indeed this is but a special case of the basic:

Lemma 2.8. Let $\mathbb{B}$ be a Banach space and consider the linear operator $\mathcal{A} : \mathbb{B} \times \mathbb{R}^\nu \to \mathbb{B} \times \mathbb{R}^\nu$ of the form:

$$\mathcal{A} \equiv \begin{pmatrix} A & B \\ C^* & D \end{pmatrix} \quad \text{where} \quad \begin{cases} A : \mathbb{B} \to \mathbb{B}\ , & B : \mathbb{R}^\nu \to \mathbb{B}\ ; \\ C^* : \mathbb{B} \to \mathbb{R}^\nu\ , & D : \mathbb{R}^\nu \to \mathbb{R}^\nu\ . \end{cases}$$

i) If A is nonsingular then $\mathcal{A}$ is nonsingular iff:

(2.8) a) $\qquad D - C^* A^{-1} B$ is nonsingular .

ii) If A is singular and

(2.8) b) $\qquad \dim \mathcal{N}(A) = \operatorname{codim} \mathcal{R}(A) = \nu$

then $\mathcal{A}$ is nonsingular iff:

(2.8) $\qquad c_0)\ \dim \mathcal{R}(B) = \nu \quad , \quad c_1)\ \mathcal{R}(B) \cap \mathcal{R}(A) = 0\ ,$

$\qquad\qquad c_2)\ \dim \mathcal{R}(C^*) = \nu \quad , \quad c_3)\ \mathcal{N}(A) \cap \mathcal{N}(C^*) = 0\ .$

iii) If A is singular and $\dim \mathcal{N}(A) > \nu$ then $\mathcal{A}$ is singular.

Proof. Not difficult to work out; it will appear elsewhere, [10]. ▩

Using Lemma 2.8 and our reformulation of (1.1) it is easy to prove a variety of bifurcation theorems which we do not report here. We only use in the present work the special case of Lemma 2.8 with $\nu = 1$. Then given (2.8b) the conditions in (2.8c) simply reduce to

(2.9) $$B \notin R(A) \text{ and } C^* \notin R(A^*) .$$

As a simple example of the use of our procedure consider the eigenvalue problem (1.2) subject to either of the normalizations

(2.10) a) $N_0(u,\lambda, s) \equiv \lambda - s$; b) $N_1(u,\lambda, s) \equiv \|u\|_2^2 - s^2$.

Using $N_0 = 0$ the predictor-corrector continuation scheme generates the trivial solution branch, $u = 0$, λ = arbitrary, starting from any point on this branch. Alternatively using $N_1 = 0$ our scheme traces out the one dimensional eigenspace belonging to any simple eigenvalue $\lambda = \lambda_0$. Switching over from one of these normalizations to the other at a simple bifurcation point $(u,\lambda) = (0,\lambda_0)$ allows us to trace out the two branches through this point. Indeed this is the key to our method for switching branches at bifurcation points in the general case, see §5.

A somewhat natural parametrization of a solution branch $[u(s), \lambda(s)]$ is to use for s a form of arclength. That is, for some $\theta \in (0,1)$:

(2.10) c) $$N_\theta(u,\lambda, s) \equiv \theta\|\dot{u}(s)\|^2 + (1-\theta)\,|\dot{\lambda}(s)|^2 - 1 = 0 .$$

This form is not the most practical one to use, even for merely proving existence theorems and so we use approximations to it. Assuming a solution of (1.1) known say $[u,\lambda] = [u_0,\lambda_0]$ we set $[u_0,\lambda_0] = [u(s_0), \lambda(s_0)]$ and define over $s_0 \le s < s_1$:

(2.10) d) $$N_2(u,\lambda, s) \equiv \theta\|u(s) - u(s_0)\|^2 + (1-\theta)\,|\lambda(s) - \lambda(s_0)|^2 - (s - s_0)^2 = 0 .$$

Alternatively if in addition to $[u_0,\lambda_0]$ we know $[\dot{u}_0,\dot{\lambda}_0]$ satisfying (2.10c) at $s = s_0$ then we can use on $s_0 \le s < s_1$:

(2.10) e) $$N_3(u,\lambda, s) \equiv \theta\,\dot{u}^*(s_0)[u(s) - u(s_0)] + (1-\theta)\,\dot{\lambda}(s_0)[\lambda(s) - \lambda(s_0)] - (s - s_0) = 0 .$$

Here $\dot{u}^*(s_0) \in \mathbb{B}^*$ is the dual element to $\dot{u}(s_0)$ such that $\dot{u}^*(s_0)\,\dot{u}(s_0) = \|\dot{u}(s_0)\|^2$. (The existence of such an element is assured by the Hahn-Banach theorem.) We call N_2 or N_3 pseudo-arclength normalizations and examine some of their properties in §3. Previous attempts to use arclength as a parameter in solving nonlinear

algebraic systems have been made in [13, 14].

3. Continuation About Regular and Limit Points.

We shall justify continuation procedures using the normalization N_3 on solution arcs composed of "regular points" or "normal limit points". Specifically let $[u_0, \lambda_0]$ be a solution of (1.1) and let $[\dot{u}_0, \dot{\lambda}_0]$ satisfy

(3.0) a) $$G_u^0 \dot{u}_0 + G_\lambda^0 \dot{\lambda}_0 = 0 \ , \quad \text{b)} \ \|\dot{u}_0\|^2 + |\dot{\lambda}_0|^2 > 0 \ .$$

Then we say that $[u_0, \lambda_0]$ is a regular solution (point) if in addition:

(3.1) $$G_u^0 \equiv G_u(u_0, \lambda_0) \text{ is nonsingular} \ .$$

We call $[u_0, \lambda_0]$ a normal limit solution (point) if (3.0) holds but in place of (3.1) we have:

(3.2) a) $$\dim \mathcal{N}(G_u^0) = \operatorname{codim} \mathcal{R}(G_u^0) = 1 \ , \quad \text{b)} \ G_\lambda^0 \notin \mathcal{R}(G_u^0) \ .$$

Theorem 3.3. Let $[u_0, \lambda_0]$ be either a regular solution or a normal limit solution. Let $G(u, \lambda)$ have two continuous derivatives in some sphere about $[u_0, \lambda_0]$. Then with $[u(s_0), \lambda(s_0)] \equiv [u_0, \lambda_0]$, $[\dot{u}(s_0), \dot{\lambda}(s_0)] \equiv [\dot{u}_0, \dot{\lambda}_0]$ and $\dot{u}^*(s_0)$ as defined after (2.10e) there exists a unique smooth arc of solutions $[u(s), \lambda(s)]$ of (2.4) using $N \equiv N_3$ on $|s - s_0| \leq \rho$ for some sufficiently small $\rho > 0$. On this solution arc the Frechet derivative $\mathcal{A}(s)$ of (2.6) is nonsingular.

