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# EXAMPLES OF RATE-THEORY CONSTITUTIVE EQUATIONS WHICH UNIFY ELASTICITY AND PLASTICITY

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Examples of rate theory types of constitutive relations are developed which allow as solutions both elastic and plastic stress-strain relations.

Elastic behavior is associated with uniqueness of solutions and plastic behavior with non-uniqueness of solutions of ordinary differential equations. Transition from one elastic solution to another elastic solution through a plastic solution allows the description both of loading and unloading by the same set of constitutive differential equations.


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# I. INTRODUCTION

In the theory of the elastic-perfectly plastic solid[1,2] two distinct types of constitutive equations are usually applied to the same material: one type is used in the elastic range and the other type is used in the plastic range. A similar statement could be made for an elastic-plastic strain-hardening material. Our purpose here is to show, by example, that a single set of equations of the rate type can describe behavior both in the elastic and in the plastic range. Furthermore, these equations can describe both loading and unloading. Indeed, the mathematical machinery needed to describe elastic-plastic behavior is contained in the classical theory of ordinary differential equations. We shall show how elastic behavior can correspond to uniqueness of solutions of such equations; how nonuniqueness of solution can correspond to plastic flow, and how the possibility of different constants of integration can explain why an elastic unloading curve can differ from a previous elastic loading curve.

# II. A ONE-DIMENSIONAL EXAMPLE

Before proceeding to our proper three dimensional example, we shall present a one-dimensional example in order to exhibit how some of the mathematical notions from ordinary differential equations theory apply. We consider a single, strain component $\gamma$ and a single non-dimensionalized stress component s which will correspond in our three-dimensional example to shear strain and shear stress respectively in a simple shearing motion. We assume that stress-rate is related to strain-rate and stress through an equation of the form

$$\frac{ds}{dt} = \frac{1}{2} \sqrt{1 - 2^{n} s^{2n}} \; \frac{d\gamma}{dt} , \tag{1}$$

where t is time and n is some positive integer.*

*We have chosen to work in terms of non-dimensional stress in order to avoid the necessity of defining symbols for material constants, which the reader then is required to keep straight. A reformulation in terms of such constants can readily be made when needed.*

[1] *Prager, W. and Hodge, P. G., "Theory of Perfectly Plastic Solids," Dover, New York, L. C. # 68-19164 (1968).*

[2] *Thomas, T. Y., "Plastic Flow and Fracture in Solids," Academic Press, New York, L. C. # 61-12277 (1961).*

In the one-dimensional case currently under discussion, we shall find it convenient to eliminate t and write merely for (1)

$$\frac{ds}{d\gamma} = \frac{1}{2}\sqrt{1 - 2^n s^{2n}} \quad . \tag{2}$$

For simplicity, we shall first discuss the case n = 1. In this case the solutions of (2) are

$$s = \frac{1}{\sqrt{2}} \sin\left(\frac{\gamma}{\sqrt{2}} + c\right), \quad c - \frac{\pi}{2} \leq \frac{\gamma}{\sqrt{2}} \leq c + \frac{\pi}{2} \tag{3}$$

$$s = \frac{1}{\sqrt{2}}, \quad s = -\frac{1}{\sqrt{2}} \quad . \tag{4}$$

The restrictions on the range of $\gamma$ in which the solutions (3) are valid are necessary since (2) requires that $ds/d\gamma$ be non-negative.* The solutions (4), in which s takes one of two constant values, may seem rather special. However, they play an essential role in our formulation.

Imagine now that starting with $\gamma = 0$, $s = 0$, one begins to increase $\gamma$. The only solution of (4) (with n - 1) possible with these initial conditions is

$$s = \frac{1}{\sqrt{2}} \sin\frac{\gamma}{\sqrt{2}} \quad . \tag{5}$$

As $\gamma$ is increased, s will be given by (5) and will continue to be given by (5) as long as $\gamma$ remains less than $\sqrt{2}\,\pi/2$. Indeed, the Piccard-Lindelof uniqueness theorem[3] assures us of this, since a Lipschitz condition will hold when $-1/\sqrt{2} < s < 1/\sqrt{2}$. Indeed, as long as $|s|$ does not reach $1/\sqrt{2}$, s will follow the elastic solution (5) whether $\gamma$ is increased, decreased or otherwise varied.

---

*Of course we can just as well let $\gamma/\sqrt{2}$ lie in a range*

$$2m\pi + c - \frac{\pi}{2} \leq \frac{\gamma}{\sqrt{2}} \leq 2m\pi + c + \frac{\pi}{2} \quad \text{for some integer } m \text{ other than zero.}$$

*However this would just introduce an unnecessary redundancy, since $2\pi m$ can be absorbed into the constant c.*

[3] *Coddington, E. A. and Levinson, N., "Theory of Ordinary Differential Equations," McGraw-Hill, New York (1955).*

Now imagine that $\gamma$ is increased until it reaches the value $\pi\sqrt{2}/2$ and is thenceforth increased further. The stress will have then reached the value $1/\sqrt{2}$ at which the Lipschitz condition fails, and so uniqueness will no longer be implied by the Piccard-Lindelof theorem. Since no real solutions of (2) (with $n = 1$) exist for $s > 1/\sqrt{2}$, and since s cannot decrease when $\gamma$ is increasing, it follows that s will remain at the value $1/\sqrt{2}$ as long as $\gamma$ is not decreasing. In other words, in loading, after s has reached $1/\sqrt{2}$, the first solution (4) takes over. This is a plastic yield solution, and it must prevail at the critical value of s, namely $s = 1/\sqrt{2}$. This process is illustrated in Figure 1.

Now suppose that yielding proceeds as described above until the strain has reached the value $\gamma_1 > \pi\sqrt{2}/2$. Suppose that at this point the strain begins to decrease. The differential equation (2) then allows the stress to follow more than one solution: It allows the solution

$$s = \frac{1}{\sqrt{2}} \sin \frac{\gamma-\gamma_1 + \pi\sqrt{2}/2}{\phantom{x}} , \qquad (6)$$

but it now also allows the solution

$$s = 1/\sqrt{2} , \qquad (7)$$

since both for (6) and (7), $ds/d\gamma \geq 0$ when $\gamma$ is decreasing and s is not increasing. However, note that for decreasing $\gamma$, (7) is an asymptotically unstable solution in the sense that a small perturbation in s will cause the solution to move onto one of the forms (3) which then takes s far from $1/\sqrt{2}$ as $\gamma$ decreases further. However, (6) is neutrally stable in the sense that a small perturbation in s will not cause an increasing divergence of the perturbed solution from (6). If we make the assumption, then, that the stable solution will be chosen, then we see that s will follow the solution (6), as shown in Figure 2. As soon as s is below $1/\sqrt{2}$, the $(s,\gamma)$ values will have again entered the region of uniqueness of the differential equation (2) and, thus, will stay on the curve (6) whether $\gamma$ is decreased or even increased again as long as the increase or decrease is not so large as to bring s to one of the yield values $\gamma = \pm 1/\sqrt{2}$. There is a return to elastic behavior about a new stress free state, that for which $\gamma = \gamma_1 - \sqrt{2}\,\pi/2$.

*In our formulation then, elastic behavior is thus related to uniqueness of solutions of a differential equation. Different elastic regimes correspond to different constants of integration. Plastic behavior is related to nonuniqueness.*

Let us now go a bit further with our example. Suppose that we continue decreasing till we reach $s = -1/\sqrt{2}$. Again there will be plastic yield (see Figure 3); s cannot decrease further and cannot increase as long as $\gamma$ is decreasing. Thus s will follow the solution $s = -1/\sqrt{2}$ until $\gamma$ is made to increase again, after which it will follow another one of the solutions (3). It will enter another elastic regime. If $\gamma$ is increased and then decreased in a periodic manner over several periods beyond the point that s reaches the critical value $1/\sqrt{2}$ on each increase and $-1/\sqrt{2}$ on each decrease, the solutions will follow a hysteresis loop (see Figure 4).

Now one objection to the above argument is that in the elastic regimes the $(\gamma, s)$ diagram follows a sine curve, whereas it would be preferable to have it follow a straight line until it is near yield and then curve abruptly to join the yield curve, say $s = 1/\sqrt{2}$. Such behavior can be realized by choosing a large enough value of n, as we shall illustrate now. Indeed, solutions to (2) are given by

$$\frac{\gamma}{\sqrt{2}} + c = \pm \frac{1}{2n} B_{2^n s^{2n}} \left( \frac{1}{2n}, \frac{1}{2} \right) \tag{8}$$

where $B_x(a,b)$ is the incomplete beta function, and where the plus sign is used if $s \geq 0$ and the minus sign if $s \leq 0$. (The function so defined is analytic for $-1 < \sqrt{2}\, s < 1$). Yield occurs at $s = \pm 1/\sqrt{2}$.

Indeed, the behavior is qualitatively the same as in the example for $n = 1$, except that linear elastic-perfectly plastic behavior joined by a transition knee is approximated for large n.

In Figures 5, 6 and 7 we show curves of stress versus strain increasing beyond yield for $n = 16$, 32 and 64.

