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THE RELATIONSHIP BETWEEN LJUSTERNIK-SCHNIRELMAN CATEGORY AND THE CONCEPT OF GENUS

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ABSTRACT

The concept of genus of an invariant, closed set A in a paracompact free G-space E is introduced for any compact Lie group G and the general result that G-genus $A = \mathrm{cat}_B A^*$ is proven where $B = E/G$, $A^* = E/G$ and cat is short for Ljusternik-Schnirelman category. As a special case, the concept of genus (Krasnoselskii) coincides with the notion of category (Ljusternik-Schnirelman) as employed in a real or complex Banach space.

AMS(MOS) Subject Classification: 55C30, 58E05

Key Words: Ljusternik-Schnirelman category, Krasnoselskii genus, Lie group, G-genus, paracompact free G-space

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SIGNIFICANCE AND EXPLANATION

A central problem in various applications of mathematics to the sciences is the determination of critical points of functions and the corresponding critical values, i.e., points where the function exhibits unusual behavior and the values of the function at these points. For example, the maximum and minimum values of the function are among its critical values. It is also important to find other critical values which might be local extremes, points of inflections, saddle points etc. Using a very special classification of subsets of the domain, Ljusternik and Schnirelman devised a technique for finding these additional critical points. Later, Krasnoselskii developed an alternative classification of subsets of the domain providing an (ostensibly) alternative method. The natural question arises as to whether the Krasnoselskii method provides a different set of critical values than the Ljusternik-Schnirelman method. The object of this note is to verify that these two methods provide precisely the same set of critical values, so that additional critical values are not obtained by combining the two methods.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

THE RELATIONSHIP BETWEEN LJUSTERNIK-SCHNIRELMAN CATEGORY

AND THE CONCEPT OF GENUS

Edward Fadell†

1. Introduction.

The Min-Max Principle in critical point theory as introduced by Ljusternik-Schnirelman [1] is based on the concept of category of a set X in an ambient space B. Krasnoselskii [2] and others [3], [4], employed the concept of genus instead of category. For example, consider the following setting. Let E denote a Banach space and observe that $\mathbb{Z}_2 = \{-1,1\}$ acts freely on $E - 0$ by scalar multiplication. Let Σ denote the closed invariant (symmetric) subsets of $E - 0$. Furthermore, let $B = E-0/\mathbb{Z}_2$ and for $A \in \Sigma$, set $A^* = A/\mathbb{Z}_2$. Then,

$$\operatorname{cat}_B A^* = k$$

is defined to mean that there exist k sets $A_1,\ldots,A_k$ in Σ such that for each i, A_i^* is contractible to a point in B and k is minimal with this property ($k = \infty$, allowed). Thus the function γ given by

$$\gamma(A) = \operatorname{cat}_B A^*$$

classifies the elements of Σ.

Alternatively, following Krasnoselskii [2], the statement

$$\text{genus } A = k$$

is defined to mean that there exists an equivariant (odd) map $f: A \to \mathbb{R}^k-0$ and k is minimal with this property ($k = \infty$ means that there is no equivariant map $f: A \to \mathbb{R}^k-0$, for any finite k and, as usual, $\mathbb{R}^k$ is Euclidean k-space).

1.1 Remark. Actually this concept of "genus" was introduced and studied earlier by Yang [5] under the name "B-index". In fact, genus A = B-index + 1.

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The function γ' given by

$$\gamma'(A) = \text{genus } A$$

also classifies the sets in Σ. Our objective in this note is to verify that these classifications are identical in general, i.e.,

(1) $$\gamma(A) = \text{cat}_M A^* = \text{genus } A = \gamma'(A), \quad A \in \Sigma .$$

A special case of (1) for compact A's is contained in Rabinowitz [3]. We will verify (1) in a very general setting as follows.

Let E denote any contractible paracompact free G-space where G is a compact Lie group. Let Σ denote the closed, invariant subsets of E and set $B = E/G$. Then for $A \in \Sigma$, $\text{cat}_B A^*$ is defined as before, where A^* is the orbit space A/G. Now, set G-genus $A = k$ if there is a G-equivariant map

(2) $$f: A \to \overbrace{G \circ G \circ \cdots \circ G}^{k}, \quad \text{(k-fold join [6])}$$

and k is minimal with this property.

