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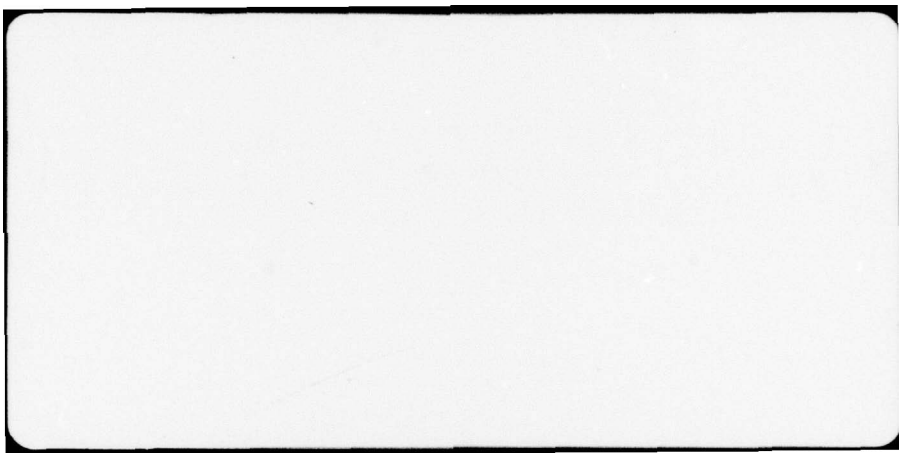

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DEPARTMENT OF OPERATIONS RESEARCH
Stanford University
Stanford, California
94305

OBSERVATIONS ON A CLASS OF NASTY LINEAR COMPLEMENTARITY PROBLEMS

by

Richard W. Cottle*

TECHNICAL REPORT 78-34

December 1978

Research and reproduction of this report were partially supported by the National Science Foundation Grant MCS76-81259 A01 and the Office of Naval Research Contract N00014-75-C-0267.

*Most of the work on this paper was carried out at the Institut für Ökonometrie und Operations Research der Universität Bonn during the author's visit there under the auspices of the Alexander von Humboldt Stiftung as a Senior U.S. Scientist Awardee.

Abstract

Earlier papers by Murty [16] and Fathi [7] have exhibited classes of linear complementarity problems for which the computational effort (number of pivot steps) required to solve them by Lemke's algorithm [13] or Murty's algorithm [15] grows exponentially with the problem size (number of variables). In this paper we consider the sequences of complementary bases that arise as these problems are solved by the aforementioned algorithms. There is a natural correspondence between these bases and binary n-vectors through which the basis sequences can be identified with particular hamiltonian paths on the unit n-cube and with the binary Gray code representations of the integers from 0 to $2^n - 1$.

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1. Introduction.

From the theoretical standpoint, the "complementary pivot methods" of Lemke [13], Murty [15], and Dantzig-Cottle [5], will always solve the nondegenerate linear complementarity problem (q, M) of order n : find w and z satisfying

$$Iw - Mz = q$$
$$w \geq 0 \ , \ z \geq 0$$
$$z^T w = 0$$

when $M \in R^{n \times n}$ is a P-matrix and $q \in R^n$ is arbitrary. Empirically it has been found that these methods work quite efficiently, often solving the problem by roughly $0(n)$ pivots. Encouraging though this may be, it need not always be the case. Indeed, Murty [16] and Fathi [7] have each exhibited classes of linear complementarity problems (indexed by the order, n) requiring either 2^n or $2^n - 1$ pivots depending on which of two complementary pivot methods is used to obtain the solution.

These contributions to the subject of computational complexity of algorithms deal with specific numerical problems (q, M) . In Murty's paper, M is a lower triangular P-matrix with positive entries below the main diagonal, and hence it is necessarily nonsymmetric. In Fathi's paper, M is symmetric, positive definite and positive. As such, these

matrices (to be specified more precisely in Section 4) look rather nice insofar as the constructive existence theory is concerned.

The primary purpose of this paper is to point out that there is something quite special about the sequences of bases through which the algorithms run when applied to the numerical examples of Murty and Fathi. A secondary purpose of the paper is to characterize the subclass of P-matrices and vectors q for which a particular pivoting sequence is followed. The necessary and sufficient conditions given here are expressed in terms of determinantal inequalities; other equivalent formulations are possible.

The plan of the paper is the following. In Section 2, we introduce some notation and background especially relevant to the main point. In Section 3, we explore some properties of a particular function related to the special pivot sequence arising in the Murty-Fathi examples. In Section 4 we analyze the aforementioned examples and show that they do indeed give rise to the pivot sequence we have in mind. In Section 5, we establish the relationship between the data and the pivot sequence followed by the complementary pivot methods of Lemke and Murty. Finally, in Section 6, we mention some interesting related literature.

2. Notation and background.

For the sake of brevity, we omit most of the customary preliminary material and refer the reader to Murty [16] and Fathi [7] for the standard concepts and notations of linear complementarity theory. To some extent, we rely on these papers for accounts of the "complementary pivot methods" to be discussed here.

However, there is one item so essential to our development that it must be reviewed. Obviously, the equation

$$Iw - Mz = q \tag{1}$$

is central to the study of (q, M). One is interested in complementary bases in the matrix $[I, -M]$. These are nonsingular matrices $B = [B_{.1}, \ldots, B_{.n}]$ such that

$$B_{.j} \in \{I_{.j}, -M_{.j}\}, \qquad j = 1, \ldots, n, \tag{2}$$

i.e., the j-th column of B is either the j-th column of I or the j-th column of $-M$. Such matrices give rise to complementary basic solutions of (1). A nonnegative complementary basic solution of (1) solves (q, M).

Given a complementary basis B, the set

$$\text{pos } B = \{q: q = Bv, v \geq 0\}$$

is called the complementary cone relative to B.

Again, suppose a complementary basis B is given. To this basis

we associate a binary row vector $c \in R^n$ as follows. For $j = 1, \ldots, n$ let

$$c_j = \begin{cases} 0 & \text{if } B_{\cdot j} = I_{\cdot j} \\ 1 & \text{if } B_{\cdot j} = -M_{\cdot j} \,. \end{cases} \tag{3}$$

When $M \in P$ (the class of matrices with positive principal minors), we must have $I_{\cdot j} \neq -M_{\cdot j}$, so there is absolutely no ambiguity in defining c_j as in (3). In other cases, one could use the definition

$$c_j = \begin{cases} 0 & \text{if } B_{\cdot j} = [I, -M]_{\cdot j} \\ 1 & \text{if } B_{\cdot j} = [I, -M]_{\cdot n+j} \,. \end{cases} \tag{3'}$$

According to (3), the vector corresponding to the identity matrix I is $(0, \ldots, 0) \in R^n$.

As is well known, there are 2^n binary vectors in R^n, and when $M \in P$ (or, more generally, when M is nondegenerate) there are 2^n complementary bases (counting multiplicities, if necessary). It is also well known that binary n-vectors can be used as a way of representing the decimal integers from 0 to 2^n-1. The binary -- i.e., base-2 -- representation comes to mind immediately. One could, therefore, use the set of all binary n-vectors appropriately ordered as a way of <u>counting</u> from 0 to 2^n-1, and by the correspondence (3) one could use the complementary bases in $[I, -M]$ for this purpose. Taking the latter scheme and, in turn, its association with complementary pivot methods into account, one wants consecutive integers to be represented by

algorithmically adjacent bases. From the standpoint of the methods under consideration here, this means the complementary bases representing consecutive integers should differ by exactly one column - or, equivalently, that their binary representations c should differ in exactly one component. This being so, the process of counting from 0 to 2^n-1 with base-2 vectors cannot correspond to an execution of either of these complementary pivot methods. For example, the base-2 representations of $2^{n-1}-1$ and 2^{n-1} differ in all n components.

