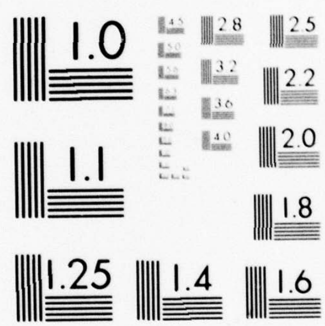
AD-A068 784 COLUMBIA UNIV DOBBS FERRY NY HUDSON LABS F/G 13/10
THE MOTION OF A NEUTRAL FLOAT AND ATTACHED DIAPHRAGM IN A PRESS--ETC(U)
FEB 61 T E POCHAPSKY N6ONR-27135
UNCLASSIFIED TM-56 NL

1 OF 1
AD
A068784

END
DATE
FILMED
7-79
DDC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

C. 2

COLUMBIA UNIVERSITY
HUDSON LABORATORIES
CONTRACT N6-ONR-27135

(1)

(6)

THE MOTION OF A NEUTRAL FLOAT AND ATTACHED DIAPHRAGM IN A PRESSURE GRADIENT*†.

(14) TM-56

by

(10) T. E. Pochapsky

(11) 10 Feb 61

(12) 27 p.

(9) Technical memo.,

(15) N6onr-27135

---

* Columbia University, Hudson Laboratories Technical Memorandum-to-File No. 56, dated 10 February 1961.

† This work was supported by the Office of Naval Research.

[redacted]

Introduction

A float that is neutrally buoyant at a fixed depth in the ocean can serve as a platform for making measurements. If the float has physical properties that are identical with those of water, a study of its motion would reveal the motion of the water. Measurements of this nature would differ from those obtained at a fixed point in space. The instruments themselves would be on an accelerating reference frame, and the meaning of their readings may not be obvious. A hydrophone inside an accelerating float could indicate pressures which do not exist. Again, no pressure fluctuations would be found under gravity waves.

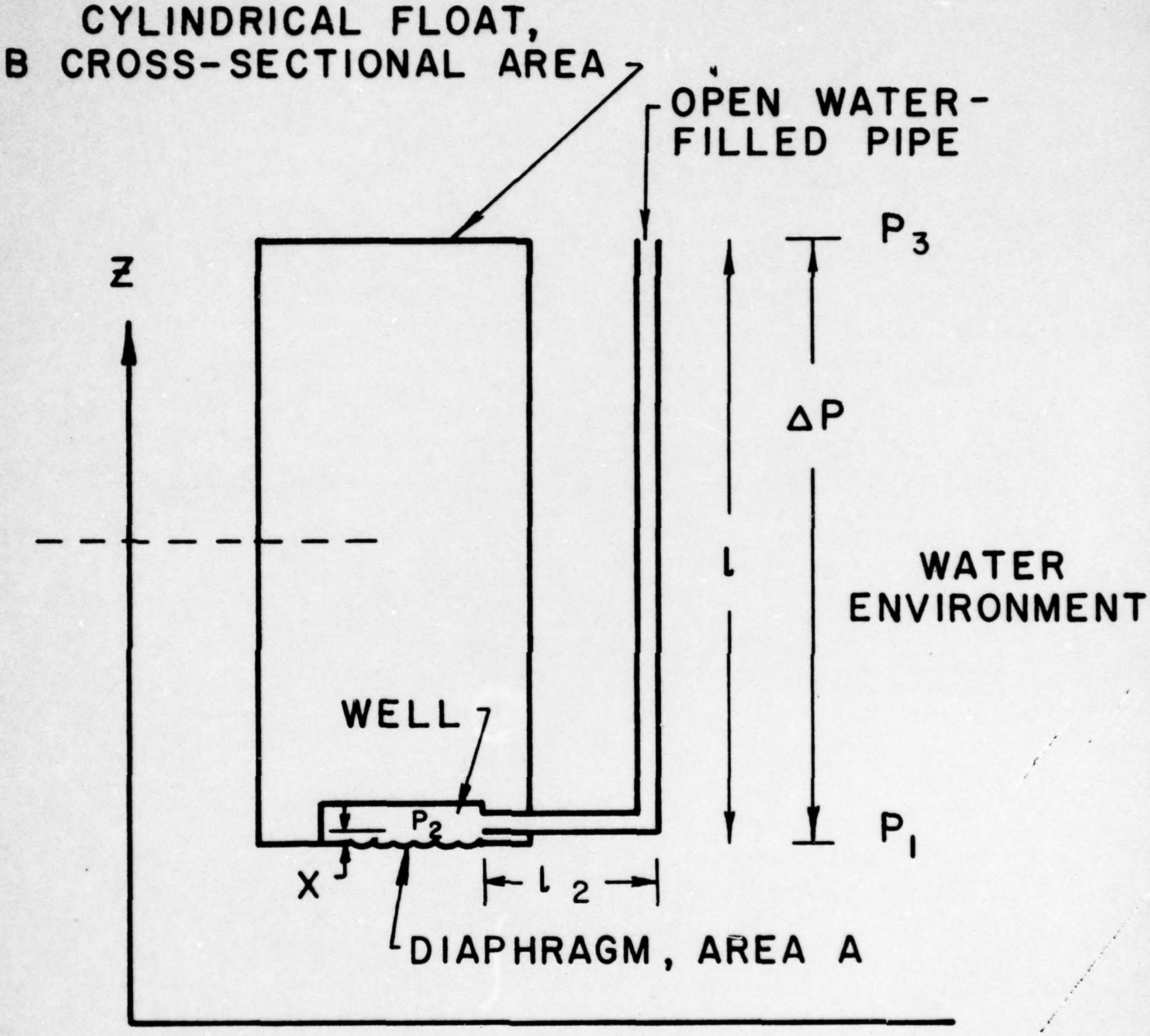
The purpose of this note is to study the response of both a float and an attached device for measuring pressure gradients. Even though the particular device is of limited interest, the study of the response of the system to oceanic motions has some fascinating features.

