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Vector Operations in a Dipole Coordinate System

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VECTOR OPERATIONS IN A DIPOLE COORDINATE SYSTEM

I. Introduction

For many physical systems it is important to follow quantities which are not easily represented in the common orthogonal coordinate systems. In studies of plasmas either in the earth's magnetosphere or in solar flares, the major component of the plasma motion is along the magnetic field line which is approximately dipolar. These are examples where resolving the flow in dipole coordinates has two major benefits. First, by using the most natural coordinate system, it preserves the physical intuition of how the system should behave. And second, use of the natural coordinate system in a numerical simulation will minimize the numerical diffusion due to interpolating large components onto other coordinates.

In the following text the authors have compiled the results of their derivations for the most common mathematical formulas and operations used in applications of a dipole coordinate system. The results are given for a right-handed coordinate system (V, L, ϕ) . Previously [1] the coordinates have been given in a slightly different order (L, V, ϕ) , where this latter system is left-handed. We note that the results for all of the operations described in the text are the same in either system.

II. Geometry of the Dipole Coordinate System

Figure 1 shows the representation of a point outside a sphere of radius r_0 in both spherical (r, θ, ϕ) and dipole (V, L, ϕ) coordinates where the relationship between the coordinates

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$$V = \frac{r_o^2 \cos \theta}{r^2}, \quad L = \frac{r}{r_o \sin^2 \theta}, \quad \phi = \phi,$$

$$r_o \leq r \leq \infty, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi,$$

defines a right-handed, orthogonal, curvilinear dipole system with coordinates in the range

$$-1 \leq V \leq 1, \quad 1 \leq L \leq \infty, \quad 0 \leq \phi \leq 2\pi.$$

By relating the unit vectors of the two systems an arbitrary vector can be transformed from one system to the other

$$\underline{e}_V = - \left(\frac{2 \cos \theta}{\delta} \underline{e}_r + \frac{\sin \theta}{\delta} \underline{e}_\theta \right), \quad \underline{e}_L = \frac{\sin \theta}{\delta} \underline{e}_r - \frac{2 \cos \theta}{\delta} \underline{e}_\theta,$$

$$\underline{e}_r = - \frac{2 \cos \theta}{\delta} \underline{e}_V + \frac{\sin \theta}{\delta} \underline{e}_L, \quad \underline{e}_\theta = - \left(\frac{\sin \theta}{\delta} \underline{e}_V + \frac{2 \cos \theta}{\delta} \underline{e}_L \right),$$

where $\delta = \sqrt{1 + 3 \cos^2 \theta}$. This dipole coordinate system has a hybrid representation where all coefficients are given in terms of spherical quantities. For many applications this is the simplest and most convenient representation.

A. Metric Coefficients for the Dipole System

$$h_V = \frac{r^3}{r_o^2 \delta}, \quad h_L = \frac{r_o \sin^3 \theta}{\delta}, \quad h_\phi = r \sin \theta.$$

B. The Differential Arc Length, Area, and Volume Elements.

$$(ds)^2 = \frac{r^4}{r_o^4 \delta^2} (dV)^2 + \frac{r_o^2 \sin^6 \theta}{\delta^2} (dL)^2 + r^2 \sin^2 \theta (d\theta)^2,$$

$$da_V = \frac{r r_o \sin^4 \theta}{\delta} dL d\phi, \quad da_L = \frac{r^4 \sin \theta}{r_o^2 \delta} dV d\phi, \quad da_\phi = \frac{r^3 \sin^3 \theta}{r_o \delta^2} dV dL,$$

$$dv = \frac{r^4 \sin^4 \theta}{r_o \delta^2} dV dL d\phi.$$

**C. The Derivatives of the Coordinates of One System
with Respect to the Other System**

$$\begin{aligned}\frac{\partial V}{\partial r} &= -\frac{2r_o^2 \cos\theta}{r^3}, & \frac{1}{r} \frac{\partial V}{\partial \theta} &= -\frac{r_o^2 \sin\theta}{r^3}, \\ \frac{\partial L}{\partial r} &= \frac{1}{r_o \sin^2\theta}, & \frac{1}{r} \frac{\partial L}{\partial \theta} &= -\frac{2\cos\theta}{r_o \sin^3\theta}, \\ \frac{r_o^2 \delta}{r^3} \frac{\partial \theta}{\partial V} &= -\frac{\sin\theta}{r\delta}, & \frac{\delta}{r_o \sin^3\theta} \frac{\partial r}{\partial L} &= \frac{\sin\theta}{\delta}, \\ \frac{r_o^2 \delta}{r^3} \frac{\partial r}{\partial V} &= -\frac{2\cos\theta}{\delta}, & \frac{\delta}{r \sin^2\theta} \frac{\partial \theta}{\partial L} &= -\frac{2\cos\theta}{r\delta}.\end{aligned}$$