The feature of adjacent complementary bases we wish to capture with binary n-vectors is called the "unit distance property", a term used in coding theory and switching theory. See [1]. One code having the unit distance property is known as Gray code - after F. Gray [11] - also known as the reflected binary code.

Gray code can be described in various ways. For example, given the integer

$$\nu \in \{0, 1, \ldots, 2^n-1\}$$

one can first write down its base-2 representation*

$$b^n(\nu) = (b_1, \ldots, b_n) \tag{4}$$

where

$$\nu = \sum_{j=1}^{n} b_j 2^{n-j} \ . \tag{5}$$

*At this point, our notation is somewhat unconventional, but it serves our purpose well.

Then the Gray code representation for ν is the binary n-vector

$$g^n(\nu) = (g_1, \ldots, g_n)$$

whose components satisfy the relations

$$\begin{cases} g_1 = b_1 \\ g_i = b_i + b_{i-1} \pmod 2 , \quad i = 2, \ldots, n . \end{cases} \tag{6}$$

Table 1 below contrasts the base-2 and Gray codes for $n = 3$. (The last column of the Table will be discussed a little later.)

ν	Base-2 Code $b^3(\nu)$	Gray Code $g^3(\nu)$	$k_3(\nu)$
0	(0, 0, 0)	(0, 0, 0)	3
1	(0, 0, 1)	(0, 0, 1)	2
2	(0, 1, 0)	(0, 1, 1)	3
3	(0, 1, 1)	(0, 1, 0)	1
4	(1, 0, 0)	(1, 1, 0)	3
5	(1, 0, 1)	(1, 1, 1)	2
6	(1, 1, 0)	(1, 0, 1)	3
7	(1, 1, 1)	(1, 0, 0)	-

Table 1.

The Gray code representation of the integers 0, 1, ..., 7 as binary 3-vectors illustrates two general properties:

$$g^n(\nu) = (0,\ g^{n-1}(\nu)) \ , \qquad \nu = 0,\ 1,\ \ldots,\ 2^{n-1}-1$$

$$g^n(\nu) = (1,\ g^{n-1}(2^n-1-\nu)) \ , \quad \nu = 2^{n-1},\ 2^{n-1}+1,\ \ldots,\ 2^n-1 \ .$$

In the case $n = 3$, we can depict the sequence $g^n(\nu)$ as a <u>hamiltonian path</u> on the unit cube in R^3.

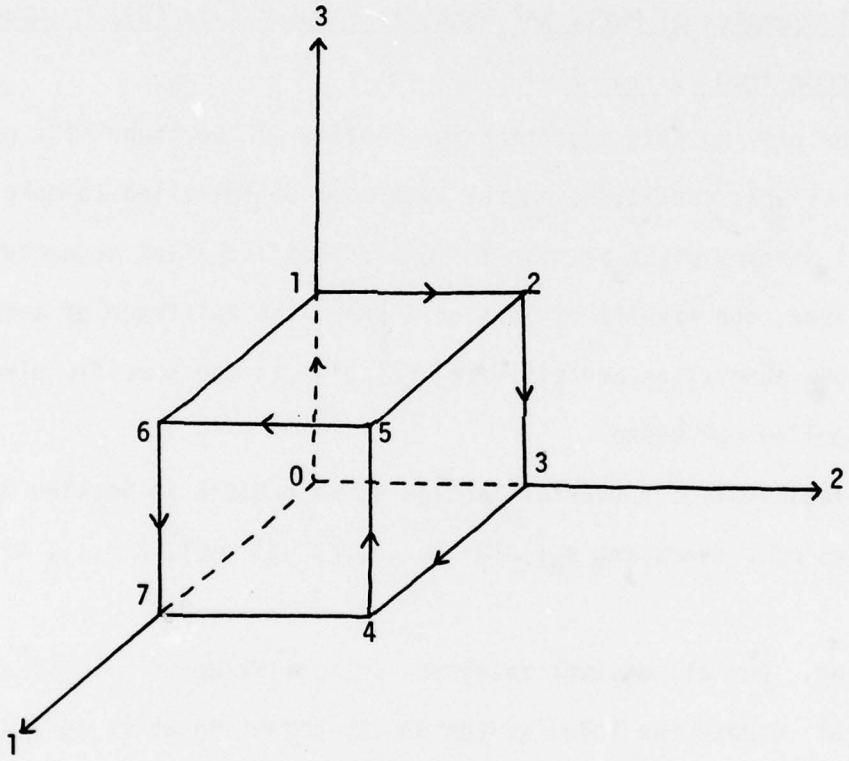

Figure 1.

Another feature of the sequence $g^3(\nu)$ illustrated in Table 1 is that every second 3-vector is obtained from its predecessor by changing the third (i.e., last) component. Again, this feature is true in general.

Our reason for devoting so much attention to the Gray code is to provide some background for the following assertion which is in fact the central point of the paper: The complementary bases which arise in the numerical examples of Murty and Fathi correspond (via (3)) to counting in Gray code from 0 to 2^n-1 .

After proving this assertion, we consider in Section 5 the general question of what conditions on the data must be fulfilled to make the two complementary pivot methods follow a specified pivot sequence. Once this is done, the results of Section 4 prove the existence of data for which these conditions are satisfied relative to the specific pivot sequence spelled out there.

To facilitate the work of Section 4, we explore in Section 3 some properties of a function k_n: $\{0, 1, \ldots, 2^n-2\} \to \{1, 2, \ldots, n\}$.

Definition. For all decimal integers ν satisfying $0 \leq \nu \leq 2^n-2$, let $k_n(\nu)$ denote the index of the one component in which $g^n(\nu)$ and $g^n(\nu+1)$ differ.

Thus, using the abbreviation $k = k_n(\nu)$ we have

$$g_k^n(\nu) + g_k^n(\nu + 1) = 1$$
$$g_j^n(\nu) + g_j^n(\nu + 1) \equiv 0 \pmod 2,\ j \neq k\ . \tag{7}$$

In terms of complementary bases, the knowledge of k_n tells us for each $\nu = 0, 1, \ldots, 2^n-2$ which column of a given basis is the next to be changed.

It is very easy to verify that for $n = 2$ we have

ν	0	1	2
$k_2(\nu)$	2	1	2

Table 2.

The function k_3 has already been displayed in Table 1. Notice that the *corresponding* sequences

k_2: 2, 1, 2

k_3: 3, 2, 3, 1, 3, 2, 3

are palindromes.

3. Properties of the function k_n .

A comparison of k_3 with k_2 shows an interesting recursive property of k_n .

Proposition 1. For all $n \geq 3$ and $0 \leq \nu \leq 2^n-2$

$$k_n(\nu) = \begin{cases} n & \text{if } \nu = 2\mu \\ k_{n-1}(\mu) & \text{if } \nu = 2\mu + 1 . \end{cases} \tag{8}$$

Proof. Let $b^n(\nu) = (b_1^n(\nu), \ldots, b_n^n(\nu))$ and $g^n(\nu) = (g_1^n(\nu), \ldots, g_n^n(\nu))$ be the base-2 and Gray code representations of ν , respectively. There are obviously two cases to be considered.

Case 1. $\nu = 2\mu$ (i.e., ν is even). This case is trivial since $b_n^n(\nu) = 0$ if and only if ν is even. It follows that $b_n^n(\nu + 1) = 1$ and $b_j^n(\nu) = b_j^n(\nu + 1)$ for $j = 1, \ldots, n-1$. Hence by (6), $g_j^n(\nu) = g_j^n(\nu + 1)$ for $j = 1, \ldots, n-1$ and $g_n^n(\nu) \neq g_n^n(\nu + 1)$. This means $k_n(\nu) = n$.