Statement of the Problem

An aluminum pipe, capped at both ends, is trimmed so that it has the same density as water at a certain depth. The water has a temperature, θ, which decreases with depth; the salinity is assumed to be constant.

There is a shallow well in the bottom of the float, Fig. 1, which is capped with a thin flexible diaphragm. A hole in the side of the well is vented into an open section of pipe which is bent to lie in the direction of the axis of the cylinder. The well is filled with sea water before putting the float in the water, and the pipe is filled with sea water at a depth of approximately 30 ft.

The question is: How do the float and diaphragm respond to pressure gradients in the water?



FLOAT DIMENSIONS
FIG. 1

To attempt to obtain an exact solution for the behavior of the cylinder in water would be a large undertaking. Instead, the equation used will ignore much of the interaction between the water and float and so will be exact only when there is no relative motion between the two. The justification for this is that a relatively simple approximate solution will point out the general features of the motion and show under what conditions the float might be expected to follow the water motion closely and where it would not. The disadvantage is that it will not describe the pressure field distortions around a moving float and these distortions are of paramount importance in determining the motion of the pressure detector. A better approximation to the correct solution could be obtained by adding the pressure field generated by a float that is accelerated in water to the initial pressure field. By a series of such successive approximations, an exact solution could probably be obtained for the cylinder geometry used. Such an exact solution, however, is not needed in the present investigation where only the order of magnitude of the response of the pressure gradient detector is desired. The equations used will be suitably qualified with the response of the detector in mind.

Equations of Motion

The Cylinder: The equation of motion to be used for the cylinder is:

$$1)\qquad \rho_F \ell B \ddot{z} + K_F \ell B z + \ell B \frac{\delta \rho_F}{\delta \theta} \theta' (z - z_W) g$$

$$= (p_1{}^F - p_3{}^F)(B-A) + (p_2 + KX - p_3{}^F) A + F(\dot{z}_W - \dot{z}) ,$$

where the quantities involved are

ρ_F,	average float density
K_F,	restoring force per unit volume of float per unit displacement from $z = 0$, the equilibrium position
z,	displacement of float from equilibrium depth in still water
z_W,	water displacement from equilibrium
B,	float cross-sectional area
ℓ,	cylinder length
θ,	water temperature
θ',	vertical temperature gradient
p_1^F,	average dynamic, or non-hydrostatic, pressure across the bottom of float
p_3^F,	average dynamic pressure at top of float
p_2,	average dynamic pressure just behind diaphragm
A,	diaphragm area
K,	diaphragm stiffness per unit area
X,	diaphragm deflection relative to float
F,	total drag on the float by water moving relative to it with unit velocity.

Dots denote differentiations with respect to time.

The first term in Eq. 1) represents the usual inertial reactance. The second term represents the stiffness or resistance to displacement of the float caused by the fact that it is less compressible than water and also by the fact that the density of sea water usually decreases with height fast enough to produce a stable stratification of the fluid. K_F is usually determined under adiabatic conditions; the third term corrects for temperature changes of the float which may be induced by water moving relative to the float.

The terms on the right of Eq. 1) represent the total force acting on the float. The side tube is assumed to be massless. The pressures p_1^F can be considered to be made up of two parts: (1) the overall pressure field which acts on both the water and float, and (2) pressures caused by relative motion between the float and adjacent water. The effect of these latter pressures on the relatively large volume of water around, but not far from, the float will be ignored.