## III. A THREE-DIMENSIONAL EXAMPLE - ELASTIC-PERFECTLY PLASTIC SOLID-PRELIMINARY ANALYSIS

We shall treat here a three-dimensional example which corresponds to an elastic-perfectly plastic solid undergoing infinitesimal strain. The solid will have a strain energy function valid for each of its elastic regimes. It will exhibit yield in the form of nonuniqueness of solutions of its governing rate-type constitutive equations. And, again, elastic behavior will be associated with uniqueness of solutions of ordinary differential equations and plastic behavior with nonuniqueness. Moreover, in the plastic regime, it will follow the Prandtl-Reuss equations, not as a new assumption, but as a consequence of the governing differential equations.

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[4] *Abramowitz, M. and Stegun, I. A., "Handbook of Mathematical Functions," U.S. Government Printing Office, Washington, D.C., L. C. # 64-60036 (1964).*

We shall write $e_{ij}$ for the infinitesmal strain tensor and $\sigma_{ij}$ for the stress tensor. The strain-deviator tensor $\varepsilon_{ij}$ and the stress-deviator tensor $s_{ij}$ are defined by

$$\varepsilon_{ij} = e_{ij} - \frac{1}{3} e_{kk}\delta_{ij} \tag{9}$$

$$s_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk}\delta_{ij} \quad . \tag{10}$$

We define the non negative quantities $\alpha$ and $\beta$ by

$$\alpha^2 = \varepsilon_{ij}\varepsilon_{ij} \tag{11}$$

$$\beta^2 = s_{ij}s_{ij} \quad . \tag{12}$$

We assume a strain energy of the form

$$W = \frac{K}{2} e_{kk}^2 + \phi(\alpha) \quad , \tag{13}$$

where $\phi$ is a function such that $\phi' > 0$ in some neighborhood of $\alpha = 0$ and $\phi'(\alpha)/\alpha$ has a well defined limit as $\alpha \to 0$, and, of course

$$\sigma_{ij} = \frac{\partial W}{\partial e_{ij}} \quad . \tag{14}$$

Note that (11) gives

$$\alpha \frac{\partial \alpha}{\partial \varepsilon_{ij}} = \varepsilon_{ij} \quad . \tag{15}$$

Furthermore, (9) gives

$$\frac{\partial \varepsilon_{ij}}{\partial \varepsilon_{k\ell}} = \delta_{ik}\delta_{j\ell} - \frac{1}{3} \delta_{k\ell}\delta_{ij} \quad . \tag{16}$$

Thus (15) and (16) yield

$$\alpha \frac{\partial \alpha}{\partial \alpha_{k\ell}} = \alpha \frac{\partial \alpha}{\partial \varepsilon_{ij}} \frac{\partial \varepsilon_{ij}}{\partial e_{k\ell}} = \varepsilon_{ij} \left[\delta_{ik}\delta_{j\ell} - \frac{1}{3} \delta_{k\ell}\delta_{ij}\right] = \varepsilon_{k\ell} = \alpha \frac{\partial \alpha}{\partial \varepsilon_{k\ell}} . \tag{17}$$

We obtain, then, from (13), (14), and (17),

$$\sigma_{ij} = Ke_{kk}\delta_{ij} + \frac{\phi'(\alpha)}{\alpha} \varepsilon_{ij} , \tag{18}$$

whence, using (9) and (10), we have

$$\sigma_{kk} = 3Ke_{kk} \tag{19}$$

$$s_{ij} = \frac{\phi'(\alpha)}{\alpha} \varepsilon_{ij} . \tag{20}$$

Squaring both sides of (20), summing and applying (12) and (13), we get

$$\beta^2 = s_{ij}s_{ij} = \frac{[\phi'(\alpha)]^2}{\alpha^2} \varepsilon_{ij} \varepsilon_{ij} = [\phi'(\alpha)]^2 \tag{21}$$

or

$$\beta = \phi'(\alpha) . \tag{22}$$

(Note, we have assumed that $\phi'(\alpha) > 0$ in some neighborhood of $\alpha = 0$ so that (22) will follow from (21) there. Remember that $\beta \geqq 0$).

Let the solution to (22) be given by

$$\alpha = \psi \beta . \tag{23}$$

Then from (20), (22), and (23), we get

$$s_{ij} = \frac{\beta}{\psi(\beta)} \varepsilon_{ij} \quad , \tag{24}$$

or

$$\frac{\psi(\beta)}{\beta} s_{ij} = \varepsilon_{ij} \quad . \tag{25}$$

Differentiation of (25) with respect to time (where dot denotes time differentiation) yields

$$\frac{\psi'(\beta)\dot{\beta}}{\beta} s_{ij} - \frac{\psi(\beta)\dot{\beta}}{\beta^2} s_{ij} + \frac{\psi(\beta)}{\beta} \dot{s}_{ij} = \dot{\varepsilon}_{ij} \quad . \tag{26}$$

Multiply both sides of (26) by $s_{ij}$ and sum to get

$$\frac{\psi'(\beta)\dot{\beta}}{\beta} s_{ij}s_{ij} - \frac{\psi(\beta)\dot{\beta}}{\beta^2} s_{ij}s_{ij} + \frac{\psi(\beta)}{\beta} s_{ij}\dot{s}_{ij} = s_{ij}\dot{\varepsilon}_{ij} \quad . \tag{27}$$

Now (12) yields by differentiation

$$\beta\dot{\beta} = s_{ij}\dot{s}_{ij} \quad . \tag{28}$$

Use (12) and (28) in (27) to obtain

$$\frac{\psi'(\beta)}{\beta} \dot{\beta} \beta^2 = s_{ij}\dot{\varepsilon}_{ij} \quad ,$$

or

$$\dot{\beta} = \frac{1}{\beta\psi'(\beta)} s_{k\ell}\dot{\varepsilon}_{k\ell} \quad . \tag{29}$$

If we now substitute the right hand side of (29) for $\dot{\beta}$ in (26) and then solve for $\dot{s}_{ij}$, we obtain

$$\dot{s}_{ij} = \frac{\beta}{\psi(\beta)} \dot{\epsilon}_{ij} + \frac{1}{\beta^2} \left[ \frac{1}{\psi'} - \frac{\beta}{\psi} \right] s_{ij} s_{k\ell} \dot{\epsilon}_{k\ell} \quad . \tag{30}$$

At this juncture, we make a fundamental new assumption which is the basis of our departure. We assume that the basic constitutive equations are (19) and (30) instead of (19) and (20). *We consider a material of the rate type for which the differential equations (30) are the governing constitutive relations between stress deviator and strain deviator.*

Thus behavior described by (20) can occur as one possible solution of (30). Other solutions are obtained by replacing $\epsilon_{k\ell}$ by $\epsilon_{k\ell} + c_{k\ell}$ in (20), where $c_{k\ell}$ is a traceless constant tensor: this can be seen immediately from (30) since the time derivatives of $\epsilon_{k\ell}$ and $\epsilon_{k\ell} + c_{k\ell}$ are the same and the strain deviator enters (30) only through its derivatives.

Furthermore, equation (29), which can be derived from (30) by multiplying both sides by $s_{ij}$ and summing, plays a critical role in the transition to the plastic regime. We shall now show, by specific examples, how with certain choices of function $\psi(\beta)$, the equation (29) will govern the onset of plastic yield.

## IV. THREE-DIMENSIONAL EXAMPLES OF ELASTIC-PERFECTLY PLASTIC SOLIDS GOVERNED BY A SINGLE EQUATION

We now seek examples of functions $\psi$, such that equation (30) will give elastic-perfectly plastic behavior. Please note that although in our derivation in Section III above, we used (29) to get to (30), in fact it is not difficult to establish that (29) also follows from (30).

Now let us observe that if we had

$$\frac{1}{\psi'(\beta)} = \sqrt{1 - \beta^2} \quad , \tag{31}$$

then (29) would read

$$\frac{d\beta^2}{dt} = 2\sqrt{1-\beta^2}\, s_{k\ell}\dot{\varepsilon}_{k\ell} \quad . \tag{32}$$

This would then imply the von Mises yield condition. Indeed if $\beta^2$ rises to the value unity, then (32) implies that as long as positive work is being done by the shearing stresses, $\beta^2$ cannot decrease and thus must remain at unity.

Now (31) gives us a differential equation for $\psi(\beta)$. Integration of this equation, with the constant of integration chosen so that $\psi(\beta)/\beta$ has a removable singularity, and is thus well defined, at $\beta = 0$ gives

$$\psi(\beta) = \sin^{-1}\beta \quad . \tag{33}$$

(To see why we want $\psi(\beta)/\beta$ to be well defined at $\beta = 0$, see how this expression figures in (25)).

If (33) is then used for $\psi$, (30) becomes

$$\dot{s}_{ij} = \frac{1}{\beta^2}\left[\sqrt{1-\beta^2} - \frac{\beta}{\sin^{-1}\beta}\right] s_{ij}s_{k\ell}\dot{\varepsilon}_{k\ell} + \frac{\beta}{\sin^{-1}\beta}\dot{\varepsilon}_{ij} \quad . \tag{34}$$

Observe then, that for an assignment of $\varepsilon_{ij}$ as a continuously differentiable function of time (or even as a piecewise smooth function of time) the Piccard-Lindelof theorem implies a unique solution $s_{ij}(t)$ of (34) provided $\beta^2 < 1$. If as remarked above, $\beta^2$ reaches unity with positive shear work continuing to be put in, then $\beta^2$ remains equal to unity, and we have plastic flow with a von Mises yield condition. During plastic yield, we put $\beta = 1$ in (34) to obtain

$$\dot{s}_{ij} = \frac{-2}{\pi} s_{ij}s_{k\ell}\dot{\varepsilon}_{k\ell} + \frac{2}{\pi}\dot{\varepsilon}_{ij} \quad . \tag{35}$$

*Note that the equations (35) are the Prandtl-Reuss equations.* (The reader is reminded that the material constants have been given specific values for simplicity. Otherwise they should appear with arbitrary values in the Prandtl-Reuss equations).