Case 2. $\nu = 2\mu + 1$ (i.e., ν is odd). We have $b_n^n(\nu) = 1$ and $b_n^n(\nu + 1) = 0$. Moreover,

$$b_{n-1}^n(\nu + 1) = b_{n-1}^n(\nu) + 1 \pmod 2 .$$

By definition,

$$g_n^n(\nu) = b_n^n(\nu) + b_{n-1}^n(\nu) = 1 + b_{n-1}^n(\nu) \pmod 2$$

and

$$g_n^n(\nu + 1) = b_n^n(\nu + 1) + b_{n-1}^n(\nu + 1) = 0 + b_{n-1}^n(\nu) + 1 \qquad (\text{mod } 2) .$$

Hence $g_n^n(\nu) = g_n^n(\nu + 1)$ which implies $k_n(\nu) < n$.

Since

$$\nu = 2\mu + 1 = \sum_{j=1}^{n} b_j^n(\nu)2^{n-j} \quad \text{and} \quad b_n^n(\nu) = 1$$

we have

$$\nu - 1 = 2\mu = \sum_{j=1}^{n-1} b_j^n(\nu)2^{n-j}$$

whence

$$\mu = \sum_{j=1}^{n-1} b_j^n(\nu)2^{n-1-j}$$

which means

$$b^{n-1}(\mu) = (b_1^{n-1}(\mu), \ldots, b_{n-1}^{n-1}(\mu)) = (b_1^n(\nu), \ldots, b_{n-1}^n(\nu)) .$$

On the other hand

$$\nu + 1 = 2(\mu + 1) = \sum_{j=1}^{n} b_j^n(\nu + 1)2^{n-j}$$

so

$$\mu + 1 = \sum_{j=1}^{n-1} b_j^n(\nu + 1)2^{n-1-j}$$

which means

$$b^{n-1}(\mu + 1) = (b_1^{n-1}(\mu + 1), \ldots, b_{n-1}^{n-1}(\mu + 1))$$

$$= (b_1^n(\nu + 1), \ldots, b_{n-1}^n(\nu + 1)) .$$

Now

$$g^n(\nu) = (b_1^n(\nu), b_2^n(\nu) + b_1^n(\nu), \ldots, b_n^n(\nu) + b_{n-1}^n(\nu)) \pmod 2$$

$$g^n(\nu+1) = (b_1^n(\nu+1), b_2^n(\nu+1) + b_1^n(\nu+1), \ldots, b_n^n(\nu+1) + b_{n-1}^n(\nu+1)) \pmod 2$$

whereas

$$g^{n-1}(\mu) = (b_1^n(\nu), b_2^n(\nu) + b_1^n(\nu), \ldots, b_{n-1}^n(\nu) + b_{n-2}^n(\nu)) \pmod 2$$

$$g^{n-1}(\mu+1) = (b_1^n(\nu+1), b_2^n(\nu+1) + b_1^n(\nu+1), \ldots, b_{n-1}^n(\nu+1) + b_{n-2}^n(\nu+1)) \pmod 2 .$$

It is now clear that $k_n(\nu) = k_{n-1}(\mu)$. □

There is no problem about evaluating $k_n(\nu)$ when ν is even. For odd values of ν we can describe the $k_n(\nu)$ as follows.

Proposition 2. If $n \geq 2$ and $0 < \nu < 2^n - 2$ is an odd number, then

$$k_n(\nu) = n - t$$

where t is smallest positive integer such that

$$\nu - (2^t - 1) = 2^t \mu_t \ , \quad \mu_t \text{ even.}$$

Proof. The assertion is trivial when $n = 2$ and $\nu = 1$. Suppose $n \geq 3$, and write $\nu = 2\mu_1 + 1$. If μ_1 is even, the assertion is true with $t = 1$ by Proposition 1. If μ_1 is odd then $\mu_1 = 2\mu_2 + 1$ and $\nu = 2^2\mu_2 + (2^2 - 1)$. If μ_2 is even, we have

$$k_n(\nu) = k_{n-1}(\mu_1) = k_{n-2}(\mu_2) = n - 2 \ .$$

So $t = 2$. If μ_2 is odd, the process can be repeated until we reach an expression of the form

$$\nu = 2^t \mu_t + (2^t - 1)$$

for which μ_t is even and minimal. In this case we have by repeated application of Proposition 1

$$k_n(\nu) = k_{n-1}(\mu_1) = k_{n-2}(\mu_2) = \cdots = k_{n-t}(\mu_t) = n - t \ ,$$

and this is the assertion. □

There is a fast way to build up k_n from its predecessor k_{n-1}.

Proposition 3. For all $n \geq 3$

$$k_n(\nu) = \begin{cases} k_{n-1}(\nu) + 1 & \text{if } 0 \leq \nu \leq 2^{n-1} - 2 \quad (9) \\ 1 & \text{if } \nu = 2^{n-1} - 1 \quad (10) \\ k_{n-1}(\nu - 2^{n-1}) + 1 & \text{if } 2^{n-1} \leq \nu \leq 2^n - 2 . \quad (11) \end{cases}$$

Moreover,

$$k_n(\nu) = k_n(\nu - 2^{n-1}) \quad \text{if } 2^{n-1} \leq \nu \leq 2^n - 2 \quad (12)$$

and

$$k_n(2^{n-1} - 1 - \nu) = k_n(2^{n-1} - 1 + \nu) \quad \text{if } 0 \leq \nu \leq 2^{n-1} - 1 . \quad (13)$$

Proof. All these claims are easily checked when $n = 3$. Assume, inductively, the statements are true for all m such that $3 \leq m \leq n - 1$.

First, suppose $0 \leq \nu \leq 2^{n-1} - 2$. All such ν belong to the domain of k_{n-1}. If ν is even, then of course

$$k_n(\nu) = n = (n - 1) + 1 = k_{n-1}(\nu) + 1 .$$

If ν is odd, there exists an integer t as in Proposition 2 such that

$$k_n(\nu) = n - t = (n-1-t) + 1 = k_{n-1}(\nu) + 1 .$$

This proves (9).

Now let $\nu = 2^{n-1} - 1$. Then $\nu = 2\mu + 1$ where $\mu = 2^{n-2} - 1$. It follows from Proposition 2 that

$$k_n(2^{n-1} - 1) = n - (n - 1) = 1$$

which proves (10).

Next, suppose $2^{n-1} \leq \nu \leq 2^2 - 2$. For all such numbers, we can write

$$\nu = 2^{n-1} + \bar{\nu} \text{ where } 0 \leq \bar{\nu} \leq 2^{n-1} - 2 .$$

This means $\bar{\nu}$ belongs to the domain of k_{n-1}. If ν is even, we have $k_n(\nu) = n$. But in this case $\bar{\nu} = \nu - 2^{n-1}$ is also even, whence $k_{n-1}(\bar{\nu}) = n - 1$. This proves (11) for all even ν. If ν is odd, we may write $\nu = 2\mu + 1$ where $2^{n-2} \leq \mu \leq 2^{n-1} - 2$. Accordingly, we have

$$\mu = 2^{n-2} + \bar{\mu} \text{ where } 0 \leq \bar{\mu} \leq 2^{n-2} - 2 .$$

By Proposition 1

$$k_n(\nu) = k_{n-1}(\mu)$$

$$k_n(\bar{\nu}) = k_{n-1}(\bar{\mu})$$

and by induction,

$$k_{n-1}(\mu) = k_{n-1}(\overline{\mu}) \ .$$

Putting these three equations together we get (12). Using (12) and induction on (11) we get

$$k_n(\nu) = k_n(\overline{\nu}) = k_{n-1}(\overline{\nu}) + 1 = k_{n-1}(\nu - 2^{n-1}) + 1$$

which gives (11) for n .