Reasonable approximations can be made to determine the pressures caused by the relative motion between the float and water. A steady flow, for instance, will generate a kinetic pressure, $1/2\rho(\dot{z}_F - \dot{z}_W)^2$ near the ends of the float. In an ideal frictionless fluid, these pressures will produce no acceleration. In a viscous fluid, they may contribute along with the viscous drag at the float surfaces to the acceleration. Their effect on the acceleration of the entire float can be considered to be included in the expression $F(\dot{z}_W - \dot{z})$ for the frictional drag. Later, when the motion of water in the side tube is considered, terms in $\rho\dot{z}^2$ will have to be considered.

When a solid is accelerated in a fluid, extra momentum is imparted to the fluid and the solid seems to have a higher inertia than its own mass would indicate. In other words, an accelerating solid has pressure differences between its ends which are proportional to the acceleration.

A sphere accelerated in water behaves as though it has an extra mass equal to one third the mass of displaced water. A cylinder would probably not have as high a correction. If the extra mass fraction is denoted by α, the average pressure drop generated along the float by relative acceleration is

$$\alpha\rho_W\ell(\ddot{z}_F - \ddot{z}_W) \quad .$$

This term must contribute to p_1^F and p_3^F along with any contribution from an externally induced field.

<u>The Diaphragm</u>: The displacement, X, of the diaphragm satisfies the equation

$$2)\qquad p_1^D = p_2 + KX + m\ddot{z} + (m+\mu)\ddot{X} \quad ,$$

where m is the mass per unit area of the diaphragm and μ is a radiation inertia term. The other quantities were described in the previous section. As implied in the previous section, the pressure on the exterior of the diaphragm, p_1^D, will not be identical with the average pressure p_1^F.

When the diaphragm moves relative to the float, it pushes water around and the radiation reactance of this motion is quite large. It can be shown that water on the exterior face of the diaphragm adds an inertial mass per unit area of $\mu = 2\rho \times$ radius of diaphragm. Inertial loading behind the diaphragm can probably be ignored because most of it takes place in the side tube whereas p_2 is measured in the well.

<u>The Venting Tube</u>: The equation used to describe the motion of fluid in the venting pipe is

$$3)\qquad p_2 - p_3^T = \rho_T(\ell+\ell_2)\,\frac{\ddot{X}A}{a} + \rho_T\ell\ddot{z} + \ell_1 K_W\left(z+\frac{XA}{a}\right) + f\,\frac{\dot{X}A}{a} \quad ,$$

where

- p_3^T, pressure at open end of pipe
- ℓ, vertical length of pipe parallel to the cylindrical float, assumed equal to the float length

ℓ_2, length of horizontal pipe

A, diaphragm area

a, pipe cross-sectional area

K_W, stiffness of water in a stratified medium

f, a friction factor

The mass of fluid in the side tube is considered constant, despite the pumping motion of the diaphragm. The "organ-pipe" end correction for ℓ is also ignored. Acceleration of the float produces no pressure drop along the horizontal pipe, so the term $\rho_T \ell_2 \ddot{z}$ does not appear above.

Approximate Solution

An external source is assumed to induce a pressure gradient near the float which is independent of horizontal position for distances large compared to the dimensions of the float. This pressure field is modified by the motion of the float in a way pointed out earlier. If the externally induced pressures at the horizontal planes through the bottom and top of the cylinder are p_1 and p_3, respectively, the pressures near the float can be written

$$
\begin{aligned}
p_1^D &= p_1 + \Delta_{1D}(\mathring{z}, \mathring{\mathring{z}}) \\
p_1^F &= p_1 + \Delta_{1F}(\mathring{z}, \mathring{\mathring{z}}) \\
p_3^F &= p_3 + \Delta_{3F}(\mathring{z}, \mathring{\mathring{z}}) \\
p_3^T &= p_3 + \Delta_{3T}(\mathring{z}, \mathring{\mathring{z}}) \quad .
\end{aligned}
\qquad 4)
$$

When these values are substituted in Eqs. 1) and 2) and p_2 eliminated between the two equations, the following relationship is found:

5) $$\rho_F \ell B \ddot{z} + K_F \ell B z + g \ell B \frac{\partial \rho_F}{\partial \theta} \theta' \ (z - z_W)$$

$$= B\Delta p + A(\Delta_{1D} - \Delta_{3F}) + (B-A)(\Delta_{1F} - \Delta_{3F}) - Am\ddot{z} - A(m+\mu)\ \ddot{X} + F(\dot{z}_W - \dot{z}).$$

Δp is written for $p_1 - p_3$.