Next let us look at the solutions to (34) in the region of uniqueness, $\beta^2 < 1$. (i.e., Let us look at solutions in the elastic regime.) If we compare (33) and (23) and (22), we get

$$\beta = \sin \alpha = \phi'(\alpha) \quad , \tag{36}$$

whence

$$\phi(\alpha) = 1 - \cos \alpha, \tag{37}$$

if we choose the constant of integration of the equation (36) for $\phi$ so that $\phi(0) = 0$. (Since $\phi(\alpha)$ figures in the strain energy function, to which one can add an arbitrary constant without affecting the stress-strain relations, this choice of the constant of integration is merely a matter of taste: it insures that $\phi(\alpha)$ gives a contribution to the strain-energy which is non negative and vanishes at $\alpha = 0$.)

Now (13) with the aid of (37) gives

$$W = \frac{K}{2} (e_{kk})^2 + 1 - \cos \alpha \tag{38}$$

whence (20) yields

$$\delta_{ij} = \frac{\sin \alpha}{\alpha} \varepsilon_{ij} \quad . \tag{39}$$

Now, (39) is a particular solution of (34). To get the general elastic solution, we can, according to the remarks at the end of Section III, replace $\varepsilon_{ij}$ by $\varepsilon_{ij} + c_{ij}$ where $c_{ij}$ is a traceless constant tensor. Thus, we obtain for the elastic solutions of (34)

$$\delta_{ij} = \frac{\sin \sqrt{(\varepsilon_{k\ell} + c_{k\ell})(\varepsilon_{k\ell} + c_{k\ell})}}{\sqrt{(\varepsilon_{k\ell} + c_{k\ell})(\varepsilon_{k\ell} + c_{k\ell})}} (\varepsilon_{ij} + c_{ij}) \quad , \tag{40}$$

where $c_{kk} = 0$. Furthermore, with the restriction $\beta^2 \leq 1$, (36) yields

$$-\frac{\pi}{2} \leq \sqrt{(\varepsilon_{k\ell} + c_{k\ell})(\varepsilon_{k\ell} + c_{k\ell})} \leq \frac{\pi}{2}$$

as a constraint on the region of validity of (40) for given $c_{k\ell}$. The strain energy corresponding to a given elastic regime is then

$$w = \frac{K}{2}(e_{kk})^2 + 1 - \cos\sqrt{(\epsilon_{k\ell} + c_{k\ell})(\epsilon_{k\ell} + c_{k\ell})} \quad . \tag{41}$$

Thus for each elastic regime, we have a strain energy function (41). The onset of plasticity is governed by equation (32). During plastic flow we have the Prandtl-Reuss equations (35). And all of these conclusions are consequences of the single set of constitutive equations (34).

We now consider simple shear for which $s_{12} = s_{21} \stackrel{\Delta}{=} s$, $s_{ij} = 0$ otherwise, $\epsilon_{12} = \epsilon_{21} = \frac{\gamma}{2}$, $\epsilon_{ij} = 0$ otherwise. We obtain from (12)

$$\beta^2 = s_{12}^2 + s_{21}^2 = 2s^2. \tag{42}$$

We also find that

$$s_{ij}\,\dot{\epsilon}_{ij} = s\dot{\gamma}. \tag{43}$$

If we put (42) and (43) into (32), we obtain equation (1) with $n = 1$, and thus the example of Section II.

For other values of $n$, let us put

$$\psi'(\beta) = \frac{1}{\sqrt{1 - \beta^{2n}}} \tag{44}$$

into (30). We then get from

$$\psi(\beta) = B_{\beta^{2n}}\left(\frac{1}{2n}, \frac{1}{2}\right) \quad . \tag{45}$$

For this our analysis holds in a manner quite similar to that developed above. Equation (29) gives

$$\frac{d}{dt}\beta^2 = 2\sqrt{1 - \beta^{2n}}\, s_{k\ell}\,\dot{\epsilon}_{k\ell} \tag{46}$$

so that again $\beta^2$ cannot increase beyond unity as long as positive shear work is being done and a von Mises yield condition holds. During yield one obtains from (30) and (44) the Prandtl-Reuss equations

$$s_{ij} = \frac{1}{\psi(1)} \dot{\epsilon}_{ij} - \frac{1}{\psi(1)} s_{ij} s_{k\ell} \dot{\epsilon}_{k\ell} \quad . \tag{47}$$

During an elastic regime, one gets a stress-strain relation of the form

$$s_{ij} = \frac{\phi' \sqrt{(\epsilon_{k\ell} + c_{k\ell})(\epsilon_{k\ell} + c_{k\ell})}}{\sqrt{\epsilon_{k\ell} \quad c_{k\ell} \quad \epsilon_{k\ell} \quad c_{k\ell}}} (\epsilon_{ij} + c_{ij}) \ , \tag{48}$$

where $c_{kk} = 0$ and $\epsilon_{ij}$ is restricted to the domain

$$\sqrt{(\epsilon_{k\ell} + c_{k\ell} + (\epsilon_{k\ell} + c_{k\ell})} < \psi(1) \quad , \tag{49}$$

and $\phi(\alpha)$ is obtained from $\psi(\beta)$ by the relations (22) and (23). The stress versus strain in a simple shear regime is shown in Figures 5, 6, and 7 for different values of n. The inverse of the elastic modulus, namely $\psi(\beta)/\beta$ is shown plotted versus $\beta$ for various values of n in Figures 8, 9 and 10. Note that for $n = 16$, our calculations show that to five decimal place accuracy, it does not deviate from unity until $\beta$ exceeds 0.8 and it achieves a maximum value of 1.0411 at yield, i.e., at $\beta = 1$. (It is constant to within 4.2% through the whole elastic range.) For $n = 32$, the inverse of the modulus remains one to five decimal places till $\beta > .92$ and it has the value 1.0207 at yield. For $n = 64$ it remains unity to five decimal places until $\beta > .96$, and it has a yield value of 1.0104. (It is constant to within about 1% in the whole elastic range.) We see then that with large enough values of n, one may approximate as closely as one wishes the type of elastic-perfectly plastic behavior in which there is a sudden break in the slope of the stress-strain curve at yield.

## V. STRAIN HARDENING, AN EXAMPLE

We shall now discuss the extension of the foregoing ideas to a strain-hardening material. To this end, consider any one of the functions $\phi(\alpha)$ discussed above and replace (13) by

$$w = \frac{K}{2} (e_{kk})^2 + \frac{\kappa}{2}\alpha^2 + \phi(\alpha) \quad , \tag{50}$$

where $\kappa$ is a material constant.

(For example, we may take $\phi(\alpha) = 1 - \cos\alpha$ in (50). Then, using (14), instead of (20), we obtain

$$\dot{s}_{ij} = \kappa \dot{\varepsilon}_{ij} + \frac{\phi'(\alpha)}{\alpha} \varepsilon_{ij} \quad . \tag{51}$$

If we define $p_{ij}$ by

$$p_{ij} = s_{ij} - \kappa\varepsilon_{ij} \quad , \tag{52}$$

then we have from (51) and (52)

$$p_{ij} = \frac{\phi'(\alpha)}{\alpha} \varepsilon_{ij} \quad . \tag{53}$$

If we now define $\zeta$ by

$$\zeta^2 = p_{ij}\, p_{ij} \quad , \tag{54}$$

we obtain instead of (30) the equation

$$\dot{p}_{ij} = \frac{\zeta}{\psi(\zeta)} \dot{\varepsilon}_{ij} + \frac{1}{\zeta^2} \left[\frac{1}{\psi'} - \frac{\zeta}{\psi}\right] p_{ij}\, p_{k\ell}\, \dot{\varepsilon}_{k\ell} \tag{55}$$

or

$$\dot{s}_{ij} = \kappa\dot{\varepsilon}_{ij} + \frac{\zeta}{\psi} \dot{\varepsilon}_{ij} + \frac{1}{\zeta^2} \left[\frac{1}{\psi'} - \frac{\zeta}{\psi}\right] (s_{ij} - \kappa\varepsilon_{ij})(s_{k\ell} - \kappa\varepsilon_{k\ell}) \dot{\varepsilon}_{k\ell} . \tag{56}$$

We obtain instead of (29)

$$\dot{\zeta} = \frac{1}{\zeta\psi'(\zeta)} p_{k\ell}\, \dot{\varepsilon}_{k\ell} \quad . \tag{57}$$

If $\psi(\zeta)$ is taken such that

$$\psi'(\zeta) = (1 - \zeta^{2n})^{-1/2} \quad , \tag{58}$$

we see that we must have $\zeta^2 \leq 1$ and yield occurs when $\zeta = \pm 1$. During yield we get the flow rule

$$\dot{s}_{ij} = \kappa \dot{\epsilon}_{ij} + \frac{1}{\psi(1)} \dot{\epsilon}_{ij} - \frac{1}{\psi(1)} (s_{ij} - \kappa \epsilon_{ij})(s_{k\ell} - \kappa \epsilon_{k\ell}) \dot{\epsilon}_{k\ell}. \quad (59)$$