Finally, equation (13) asserts that

$$k_n(0),\ k_n(1),\ \ldots,\ k_n(2^n - 3),\ k_n(2^n - 2) \qquad (14)$$

is a palindrome. This has already been seen for $n = 2$ (and $n = 3$) and since each sequence (14) is obtained from its predecessor

$$k_{n-1}(0),\ k_{n-1}(1),\ \ldots,\ k_{n-1}(2^{n-1} - 3),\ k_{n-1}(2^{n-1} - 2)$$

by putting the number n at the beginning and end of the sequence as well as between every consecutive pair of its terms, it follows (by induction) that (14) is indeed a palindrome. □

4. Application to specific numerical examples.

In this section we aim to prove that solving the specific linear complementarity problems of Murty and Fathi by the complementary pivot methods of Lemke and Murty (called complementary pivot methods I and II, respectively) gives rise to a special pivot sequence, namely the one corresponding to the function k_n studied in Section 3.

4.1 Murty's examples.

To exhibit exponential computational requirements with complementary pivot method I, Murty defines $q = q(n)$ and $M = M(n)$ as follows:

$$q_i(n) = -\sum_{j=n+1-i}^{n} 2^j \qquad m_{ij}(n) = \begin{cases} 0 & \text{if } i < j \\ 1 & \text{if } i = j \\ 2 & \text{if } i > j \end{cases} \tag{15}$$

Remark 1. For all $n \geq 2$, the matrix $M(n)$ is lower triangular and has a positive diagonal. It is therefore trivial to verify that $M(n) \in P$. A significant fact about problems of this sort is that there is a natural (greedy) way to solve (q, M) for any q. For $i = 1, \ldots, n$ sequentially define

$$\hat{z}_i = \max \{0, -(q_i + \sum_{j<i} m_{ij}\hat{z}_j)/m_{ii}\} . \tag{16}$$

The solution to (q, M) is then $q + M\hat{z}, \hat{z}$. For all of the problems $(q(n), M(n))$ given by (15), it is immediate that $\hat{z}_1 = 2^n$ and $\hat{z}_i = 0$ $i = 2, \ldots, n$.

Remark 2. According to the definition (15)

$$q_n(n) = -(2^{n+1} - 2) = -2(2^n - 1) .$$

This fact comes into play later on.

Remark 3. Murty solves $(q(3), M(3))$ where

$$q(3) = \begin{pmatrix} -8 \\ -12 \\ -14 \end{pmatrix} \qquad M(3) = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}$$

in complete detail by complementary pivot method I. After modifying this algorithm to its equivalent parametric (principal pivoting) form (see [15], [7], [13]) one can see that the binary representations (via (3)) of the $8 = 2^3$ bases visited are precisely the Gray code representations of the numbers $0, 1, \ldots, 7$. One can also see that the pivot sequence is k_3 , that is $3, 2, 3, 1, 3, 2, 3$. Analogous statements for $(q(2), M(2))$ are even easier to check, and in this case one can draw a figure which nicely illustrates the solution process.

Our goal now is to generalize the observations made in Remark 3 to $(q(n), M(n))$ for all $n \geq 3$. Formally, we have

Theorem 1. The parametric form of complementary pivot method I applied to $(q(n), M(n))$ as defined in (15) will generate the pivot sequence $k_n(0), k_n(1), \ldots, k_n(2^n - 2)$.

Proof. The assertion is true for $n = 2$ (as well as $n = 3$). Assume

it is true for $n - 1 \geq 2$ and consider the problem $(q(n), M(n))$.

An additional, easily-proved fact is that when $n = 2$, the critical values of z_0 are $z_0^0 = 0$, $z_0^1 = 2$, $z_0^2 = 4$, and $z_0^3 = 6$. In other words, $z_0^\nu = 2\nu$. (The same can be shown when $n = 3$.) We adopt this as another part of the inductive hypothesis.

For $(q(n), M(n))$ let $\bar{z}_0 = 2^{n+1} - 2 = 2(2^n - 1)$. Then define

$$e(n) = (1, \ldots, 1)^T \in R^n$$

and

$$\bar{q}(n) = q(n) + \bar{z}_0 e(n) \ .$$

It is very easy to verify that

$$\bar{q}(n) = \begin{pmatrix} 2^n - 2 \\ \bar{q}(n-1) \end{pmatrix} \ . \tag{17}$$

The task is now to solve $(\bar{q}(n)-z_0e(n), M(n))$ for all $z_0 \in [0, \bar{z}_0]$. To do this, we write

$$[0, 2^{n+1}-2] = [0, 2^n-2] \cup (2^n-2, 2^n) \cup [2^n, 2^{n+1}-2] \ .$$

Now obviously for all $z_0 \in [0, 2^n-2]$ we must have $z_1 = 0$ in the solution of $(\bar{q}(n)-z_0e(n), M(n))$. Looking at the structure of

$$M(n) = \begin{pmatrix} 1 & 0 \\ 2e(n-1) & M(n-1) \end{pmatrix}$$

and taking note of (17), we see that for $z_0 \in [0, 2^n-2]$ we must be solving $(q(n-1), M(n-1)$ by the same method. By the inductive hypothesis we obtain the pivot sequence $k_{n-1}(\nu)$ for $\nu = 0, 1, \ldots, 2^{n-1} - 2$. For $(q(n), M(n))$, this (partial) pivot sequence translates by (9) into $k_n(\nu) = k_{n-1}(\nu) + 1$ for $\nu = 0, 1, \ldots, 2^{n-1} - 2$.

Next we note that for all $z_0 \in [2^{n-1}-2, 2^n-2]$ the solution of $(\bar{q}(n)-z_0e(n), M(n))$ must have $z_1 > 0$ and $w_1 = 0$. (See Remark 1.) By making z_1 basic in place of w_1, we have a pivot sequence which agrees with k_n up to $\nu = 2^{n-1} - 1$. For, by (10) $k_n(2^{n-1}-1) = 1$. Thus,

$$g^n(2^{n-1}) = (1, 1, 0, \ldots, 0)$$

corresponds to the complementary basis $B^{2^{n-1}}$. This is the complementary basis for $(\bar{q}(n)-z_0e(n), M(n))$ for all $z_0 \in (2^{n-1}-2, 2^n)$.

We now want to solve the problem for $z_0 \in [2^n, 2^{n+1}-2]$. Since $z_1 > 0$ and $w_1 = q_1(n) - z_0 + z_1 = 0$ when z_0 is in this interval, we have

$$z_1 = z_0 - (2^n - 2) \tag{18}$$

and substituting for z_1 via (18), the values of $w_2, \ldots, w_n, z_2, \ldots, z_n$ must solve

$$(\bar{q}(n-1) + 2(z_0-2^n+2)e(n-1) - z_0e(n-1), M(n-1))$$

for all $z_0 \in [2^n, 2^{n+1}-2]$. This can be written as

$$(\bar{q}(n-1) + 2e(n-1) - (2^{n+1}-2-z_0)e(n-1), M(n-1)) \ . \qquad (19)$$

Notice that for $z_0 \in [2^n, 2^{n+1}-2]$ we have

$$2^n - 2 \geq (2^{n+1} - 2 - z_0) \geq 0 \ .$$

So (19) is just

$$(\bar{q}(n-1) + 2e(n-1) - \varsigma e(n-1), M(n-1)) \ , \quad 2^n - 2 \geq \varsigma \geq 0 \ . \qquad (20)$$

In solving (20), we get the same sequence of complementary <u>bases</u> as in solving $(\bar{q}(n-1) - z_0 e(n-1), M(n-1))$ for $z_0 \in [0, 2^n-2]$ except that they occur in reverse order. By (13) and the inductive hypothesis we obtain the remainder of the sequence k_n . □

This argument proves Murty's result: There are 2^n-1 basis changes when the parametric form of complementary pivot method I is applied to $(q(n), M(n))$. It also shows that the complementary bases encountered correspond to the Gray code representations of the numbers $\nu = 0, 1, \ldots, 2^n-1$. As noted by Murty, the critical values of the parameter z_0 are of the form 2ν for $\nu = 0, 1, \ldots, 2^n-1$ and the half line of points $\bar{q}(n) - z_0 e(n)$ where $0 \leq z_0 \leq 2^{n+1}-2$ passes through all 2^n complementary cones.