D. Christoffel Symbols

Certain vector operations are simplified by the introduction of these symbols. For the dipole coordinates, there are four independent nonvanishing components.

$$\begin{aligned}\Gamma_{VL}^V &= -\Gamma_{LV}^V = \frac{3\sin\theta}{r\delta^3} (1 + \cos^2\theta), & \Gamma_{VL}^L &= -\Gamma_{LV}^L = \frac{6\cos\theta}{r\delta^3} (1 + \cos^2\theta), \\ \Gamma_{V\phi}^\phi &= -\Gamma_{\phi V}^\phi = \frac{3\cos\theta}{r\delta}, & \Gamma_{\phi L}^\phi &= -\Gamma_{L\phi}^\phi = \frac{1}{r \sin\theta\delta} (1 - 3\cos^2\theta).\end{aligned}$$

III. Vector Operations in the Dipole Coordinate System

For the following operations f is a scalar function,

$$f = f(V, L, \phi),$$

$\underline{A}$ and $\underline{B}$ are vector functions,

$$\begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} A_V \\ B_V \end{Bmatrix} (V, L, \phi) \underline{e}_V + \begin{Bmatrix} A_L \\ B_L \end{Bmatrix} (V, L, \phi) \underline{e}_L + \begin{Bmatrix} A_\phi \\ B_\phi \end{Bmatrix} (V, L, \phi) \underline{e}_\phi,$$

and $\underline{T}$ is a corresponding tensor function,

$$\underline{T} = \underline{T}(V, L, \phi).$$

A. Divergence of a Vector

$$\nabla \cdot \underline{A} = \frac{r_o^2 \delta^2}{r^4 \sin^4 \theta} \frac{\partial}{\partial V} \left[\frac{r \sin^4 \theta}{\delta} A_V \right] + \frac{\delta^2}{r_o r^4 \sin^4 \theta} \frac{\partial}{\partial L} \left[\frac{r^4 \sin \theta}{\delta} A_L \right] + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}$$

B. Gradient of a Scalar

$$(\nabla f)_V = \frac{r_o^2 \delta}{r^3} \frac{\partial f}{\partial V}, \quad (\nabla f)_L = \frac{\delta}{r_o \sin^3 \theta} \frac{\partial f}{\partial L}, \quad (\nabla f)_\phi = \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi}$$

C. Laplacian of a Scalar

$$\nabla^2 f = \frac{r_o^4 \delta^2}{r^4 \sin^4 \theta} \frac{\partial}{\partial V} \left[\frac{\sin^4 \theta}{r^2} \frac{\partial f}{\partial V} \right] + \frac{\delta^2}{r_o^2 r^4 \sin^4 \theta} \frac{\partial}{\partial L} \left[\frac{r^4}{\sin^2 \theta} \frac{\partial f}{\partial L} \right] + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}$$

D. Curl of a Vector

$$\begin{aligned} (\nabla \times \underline{A})_V &= \frac{\delta}{r r_o \sin^4 \theta} \frac{\partial}{\partial L} (r \sin \theta A_\phi) - \frac{1}{r \sin \theta} \frac{\partial A_L}{\partial \phi} \\ (\nabla \times \underline{A})_L &= \frac{1}{r \sin \theta} \frac{\partial A_V}{\partial \phi} - \frac{r_o^2 \delta}{r^4 \sin \theta} \frac{\partial}{\partial V} (r \sin \theta A_\phi) \\ (\nabla \times \underline{A})_\phi &= \frac{r_o^2 \delta^2}{r^3 \sin^3 \theta} \frac{\partial}{\partial V} \left[\frac{\sin^3 \theta}{\delta} A_L \right] - \frac{\delta^2}{r_o r^3 \sin^3 \theta} \frac{\partial}{\partial L} \left[\frac{r^3}{\delta} A_V \right] \end{aligned}$$