Similarly, Eqs. 2) and 3) yield

6) $$\rho_T(\ell + \ell_2)\ \frac{\ddot{X}A}{a} + \rho_T \ell \ddot{z} + \ell K_W (z + \frac{XA}{a}) + f\ \frac{\dot{X}A}{a}$$

$$= \Delta p + \Delta_{1D} - \Delta_{3T} - KX - m\ddot{z} - (m+\mu)\ \ddot{X}\ .$$

Now assume that $\Delta p(t)$ is of the form $\Delta p e^{i\omega t}$. Equation 5) becomes

7) $$\left(K_F \ell B - \omega^2 \rho_F \ell B - Am\omega^2 + i\omega F + g\ell B\ \frac{\partial \rho_F}{\partial \theta}\ \theta'\right) z$$

$$= B\Delta p + A(m+\mu)\omega^2 X + i\omega F z_W + g\ell B\ \frac{\partial \rho_F}{\partial \theta}\ \theta'\ z_W$$

$$+ A(\Delta_{1D} - \Delta_{3F}) + (B-A)(\Delta_{1F} - \Delta_{3F})\ ,$$

and Eq. 6) becomes

8) $$(\omega^2 \rho_T \ell - \ell K_W + \omega^2 m)\ z$$

$$= -\Delta p - \Delta_{1D} + \Delta_{3T} - \omega^2 \rho_T\ (\ell + \ell_2)\ \frac{A}{a}\ X$$

$$+ \ell K_W\ \frac{A}{a}\ X + KX - \omega^2 (m+\mu)\ X + i\omega f\ \frac{A}{a}\ X\ .$$

When Eqs. 8) and 7) are combined,

9)

$$(\omega^2 \rho_T \ell - \ell K_W + \omega^2 m) \frac{[\Delta p + (m+\mu)\omega^2 XA/B + i\omega F z_W/B + g\ell \frac{\partial \rho_F}{\partial \theta} \theta' z_W + \frac{A}{B}(\Delta_{1D} - \Delta_{3F}) + \frac{(B-A)}{B}(\Delta_{1F} - \Delta_{3F})]}{K_F \ell - \omega^2 \rho_F \ell - Am\omega^2/B + i\omega F/B + g\ell \frac{\partial \rho_F}{\partial \theta} \theta'}$$

$$= X[K + \ell K_W \frac{A}{a} - \omega^2 \rho_T (\ell + \ell_2) \frac{A}{a} - \omega^2 (m+\mu) + i\omega f \frac{A}{a}] - \Delta p - \Delta_{1D} + \Delta_{3T} .$$

Fquation 9) will not be graced with the title "solution." It and Eq. 8), however, form a useful guide for determining motions in circumstances where conditions are particularly simple. A major reason for this analysis is to determine the response of the diaphragm under a gravity wave. Experiments with floats showed approximately a 1-percent response to gravity wave pressure gradients at various depths. Maximum expected values for the constants in the above equations should allow determining whether such a response is reasonable.

The simplifying condition in the pressure gradient measurements was that most of the response of the diaphragm was sharply peaked at $\omega = 1.5$ rad sec^{-1}. It will be shown that at this frequency $\omega^2 \rho_T \gg K_W \ell$. In fact, $\omega^2 \rho_T$ can be shown large compared to any of the other coefficients with the possible exception of the Δ's. Now consider the left side of Eq. 9), excluding for a while the bracketed part of the expression. This can be written

10)

$$= \frac{1 - \frac{K_W}{\omega^2 \rho_T} + \frac{m}{\rho_T \ell}}{1 + \frac{\delta p}{\rho_T} - \frac{K_F}{\omega^2 \rho_T} + \frac{A}{B}\frac{m}{\rho_T \ell} - \frac{iF}{B\omega \rho_T \ell} - \frac{g \frac{\partial \rho_F}{\partial \theta} \theta'}{\omega^2 \rho_T}}$$

where all the parameters are small compared with unity and $\rho_F = \rho_T + \delta\rho$.

The water displacement, z_W, for the case $\omega^2 \rho_T \gg K_W \ell$ can be written

$$z_W = -\frac{\Delta p}{\omega^2 \rho_W \ell} \qquad (11)$$

With approximations of the type $(1+a)^{-1} = 1 - a$, the left side of Eq. 9) can now be written

$$= \left(1 - \frac{K_W}{\omega^2 \rho_T} + \frac{m}{\rho_T \ell} - \frac{\delta\rho}{\rho_T} + \frac{K_F}{\omega^2 \rho_T} - \frac{A}{B}\frac{m}{\rho_T \ell} + \frac{iF}{B\omega\rho_T \ell} + \frac{g\frac{\partial \rho_F}{\partial \theta}\theta'}{\omega^2 \rho_T}\right)\Delta p \qquad (12)$$

$$- \left((m+\mu)\omega^2 X \frac{A}{B} - \frac{iF\Delta p}{B\omega\rho_W \ell} - g\frac{\partial \rho_F}{\partial \theta}\theta' \frac{\Delta p}{\omega^2 \rho_W} - \frac{A}{B}(\Delta_{1D} - \Delta_{3F}) - \frac{(B-A)}{A}(\Delta_{1F} - \Delta_{3F})\right).$$