<u>Theorem</u>. For $A \in \Sigma$ we have

(3) $$\text{cat}_B A^* = \text{G-genus } A.$$

Note that (1) is a corollary of (3) by taking $G = \mathbb{Z}_2$ and observing that the k-fold join of the 0-sphere S^0 is just S^{k-1} which is the unit sphere in $\mathbb{R}^k$. The corresponding result to (1) for complex Banach spaces is obtained by taking $G = S^1$, unit circle of complex numbers of norm 1. We should also remark that the idea of using (2) for an "index theory" appears briefly in [7].

2. Preliminaries.

Throughout G will denote a compact Lie Group and $\mathfrak{F}$ will denote the category of free paracompact G-spaces. An object $X \in \mathfrak{F}$ may be identified with the principal bundle $p : X \to X/G$, where p is the natural projection to the orbit space X/G. Hence, the general theory of principal bundles over a paracompact base applies (see [8]). We will also find the following definitions convenient.

2.1 Definition. A free G-space $Y \in \mathfrak{F}$ is called a G-ENR (Euclidean Neighborhood Retract G-space) if

a) there is a real representation $\varphi: G \to O(n)$ of G as orthogonal matrices for some n;

b) there is an equivariant imbedding $h:Y \to \mathbb{R}^n$ of Y in $\mathbb{R}^n$, i.e. $h(gy) = \varphi(g)\, h(y)$;

c) there is an invariant neighborhood U of $f(Y) \subseteq \mathbb{R}^n$ and an equivariant retraction of U onto $f(Y)$, i.e. there is a map $r:U \to h(Y)$ such that $r(u) = u$ when $u \in f(Y)$ and $r(\varphi(g)u) = \varphi(g)\, r(u)$.

2.2 Proposition. Let $X \in \mathfrak{F}$, A a closed invariant subspace of X and Y a G-ENR. Then any equivariant map $f:A \to Y$ has an equivariant extension $\bar{f}:V \to Y$, where V is an invariant neighborhood of A in X.

Proof. We assume without loss that $Y \subseteq \mathbb{R}^n$ and $G \subseteq O(n)$. Then, employing the Tietze-Gleason Extension Theorem [9], there is an equivariant extension $F:X \to \mathbb{R}^n$. Let U denote the invariant neighborhood of Y which admits an equivariant retraction $r:U \to Y$. Then, if $V = r^{-1}(U)$, $f = r \circ (F|V)$ is the required extension: $V \to Y$.

2.3 Remark. The compact Lie group G is a G-ENR [9]. In fact, every compact smooth G-manifold is a G-ENR [9]. Hence, the neighborhood extension theorem (Proposition 2.2) applies for maps into these spaces. Palais [9] defines a G-ANR as a space Y which satisfies Proposition 2.2 for normal spaces X, so that every G-ENR is a G-ANR.

We also recall the notion of join. Let $Y_1, Y_2, \ldots, Y_k$ denote G-spaces and consider the space

(4) $$(I \times Y_1) \times (I \times Y_2) \times \ldots \times (I \times Y_k)$$

a point of which is designated by

(5) $$(t_1y_1, t_2y_2, \ldots, t_ky_k) .$$

Let J denote the subset of (4) consisting of points (5) with the added condition that $\Sigma t_j = 1$. Define an equivalence relation $\sim$ by setting

$$(t_1y_1, t_2y_2, \ldots, t_ky_k) = (t_1'y_1', t_2'y_2', \ldots, y_k'y_k')$$

if $t_j = t_j'$ for all j and $y_j = y_j'$ whenever $t_j \neq 0$. Then we set

(6) $$Y_1 \circ Y_2 \circ \ldots \circ Y_k = J/\sim$$

employing the identification topology. The action

$$G \times (Y_1 \circ \ldots \circ Y_k) \to Y_1 \circ \ldots \circ Y_k$$

given by

$$g[t_1y_1, \ldots, t_ky_k] = [t_1gy_1, \ldots, t_kgy_k]$$

is continuous whenever the Y_j's are compact [6].