Proof. All of these results will follow from the implicit function theorem applied to (2.4) at $[u, \lambda, s] = [u_0, \lambda_0, s_0]$ if $\mathcal{A}(s_0)$ is nonsingular. We first consider $[u_0, \lambda_0]$ to be a normal limit solution. By (3.2b) in (3.0a) it then follows that $\dot{\lambda}_0 = 0$ and hence $\dot{u}_0 \in \mathcal{N}(G_u^0)$. Now (3.2a) implies since $\dot{u}_0^* \dot{u}_0 \neq 0$ that $\dot{u}_0^* \notin \mathcal{R}(G_u^{0*})$. This result and (3.2) now yield with $\nu = 1$ in Part ii) of Lemma 2.8 that

$$\mathcal{A}(s_0) = \begin{pmatrix} G_u^0 & G_\lambda^0 \\ \theta \dot{u}_0^* & (1-\theta)\dot{\lambda}_0 \end{pmatrix}$$

is nonsingular [use the form in (2.9)].

Next let $[u_0, \lambda_0]$ be a regular solution. If $\dot{\lambda}_0 = 0$ then by (3.0a)

and (3.1), $\dot{u}_0 = 0$. This contradicts (3.0b) so $\dot{\lambda}_0 \neq 0$ at a regular point. Then (3.0a) and (3.1) imply

$$\dot{u}_0/\dot{\lambda}_0 = -(G_u^0)^{-1} G_\lambda^0 \ .$$

Now by Part i) of Lemma 2.8. $A(s_0)$ above is nonsingular iff

$$(1-\theta)\dot{\lambda}_0 - \theta\, \dot{u}_0^{*}(G_u^0)^{-1} G_\lambda^0 \neq 0 \ .$$

That is, using the above, iff

$$[\,\theta\, \dot{u}_0^{*}\dot{u}_0 + (1-\theta)\dot{\lambda}_0^2\,]/\dot{\lambda}_0 = [\,\theta\|\dot{u}_0\|^2 + (1-\theta)\,|\dot{\lambda}_0|^2\,]/\dot{\lambda}_0 \neq 0 \ .$$

Since $\dot{\lambda}_0 \neq 0$ and $\|\dot{u}_0\| \neq 0$ it follows that $A(s_0)$ is nonsingular. ▨

Clearly any smooth branch of solutions composed of regular points or normal limit points can be determined using, say, Euler-Newton continuation on (2.4) with the normalization $N \equiv N_3$. We could easily justify the normalization $N = N_2$ for smooth arcs since on them:

$$u(s) - u(s_0) = \dot{u}(s_0)(s - s_0) + O(|s - s_0|^2) \ ,$$
$$\lambda(s) - \lambda(s_0) = \dot{\lambda}(s_0)(s - s_0) + O(|s - s_0|^2) \ .$$

When using (2.10e) over a sequence of intervals $[\,s_0, s_1\,]$, $[\,s_1, s_2\,]$, ... it is a good policy to impose the arclength condition

$$\theta\|\dot{u}(s)\|^2 + (1-\theta)\,|\dot{\lambda}(s)|^2 = 1$$

periodically, say at each joint, $s = s_k$. The resulting arc is then only piecewise smooth in s, with jump discontinuities in the length of the tangent vector $[\,\dot{u}(s), \dot{\lambda}(s)\,]$ at $s = s_k$. Specifically if $[\,\dot{u}(s_k^-), \dot{\lambda}(s_k^-)\,]$ is the limit as $s \uparrow s_k$ then we use on $[\,s_k, s_{k+1}\,]$

$$[\,\dot{u}(s_k), \dot{\lambda}(s_k)\,] = c[\,\dot{u}(s_k^-), \dot{\lambda}(s_k^-)\,] \ ,$$

$$c^{-2} = \theta\|\dot{u}(s_k^-)\|^2 + (1-\theta)\,|\dot{\lambda}(s_k^-)|^2 \ .$$

This renormalization allows more uniform steps in s to be taken during the continuation process.

4. Continuation Past Singular Points.

A solution $x(s) = [\,u(s), \lambda(s)\,]$ of (2.4) is said to be singular or a singular point if $A(s)$ in (2.6) is singular. We will consider smooth arcs of solutions $x(s)$ for $s_a \leq s \leq s_b$ on which only $x(s_0)$ for some

$s_0 \in (s_a, s_b)$ is singular. Under mild smoothness conditions various continuation procedures can "jump" over the singular point in going from s_a to s_b. A simple way to do this is by using Euler-Chord or Euler-Newton continuation, as we proceed to show.

We assume, as stated above, that

(4.0) $\mathcal{A}(s) \equiv P_x(x(s), s)$ *is nonsingular for* $s \in [s_a, s_b] - \{s_0\}$.

Then the tangent, $\dot{x}(s_a)$, is uniquely defined by

(4.1) a) $$\mathcal{A}(s_a)\,\dot{x}(s_a) = -P_s(x(s_a), s_a) ,$$

and an approximation to $x(s)$ is

(4.1) b) $$x^0(s) \equiv x(s_a) + [s - s_a]\,\dot{x}(s_a) .$$

Using this approximation we consider the chord (or special Newton) method:

(4.2)
$$\text{a)} \quad \mathcal{A}^0(s) \equiv P_x(x^0(s), s) ,$$
$$\text{b)} \quad \mathcal{A}^0(s)\left[x^{\nu+1}(s) - x^{\nu}(s)\right] = -P(x^{\nu}(s), s) , \quad \nu = 0, 1, \ldots .$$

We could also try Newton's method, with $y^0(s) \equiv x^0(s)$:

(4.3)
$$\left.\begin{aligned} &\text{a)} \quad \mathcal{A}^{\nu}(s) \equiv P_x(y^{\nu}(s), s) \\ &\text{b)} \quad \mathcal{A}^{\nu}(s)\left[y^{\nu+1}(s) - y^{\nu}(s)\right] = -P(y^{\nu}(s), s) \end{aligned}\right\} \nu = 0, 1, \ldots .$$

To get convergence we need only show that $x^0(s)$ is in the appropriate domain of attraction about $x(s)$ and that the $\mathcal{A}^{\nu}(s)$ are all nonsingular. We do this in

Theorem 4.4. Let $x(s)$ be a twice continuously differentiable arc of solutions of (2.5) on $[s_a, s_b]$. Let (4.0) hold and for some positive functions $K(s)$, $\mathcal{K}(s)$ and $\rho(s)$ defined on $[s_a, s_b]$:

(4.4)
$$\text{a)} \quad \|P_x(y, s) - P_x(x(s), s)\| \le K(s)\,\|y - x(s)\| \;\forall\; y \text{ in}$$
$$\|y - x(s)\| \le \rho(s) ;$$
$$\text{b)} \quad \max_{s_a \le t \le s} \|\dot{x}(t)\| \le \mathcal{K}(s) .$$

For $s \in [s_a, s_b] - \{s_0\}$ *define*

(4.4) c) $$M(s) \equiv \|\mathcal{A}^{-1}(s)\| \ ,$$

and for some positive $\theta(s) < 1/3$ define

(4.4) d) $$r(s) \equiv \min\left[\rho(s), \frac{\theta(s)}{M(s)\,K(s)}\right] \ .$$