Murty applies complementary pivot method II to the problem $(-e(n), M(n))$.

Theorem 2. Complementary pivot method II applied to $(-e(n), M(n))$ with $M(n)$ as defined in (15) will generate the pivot sequence $k_n(0), k_n(1), \ldots, k_n(2^n-2)$.

Proof. This is easily verified in the case where $n = 2$. Now suppose it is true for $n - 1 \geq 2$. We have

$$-e(n) = \begin{pmatrix} -1 \\ -e(n-1) \end{pmatrix} \quad \text{and} \quad M(n) = \begin{pmatrix} 1 & 0 \\ 2e(n-1) & M(n-1) \end{pmatrix} .$$

By the way the method is defined, we must have $z_1 = 0$ (non basic) until $(-e(n-1), M(n-1))$ is solved. By the inductive hypothesis, this generates the pivot sequence $k_{n-1}(\nu)$ for $\nu = 0, 1, \ldots, 2^{n-1}-2$. At the termination of the latter solution process we have - in terms of the problem $(-e(n), M(n))$ - the variable z_2 basic and all other z_i non basic. The next variable to become basic is z_1 , so by Proposition 3 we have the pivot sequence $k_n(\nu)$ for $0 \leq \nu \leq 2^n-1$. Now with z_1 and z_2 basic and z_i non basic for $3 \leq i \leq n$ we must once again have to solve the problem $(-e(n-1), M(n-1))$. (We remark that w_1 will never become basic again.) By the inductive hypothesis, this requires the sequence $k_{n-1}(\nu)$ for $0 \leq \nu \leq 2^{n-1}-2$. Since this sequence is a palindrome, it follows that the entire pivot sequence for solving $(-e(n), M(n))$ is just k_n . □

Here too we get a sequence of 2^n-1 pivots. Moreover, the comple-

mentary bases used correspond to the Gray-code representations of the numbers $0, 1, \ldots, 2^n-1$.

4.2 Fathi's examples.

The main point of Fathi's paper is that there are symmetric P-matrices (which of course, must therefore be positive definite) for which the same computational effort is required as in the solution of Murty's examples by the same methods. Indeed, Fathi constructs a special matrix out of Murty's matrix $M(n)$. Before defining this matrix, we wish to call the reader's attention to the fact that there are some significant notational differences between the following development and what is to be found in Fathi's paper. So, to begin with, let

$$F(n) := M(n)M(n)^T \quad . \tag{21}$$

This matrix is obviously symmetric and positive definite. Furthermore, it has the following suggestive structure:

$$F(n) = \begin{pmatrix} 1 & 2e(n-1)^T \\ 2e(n-1) & F(n-1) + 4E(n-1) \end{pmatrix} \tag{22}$$

where

$$E(n) := e(n)e(n)^T \quad .$$

Fathi's analysis (and ours) makes use of two related matrices, namely

$$\bar{F}(n) := F(n) + 4E(n) \tag{23}$$

and

$$\bar{\bar{F}}(n) := F(n) - 4G(n) \tag{24}$$

where

$$G(n) := [0,\ e(n)e(n-1)^T] \ . \tag{25}$$

We notice at once that $F(n-1)$ is a principal submatrix of $F(n)$.

For reasons of simplicity, we reverse the order of the complementary pivot methods considered and look at the solution of Fathi's problem $(-e(n), F(n))$ by means of complementary pivot method II.

Examining Fathi's solution of $(-e(3), F(3))$ we see that complementary pivot method II generates the pivot sequence k_3 . One can easily verify the analogous statement $(-e(2), F(2))$. In Theorem 3 we prove this in general. The proof (naturally an inductive one) makes critical use of an observation of Fathi's. We state this without proof as

Lemma 1. For all $n \geq 2$, the pivot sequences generated by complementary pivot method II in the solution of $(-e(n), F(n))$, $(-e(n), \bar{F}(n))$, and $(-e(n), \bar{\bar{F}}(n))$ are the same.

Now we have

Theorem 3. Complementary pivot method II applied to $(-e(n), F(n))$ will generate the pivot sequence $k_n(0), \ldots, k_n(2^n-2)$.

Proof. The statement is true for $n = 2$, so assume it is true for $n - 1 \geq 2$. To solve $(-e(n), F(n))$ one must first solve the subproblem $(-e(n-1), \overline{F}(n-1))$ before z_1 can be made basic. By the inductive hypothesis and Lemma 1 we see that the solution of $(-e(n), F(n))$ by complementary pivot method II begins with $2^{n-1}-1$ pivots given by $k_n(\nu)$ for $0 \leq \nu \leq 2^{n-1}-2$. The next pivot must be the exchange of w_1 and z_1. Hence we have agreement with $k_n(2^{n-1}-1)$. When this stage is reached, $z_1, z_2, w_3, \ldots, w_n$ are basic variables. The corresponding tableau has the form

	1	w_1	w_2	z_3	$\cdots$	z_n
z_1	3	5	-2	2	$\cdots$	2
z_2	-1	-2	1	-2	$\cdots$	-2
w_3	-1	-2	2			
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$F(n-2)$	
w_w	-1	-2	2			

According to the rules of the method, we must solve the subproblem corresponding to the indices $2, 3, \ldots, n$. This is easily recognized as $(-e(n-1), \overline{F}(n-1))$. By the inductive hypothesis and Lemma 1, we see that pivot sequence for this part of the problem is just the last $2^{n-1}-1$ terms of the sequence k_n, i.e., $k_n(\nu)$ for $2^{n-1} \leq \nu \leq 2^n-2$. After these pivots, the problem is solved. Hence complementary pivot

method II applied to $(-e(n), F(n))$ generates the pivot sequence $k_n(\nu)$ for $0 \leq \nu \leq 2^n-2$. $\square$

To obtain a computational complexity result for complementary pivot method I, Fathi uses the matrix we have called $F(n)$, but instead of $q(n)$ as defined in (15), he considers a more general class of vectors. The central idea in this instance is to show the existence of a line $L(n)$ which passes through all 2^n complementary cones defined relative to $F(n)$.

The construction goes as follows. Let $\bar{p}(n) \in R^n$ satisfy

$$0 < \bar{p}_n(n) < \cdots < \bar{p}_1(n) \ . \tag{26}$$

Note that $\bar{p}(n) \in \text{int } R^n_+$. For purposes of the proof, it would suffice to let $\bar{p}_n(n)$ equal 0 in (26). The desired line is

$$L(n) := \{\bar{p}(n) - z_0 e(n): z_0 \in R\} \ . \tag{27}$$

A figure corresponding to the parametric linear complementarity problem $(p(2)-z_0e(2), F(2))$ shows that $L(2)$ passes through the four complementary cones defined relative to $F(2)$. It also shows that if $\bar{z}_0$ is suitably large and positive, then $p(2) := \bar{p}(2) - \bar{z}_0 e(2)$ lies in the interior of $\text{pos } [-F_{\cdot 1}, I_{\cdot 2}]$. Moreover, the line segment between $p(2)$ and $\bar{p}(2)$ meets all four complementary cones relative to $F(2)$. Indeed, solving the linear complementarity problem $(p(2), F(2))$ by complementary pivot method I in parametric form yields the pivot sequence k_2 .