Now, certain terms would drop out of this expression if $\rho_W = \rho_T$. This equality may not always hold but the densities should never differ by more than 1 part in 100. The coefficients of such terms in ρ are

$$\frac{F}{B\omega\rho_W \ell} \approx \frac{10^2}{380 \times 1.5 \times 1 \times 10^2} \approx 2 \times 10^{-3}$$

$$\frac{g \frac{\partial\rho_F}{\partial\theta} \theta'}{\omega^2 \rho_W} \approx \frac{10^3 \times 2.5 \times 10^{-4} \times 4 \times 10^{-4}}{2.3} \approx 4 \times 10^{-5}$$

$$\frac{K_F}{\omega^2 \rho_W} \approx \frac{14 \times 10^{-5}}{2.3} \approx 6 \times 10^{-5} \quad .$$

The values of the constants used here can be obtained from the Appendix to this report. Difference terms involving these coefficients and the densities, Eq. 12), should then be zero to order 10^{-5}. To within this accuracy, Eq. 9) can be written, after considering Eq. 12)

13) $$[\ldots] X = \Delta_{1D} - \Delta_{3T} - \frac{A(\Delta_{1D} - \Delta_{3F})}{B} - \frac{(B-A)(\Delta_{1F} - \Delta_{3F})}{B}$$

$$+ \left[\frac{K_W - K_F}{\omega^2 \rho_T} - \frac{B-A}{B} \frac{m}{\rho_T \ell} + \frac{\delta_\rho}{\rho_T}\right] \Delta p - (m+\mu)\omega^2 X \frac{A}{B} \quad .$$

The later terms have magnitudes

$$\frac{K_F}{\omega^2 \rho} \approx \frac{K_W}{\omega^2 \rho} \approx \frac{12 \times 10^{-5}}{2.3} \qquad 5 \times 10^{-5}$$

$$\frac{m}{\rho_T \ell} \frac{B-A}{B} \approx .004 \times 10^{-2} \times 1 \approx 4 \times 10^{-5} \text{ for aluminum}$$

$$\approx 6 \times 10^{-4} \text{ for platinum}$$

$$(m+\mu)\omega^2 X \frac{A}{B} \approx (.056+5) \times 2.3 \times 10^{-4} \times \frac{20}{380} \approx 7 \times 10^{-5} \quad .$$

Then

$$14) \qquad [\ldots]\,X = \Delta_{3F} - \Delta_{3T} + (\Delta_{1D} - \Delta_{3F})(1 - \frac{A}{B}) + \frac{\delta\rho}{\rho_T}\Delta p$$

to an accuracy of 1 part in 10^4, excluding the case of a heavy platinum diaphragm. Again, $\delta\rho = \rho_F - \rho_T$.

Earlier, it was shown that the Δ's have the form

$$15) \qquad \Delta = \alpha\rho_W \ell\,(\ddot{z} - \ddot{z}_W) + \beta\rho_W\,(\dot{z} - \dot{z}_W)^2 \quad ,$$

where α and β are constants with values less than unity. For $\omega = 1.5$ and $\ell = 10^2$ cm , the first of the expressions on the right is dominant. The order of magnitude of the difference between z and z_W can be found by writing Eq. 5) in the approximate form

$$16) \qquad \rho_F \ell B\ddot{z} + K_F \ell Bz + g\ell B\,\frac{\partial\rho_F}{\partial\theta}\,\theta'\,(z - z_W) = B\Delta p + A\alpha'\rho_W\ell\,(\ddot{z} - \ddot{z}_W)$$

and noting that

$$17) \qquad \rho_W\,\ddot{z}_W + K_W\,z_W = \frac{\Delta p}{\ell}$$

for a large volume of water relative to the size of the float. It can be verified from these that

18) $$\frac{z_W}{z} \approx 1 + \frac{\rho_F - \rho_W}{\rho_W} + \frac{K_F - K_W}{\rho_F \omega^2}$$

$$\approx 1 + \frac{\rho_F - \rho_W}{\rho_W}$$

and from Eqs. 15), 17), and 18)

19) $$\Delta \approx a \rho_W \ell \omega^2 z \left(\frac{\rho_W - \rho_F}{\rho_W} \right)$$

$$\approx a \left(\frac{\rho_W - \rho_F}{\rho_W} \right) \Delta p \quad .$$

It is consequently apparent that the diaphragm response would be controlled by mismatches in density $\Delta\rho$ between the float, water in the side tube and water external to the float. Eq. 14) will then be written in the form of an inequality

20) $$[\ldots] X < \left[\frac{3a\,|\rho_F - \rho_W|}{\rho} + \frac{2\,|\rho_T - \rho_W|}{\rho} \right] \Delta p$$

where the coefficient of X is the same as in Eq. 9).