Then if

(4.4) e) $$|s - s_a|^2 \ \mathcal{K}(s) \leq 2r(s) \ , \quad s \neq s_0 \ ,$$

the Chord iterates $x^\nu(s) \to x(s)$ with geometric convergence factor

$$\frac{2\theta(s)}{1 - \theta(s)} \ .$$

Proof. It easily follows from (4.1) and (4.4) that

$$\|x^0(s) - x(s)\| \leq \tfrac{1}{2}|s - s_a|^2 \ \mathcal{K}(s) \ .$$

Then (4.4d, e) imply that $\|x^0(s) - x(s)\| \leq r(s)$. Now (4.4a, c) yield

$$\|\mathcal{A}^{-1}(s)[\mathcal{A}^0(s) - \mathcal{A}(s)]\| \leq M(s)\,K(s)\,r(s) \leq \theta(s) \ .$$

Thus the Banach Lemma insures that $\mathcal{A}^0(s)$ is nonsingular with

$$\|\mathcal{A}^0(s)^{-1}\| \leq \frac{M(s)}{1 - \theta(s)} \ .$$

We can now define

$$H(y, s) \equiv y - \mathcal{A}^0(s)^{-1}\,P(y, s)$$

and for all y, z in $\|y - x(s)\| \leq r(s)$:

$$\|H(y, s) - H(z, s)\| = \|\mathcal{A}^0(s)^{-1}[\mathcal{A}^0(s)(y-z) - (P(y, s) - P(z, s)]\| \ ,$$

$$\leq \frac{M(s)}{1 - \theta(s)}\,K(s)\,2r(s)\,\|y-z\| \ ,$$

$$\leq \frac{2\theta(s)}{1 - \theta(s)}\,\|y - z\| \ .$$

Since $\theta(s) < 1/3$ we get that $H(y, s)$ is contracting on $\|y-x(s)\| \leq r(s)$ and the theorem follows. ∎

Note that $M(s)$ must become unbounded and thus $r(s) \to 0$ as $s \to s_0$. Our result thus uses the fact (see figure 2) that there is a cone about $x(s)$ with vertex at $x(s_0)$ and interior to this cone the chord method converges. To jump over a singular point the tangent to $x(s)$ at $x(s_a)$ need only penetrate the cone for some $s > s_0$ for which (4.4e) holds. Clearly if the curvature of the solution arc is not

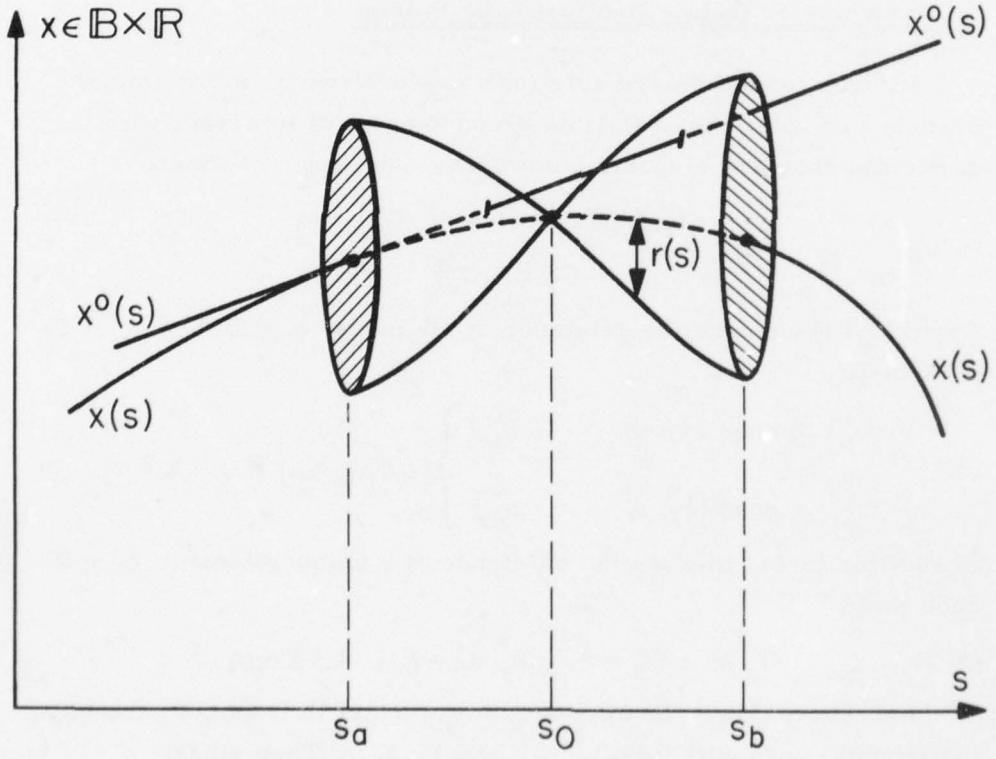


Figure 2

too great over $[s_a, s_b]$ this can be achieved.

Newton's method can also be shown to converge in the same cone. However Newton's method may even converge in a much larger region, including a cylindrical tube about $x(s)$. Thus the singular solution $x(s_0)$ can be determined directly in such cases not only by bisection. Unfortunately the status of the convergence of Newton's method at singular points is not completely clear at the present time. The main idea and no doubt the behavior to be expected in many cases is explained in the basic paper of Rall [16]. But the details of a proof with reasonable sufficient conditions seem to be lacking. Progress in this direction has recently been made by Reddien [17].

5. Switching Branches at Bifurcation Points.

Bifurcation points are solutions at which two or more smooth branches of solutions of (1.1) have non-tangential intersections. In particular they are singular points, say $[u_0, \lambda_0]$, at which:

$$\text{(5.0)}\quad \begin{aligned} &\text{a)} \quad \dim \mathcal{N}(G_u^0) = \operatorname{codim} \mathcal{R}(G_u^0) = m \ , \\ &\text{b)} \quad G_\lambda^0 \in \mathcal{R}(G_u^0) \ . \end{aligned}$$

From (5.0a) we have the existence of elements $\phi_j \in \mathbb{B}$ and $\psi_j^* \in \mathbb{B}^*$ such that:

$$\text{(5.1)}\quad \left.\begin{aligned} \mathcal{N}(G_u^0) &= \operatorname{span}\{\phi_1, \phi_2, \ldots, \phi_m\} \\ \mathcal{N}(G_u^{0\,*}) &= \operatorname{span}\{\psi_1^*, \psi_2^*, \ldots, \psi_m^*\} \end{aligned}\right\} \psi_i^* \phi_j = \delta_{ij} \ ; \ i, j = 1, 2, \ldots, m.$$

In addition (5.0b) implies the existence of a unique element $\phi_0 \in \mathbb{B}$ such that:

$$\text{(5.2)}\quad G_u^0 \phi_0 + G_\lambda^0 = 0 \ ; \ \psi_j^* \phi_0 = 0 \ , \ 1 \leq j \leq m \ .$$

Let $[u(s), \lambda(s)]$ be any smooth branch of (1.1) through the bifurcation point, say with $u(s_0) = u_0$, $\lambda(s_0) = \lambda_0$. Then since

$$\text{(5.3) a)}\quad G_u^0 \dot{u}(s_0) + G_\lambda^0 \dot{\lambda}(s_0) = 0$$

it follows from (5.1)-(5.2) that

$$\text{(5.3) b)}\quad \dot{u}(s_0) = \sum_{j=0}^{m} \alpha_j \phi_j$$

where

(5.3) c) $\alpha_0 = \lambda(s_0)$; $\alpha_j = \psi_j^* \dot{u}(s_0)$, $1 \leq j \leq m$.