Uniqueness of solutions of (56) holds for $\zeta^2 < 1$. These solutions are of the form

$$s_{ij} = \kappa \epsilon_{ij} + \frac{\phi'\left(\sqrt{(\epsilon_{k\ell} + c_{k\ell})(\epsilon_{k\ell} + c_{k\ell})}\right)}{\sqrt{(\epsilon_{k\ell} + c_{k\ell})(\epsilon_{k\ell} + c_{k\ell})}} (\epsilon_{ij} + c_{ij}) \, , \quad (60)$$

where $c_{kk} = 0$. The strain energy has the form

$$W = \frac{k}{2} (e_{kk})^2 + \frac{\kappa}{2} \epsilon_{ij}\epsilon_{ij} + \phi \left(\sqrt{(\epsilon_{kl} + c_{kl})(\epsilon_{kl} + c_{kl})}\right). \quad (61)$$

The yield condition $\zeta^2 = 1$ as implied by (57) and (58) is then

$$(s_{ij} - \kappa \epsilon_{ij}) \, (s_{ij} - \kappa \epsilon_{ij}) = 1 \quad . \quad (62)$$

For an example of how this differs from the perfectly plastic solid, let us consider simple shear with $\kappa = 1/2$ and $n = 1$. We then obtain

$$\zeta = \sqrt{2} \, \left| s - \frac{\gamma}{4} \right| \, , \quad (63)$$

so that the plastic yield solutions are

$$s = \pm \frac{1}{\sqrt{2}} + \frac{\gamma}{4} \quad . \quad (64)$$

The elastic solutions are

$$s = \frac{\gamma}{4} + \frac{1}{\sqrt{2}} \sin \left( \frac{\gamma}{\sqrt{2}} + c \right) , \quad \frac{-\pi}{2} - c < \frac{\gamma}{\sqrt{2}} < \frac{\pi}{2} + c \, . \quad (65)$$

A plot of the stress response to a strain in which $\gamma$ is first increased beyond yield and then decreased until yield again occurs is shown in Figure 11.

## VI. DISCUSSION AND CONCLUSIONS

Our purpose has been to show by example that elastic-plastic behavior can be embodied in a single set of constitutive relations of the rate type. Elasticity is then associated with uniqueness and plasticity with nonuniqueness of solutions of the differential equations which the constitutive relations comprise. We wish now to make explicit a point which has up to now only been implicity stated: the plastic solutions do not arise as asymptotic solutions of the equations. They are reached at definite finite values of strain during any given loading. There is no need to replace any solution by an asymptotic solution: the plastic and the elastic solutions are exact, both for loading and unloading.

In closing, we ask the reader to bear in mind that in this report we are merely presenting examples to show that it is possible to unify elasticity and plasticity in a single rate-type theory. We hope that in the future we can state the notions herein contained more generally and abstractly so that their application can be extended to more types of materials than those discussed herein.

## ACKNOWLEDGEMENTS

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We also wish to acknowledge that the germ of the idea on which this work was based arose in discussions with Dr. S. J. Chang of Oak Ridge National Laboratories just prior to the beginning of our tour of duty at the Ballistic Research Laboratory.

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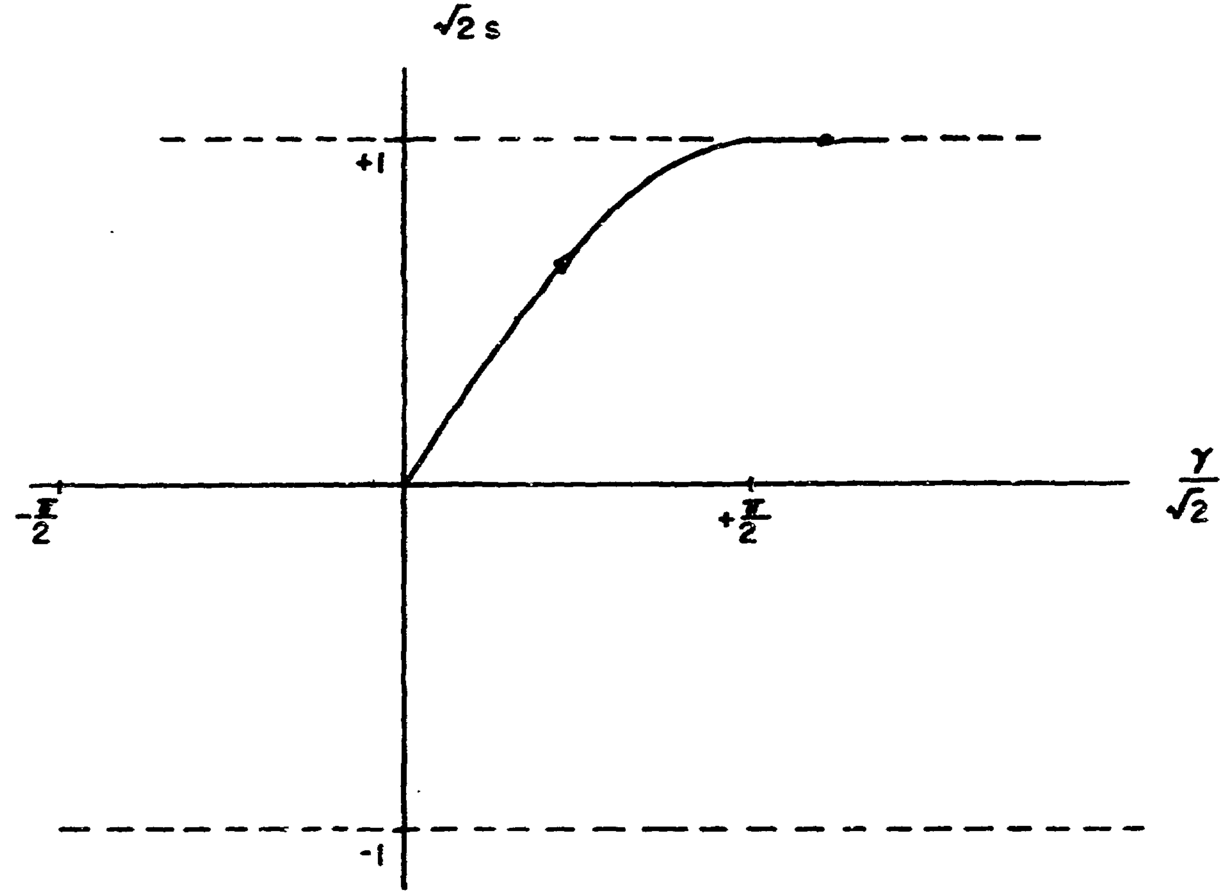


Figure 1. Theoretical Stress-Strain Curve During Loading Obtained as an Exact Solution of (1) with $n = 1$. Elastic Strain Given By (5) is Followed by Plastic Yield Given By (7). Shear Stress is $s$ and Shear Strain is $\gamma$.