There are probably only three major reasons for a density mismatch. First, the water in the side tube may differ from that outside the float. Secondly, the water may have temperature fluctuations about the average gradient and, lastly, float motions outside the examined frequency range may produce a mismatch in density.

Water in the side tube is sealed in near the surface. The salinity difference between this and deeper water would certainly be less than 2 $^o/oo$, so that the maximum possible density difference would be less than 1.5×10^{-3} gm cm^{-3}. Temperature fluctuations about the average would probably not exceed 1°C, for which the density change is 3×10^{-4} gm cm^{-3}.

At very low frequencies, the float amplitude is controlled by K_F, Eq. 5), and the water amplitude is controlled by K_W, Eq. 17). These have values of 14×10^{-5} dyne cm^{-1} and 12×10^{-5} dyne cm^{-1}, respectively, so that the amplitudes would have the ratio 0.84. It is conceivable that low period internal waves were present and had an amplitude of 15 m. When the water gradient is 4×10^{-4} °C/cm, a failure of the float to follow the motion of such waves could result only in a trivial density mismatch.

The result of the above analysis shows that the diaphragm should respond to less than a few tenths of a percent of the pressure gradient associated with waves having a wavelength much larger than the largest dimension of the float. The approximations were generous in trying to get a response; in practice, a 0.1 percent limit should probably be applied.

Response to Gravity Waves

In a steady 15-knot wind, surface waves develop to a 27.3 cm average amplitude or to a 55.5 cm average height; the average period of these waves is 7.5 seconds and the wavelength is 88 m (J. Darbyshire, 1959). The characteristics of a progressive gravity wave having these average features are shown as a function of depth in the following table:

Gravity Wave Pressures

Depth, meters	Amplitude, cm	Particle velocity, cm sec^{-1}	Pressure, dynes cm^{-2}	Pressure gradient, dyne cm^{-3}	Expected maximum instrument response dyne cm^{-2}
0	27.3	23	2.67×10^4	19.1	1.9
20	4.0	3.4	3.9×10^3	2.77	.3
60	0.38	.32	3.7×10^2	0.26	.03
100	0.02	.02	2.1×10	0.015	.002

The expected instrument response is based on measuring 0.1 percent of the pressure drop over a meter distance. The approximate rms responses actually obtained under different conditions are shown in the next table.

Instrument Response

Depth, meters	Mirror response, mm.	Minimum probable pressure	Location in Atlantic Ocean
13	0.3	0.2	Off shelf, New York
17	1.3	0.7	"
21	3.3	2.0	"
64	.6	0.4	Bermuda
98	.0	.0	Virgin Islands
102	.1?	.06	"
109	.1?	.06	Bermuda

Pressures at depths between 190 m and 700 m were too small to detect. The minimum probable pressure was obtained by using an average d. c. sensitivity of 3 dynes cm^{-2} mm^{-1} and then allowing a fivefold increase in sensitivity to allow for instrument resonances. This procedure probably underestimates the pressures acting on the diaphragm. The results near 100 m are questioned because (1) the float at Bermuda was not completely at rest and (2) the Virgin Islands result was obtained with a platinum diaphragm which may have responded to temperature fluctuations.

These results show that the maximum probable observed pressures were somewhat larger than the maximum expected to be measured. This leads to some reluctance to attribute the pressures to gravity waves, but some strong counter arguments can be made: (1) Responses increase with depth near the surface as would be expected if strongly saline surface waters fill the measuring pipe; (2) the measurements near 100 m are questionable; (3) the 100 m measurement near the Virgin Islands was performed with a critically damped system and the predominant frequency had a period near that of the surface waves, 10 sec; (4) there were too few shallow tests in too many locations to give a good description of the depth effect being measured; (5) a frequency analysis showed a considerably broader spectrum than expected from the characteristics of the instrument; higher frequencies could have been introduced by diaphragm resonances and lower ones by gravity waves.

After all this, one is still left with the original discrepancy. The first reaction is that the instrument measures pressure gradients associated with gravity waves, but this is on a shaky quantitative ground. A definite answer is needed to know whether gravity waves or turbulent and internal wave motions are the sources of excitation.

Motion of the Float

For most purposes, the motion of the float and surrounding water is adequately described by Eqs. 16) and 17). It would be useful to make only a few general observations based on these equations.

Equation 17) shows that water motion can be excited to high amplitudes at a frequency given by

$$\omega = \sqrt{\frac{K_W}{\rho_W}} \quad .$$

This frequency, called the Väisälä or stability frequency, has an associated period ranging from about 10 minutes to many hours, depending on K_W. Internal waves will propagate only at frequencies lower than this Väisälä frequency.

Similarly, the float has a resonance which varies with K_F. In the more stable waters, K_F is only slightly greater than K_W and the float will have approximately the same resonant period as the water. K_F, however, is determined by the low compressibility of the aluminum shell as well as by the stability of the surrounding water and so it varies less with water conditions than does the Väisälä frequency. The float resonant frequency can change only by a factor of about two throughout the ocean.