As in the previously considered cases, this fact points to an inductive proof of the natural generalization of this observation. We shall not give a formal proof of the following result, but simply remark that using the obviously true case of $n = 2$ as an inductive hypothesis with Fathi's own inductive proof of his Theorem 3.4 we have

Theorem 4. If $p(n) \in L(n)$ lies in the interior of the complementary cone pos $[-F_{.1}, I_{.2}, \ldots, I_{.n}]$, then the parametric version of complementary pivot method I applied to $(p(n), F(n))$ will generate the pivot sequence $k_n(0), \ldots, k_n(2^n-2)$.

Thus, the two complementary pivot methods applied to the linear complementarity problems set down by Murty and Fathi give rise to complementary bases which correspond to the Gray-code representations of the integers $0, 1, \ldots, 2^n-1$.

5. Conditions for achieving a prescribed pivot sequence.

In this section, we are concerned with the relationship between the data q and M of a linear complementarity problem and the pivot sequence followed in the solution of (q, M) by complementary pivot methods I and II.

Throughout this section we assume that $M \in R^{n\times n}$ is a P-matrix. We know then that for all $q \in R^n$, the problem (q, M) has a unique solution [17] and either of the aforementioned methods will find it.

The "results" of this section serve two purposes. First, they provide a brief review of two complementary pivot methods under consideration here. (This may be helpful for those not already conversant with them.) Second, and no less important, they suggest the possibility of constructing other nasty problems, about which more will be said in Section 6.

Both of these complementary pivot methods use an algebraic operation know as principal pivoting. In general (i.e., for M not necessarily in P), if $m_{ii} \neq 0$, the principal pivotal transform of M obtained by pivoting on m_{ii} is the matrix M' with elements m'_{ij} satisfying the rules

$$\left.\begin{aligned} m'_{ii} &= m_{ii}^{-1} \\ m'_{ij} &= -m_{ii}^{-1}m_{ij} \\ m'_{ji} &= m_{ji}m_{ii}^{-1} \end{aligned}\right\} \quad j \neq i$$

$$m'_{j\ell} = m_{j\ell} - m_{ji}m_{ii}^{-1}m_{i\ell}, \quad j \neq i \neq \ell .$$

A salient point in the present discussion is a result due to Tucker [19]

which states that if $M \in P$ then so is every principal pivotal transform M'. Thus the assumption that $M \in P$ guarantees that all the indicated pivot operations are possible.

5.1 Complementary pivot method I (Lemke [13]).

In the present circumstances, Lemke's method can be put into parametric form as follows. Given a problem (q, M) of order n with $q \not\geq 0$ (for nontriviality) define the positive real number

$$\bar{z}_0 = - \min_{1 \leq i \leq n} q_i$$

and let $e = (1, \ldots, 1)^T \in R^n$. Then

$$\bar{q} := q + \bar{z}_0 e \geq 0 .$$

(Recall what was done in the proof of Theorem 1.) To solve (q, M) we may consider the parametric linear complementarity problem

$$(\bar{q} - z_0 e, M) , \qquad 0 \leq z_0 \leq \bar{z}_0 . \tag{28}$$

For all $z_0 \in [0, \bar{z}_0]$, the problem $(\bar{q} - z_0 e, M)$ has a unique solution. For $z_0 = 0$, the solution is obvious. What we want is the solution for $z_0 = \bar{z}_0$. This can be thought of as the homotopic approach to the problem.

As z_0 runs from 0 to $\bar{z}_0$, the vector $\bar{q} - z_0 e$ passes through several complementary cones along the line segment from $\bar{q}$ to q. The critical values z_0^ν of z_0 correspond to points where this line

segment meets the boundaries of the complementary cones. For example, $z_0^0 = 0$ is the largest value for which $\bar{q} - z_0 e \in \text{pos } B^0$ where $B^0 = I$. Thus $z_0^0 = 0$ is a critical value of z_0. For any larger value of z_0, $\bar{q} - z_0 e$ belongs to a different complementary cone. In the nondegenerate case, this cone is uniquely determined. So, under this assumption, for the critical value z_0^0, there will be a unique index $k(0)$ such that

$$(\bar{q} - z_0^0 e)_{k(0)} = 0 . \tag{29}$$

In the degenerate case, several components of $\bar{q} - z_0^0 e$ may vanish. One can make the choice unique and the method finite by using a least-index rule [3]. We shall say more about this later; for the moment we retain the nondegeneracy assumption.

According to (29), the line segment from $\bar{q}$ to q first enters the complementary cone pos B^1 where

$$B^1 = [I_{\cdot 1}, \ldots, I_{\cdot k(0)-1}, -M_{\cdot k(0)}, I_{\cdot k(0)+1}, \ldots, I_{\cdot n}] .$$

The process then continues after the system is put into canonical form with respect to the complementary basis B^1. This corresponds to a principal pivot on $m_{k(0),k(0)}$. Equivalently, we take the equation

$$Iw - Mz = \bar{q} - z_0 e$$

and multiply through by $(B^1)^{-1}$ to obtain the up-dated system. We must next determine the critical value z_0^1, that is, the largest value

of z_0 for which $\bar{q} - z_0 e \in \text{pos } B^1$. However, in this and all subsequent stages, care must be taken to make sure that the next critical value does not exceed $\bar{z}_0$. In particular if $z_0^\nu \leq \bar{z}_0 < z_0^{\nu+1}$, then $q \in \text{pos } B^\nu$ and the process stops as soon as the easily found solution of $(\bar{q}-\bar{z}_0 e, M) = (q, M)$ is written down.

Suppose the solution process requires exactly ℓ pivots (basis changes). This means sequences of $\ell + 1$ complementary bases $B^0, B^1, \ldots, B^\ell$ and critical values $z_0^0, z_0^1, \ldots, z_0^\ell$ are determined in the manner sketched above. Each basis is obtained from its predecessor by a simple column change. For $\nu = 0, 1, \ldots, \ell-1$ let $k(\nu)$ denote the index of the column in B^ν to be exchanged. Thus

$$B^{\nu+1}_{\cdot k(\nu)} = \{I_{\cdot k(\nu)}, -M_{\cdot k(\nu)}\} \setminus \{B^\nu_{\cdot k(\nu)}\}$$
$$B^{\nu+1}_{\cdot j} = B^\nu_{\cdot j}, \qquad j \neq k(\nu). \tag{30}$$

If we have $B^0 = I$, and we know the <u>pivot sequence</u> (function) $k(0), k(1), \ldots, k(\ell-1)$, then we know all the complementary bases used in the process.

What we seek is the connection between the data of (q, M) and a given pivot sequence k. The question is: When will complementary pivot method I (in parametric form) generate a particular sequence of pivots?