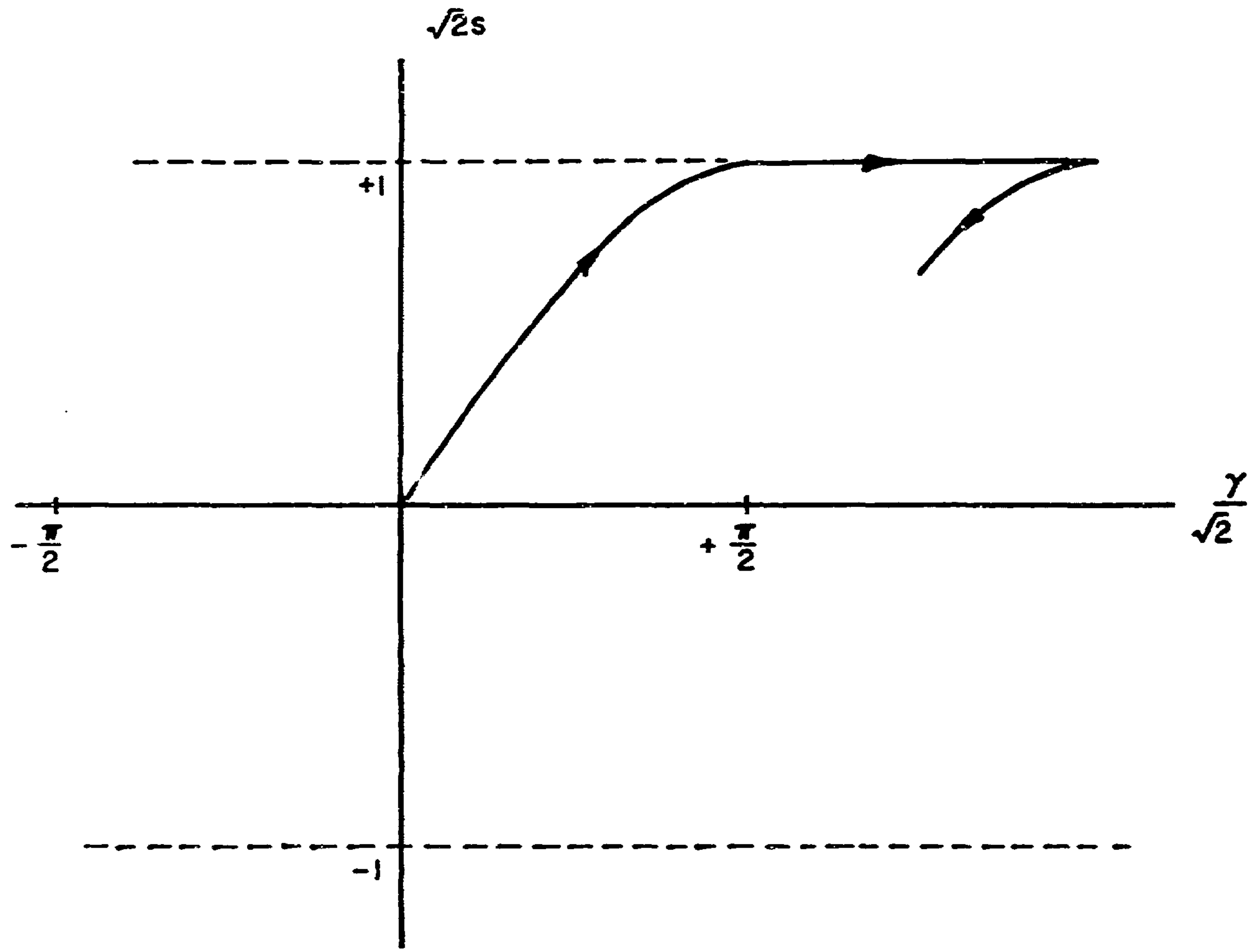


Figure 2. Theoretical Stress-Strain Curve Obtained as an Exact Solution to (1) with n = 1. The Loading Shown in Figure 1 is Followed by an Elastic Unloading Regime Given By (6). Shear Stress is $s$ and Shear Strain is $\gamma$.

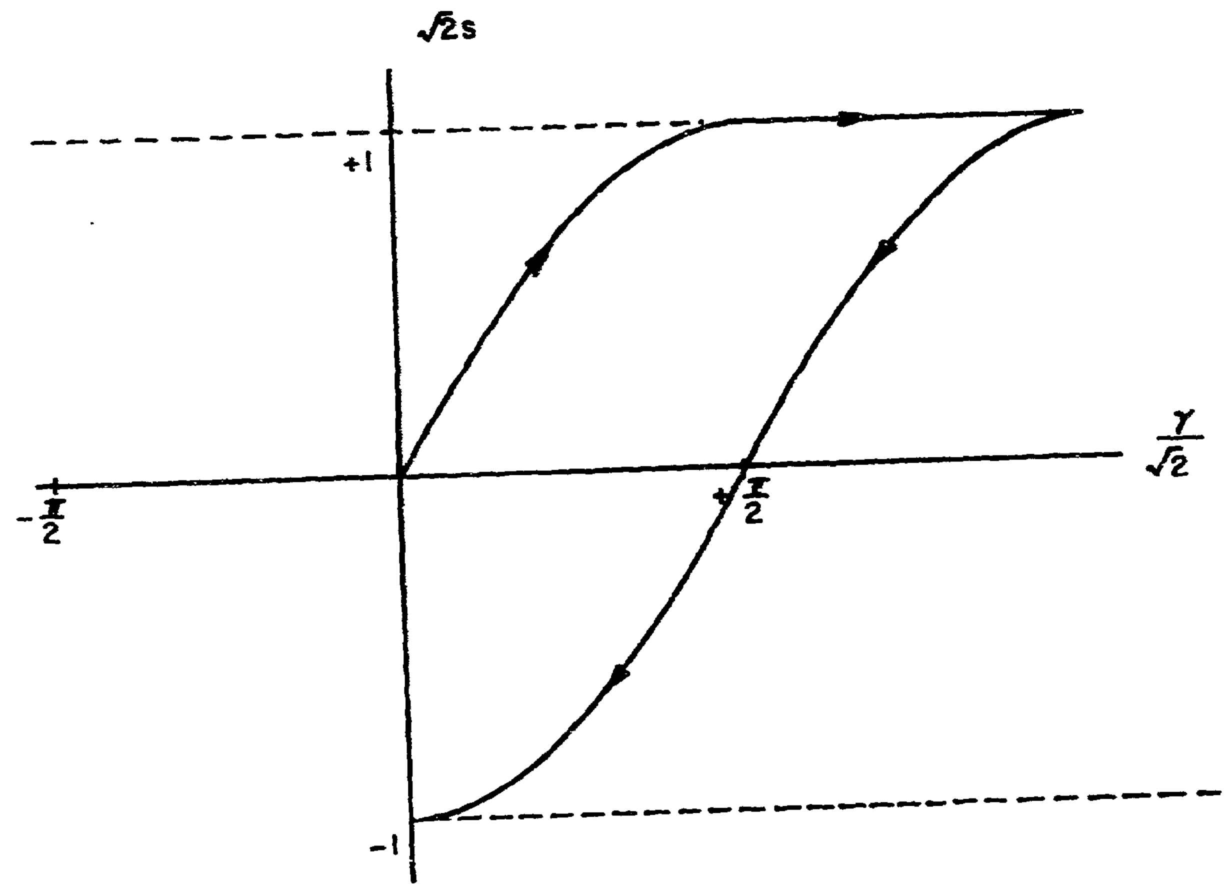


Figure 3. The Solutions of (1) with n = 1 Shown in Figures 1 and 2 Continued Until Yield Again Sets in at Shear Stress $s = -1/\sqrt{2}$. Shear Strain is $\gamma$.

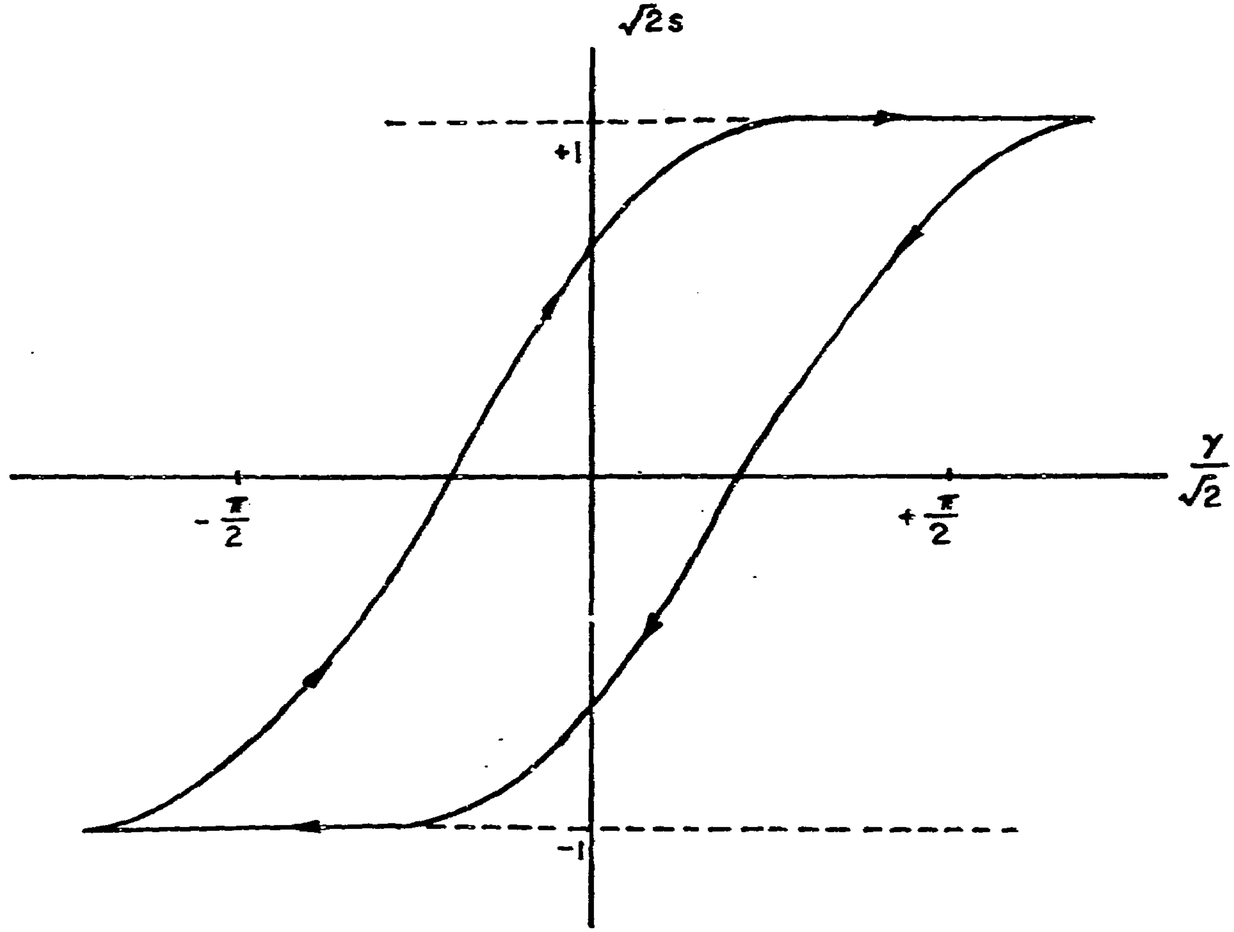


Figure 4. Hysteresis Loop Which Can Arise as an Exact Solution of (1) with $n = 1$ by Periodically Straining Beyond Yield in Both Directions. Shear Stress is $s$ and Shear Strain is $\gamma$.