In the previous sections, interest was centered on frequencies well above the resonant frequencies of the float and water where the equality of density allowed the float to behave mechanically in the same way as an equivalent amount of water. Stable water, with a large temperature gradient, formed the environment and the K's for the water and shell were sufficiently close for the float to follow water motions to a large extent at frequencies well below resonance as well. Motion near the resonant frequency is a special case which need not be discussed here.

These conditions change in deep water where the Väisälä frequency is very low and $K_F \gg K_W$. At the low frequency end, the float would hardly respond whereas large water motions are likely to take place. Sizeable vertical motions of the float could only take place near and above its resonance frequency, a region where little kinetic energy is likely to be present.

Appendix

This section deals with evaluating the stiffness constants K_F and K_W for displacements in a stably stratified medium.

A stratified medium is one in which the density, or some other property, of the fluid depends only on the distance, z, from some horizontal reference plane. z will be measured upward. Then, the density difference between two planes Δz apart can be written

$$\text{1A)} \qquad \frac{d\rho}{dz} \Delta z \ .$$

When a unit volume of fluid is moved through the distance Δz, it will undergo a density change as a result of the pressure difference between the two levels. The thermal diffusivity of the medium is assumed sufficiently small for the change to take place adiabatically. The resulting density change can be written

$$\text{2A)} \qquad \left(\frac{\partial \rho}{\partial p}\right)_\eta dp = -\left(\frac{\partial \rho}{\partial p}\right)_\eta \rho g \Delta z = -\frac{1}{c^2} \rho g \Delta z \quad ,$$

where η is the entropy; ρ, the density; p, the pressure; g, the acceleration of gravity; and c, the velocity of sound in the medium.

In general, the density of displaced fluid will differ from that of the fluid on the new level and there will be a net <u>buoyant</u> force per unit volume given by

$$\text{3A)} \qquad + g \left(\frac{d\rho}{dz} - \left(\frac{\partial \rho}{\partial p}\right)_\eta \frac{dp}{dz}\right) \Delta z \ .$$

This expression can be examined in more detail by expressing the density as a function of the temperature, θ , and the pressure.

4A) $$\frac{d\rho}{dz}\Delta z = \left(\frac{\partial\rho}{\partial\theta}\right)_p \frac{d\theta}{dz}\Delta z + \left(\frac{\partial\rho}{\partial p}\right)_T \frac{dp}{dz}\Delta z \quad .$$

It is then apparent that the buoyancy depends on the thermal expansion of the medium in a temperature gradient and on the dif-ference between the isothermal and adiabatic compressibilities of the fluid. The latter quantity is usually very small, but it may be the most important factor in deep water.

Compressibilities

The difference between the adiabatic and isothermal compressibilities can be obtained thermodynamically. Application of the result to a cylindrical container is irregular, so that a complete treatment of the problem will be given here.

The volume of the float, or of water around it, can be expressed as a function of two thermodynamic variables. It can be expressed as $V(\alpha, \beta)$ for the variables α, β or $V'(a, b)$ for the variables a, b . Derivatives in terms of the first variables can be expressed in terms of the second, so that

5A) $$\left(\frac{\partial V}{\partial \alpha}\right)_\beta = \left(\frac{\partial V'}{\partial a}\right)_b \left(\frac{da}{d\alpha}\right)_\beta + \left(\frac{\partial V'}{\partial b}\right)_a \left(\frac{db}{d\alpha}\right)_\beta \quad .$$

It is then correct to write

6A) $$\left(\frac{\partial V}{\partial p}\right)_\eta = \left(\frac{\partial V}{\partial p}\right)_\theta + \left(\frac{\partial V}{\partial \theta}\right)_p \left(\frac{d\theta}{dp}\right)_\eta ,$$

where the pressure and entropy, p and η, are one set of variables and the pressure and temperature, p and θ, are the second set. The isentropic volume change is then seen to be caused by the isothermal compressibility and by thermal expansion associated with the temperature change in the adiabatic process.

By starting with the enthalpy equation and using $\eta(p, \theta)$, it is possible to show

7A) $$d\eta = c'_p \frac{d\theta}{\theta} - \left(\frac{\partial V}{\partial \theta}\right)_p dp$$

or

$$\left(\frac{d\theta}{dp}\right)_\eta = \frac{\theta}{c'_p} \left(\frac{\partial V}{\partial \theta}\right)_p .$$

Then,

$$\left(\frac{\partial V}{\partial p}\right)_\eta = \left(\frac{\partial V}{\partial p}\right)_\theta + \frac{\theta}{c'_p} \left(\frac{\partial V}{\partial \theta}\right)_p^2 .$$

c'_p is the heat capacity at constant pressure of the entire volume V. If this last equation is divided by V, the derivatives can be replaced by effective compressibility, β, or thermal expansion, α, terms in the following way:

8A) $$\beta_\eta = \beta_\theta - \frac{V\theta}{c'_p} \alpha^2 .$$

Note that $\beta = -\frac{1}{V} \left(\frac{dV}{dp}\right)$.