This question has an obvious answer in terms of some determinantal inequalities. To facilitate their expression, we introduce the following notation. Suppose A is an $m \times n$ matrix, b is an m-vector, and $k \in \{1, \ldots, n\}$. Then

$$A(k;\, b) := [A_{\cdot 1}, \ldots, A_{\cdot k-1}, b, A_{\cdot k+1}, \ldots, A_{\cdot n}]$$

is just the matrix A but with b in place of $A_{\cdot k}$. Note that if $m = n$ and A is nonsingular, then by Cramer's rule for solving the linear algebraic system $Ax = b$ we have

$$x_k = \det A(k;\, b)/\det A\,, \quad k = 1, \ldots, n\,.$$

Concerned as we are with <u>complementary</u> bases relative to $[I, -M]$ where $M \in P$, it should be clear that given such a basis, B, its determinant is positive or negative according to whether B contains an even or odd number of columns of $-M$. Moreover, given $B^0 = I$ and a pivot sequence $k(0), \ldots, k(\ell-1)$, the complementary bases B it generates will contain an even or odd number of columns from $-M$ according to whether ν is even or odd. Thus

$$(-1)^\nu \det B^\nu > 0\,, \quad 0 \leq \nu \leq \ell\,.$$

We use this fact in the proof of the theorem below.

<u>Theorem</u> 5. If $M \in P$, and the solution of (q, M) by (the parametric form of) complementary pivot method I generates the pivot sequence $k(0), \ldots, k(\ell-1)$ and corresponding complementary bases $B^0, \ldots, B^\ell$, then for $\nu = 0, \ldots, \ell-1$

(i) $\qquad (-1)^{\nu} \det B^{\nu}(k(\nu); e) > 0 ;$

(ii) $\qquad (-1)^{\nu} \det B^{\nu}(k(\nu); q) \leq 0 ;$

(iii) $$\frac{\det B^{\nu}(k(\nu); q)}{\det B^{\nu}(k(\nu); e)} = \min_{i \leq j \leq n} \left\{ \frac{\det B^{\nu}(j; q)}{\det B^{\nu}(j; e)} \;\middle|\; (-1)^{\nu} \det B^{\nu}(j; e) > 0 \right\}$$

and for $\nu = \ell$

(iv) either

$$\max_{1 \leq j \leq n} (-1)^{\ell} \det B^{\ell}(j; e) \leq 0$$

or else there exists an index $k(\ell)$ for which $(-1)^{\ell} \det B^{\ell}(k(\ell); e) > 0$, (iii) holds for $\nu = \ell$ and $(-1)^{\ell} \det B^{\ell}(k(\ell); q) \geq 0$.

Conversely, if (i) - (iv) hold and for all $z_0 \in [0, \bar{z}_0]$ there are at most $n + 1$ zero components in the complementary basic solution to $(\bar{q} - z_0 e, M)$, then the sequence of pivots must be given by the integers $k(0), \ldots, k(\ell-1)$.

Proof. Suppose the pivot sequence $k(0), \ldots, k(\ell-1)$ is generated. Then for each $\nu = 0, \ldots, \ell-1$, the $k(\nu)$-th basic variable in the ν-th basic set of variables must decrease as z_0 increases. To simplify the notation, let $k = k(\nu)$, and let w_k^{ν} be the k-th basic variable relative to the ν-th basis B^{ν}. Then

$$\frac{\partial w_k^\nu}{\partial z_0} = ((B^\nu)^{-1}(-e))_k = \frac{\det B^\nu(k;\ -e)}{\det B^\nu} < 0 \ .$$

Since $M \in P$, we have $(-1)^\nu \det B^\nu > 0$, the inequality (i) now follows.

Next, consider the calculation of the critical value z_0^ν. The up-dated right-hand side of the equation

$$Iw - Mz = \bar{q} - z_0 e$$

is of the form

$$(B^\nu)^{-1}\bar{q} - z_0(B^\nu)^{-1}e \ .$$

The critical value z_0^ν makes the k-th component of the vector in (31) equal zero. Hence for each $\nu = 0, \ldots, \ell-1$

$$z_0^\nu = \frac{((B^\nu)^{-1}\bar{q})_{k(\nu)}}{((B^\nu)^{-1}e)_{k(\nu)}} = \frac{((B^\nu)^{-1}q)_{k(\nu)} + \bar{z}_0((B^\nu)^{-1}e)_{k(\nu)}}{((B^\nu)^{-1}e)_{k(\nu)}}$$

$$= \frac{((B^\nu)^{-1}q)_{k(\nu)}}{((B^\nu)^{-1}e)_{k(\nu)}} + \bar{z}_0 \ .$$

However, for these values of ν, $z_0^\nu \leq \bar{z}_0$. The inequality (ii) now follows from (i). The fact that z_0^ν is the critical value for the B^ν implies (iii).

When the last basis B^ℓ is obtained, it must be the case that either $(B^\ell)^{-1}(-e) \geq 0$ in which case

$$\max_{1 \le j \le n} \det B^{\ell}(j; e) \le 0$$

and none of the variables w_j^{ℓ} decreases as z_0 increases from $z_0^{\ell-1}$ to $\bar{z}_0$ or else $(B^{\ell})^{-1}(-e)$ has a negative component and it is possible to define a new critical value $z_0^{\ell} \ge \bar{z}_0$ as above. In particular, there must exist an index $k(\ell)$ for which (iii) holds and

$$(-1)^{\ell} \det B^{\ell}(k(\ell); q) \ge 0 .$$

Since $(B^{\ell})^{-1}(\bar{q} - z_0 e) \ge 0$ for all $z_0 \in [z_0^{\ell-1}, z_0]$, it is not necessary to execute another pivot; hence $k(\ell)$ is not part of the pivot sequence.

The proof of the converse is clear form the nondegeneracy assumption. Indeed the pivot sequence is uniquely determined by the inequalities (i) - (iv). □

Remark. The necessary conditions stated in Theorem 5 are valid even in the degenerate case, but in that situation, the statement of the converse must be modified. If we use a least-index rule to choose the pivot element, then (i) - (iv) are sufficient provided the $k(\nu)$ are the least integers (indices) for which those conditions hold.

5.2 Complementary pivot method II (Murty [15], [16]).

In [15] and [16], Murty develops variants of the so-called Bard method* for solving (q, M) where $M \in P$. The Bard procedure is a

*This sort of method is related to an earlier one of Zoutendijk [20, pp. 83-90].

type of principal pivoting method which starts with the equation (1) and the complementary basis $B^0 = I$. At the ν-th step, one has a complementary basis B^ν and the vector

$$q^\nu := (B^\nu)^{-1} q .$$

If $q^\nu \geq 0$, the solution has been formed. Otherwise <u>any</u> index $k = k(\nu)$ such that $q_k^\nu < 0$ is chosen, a new complementary basis $B^{\nu+1}$ is determined according to (30), and by principal pivoting (or up-dating) one has the new vector

$$q^{\nu+1} := (B^{\nu+1})^{-1} q .$$

At this point, the steps outlined above are repeated.

It is known that in the degenerate case this arbitrary way of selecting $k(\nu)$ can lead to circling (alias cycling): the repetition of a sequence of complementary bases. The essence of Murty's variant is a specific rule for choosing $k = k(\nu)$ which makes the method finite. In [14], he uses the rule

$$k(\nu) = \arg\min \{k: q_k^\nu < 0\} \tag{32}$$

whereas in [16] he uses the largest-index rule

$$k(\nu) = \arg\max \{k: q_k^\nu < 0\} . \tag{33}$$

Under (32) or (33) or indeed under any other specific rule, k becomes a function of ν and the data q and M.

So, complementary pivot method II is Bard's method with the largest index pivot selection rule (33). It seems that Murty's motivation for using (33) rather than (32) in [16] was to exhibit the required exponential computational effort of <u>both</u> complementary pivot methods with the same matrix. Fathi [7] also uses (33) as part of complementary pivot method II.