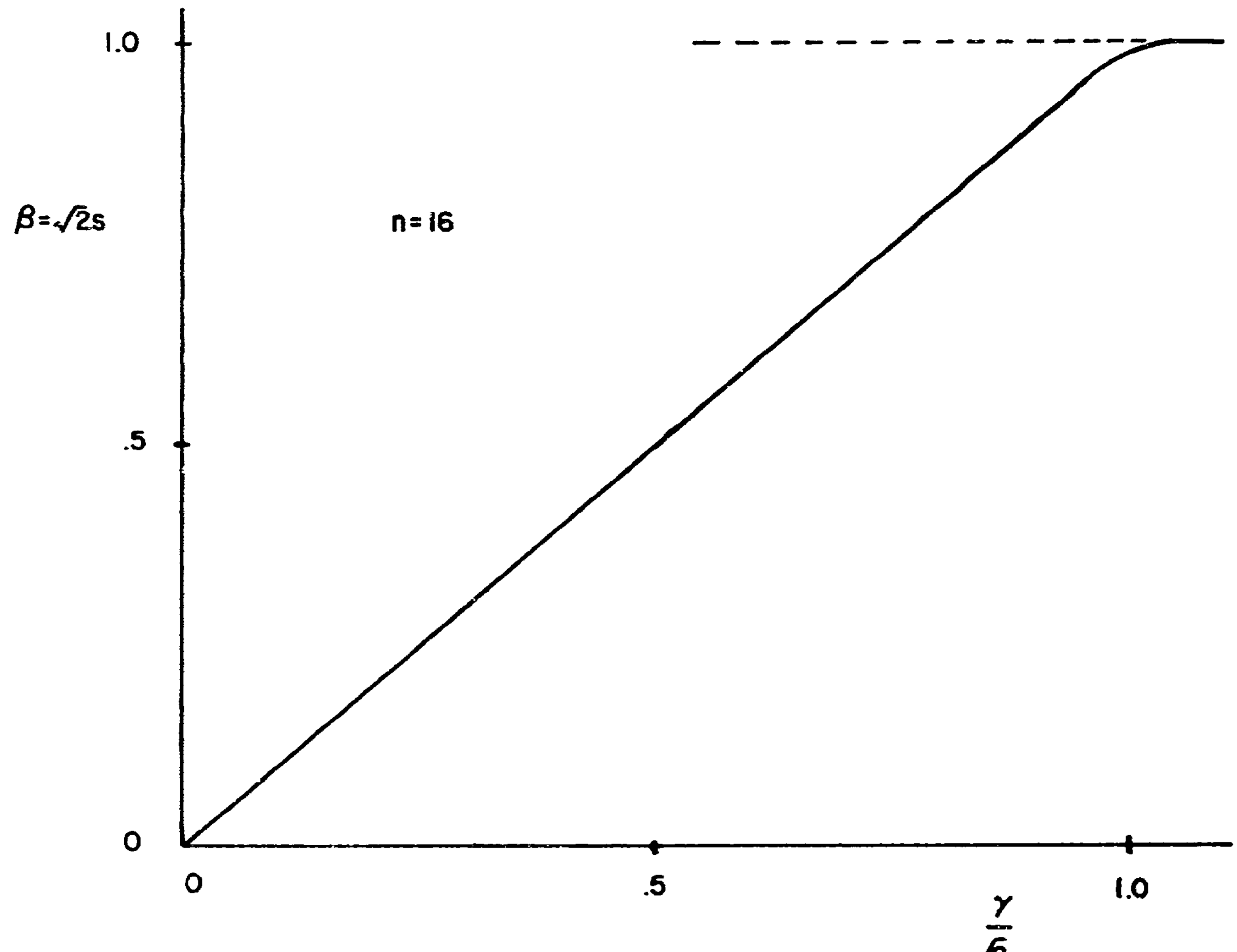


Figure 5. Theoretical Stress Strain Curve During Loading Obtained as an Exact Solution of (1) with n = 16 Showing Elastic Response Followed by Yield. Shear Stress is $s$ and Shear Strain is $\gamma$.

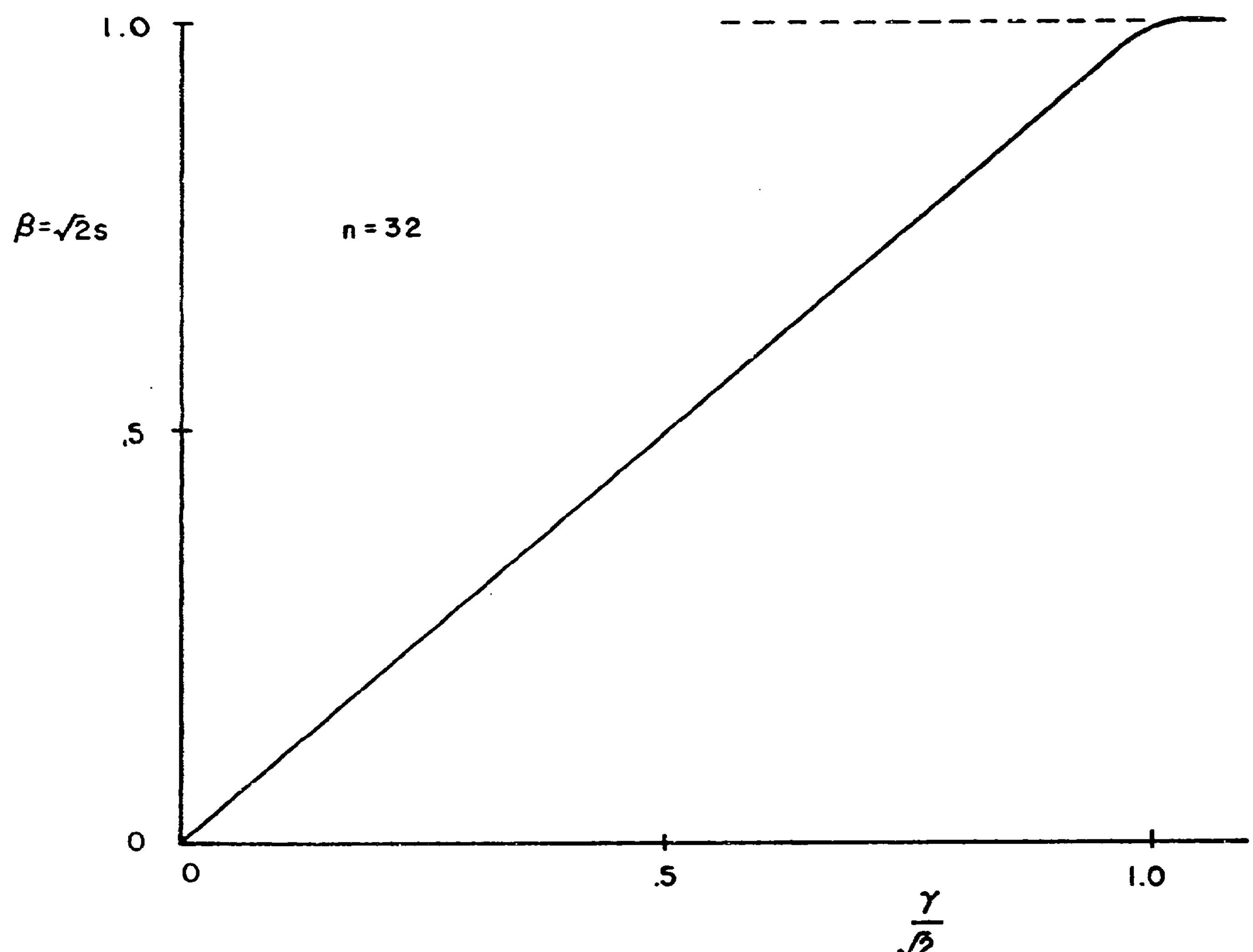


Figure 6. Theoretical Stress-Strain Curve During Loading Obtained as an Exact Solution of (1) with $n = 32$ Showing Elastic Response Followed by Yield. Shear Stress is $s$ and Shear Strain is $\gamma$.