2.4 <u>Lemma</u>. Suppose Y is a free G-space, with $Y \subset \mathbb{R}^n$ and $G \subset O(n)$. Then, there is an equivariant imbedding

$$f: Y \to \mathbb{R}^{n+1}$$

with the additional property that $y_1 \neq y_2$ implies $f(y_1)$ and $f(y_2)$ are independent, i.e. they do not lie on a line thru the origin.

<u>Proof</u>. Set $f(y) = (y, \|y\|^2)$, $y \in \mathbb{R}^n$, $\|y\| = \text{norm } y$.

This lemma is used to prove the following proposition which is essentially Lemma 2.7.9 of [9].

2.5 <u>Proposition</u>. If $Y_1, \ldots, Y_k$ are compact G-ENR's, so is the k-fold join

$$Y_1 \circ \ldots \circ Y_k .$$

Proof. We need only show this for $k = 2$. Clearly $Y_1 \circ Y_2$ is compact. We may assume without loss, that Y_1 is a closed G_1 - subspace of $\mathbb{R}^p$, where $G_1 \subseteq O(p)$ and G_1 is isomorphic to G, say by $\varphi_1 : G_1 \to G_1$. Similarly, we may assume that there is an isomorphism $\varphi_2 : G \to G_2 \subseteq O(q)$ and Y_2 is a G_2-subspace of $\mathbb{R}^q$.

Then, there is a natural equivariant map $\eta : Y_1 \circ Y_2 \to \mathbb{R}^p \oplus \mathbb{R}^q$ given by

$$\eta : [t_1 y_1, t_2 y_2] \to t_1 y_1 \oplus t_2 y_2$$

where G acts on $\mathbb{R}^p \oplus \mathbb{R}^q$ via the diagonal action

$$g(y_1, y_2) = (\varphi_1(g) y_1, \varphi_2(g) y_2).$$

Now, if we use Lemma 2.4 we may also assume that distinct points y_1, y_1' of Y_1 are independent vectors and similarly for Y_2. Then, if

$$t_1 y_1 \oplus t_2 y_2 = t_1' y_1' \oplus t_2' y_2'$$

we have $t_1 y_1 = t_1' y_1'$ and $t_2 y_2 = t_2' y_2'$. This forces

$$[t_1 y_1, t_2 y_2] = [t_1' y_1', t_2' y_2']$$

and η is injective, hence an imbedding. Now, suppose

$$\rho_i : U_i \to Y, \quad i = 1,2$$

are invariant retractions where U_1, U_2 are invariant neighborhoods of Y_1 and Y_2 in $\mathbb{R}^p, \mathbb{R}^q$, respectively. Now, let U denote the union of all lines $L(u_1, u_2)$, $u_i \in U_i$. Thus a point $u \in U$ has the form

$$(1-t)u_1 + tu_2, \quad -\infty < t < \infty .$$

Set

$$\rho((1-t)u_1 + tu_2) = \begin{cases} \rho_1(u_1), & \text{if } t \le 0 \\ (1-t)\rho_1(u_1) + t\rho_2(u_2), & \text{if } 0 \le t \le 1 \\ \rho_2(u_2), & \text{if } t \ge 1 . \end{cases}$$

$\rho : U \to \eta(Y_1 \circ Y_2)$ is an equivariant retraction and hence $Y_1 \circ Y_2$ is a G-ENR.

The following proposition uses the obvious fact that L - S category is subadditive, i.e. if $Y = Y_1 \cup Y_2 \subset M$, where Y_i are closed in M , $i = 1,2$, then

$$\mathrm{cat}_M Y \leq \mathrm{cat}_M Y_1 + \mathrm{cat}_M Y_2 .$$

2.6 <u>Proposition</u>. Suppose Y_1, Y_2 are compact invariant subspaces contained in a free G-space E , and let $Y = Y_1 \circ Y_2$. Then,

$$\mathrm{cat}_{Y^*} Y^* \leq \mathrm{cat}_{Y_1^*} Y_1^* + \mathrm{cat}_{Y_2^*} Y_2^*$$

where $A^* = A/G$.

<u>Proof</u>. $Y_1 \circ Y_2$ splits into two pieces

$$X_1 = \{[y_1,t,y_2] , t \leq \frac{1}{2}\}$$

$$X_2 = \{[y_1,t,y_2], t \leq \frac{1}{2}\}$$

with Y_i a strong deformation retract of X_i (equivariantly). Thus Y_i^* is a strong deformation of X_i^* and since

$$\mathrm{cat}_{Y^*} Y^* \leq \mathrm{cat}_{X_1} X_1^* + \mathrm{cat}_{X_2^*} X_2^*$$

we have the desired result.