We get by differentiation of $G(u(s), \lambda(s)) = 0$ at $s = s_0$:

(5.4) $$G_u^0 \ddot{u}_0 = -[G_{uu}^0 \dot{u}(s_0)\dot{u}(s_0) + 2G_{u\lambda}^0 \dot{u}(s_0)\dot{\lambda}(s_0) + G_{\lambda\lambda}^0 \dot{\lambda}(s_0)\dot{\lambda}(s_0)] - G_\lambda^0 \ddot{\lambda}(s_0) .$$

Since $G_u^0 \ddot{u}(s_0) \in R(G_u^0)$ and $G_\lambda^0 \in R(G_u^0)$ the bracketed term on the right side of (5.4) is also in $R(G_u^0)$. Then $\psi_i^*[\ldots] = 0$ must hold for this term and so using (5.3) it follows that the $m+1$ scalars $\{\alpha_0, \ldots, \alpha_m\}$ must satisfy the quadratic system:

(5.5) a) $$\sum_{j=1}^{m}\sum_{k=1}^{m} a_{ijk}\alpha_j\alpha_k + 2\sum_{j=1}^{m} b_{ij}\alpha_j\alpha_0 + c_i\alpha_0^2 = 0 , \quad 1 \leq i \leq m ;$$

where

(5.5) b) $$a_{ijk} \equiv \psi_i^* G_{uu}^0 \phi_j\phi_k , \quad b_{ij} \equiv \psi_i^*[G_{uu}^0\phi_0 + G_{u\lambda}^0]\phi_j , \quad 1 \leq j, k \leq m ;$$

$$c_i \equiv \psi_i^*(G_{uu}^0\phi_0\phi_0 + 2G_{u\lambda}^0\phi_0 + G_{\lambda\lambda}^0) .$$

Thus the tangent, $[\dot{u}(s_0), \dot{\lambda}(s_0)]$, to every smooth branch through the bifurcation point $[u_0, \lambda_0]$ must have the form (5.3b, c) and satisfy (5.5). Conversely if at a solution $[u_0, \lambda_0]$ of (1.1) conditions (5.0) hold and (5.5) has $r \geq 2$ distinct nontrivial roots† then $[u_0, \lambda_0]$ is a bifurcation point with at least r smooth solution branches of (1.1) intersecting there. Essentially this result is proven in [11]. In the important special case $m = 1$, the algebraic bifurcation equations (5.5) reduce to the single quadratic

(5.6) a) $$a_{111}\alpha_1^2 + 2b_{11}\alpha_1\alpha_0 + c_1\alpha_0^2 = 0 .$$

If $[\alpha_0, \alpha_1]$ is one nontrivial root of this quadratic then the other root is distinct provided

(5.6) b) $$a_{111}\alpha_1 + b_{11}\alpha_0 \neq 0 .$$

If $[u_1(s), \lambda_1(s)]$ is a smooth branch of solutions through the

† Since (5.5a) is homogeneous a root is determined only to within a multiplicative constant. Roots are said to be distinct if they are not scalar multiples of each other.

bifurcation point with tangent $[\dot{u}_1(s_0), \dot{\lambda}_1(s_0)]$ determined by $[\alpha_0, \alpha_1]$ in (5.3b, c) with $m = 1$ then the condition (5.6b) can be written as:

$$\text{(5.6) c)} \qquad \psi_1^* [G^0_{uu}\, \dot{u}_1(s_0) + G^0_{u\lambda}\, \dot{\lambda}_1(s_0)]\, \phi_1 \neq 0 \ .$$

This is essentially the form of the bifurcation condition given by Crandall and Rabinowitz [5]. However it is seldom pointed out that this condition insures the existence of two distinct roots of a quadratic†; Rheinboldt [18] makes this quite explicit for the class of problems he treats.

Method I. An obvious way to determine branches bifurcating at $[u_0, \lambda_0]$ is to determine several distinct roots of (5.5), use them in (5.3b, c) to construct several distinct tangent vectors $[\dot{u}_k(s_0), \dot{\lambda}_k(s_0)]$, $k = 1, 2, \ldots,$ then use each of these tangents in $N_3 = 0$ of (2.10e) and proceed as previously indicated. For $m = 1$, simple bifurcation, similar devices have been used and suggested in analytical and perturbation studies [7, 18, 21]. Rheinboldt [18] uses an approximation to the second root in some of his numerical methods to predict a point on the "second"branch. This idea can be used quite generally to obtain approximations to the coefficients $\{a_{ijk}, b_{ij}, c_i\}$ in (5.5) if the ϕ_j and ψ_j^* are known (or sufficiently well approximated). Thus we define:

$$\text{(5.7) a)} \qquad a_{ijk}(\epsilon) \equiv \psi_i^* \frac{1}{\epsilon} [G_u(u_0 + \epsilon\phi_j, \lambda_0) - G^0_u]\, \phi_k \ ,$$

$$\text{b)} \qquad b_{ij}(\epsilon) \equiv \psi_i^* \frac{1}{\epsilon} \{[G_u(u_0 + \epsilon\phi_j, \lambda_0) - G^0_u]\, \phi_0 + [G_\lambda(u_0 + \epsilon\phi_j, \lambda_0) - G^0_\lambda]\}$$

$$\text{c)} \qquad c_i(\epsilon) \equiv \psi_i^* \frac{1}{\epsilon} \{[G_u(u_0 + \epsilon\phi_0, \lambda_0) - G^0_u]\, \phi_0 + 2[G_\lambda(u_0 + \epsilon\phi_0, \lambda_0) - G^0_\lambda] + [G_\lambda(u_0, \lambda_0 + \epsilon) - G^0_\lambda]\} \ .$$

† In bifurcation from the trivial solution $[u_1(s), \lambda_1(s)] = [0, s]$ it follows that $\phi_0 \equiv 0$, since $G^0_\lambda = 0$. Hence $c_i \equiv 0$ and the quadratic (5.6a), in the case $m=1$ of simple bifurcation, has the two solutions: $\alpha_1/\alpha_0 = 0$, $\alpha_1/\alpha_0 = -2b_{11}/a_{111}$. Clearly $b_{11} \neq 0$ if (5.6c) holds since $\dot{u}_1(s_0) = 0$.

Clearly $\{a_{ijk}(\epsilon),\ b_{ij}(\epsilon),\ c_i(\epsilon)\} \to \{a_{ijk},\ b_{ij},\ c_i\}$ as $\epsilon \to 0$. The above scheme avoids the need for determining second Fréchet derivatives.