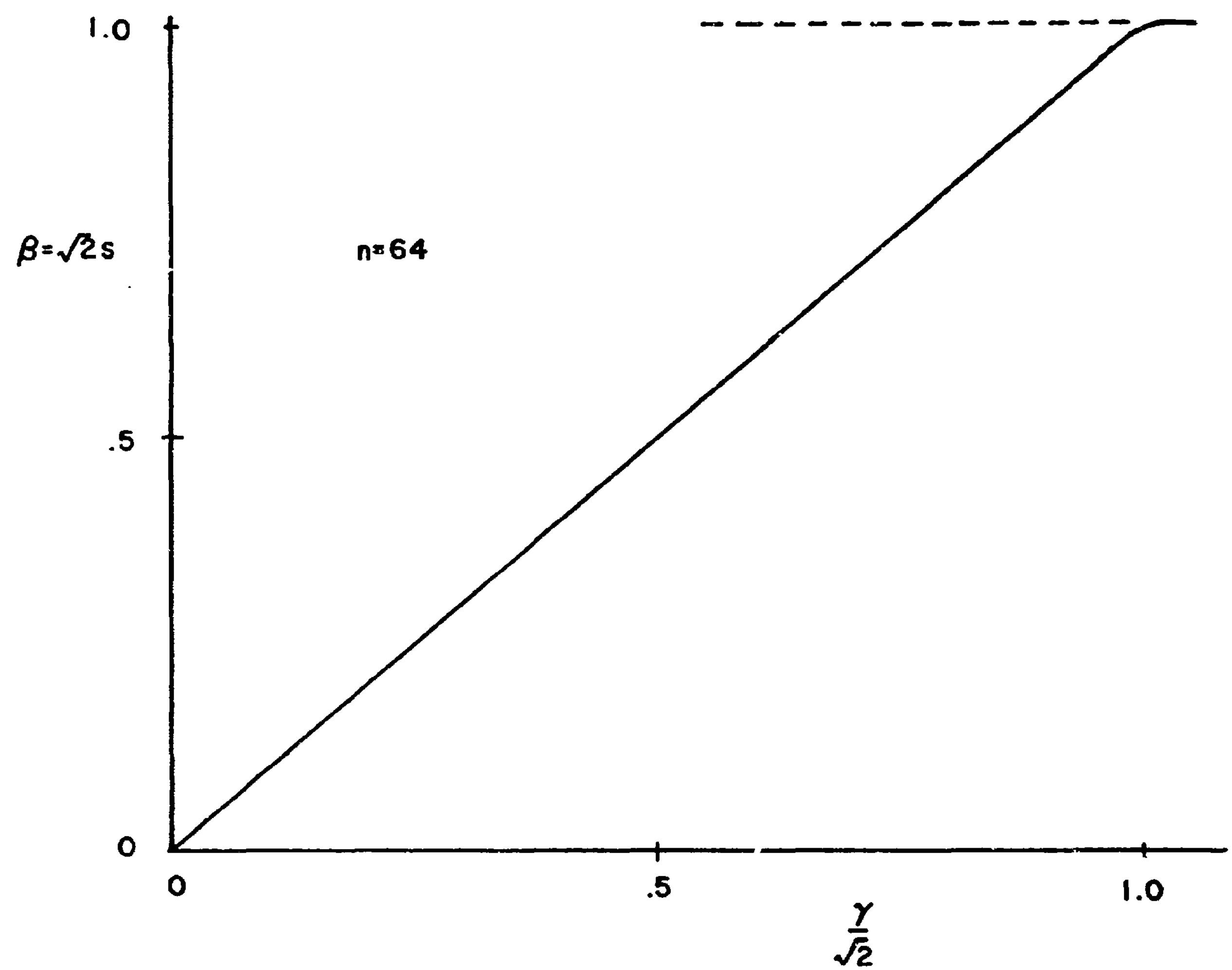


Figure 7. Theoretical Stress-Strain Curve During Loading Obtained as an Exact Solution of (1) with n = 64 Showing Elastic Response Followed by Yield. Shear Stress is $s$ and Shear Strain is $\gamma$.

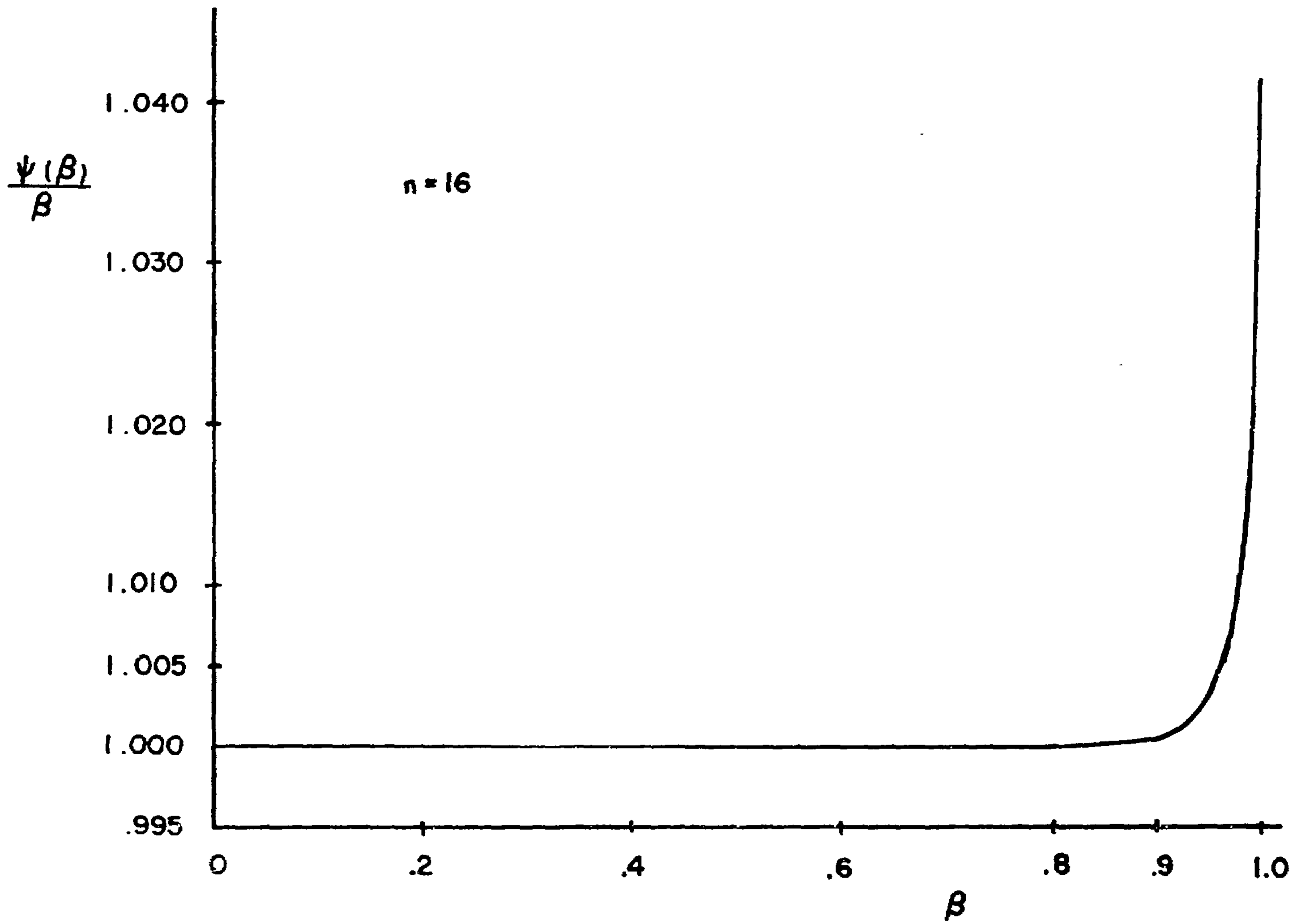


Figure 8. Inverse of Half the Elastic Shear Modulus as a Function of the Stress Invariant $\beta$ for $\psi$ Given By (45) with $n = 16$. Note, it is Constant to Within 4.2%. Its Value at Yield ($\beta = 1$) being 1.0411.

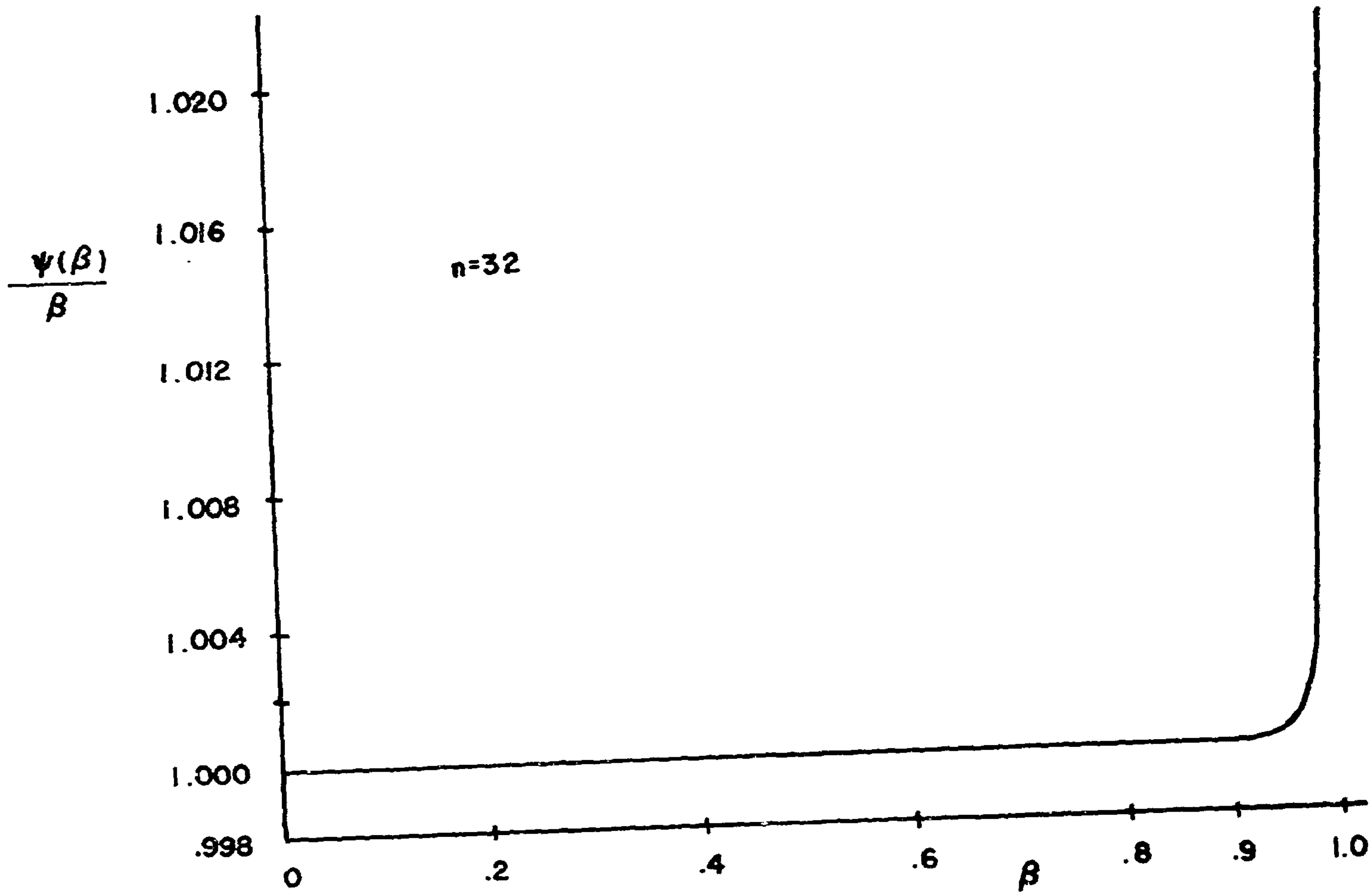


Figure 9. Inverse of Half the Elastic Shear Modulus as a Function of the Stress Invariant $\beta$ for $\psi$ Given by (45) with $n = 32$. Note that it is Constant to Within 2.1%. Its Value at Yield ($\beta = 1$) is 1.0207.