E. Laplacian of a Vector

$$\begin{aligned} (\nabla^2 \underline{A})_V &= \nabla^2 A_V + \frac{6 r_o^2 \sin \theta}{r^4 \delta^2} (1 + \cos^2 \theta) \frac{\partial A_L}{\partial V} + \frac{12 \cos \theta}{r r_o \sin^3 \theta \delta} (1 + \cos^2 \theta) \frac{\partial A_L}{\partial L} + \frac{6 \cos \theta}{r^2 \sin \theta \delta} \frac{\partial A_\phi}{\partial \phi} \\ &\quad - \left[\frac{9}{r^2 \delta^4} (1 + \cos^2 \theta)^2 + \frac{9}{r^2 \delta^2} \cos^2 \theta \right] A_V - \frac{12 \cos \theta}{r^2 \sin \theta \delta^4} (1 + 3 \cos^4 \theta) A_L \\ (\nabla^2 \underline{A})_L &= \nabla^2 A_L - \frac{6 r_o^2 \sin \theta}{r^4 \delta^2} (1 + \cos^2 \theta) \frac{\partial A_V}{\partial V} - \frac{12 \cos \theta}{r r_o \sin^3 \theta \delta} (1 + \cos^2 \theta) \frac{\partial A_V}{\partial L} \\ &\quad - \frac{2}{r^2 \sin^2 \theta \delta} (1 - 3 \cos^2 \theta) \frac{\partial A_\phi}{\partial \phi} + \frac{18 \sin \theta \cos \theta}{r^2 \delta^4} (1 + \cos^2 \theta) A_V \\ &\quad - \left[\frac{9}{r^2 \delta^4} (1 + \cos^2 \theta)^2 + \frac{1}{r^2 \delta^2 \sin^2 \theta} (1 - 3 \cos^2 \theta)^2 \right] A_L \\ (\nabla^2 \underline{A})_\phi &= \nabla^2 A_\phi - \frac{6 \cos \theta}{r^2 \sin \theta \delta} \frac{\partial A_V}{\partial \phi} + \frac{2}{r^2 \sin^2 \theta \delta} (1 - 3 \cos^2 \theta) \frac{\partial A_L}{\partial \phi} - \frac{A_\phi}{r^2 \sin^2 \theta} \end{aligned}$$

F. Directional Derivative of a Vector

$$\begin{aligned}
[(\underline{B} \cdot \underline{\nabla}) \underline{A}]_V &= (\underline{B} \cdot \underline{\nabla}) A_V + \frac{3 \sin \theta}{r \delta^3} (1 + \cos^2 \theta) B_V A_L + \frac{6 \cos \theta}{r \delta^3} (1 + \cos^2 \theta) B_L A_L \\
&\quad + \frac{3 \cos \theta}{r \delta} B_\phi A_\phi \\
[(\underline{B} \cdot \underline{\nabla}) \underline{A}]_L &= (\underline{B} \cdot \underline{\nabla}) A_L - \frac{3 \sin \theta}{r \delta^3} (1 + \cos^2 \theta) B_V A_V - \frac{6 \cos \theta}{r \delta^3} (1 + \cos^2 \theta) B_L A_V \\
&\quad - \frac{1}{r \sin \theta \delta} (1 - 3 \cos^2 \theta) B_\phi A_\phi \\
[(\underline{B} \cdot \underline{\nabla}) \underline{A}]_\phi &= (\underline{B} \cdot \underline{\nabla}) A_\phi - \frac{3 \cos \theta}{r \delta} B_\phi A_V + \frac{1}{r \sin \theta \delta} (1 - 3 \cos^2 \theta) B_\phi A_L
\end{aligned}$$

G. Divergence of a Tensor

$$\begin{aligned}
(\underline{\nabla} \cdot \underline{T})_V &= \frac{r_o^2 \delta^2}{r^4 \sin^4 \theta} \frac{\partial}{\partial V} \left(\frac{r \sin^4 \theta}{\delta} T_{VV} \right) + \frac{\delta^2}{r_o r^4 \sin^4 \theta} \frac{\partial}{\partial L} \left(\frac{r^4 \sin \theta}{\delta} T_{LV} \right) + \frac{1}{r \sin \theta} \frac{\partial T_{\phi V}}{\partial \phi} \\
&\quad + \frac{3 \sin \theta}{r \delta^3} (1 + \cos^2 \theta) T_{VL} + \frac{6 \cos \theta}{r \delta^3} (1 + \cos^2 \theta) T_{LL} + \frac{3 \cos \theta}{r \delta} T_{\phi \phi} \\
(\underline{\nabla} \cdot \underline{T})_L &= \frac{r_o^2 \delta^2}{r^4 \sin^4 \theta} \frac{\partial}{\partial V} \left(\frac{r \sin^4 \theta}{\delta} T_{VL} \right) + \frac{\delta^2}{r_o r^4 \sin^4 \theta} \frac{\partial}{\partial L} \left(\frac{r^4 \sin \theta}{\delta} T_{LL} \right) + \frac{1}{r \sin \theta} \frac{\partial T_{\phi L}}{\partial \phi} \\
&\quad - \frac{3 \sin \theta}{r \delta^3} (1 + \cos^2 \theta) T_{VV} - \frac{6 \cos \theta}{r \delta^3} (1 + \cos^2 \theta) T_{LV} \\
&\quad - \frac{1}{r \sin \theta \delta} (1 - 3 \cos^2 \theta) T_{\phi \phi} \\
(\underline{\nabla} \cdot \underline{T})_\phi &= \frac{r_o^2 \delta^2}{r^4 \sin^4 \theta} \frac{\partial}{\partial V} \left(\frac{r \sin^4 \theta}{\delta} T_{V\phi} \right) + \frac{\delta^2}{r_o r^4 \sin^4 \theta} \frac{\partial}{\partial L} \left(\frac{r^4 \sin \theta}{\delta} T_{L\phi} \right) + \frac{1}{r \sin \theta} \frac{\partial T_{\phi \phi}}{\partial \phi} \\
&\quad - \frac{3 \cos \theta}{r \delta} T_{\phi V} + \frac{1}{r \sin \theta \delta} (1 - 3 \cos^2 \theta) T_{\phi L}
\end{aligned}$$