Method II. There are other devices which avoid the need to evaluate the coefficients or roots of (5.5). This assumes that one branch through the bifurcation point has been determined. Then the tangent $[\dot{u}(s_0),\ \dot{\lambda}(s_0)]$ can also be assumed known on this branch. The idea is simply to seek solutions on some subset "parallel" to the tangent but displaced from the bifurcation point in some direction "normal" to the tangent.

For example in the case $m = 1$ of simple bifurcation the known solution branch $[u_1(s),\ \lambda_1(s)]$ has the tangent at the bifurcation, $s = s_0$ given by (5.3b, c). An "orthogonal" to this tangent in the plane spanned by $[\phi_1, 0]$ and $[\phi_0, 1]$ is also given by (5.3b, c) but with α_1 and α_0 replaced by:

(5.8) a) $$\hat{\alpha}_1 = \alpha_0(1 + \|\phi_0\|^2)\ ,\quad \hat{\alpha}_0 = -\alpha_1\|\phi_1\|^2\ .$$

Then we seek solutions in the form:

(5.8) b) $$u_2 = u_1(s_0) + \epsilon[\hat{\alpha}_0\phi_0 + \hat{\alpha}_1\phi_1] + v\ ,$$
$$\lambda_2 = \lambda_1(s_0) + \epsilon\hat{\alpha}_0 + \eta\ .$$

These are to satisfy:

(5.9) $$G(u_2,\ \lambda_2) = 0\ ,$$
$$N(u_2,\ \lambda_2) \equiv (\hat{\alpha}_0\phi_0^* + \hat{\alpha}_1\phi_1^*)v + \hat{\alpha}_0\eta = 0\ .$$

We use Newton's method to solve (5.9) for $v \in \mathbb{B}$ and $\eta \in \mathbb{R}$ with the initial estimate $(v^0, \eta^0) = (0, 0)$. Here ϵ must be taken sufficiently large so that the scheme does not return to $(u_1(s_0), \lambda_1(s_0))$ as the solution.

Method III. Another way to determine a branch bifurcating from a known branch $[u(s),\ \lambda(s)]$ at $s = s_0$ is to apply a constructive existence theory, using iterations, say as in [8, 11]. To sketch the basic idea we consider simple bifurcation and seek the bifurcated branch in the form

(5.10) a) $$u = u_1(\sigma) + \epsilon[\phi_1 + v] \quad , \quad \psi_1^* v = 0 \ ;$$
$$\lambda = \lambda_1(\sigma) \ .$$

Then (1.1) is written, using (5.10a), as

(5.10) b) $$G_u^0 v = G_u^0 v - \frac{1}{\epsilon} G(u_1(\sigma) + \epsilon[\phi_1 + v], \lambda_1(\sigma)) \ ; \ \psi_1^* v = 0 \ .$$

To insure that the right hand side is in $\mathcal{R}(G_u^0)$ we try to pick $\sigma = s$ such that $h(s; \epsilon, v) = 0$ where

(5.10) c) $$h(s;\epsilon, v) \equiv \begin{cases} \psi_1^*[G_u^0 v - \frac{1}{\epsilon} G(u_1(s) + \epsilon[\phi_1 + v], \lambda_1(s))], & \epsilon \neq 0 \ , \\ \psi_1^*[G_u^0 v - G_u(u_1(s), \lambda_1(s))[\phi_1 + v]] \ , & \epsilon = 0 \ . \end{cases}$$

It easily follows that $h(s_0; 0, 0) = 0$ and

(5.10) d) $$h_s^0 \equiv h_s(s_0, 0, 0) = -\psi_1^*[G_{uu}^0 \dot{u}_1(s_0) + G_{u\lambda}^0 \dot{\lambda}_1(s_0)]\phi_1 .$$

Thus as in (5.6), $h_s(s_0; 0, 0) \neq 0$ and so the implicit function theorem yields a root, $s = \sigma(\epsilon, v)$, of $h(s;\epsilon, v) = 0$. We use this root in (5.10b) and then, by contraction maps, it is easily shown that (5.10b) has a unique solution $v = v(\epsilon)$ for $|\epsilon|$ sufficiently small [10].

The main difficulty in applying the above procedure is in solving $h(s;\epsilon, v) = 0$ for s at each iterate $v = v^\nu$, say. Of course if λ occurs linearly in the problem and it is used as the parameter, s, then this is trivial. But when λ occurs nonlinearly as it must for secondary bifurcation, modifications must be introduced. Several have been proposed in [4, 6, 15, 19]. For example given the ν^{th} iterate, (σ^ν, v^ν), we could use the chord method to define $\sigma^{\nu+1}$ as in

(5.11) a) $$m^0 \sigma^{\nu+1} = m^0 \sigma^\nu - h(\sigma^\nu; \epsilon, v^\epsilon) \ , \quad m^0 \equiv \psi_1^* B \ ;$$

and then $v^{\nu+1}$ is obtained from

(5.11) b) $$G_u^0 v^{\nu+1} = G_u^0 v^\nu - \frac{1}{\epsilon} G(u_1(\sigma^\nu) + \epsilon[\phi_1 + v^\nu], \lambda_1(\sigma^\nu)) - B[\sigma^{\nu+1} - \sigma^\nu] \ ; \ \psi_1^* v^{\nu+1} = 0 \ .$$

Applying ψ_1^* to the right hand side above we see that it is in $\mathcal{R}(G_u^0)$. Furthermore with the choice

(5.12) a) $$B = B^0 \equiv [G_{uu}^0 \dot{u}_1(s_0) + G_{u\lambda}^0 \dot{\lambda}_1(s_0)] \ \phi_1$$

it follows from (5.10d) that $m^0 = h_s^0 \neq 0$. There is no difficulty in showing convergence of the above scheme. To avoid the evaluation of

second derivatives we can use the trick in (5.7) and take:

$$(5.12)\ b) \qquad B = B(\epsilon) \equiv G_u(u_1(s_0) + \epsilon\phi_1, \lambda_1(s_0))\, \dot{u}_1(s_0) + G_\lambda(u_1(s_0) + \epsilon\phi_1, \lambda_1(s_0))\, \dot{\lambda}_1(s_0) \ .$$

Clearly $B(\epsilon) = B^0 + \mathcal{O}(\epsilon)$ so that $\psi_1^* B(\epsilon) = h_s^0 + \mathcal{O}(\epsilon)$ and the proofs proceed as before. This modification is due to Rheinboldt [19] where it is justified for finite dimensional problems. He also shows that if the solution branch $[u_1(s), \lambda_1(s)]$ in (5.11) is replaced by the "parabolic" approximation: †

$$(5.13) \qquad \begin{aligned} \hat{u}_1(s) &= u_1(s_0) + (s-s_0)\, \dot{u}_1(s_0) + \tfrac{1}{2}(s-s_0)^2\, \ddot{u}_1(s_0) \\ \hat{\lambda}_1(s) &= \lambda_1(s_0) + (s-s_0)\, \dot{\lambda}_1(s_0) + \tfrac{1}{2}(s-s_0)^2\, \ddot{\lambda}_1(s_0) \end{aligned}$$

the procedure still converges to a bifurcated solution.