<u>Theorem</u> 6. If $M \in P \cap R^{n\times n}$, the solution of (q, M) by complementary pivot method II generates the pivot sequence $k(0), k(1), \ldots, k(\ell-1)$ and corresponding complementary bases $B^0, B^1, \ldots, B^\ell$ if and only if

(i) for all $\nu = 0, 1, \ldots, \ell-1$

$$\min_{1\le k\le n} (-1)^\nu \det B^\nu(k; q) < 0$$

and

$$k(\nu) = \arg\max \{k: (-1)^\nu \det B^\nu(k; q) < 0\} ;$$

(ii) for $\nu = \ell$

$$(-1)^\ell \det B^\ell(k; q) \ge 0 , \quad k = 1, \ldots, n .$$

Moreover, the bound $\ell \le 2^n - 1$ is sharp.

<u>Proof</u>. Given a complementary basis B^ν we have

$$q_k^\nu = ((B^\nu)^{-1} q)_k = \det B^\nu(k; q)/\det B^\nu .$$

Since $(-1)^{\nu}\det B^{\nu} > 0$ for all complementary bases generated by complementary pivot method II, conditions (i) and (ii) are just a paraphrase of the tests of the algorithm. The bound $\ell \leq 2^n - 1$ is a consequence of the fact that there are 2^n complementary bases which (by Murty's finiteness proof) cannot be repeated in the solution process. The sharpness of the bound is shown by Murty's problem $(-e(n), M(n))$. □

6. Some connections with other literature.

In section 4, we established the correspondence between the numerical examples of Murty and Fathi and the Gray code representations, $g^n(\nu)$, of the integers $\nu = 0, 1, \ldots, 2^n-1$. This sequence describes a hamiltonian path on the unit n-code.

As may be expected, there is a sizable literature on Gray codes and related matters. We mention just a little of it here. Some papers are concerned with Gray codes in bases other than 2 and conversions between number systems. See [4], [6], and [8]. Other papers treat the problem of describing paths on n-cubes. See [10], for example. The latter are related to "snake-in-a-box" problems and codes. The Introduction (in Volume I) and Bibliography (in Volume II) of the book [14] by MacWilliams and Sloane is a rich source of references on this literature. See also [2].

By far the most amusing article on the subject of Gray codes is the one by Martin Gardner [9] who points out that the binary Gray code is the key to the solution of the Chinese ring puzzle and the well-known Tower of Hanoi problem. To these may be added a recently-marketed puzzle called "The Brain." It requires the solver to move a set of eight rods from one extremal position (e.g., all "out") to the other (all "in"). The puzzle is constructed in such a way that at any stage, only two of the rods will move. It takes 170 moves to solve the puzzle, for (in the notation of Section 2)

$$g^8(0) = (0, 0, 0, 0, 0, 0, 0, 0)$$

and

$$g^8(170) = (1, 1, 1, 1, 1, 1, 1, 1) .$$

Remembering that the papers by Murty and Fathi are concerned with the computational complexity of complementary pivot methods and that problems requiring $2^n - 1$ pivots are exhibited, one may ask whether these have a connection with the Klee-Minty result [12] on the simplex method of linear programming. In their paper, Klee and Minty devise a class of linear programs in which the feasible region is a specially perturbed unit n-cube . The objective function is just one coordinate, and the simplex method with the customary pivot selection rules runs through $2^n - 1$ pivots in finding the solution. In so doing, it generates a hamiltonian path on the "cube" which can be associated in a simple way with the path that arises from the Gray code. Their example [12, p. 163] with $\varepsilon = 0$ and the coordinates of the vertices written in reverse order illustrates this point. It is not clear (to the author at least) that a stronger link between these phenomena can be found.

For the combinatorial theorist, an appealing question is: What "solution paths" on the unit n-cube can be realized - through the correspondence (3) - when a linear complementarity problem (q, M) with $M \in P$ is solved by methods of the type discussed here?

Very recently, Stickney and Watson [18] have published a paper focusing attention on Bard-type algorithms for the P-matrix case. For a given problem (q, M) they define a directed graph G(M/q) whose vertices correspond to complementary bases of [I, -M] . The graph has a directed edge from vertex A to vertex B if there exist an index $k \in \{1, \ldots, n\}$ such that

$$\text{(i)}\quad (A^{-1}q)_k < 0 \qquad (34)$$

$$\text{(ii)}\quad A_{\cdot k} \in \{I_{\cdot k}, -M_{\cdot k}\} \backslash B_{\cdot k}$$

$$\text{(iii)}\quad A_{\cdot j} = B_{\cdot j}, \qquad j \neq k .$$

Clearly, the vertices of $G(M/q)$ correspond to those of the unit n-cube via (3). The edges of $G(M/q)$ being defined by the conditions (34) are just those of the unit n-cube apart from their orientation. That is, when $M \in P$, the total degree of a vertex in $G(M/q)$ is n. See [18, p. 324].

One of the main results of the Stickney-Watson paper is that when $M \in P$, a linear complementarity problem (q, M) with q belonging to a complementary cone spanned by k columns of $-M$ and $n - k$ columns of I can be solved by a sequence of k Bard-type pivots. Unfortunately, in the general case, one still has no clue as to <u>which</u> Bard-type pivots ought to be executed. Perhaps future studies will shed more light on this. Clearly, for this insight to have practical algorithmic value, it will be necessary to identify the "right" pivots without an excessive amount of auxiliary work.

Stickney and Watson also comment on the problem of circling in Bard-type algorithms and about the effect of scaling in this regard. In view of the relationship between complementary bases and vertices of the unit n-cube, and hence between principal pivoting algorithms and <u>paths</u> on the unit n-cube, it is tempting to speculate about the interpretation of these vertices (binary n-vectors) as encoded numbers. In particular, can the sequence $\{B^\nu\}$ of complementary bases be chosen in such a way that the corresponding sequence $\{c(B^\nu)\}$ of binary

n-vectors is strictly increasing in the sense of some binary number system? If so, this might be a way to avoid circling.

Acknowledgement.

The author warmly thanks many colleagues - especially B. C. Eaves, D. Gale, G. H. Golub, D. Knuth, C. E. Lemke, R. von Randow, and M. H. Wright - for stimulating discussions and helpful information, much of which is included in this paper.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE	READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER: TR 78-34	2. GOVT ACCESSION NO. / 3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle): OBSERVATIONS ON A CLASS OF NASTY LINEAR COMPLEMENTARITY PROBLEMS	5. TYPE OF REPORT & PERIOD COVERED: Technical Report, 6. PERFORMING ORG. REPORT NUMBER: 78-34
7. AUTHOR(s): Richard W. Cottle	8. CONTRACT OR GRANT NUMBER(s): N00014-75-C-0267, NSF-MCS76-81259
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Department of Operations Research, Stanford University, Stanford, CA 94305	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: NR-047-143
11. CONTROLLING OFFICE NAME AND ADDRESS: Operations Research Program -- ONR, Department of the Navy, 800 N. Quincy St., Arlington, VA 22217	12. REPORT DATE: December 1978; 13. NUMBER OF PAGES: 44
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) 48p.	15. SECURITY CLASS. (of this report): Unclassified; 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

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OBSERVATIONS ON A CLASS OF NASTY LINEAR COMPLEMENTARITY PROBLEMS

78-34 by Richard W. Cottle

Abstract

Earlier papers by Murty [16] and Fathi [7] have exhibited classes of linear complementarity problems for which the computational effort (number of pivot steps) required to solve them by Lemke's algorithm [13] or Murty's algorithm [15] grows exponentially with the problem size (number of variables). In this paper we consider the sequences of complementary bases that arise as these problems are solved by the aforementioned algorithms. There is a natural correspondence between these bases and binary n-vectors through which the basis sequences can be identified with particular hamiltonian paths on the unit n-cube and with the binary Gray code representations of the integers from 0 to ~~$2^n - 1$.~~ 2 to the n minus 1.