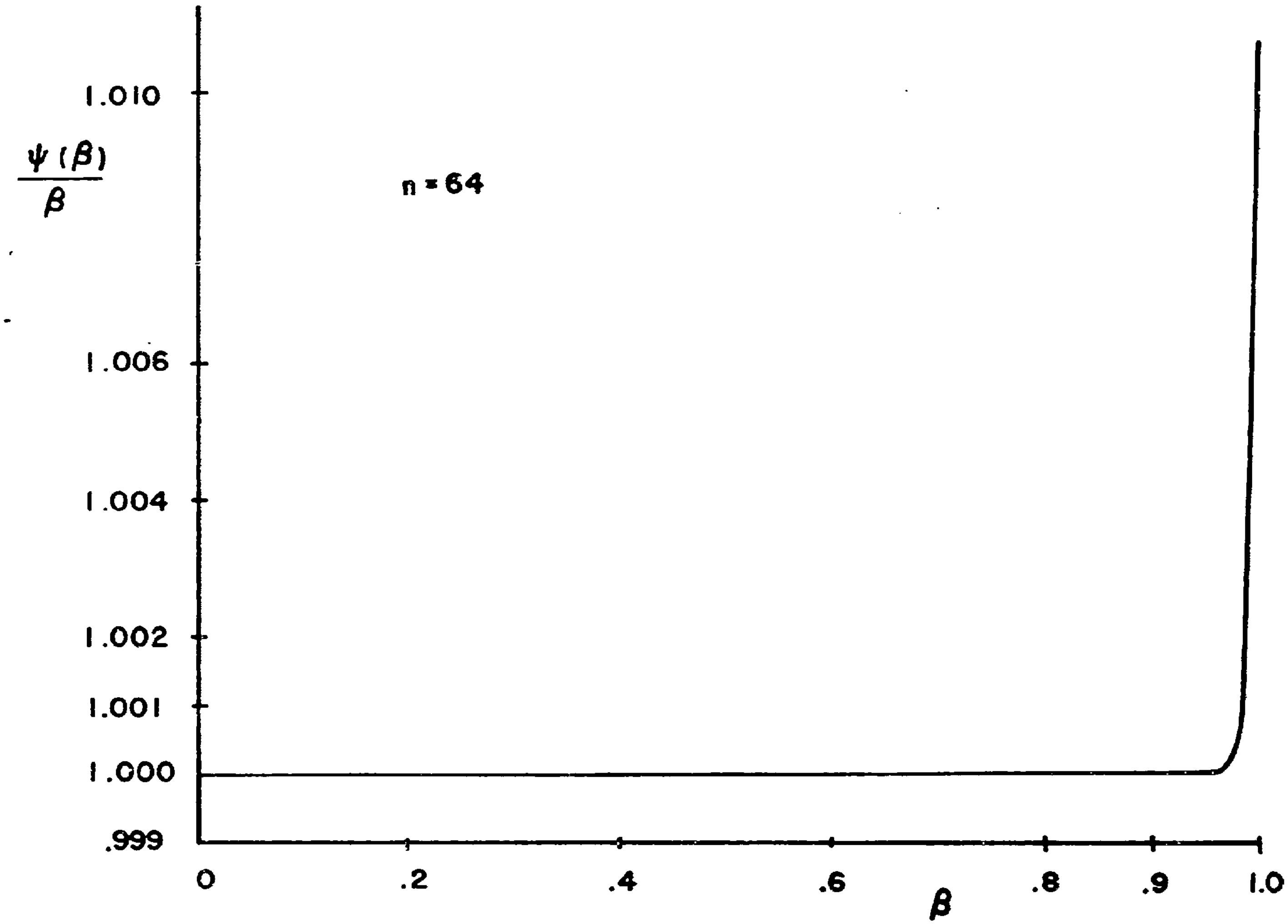


Figure 10. Inverse of Half the Elastic Shear Modulus as a Function of the Stress Invariant $\beta$ for $\psi$ Given by (45) with $n = 64$. Note that it is Constant to Within 1.1%. Its Value at Yield ($\beta = 1$) is 1.0104.

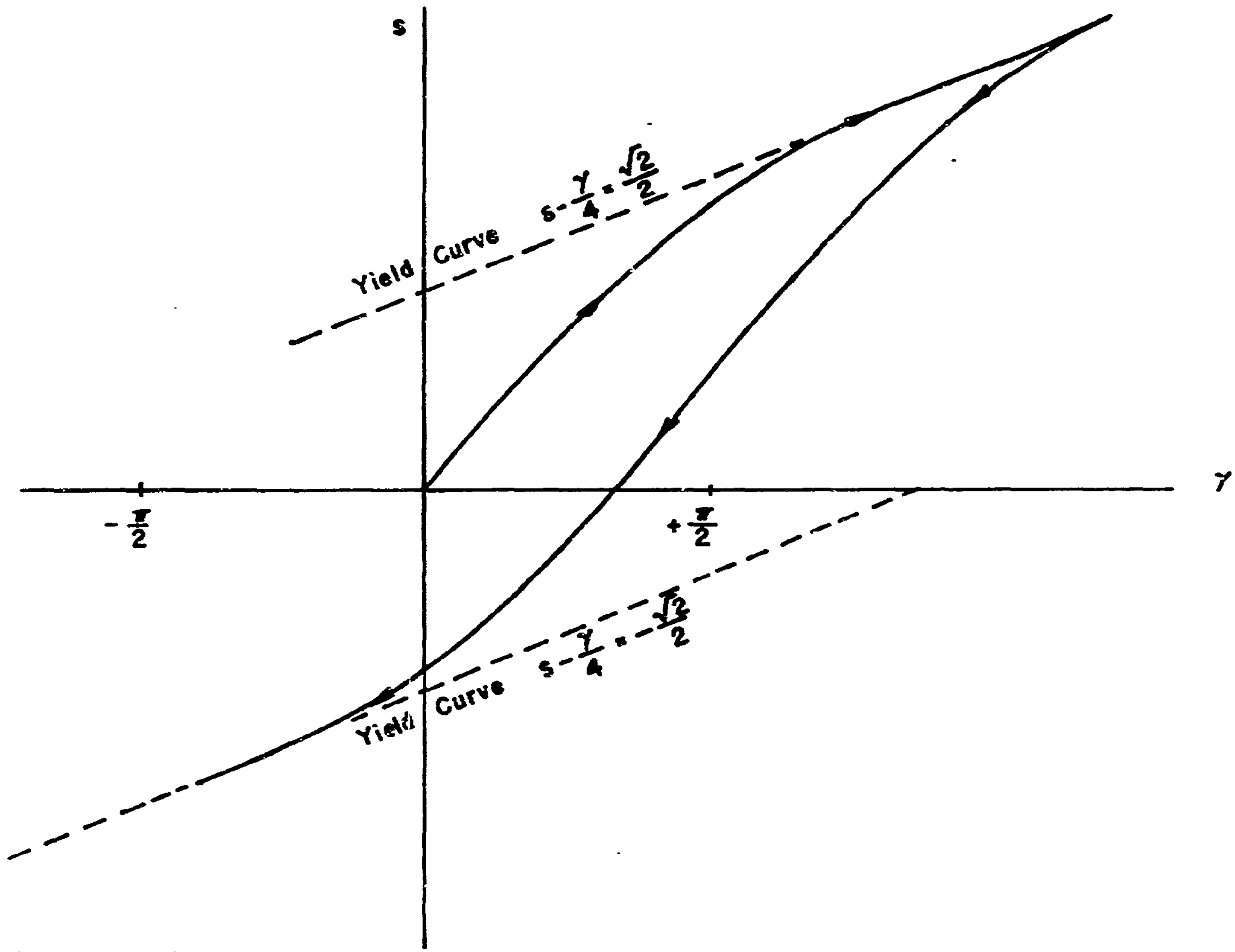


Figure 11. Theoretical Stress-Strain Curves Obtained as Exact Solutions in Simple Shear of a Strain-Hardening Material Given by (56) and (58) with $n = 1$, $\kappa = 1/2$. Shear stress is $s$ and Shear Strain is $\gamma$.

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