Method IV. A final method for determining the bifurcating branch at a simple eigenvalue is to use a technique based on a modification [10] of the Crandall and Rabinowitz [5] proof of bifurcation. Thus we again seek solutions of the form (5.10a) and define

$$(5.14)\ a) \quad g(v, s;\epsilon) \equiv \begin{cases} \dfrac{1}{\epsilon} G(u_1(s) + \epsilon[\phi_1 + v], \lambda_1(s)) \ , & \epsilon \neq 0 \ ; \\ G_u(u_1(s), \lambda_1(s))[\phi_1 + v] \ , & \epsilon = 0 \ ; \end{cases}$$

$$b) \quad N(v, s;\epsilon) \equiv \psi_1^* v \ .$$

Now we note that

$$(5.15)\ a) \qquad g(0, s_0;0) = 0 \ , \quad N(0, s_0;0) = 0$$

and the Frechét derivative at $(v, s;\epsilon) = (0, s_0;0)$ is

$$(5.15)\ b) \qquad \mathcal{A}^0 = \frac{\partial(g, N)}{\partial(v, s)}\bigg|_{(0, s_0;0)} = \begin{pmatrix} G_u^0 & B^0 \\ \psi_1^* & 0 \end{pmatrix} \ ,$$

where B^0 is given in (5.12a). If (5.6c) holds then by our Lemma 2.8 it follows that $\mathcal{A}^0$ is nonsingular. Now the usual implicit function theorem shows that

$$(5.15)\ c) \qquad \begin{cases} g(v, s;\epsilon) = 0 \ , \\ N(v, s;\epsilon) = 0 \ , \end{cases}$$

† In [19] the parameter $s \equiv \lambda$ is employed but the above indicated generalization can be shown to work with no additional difficulties.

has a smooth solution $(v(\epsilon), \sigma(\epsilon))$ for each $|\epsilon| \leq \epsilon_0$ and using this solution in (5.10a) yields the bifurcating branch of solutions.

In solving (5.15c) we never use $\epsilon = 0$ so that even when employing Newton's method second derivatives need not be computed. Further we can use the device of Rheinboldt and replace $[u_1(s), \lambda_1(s)]$ in (5.10a) and (5.14a) by $[\hat{u}_1(s), \hat{\lambda}_1(s)]$ of (5.13). The implicit function theorem still holds as above and now we get a solution $(\hat{v}(\epsilon), \hat{\sigma}(\epsilon))$ of the modified (5.15c) to use in the modified form of (5.10a). The rigorous justification of this procedure is straightforward and obviously the method can be made constructive, [10].

6. Numerical Methods

To apply the previously indicated procedures we must assume that stable, convergent numerical methods are known for approximating the solutions of the linearized problems which arise in Newton's method or in the chord method. Indeed even more is required to rigorously justify the numerical methods used in switching branches at a bifurcation point. We must be assured that bases for $\mathcal{N}(G_u^0)$ and $\mathcal{N}(G_u^{0*})$ can be accurately determined. For the case of simple bifurcation this is not very difficult since the theory of numerical methods for computing eigenfunctions belonging to simple eigenvalues for broad classes of linear operators is well developed. Indeed the only works thus far to justify numerical methods at bifurcation points consider simple bifurcation from a tirvial branch, [2, 22, 23]. We shall not present a general convergence theory here but rather indicate the practical aspects in actually carrying out our procedures.

Basically the problem (2.4) is discretized in some form which we indicate by:

(6.1) a) $$G_h(u_h, \lambda_h) = 0 \quad , \quad N_h(u_h, \lambda_h, s) = 0 \quad .$$

Here (u_h, λ_h) represents the approximation to (u, λ) on the net or family of nets which is parametrized by h. If (2.4) has a smooth isolated solution then under modest assumptions on the consistency of $[G_h, N_h]$ with $[G, N]$ and on the stability and Lipschitz continuity of the linearized difference operators, say $\mathcal{A}_h$, the general theory in [9] assures us that (6.1) has a unique solution which can be computed

by Newton's method. The bulk of the computations occur in this case and we write the difference operators linearized about (u_h, λ_h) say in the form:

$$\mathcal{A}_h \equiv \begin{pmatrix} A_h & b_h \\ c_h^* & d_h \end{pmatrix} \quad . \tag{6.2}$$

From the comparison with $\mathcal{A}$ in (2.6) we see that A_h is a form of difference approximation to G_u. Indeed. A_h is in general a matrix of (large) order h^{-m} when the basic problem is formulated in E^m. Similarly b_h and c_h^* are column and row vectors approximating G_λ and N_u, respectively, while d_h is a scalar approximating N_λ.

The basic computational problem is to solve linear systems in the form

$$\mathcal{A}_h \begin{pmatrix} \delta u_h \\ \delta \lambda_h \end{pmatrix} = \begin{pmatrix} r_h \\ \rho_h \end{pmatrix} \quad . \tag{6.3}$$

To do this we need only determine y_h and z_h satisfying

$$\text{(6.4) a)} \quad A_h y_h = b_h \ , \quad \text{b)} \quad A_h z_h = r_h \ ;$$

and then:

$$\text{(6.4) c)} \quad \delta\lambda_h = (c_h^* z_h + \rho_h)/(d_h - c_h^* y_h) \ , \quad \text{d)} \quad \delta u_h = z_h - \delta\lambda_h y_h \ .$$

Of course we solve (6.4a, b) by some Gaussian elimination procedure with some form of pivoting to get (neglecting the permutations for clarity of presentation)

$$\text{(6.4) e)} \quad A_h = L_h U_h \ .$$

Since the bulk of the computations occur in determining the factorization (6.4e) it follows that our normalization procedure costs very little extra effort. Now we sketch an algorithm for generating "all" the solution branches of (1.1) using the indicated techniques and assuming that only simple bifurcation occurs.

ALGORITHM

i) Using Euler-Newton continuation generate an approximate

solution arc: $x_{1h}(s) \equiv (u_{1h}(s), \lambda_{1h}(s))$, skipping over any "singular" points that are encountered.

ii) Return to the neighborhood of each singular point and locate it accurately (i.e. use false position or bisection to determine the zero, s_{0h}, of $\det A_h(s)$). In particular simple roots or more generally odd order roots are easily sensed by the sign change in $\det A_h$. However one must remember to account for the row or column interchanges in the LU-decomposition.

iii) Test for limit point or bifurcation at each singularity. To do this we need an approximation ψ_{1h}^* to ψ_1^*, the null vector of G_h^{0*}. We do this as we also compute ϕ_{1h} an approximation to ϕ_1 the null vector of G_h^0. This is easily and efficiently done by inverse iteration. In fact if we are really close to a singular point then it suffices to use, say:

$$\text{(6.5)} \quad \begin{aligned} &\text{a)} \quad A_h \phi_h = \phi_h^0 \ , \\ &\text{b)} \quad A_h^* \psi_h = U_h^* L_h^* \psi_h = \phi_h \ ; \end{aligned}$$

where $\phi_h^0 = \delta u_h$ is the last correction in the Newton scheme used to compute $u_h(s_{0h})$. As we already have the LU-decomposition in (6.4) these calculations are not costly.