IV. Conclusion

Most of these expressions have been used and tested in large numerical simulations of the ionosphere and magnetosphere. We hope that by writing the operations out in manual form the tedious job of re-deriving them can be avoided in the future.

REFERENCES

1. P.C. Kendall, "Geomagnetic Control of Diffusion in the F2-Region of the Ionosphere-I," *J. Atmospheric Terrest. Phys.*, 24, 805 (1962).

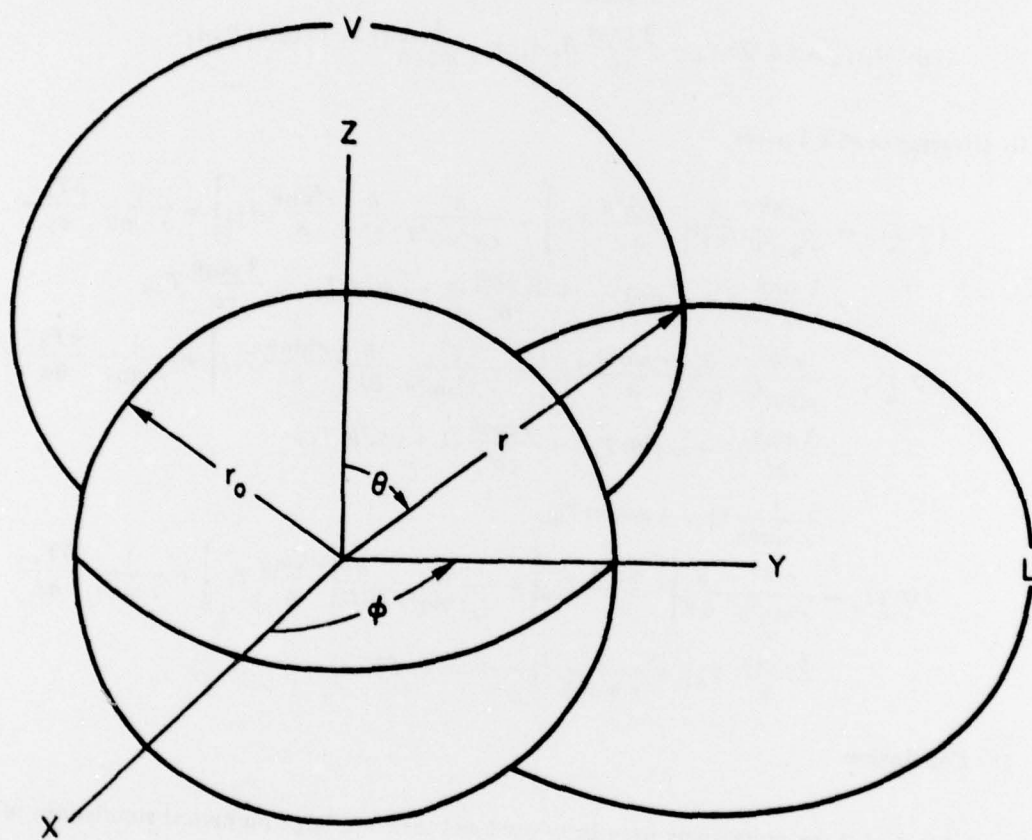


Figure 1 - The representation of a point outside a sphere of radius r_0 in both spherical and dipole coordinates.

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