The total heat capacity of the float can be written

9A) $$c'_p = \Sigma m_i c_{ip} \quad ,$$

where c_{ip} is the heat capacity per gram of material i present in a mass m_i in the float. It will be assumed that m_i consists only of aluminum, for which $c_p = 0.21$ cal gm^{-1} oC^{-1} . The ratio $\frac{m}{V} = \rho_W$, the density of surrounding water when the float is neutrally balanced. β_θ for the float is known to be 1.8×10^{-11} dyne^{-1} cm^2 ; the compressibility of solid aluminum is 1.42×10^{-12} . For a float under typical conditions, Eq. 8A) shows

$$\beta - \beta_\theta = - \frac{298 \times (69 \times 10^{-6})^2}{1.03 \times 0.21 \times 4.18 \times 10^7}$$

$$= - 1.6 \times 10^{-13}$$

or

$$\left(\frac{\beta - \beta_\theta}{\beta_\theta}\right)_{\text{Float}} = - 9 \times 10^{-3} \quad .$$

The same ratio for solid aluminum is .04.

Similarly, for water

$$\beta - \beta_\theta = - \frac{298 \times (300 \times 10^{-6})^2}{1.03 \times 4.18 \times 10^7}$$

$$= - 6.2 \times 10^{-13}$$

and

$$\frac{\beta - \beta_\theta}{\beta_\theta}\ \text{water} = - \frac{6.2 \times 10^{-13}}{4.3 \times 10^{-11}} = 1.5 \times 10^{-2} \ .$$

This ratio for water is only indicative. It varies appreciably with water conditions because of the unusual thermal expansion properties of water.

Evaluation of Stiffness Constants

The restoring force on a unit volume of material displaced adiabatically in a stratified medium was shown to be, Eqs. 3A) and 4A),

$$K\Delta z = -g \left[\left(\frac{\partial \rho}{\partial \theta}\right)_p \frac{d\theta}{dz} - \left(\left(\frac{\partial \rho}{\partial p}\right)_T - \left(\frac{\partial \rho}{\partial p}\right)'_\eta \right) \rho g \right] \Delta z$$

or

10A) $$= +g\rho \left[\alpha_p \frac{d\theta}{dz} - \left(\beta'_\eta - \beta_T\right) \rho g \right] \Delta z \ .$$

α_p and β_T refer to the medium; β'_η refers to the substance displaced.

The above expression will be evaluated first for water motions only.

Experiments were performed at locations where the top 100 meters of water had a gradient of approximately 4×10^{-4} °C cm^{-1}. The temperature gradient at a 1000-meter depth was about a factor of ten less. The thermal expansion of the surface waters, which were at a temperature near 25°C, was 3×10^{-4} °C^{-1}. At a depth of 1000 m, the expansion coefficient should have been 1.4×10^{-4} $^{o}C^{-1}$. These values for the surface waters and the values of the compressibility difference found in the preceding section can be substituted in Eq. 10A) to find K_W :

$$K_W = 980\left[3 \times 10^{-4} \times 4 \times 10^{-4} + 6.2 \times 10^{-13} \times 1.03 \times 980\right] 1.03$$

$$= 12 \times 10^{-5} \text{ dyne cm}^{-4} .$$

The effect of the difference in compressibilities is obviously negligible at this depth and stability is caused by the temperature gradient. It affects the value of K_W by approximately 10 percent at the 1000-m level. In the absence of a temperature gradient, the compressibility term alone would produce a K_W of approximately 6×10^{-7} dyne cm^{-4} .

The stiffness associated with displacements of the float can now be calculated for the near surface case:

$$K_F = 980\left[3\times 10^{-4}\times 4\times 10^{-4} - (1.8\times 10^{-11} - 4.3\times 10^{-11})\ 1.03\times 980\right] 1.03$$

$$= 980\left[12\times 10^{-8} + 2.5\times 10^{-8}\right]$$

$$= 14\times 10^{-5} \text{ dyne cm}^{-4} .$$

Again, the stability of the float in surface waters is controlled by the temperature gradient in the water, but now the float incompressibility makes a contribution which becomes dominant in smaller temperature gradients.

The relatively small float will not respond adiabatically in slow motions but will assume the temperature of nearby water which does react adiabatically. For most purposes, the difference between adiabatic and isothermal response is small enough to ignore, but correction must be made if water washes over the float so as to change its temperature. It is for this reason that the term in $\theta'(z-z_W)\,\partial\rho_F/\delta\theta$ is included in Eq. 1).