The test is in the form:

$$\text{(6.6)} \quad \psi_{1h}^* b_h \begin{cases} \doteq 0 \ , & \text{seek bifurcation} \ ; \\ \neq 0 \ , & \text{a limit point.} \end{cases}$$

iv) To switch over to a bifurcating branch we must also compute $[\dot{u}_{1h}(s), \dot{\lambda}_{1h}(s)]$, an approximation to the tangent to the solution branch, $x_{1h}(s)$, at the point s_{0h} best approximating the bifurcation point. However, this will have been computed in step i) or ii) if, as we assume, the normalization N_3 of (2.10e) has been employed. Then we can easily determine ϕ_{0h} an approximation to ϕ_0 of (5.2) as follows. We set

$$\text{(6.7) a)} \quad \alpha_{0h} = \dot{\lambda}_{1h}(s_{0h}) \ , \quad \text{b)} \quad \alpha_{1h} = \psi_{1h}^* \dot{u}_{1h}(s_{0h}) / \psi_{1h}^* \phi_{1h} \ ,$$

and then compute

$$\text{(6.7) c)} \quad \phi_{0h} = \frac{1}{\alpha_{0h}} [\dot{u}_{1h}(s_{0h}) - \alpha_{1h} \phi_{1h}] \ .$$

[Of course if $\alpha_{0h} = 0$ we cannot use this simple procedure. In its place we must use a solution of (6.4a) at $s = s_{0h}$ and subtract a multiple of ϕ_{0h} so that $\psi_{1h}^{*}\,\phi_{0h} = 0$.]

Now to use Method I we approximate $a_{111}(\delta)$ and $b_{11}(\delta)$ of (5.7b) by:

$$\text{a)}\qquad a_{111h}(\delta) = \psi_{1h}^{*}\,\frac{1}{\delta}\big[\,A_h(u_{1h}(s_{0h}) + \delta\phi_{1h},\ \lambda_{1h}(s_{0h})) - A_h(u_{1h}(s_{0h}),\ \lambda_{1h}(s_{0h}))\big]\,\phi_{1h}$$

$$\text{b)}\qquad b_{11h}(\delta) = \psi_{1h}^{*}\,\frac{1}{\delta}\big[\,A_h(u_{1h}(s_{0h}) + \delta\phi_{1h},\ \lambda_{1h}(s_{0h})) - A_h(u_{1h}(s_{0h}),\ \lambda_{1h}(s_{0h}))\big]\,\phi_{0h} + \psi_{1h}^{*}\,\frac{1}{\delta}\big[\,b_h(u_{1h}(s_{0h})+\delta\phi_{1h},\ \lambda_{1h}(s_{0h})) - b_h(u_{1h}(s_{0h}),\ \lambda(s_{0h}))\big]\ . \tag{6.8}$$

Then we approximate the other root of (5.6a) by $[\bar{\alpha}_{0h},\ \bar{\alpha}_{1h}]$ where

$$\text{(6.8) c)}\qquad \bar{\alpha}_{1h}/\bar{\alpha}_{0h} = -\left(\frac{\alpha_{1h}}{\alpha_{0h}} + \frac{2b_{11h}(\delta)}{a_{111h}(\delta)}\right)\ .$$

The tangent to the bifurcated branch is approximated by:

$$\text{(6.9)}\qquad \dot{u}_{2h}(s_{0h}) = \bar{\alpha}_{1h}\,\phi_{1h} + \bar{\alpha}_{0h}\,\phi_{0h}\ ,\quad \dot{\lambda}_{2h}(s_{0h}) = \bar{\alpha}_{0h}\ .$$

Using (6.9) in the normalization N_3 we simply return to step i) and generate the bifurcating branch.

To use Method II we proceed as before but do not bother with (6.8). Rather we approximate $[\hat{\alpha}_0,\ \hat{\alpha}_1]$ of (5.8a) by:

$$\text{(6.10) a)}\qquad \hat{\alpha}_{0h} = -\,\alpha_{1h}\|\phi_{1h}\|^2\ ,\quad \hat{\alpha}_{1h} = \alpha_{0h}(1 + \|\phi_{0h}\|^2)\ .$$

Then we seek a solution of (6.1) in the form:

$$\text{(6.10) b)}\qquad \begin{aligned} u_h &= u_{1h}(s_{0h}) + \epsilon[\,\hat{\alpha}_{0h}\phi_{0h} + \hat{\alpha}_{1h}\phi_{1h}] + v_h\ ,\\ \lambda_h &= \lambda_{1h}(s_{0h}) + \epsilon\,\hat{\alpha}_{0h} + \eta_h\ ; \end{aligned}$$

where $N_h(\cdot)$ is taken as:

$$\text{(6.10) c)}\qquad N_h(u_h, \lambda_h) \equiv (\hat{\alpha}_{0h}\phi_{0h}^{*} + \hat{\alpha}_{1h}\phi_{1h}^{*})v_h + \hat{\alpha}_{0h}\,\eta_h\ .$$

Once a solution $[v_h,\ \eta_h]$ is obtained a new tangent vector is computed and we return to step i) using the normalization N_3. Indeed the above indicated computations involve but minor modifications from the

procedure of step i).

Method III has been discussed in more detail by Rheinboldt [19]. It has also been used for bifurcation from the trivial solution in [15, 22, 23].

To our knowledge the new Method IV has not yet been used in actual calculations. However it is in the process of being tested at the present time.

7. A Simple Example.

We have used the procedures of §6 on several examples of the form:

$$u_{xx} + f(x, u;\lambda) = 0 \ , \quad u(0) = u(1) = 0 \ , \tag{7.1 a}$$

where

$$f(x, u;\lambda) \equiv 2\, q(\lambda) + \pi^2 \lambda\, p(u - q(\lambda)x(1-x)) \ . \tag{7.1 b}$$

If $p(0) = 0$ then a solution of (7.1) is given by

$$u_1(x, \lambda) = q(\lambda)\, x(1-x) \ , \quad \lambda = \text{arb} \ . \tag{7.2}$$

The linearized problem about $u_1(x)$ is

$$\phi_{xx} + \pi^2 \lambda p_u(0)\phi = 0 \ , \quad \phi(0) = \phi(1) = 0 \ . \tag{7.3}$$

Thus if $p_u(0) = 1$ the eigenvalues of (7.3) are

$$\lambda_k = k^2 \ , \quad k = 1, 2, \ldots \ . \tag{7.4}$$

We have used several choices for $q(\lambda)$ and $p(z)$ but we show here, in figure 3, the computed results for the choice

$$q(\lambda) \equiv \lambda^2\, e^{-\lambda/2} \ , \quad p(z) \equiv z + z^2 \ . \tag{7.5}$$

The difference scheme used was the Collatz Mehrstellenverfahren which is $\mathcal{O}(h^4)$ accurate and the net spacing was $h = 1/20$. The procedures of §6 were applied using the pseudoarclength normalizations N_2 and N_3 and Method II was used to switch branches at bifurcation points. (The norm, $\|x\|_2$ was allowed to go "negative" if it went to zero along a branch; thus Γ_1 is a smooth curve on figure 3.) The only initial guess required was $[u, \lambda] \equiv [0, 0]$ employed near $\lambda = 0$ to start the Euler-Newton continuation on the branch Γ_0 which is the

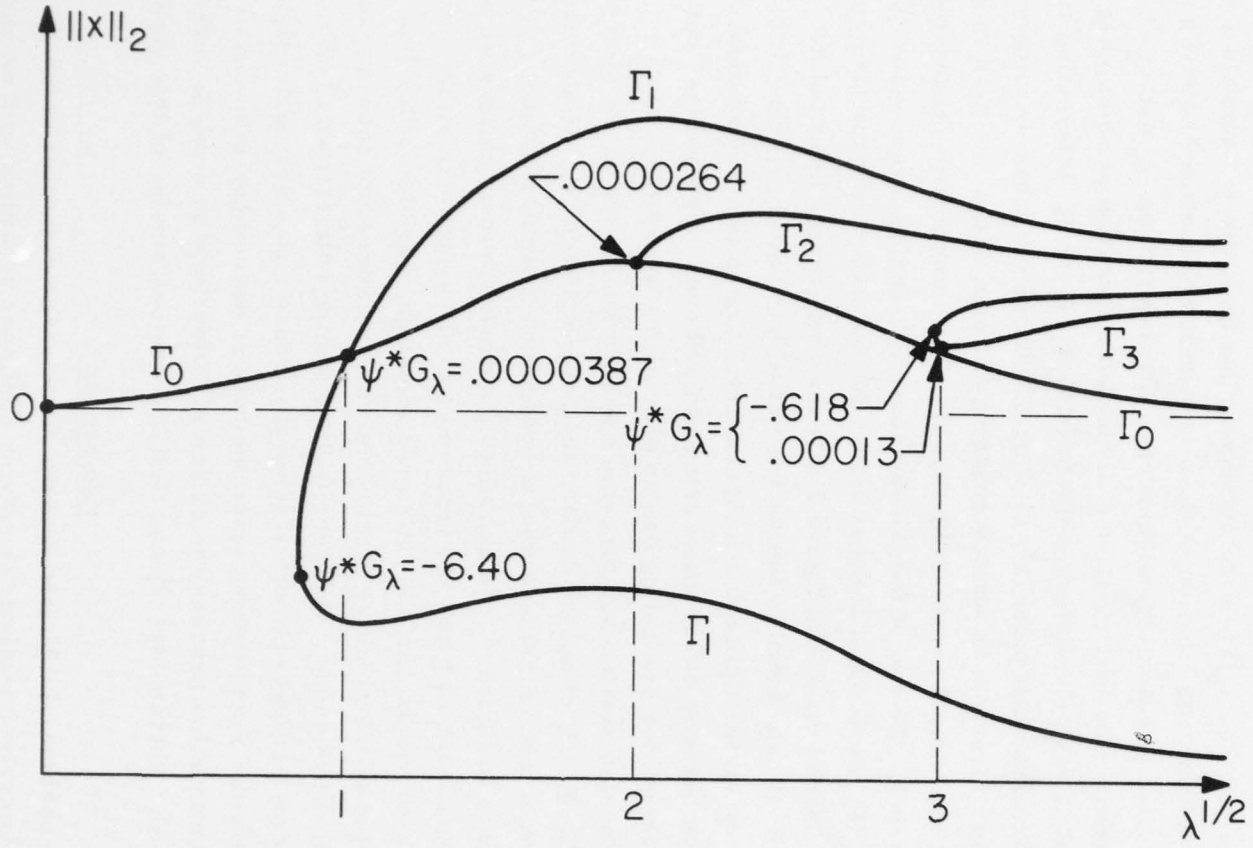

$\|x\|_2$
Γ_1
.0000264
Γ_2
Γ_3
Γ_0
$\psi^* G_\lambda = .0000387$
0
$\psi^* G_\lambda = \begin{cases} -.618 \\ .00013 \end{cases}$
Γ_0
$\psi^* G_\lambda = -6.40$
Γ_1
1
2
3
$\lambda^{1/2}$

Figure 3

basic solution $u_1(x, \lambda)$. This branch was easily computed and sign changes in $\det A_h(\lambda)$ were noted near the first three eigenvalues. Upon refining the location of these potential bifurcation or limit points, using false position, the eigenvector approximations ϕ_h and ψ_h were computed. The test $\psi^* G_\lambda = 0$ indicated bifurcation at each point and Method II easily switched to the branch Γ_1, Γ_2 or Γ_3 bifurcating from the corresponding point $\lambda^{\frac{1}{2}} = 1, 2$ or 3. These branches were extended in either direction by simply changing the sign of the ϵ used in defining the normal vector in Method II. Then continuation generated the branch. On two of these extensions, Γ_1 and Γ_3, new zeros of $\det A_h(\lambda)$ were found but they failed the necessary test for bifurcation. Indeed we see in figure 3 that they are simple limit points. The branch Γ_2 as shown in figure 3 is actually covered twice but it does not show on the figure due to symmetry. There is a fundamental difference between bifurcations from odd and even "eigenvalues" but our scheme for computing has no difficulties with either case.

We also started our procedure at a remote point on the bifurcated branch Γ_1. It of course located the basic solution Γ_0 as a bifurcation from this branch and then proceeded to find the remainder of the branches in figure 3. To completely automate our procedure we would have to devise step control techniques to allow optimum steps in the arclength parameter, s. Also net selection, variable order (via deferred corrections or Richardson extrapolation) and accuracy tests should be included. However further testing with Methods I-IV should be carried out before general purpose codes are seriously contemplated. Furthermore since the bulk of the computations occur in the continuation process the choice of Euler-Newton must be reconsidered. Rheinboldt [18] has initiated serious studies of this question.

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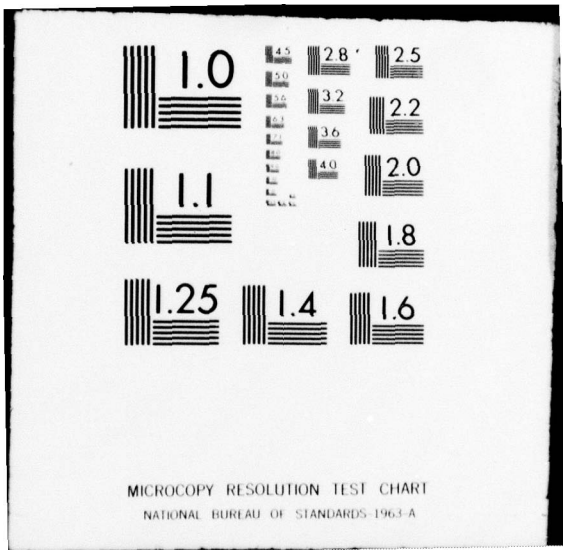
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