2.7 <u>Corollary</u>. If $Y = \overbrace{G \circ \ldots \circ G}^{k}$, then $\mathrm{cat}_{Y^*} Y^* \leq k$.

The next proposition establishes that G-genus is also subadditive.

2.8 <u>Proposition</u>. If $Y \in \mathfrak{Z}$ and $Y = Y_1 \cup Y_2$ where Y_1 and Y_2 are closed invariant subspaces, then

$$\text{G-genus } Y \leq \text{G-genus } Y_1 + \text{G-genus } Y_2 .$$

<u>Proof</u>. Suppose G-genus $Y_1 = k_1$ and G-genus $Y_2 = k_2$. Let

$$H_1 = \overbrace{G \circ \ldots \circ G}^{k_1}, \quad H_2 = \overbrace{G \circ \ldots \circ G}^{k_2}$$

and observe that H_1 and H_2 are compact G-ENR's (Proposition 2.5). Suppose

$$f_i : Y_i \to H_i , \quad i = 1,2$$

are equivariant maps. Then f_i extends to an equivariant map

$$f_i' : U_i \to H_i \ , \quad i = 1,2$$

where U_i is an invariant open set containing Y_i. Select an equivariant partition of unity $\varphi_i : Y \to [0,1]$ so that

$$Y_i \subset \varphi_i^{-1}((0,1]) \subset U_i \ , \qquad i = 1,2 \ .$$

Then, define an equivariant map

$$f : Y \to H_1 \circ H_2$$

by setting

$$f(y) = [\varphi_1(y) f_1'(y), \ \varphi_2(y) f_2'(y)]$$

and the result follows.

2.9 Remark. Let us recall that if we set $Y_k = \overbrace{G \circ \ldots \circ G}^{k}$ and $Y_k^* = Y_k/G$, we have natural imbeddings

$$\begin{array}{ccccccc} G & \to \cdots \to & Y_k & \to & Y_{k+1} & \to \cdots \\ \downarrow & & \downarrow & & \downarrow & \\ * & \to \cdots & Y_k^* & \to & Y_{k+1}^* & \to \cdots \end{array}$$

and the direct limit yields the Milnor universal bundle [6] (E_G, p_G, B_G) for G. Now, if E is a contractible, paracompact free G-space, and if $E/G = B$, then (E,p,B) is also a universal bundle for G-bundles over paracompact spaces [10].

As we have seen, G-genus is subadditive but the proof was more substantial than the corresponding trivial result for L - S category. Just the opposite occurs for the "monotone" property. If $\varphi : X \to Y$ is an equivariant map (in $\mathfrak{F}$), then it is immediate that

$$G\text{-genus } X \leq G\text{-genus } Y \ .$$

However, the corresponding result for L - S category requires some details - and makes use of the classification theorem for G-bundles.

2.10 **Proposition.** Suppose X_1 and X_2 are closed invariant subspaces of paracompact free G-spaces E_1 and E_2, respectively. Then, if $\varphi: X_1 \to X_2$ is an equivariant map and if

$$X_1^* = X_1/G,\ X_2^* = X_2/G,\ B_1 = E_1/G,\ B_2 = E_2/G\ ,$$

then

$$\operatorname{cat}_{B_1} X_1^* \le \operatorname{cat}_{B_2} X_2^*\ .$$

Proof. The bundles (E_i, p_i, B_i) $i = 1,2$ are universal bundles and hence we have the following diagram of bundle maps

$$\begin{array}{ccccccc} X_1 & \xrightarrow{\varphi} & X_2 & \xrightarrow{i_2} & E_2 & \xrightarrow{\alpha} & E_1 \\ {\scriptstyle q_1}\downarrow & & {\scriptstyle q_2}\downarrow & & {\scriptstyle p_2}\downarrow & & {\scriptstyle p_1}\downarrow \\ X_1^* & \xrightarrow{\bar\varphi} & X_2^* & \xrightarrow{\bar i_2} & B_2 & \xrightarrow{\bar\alpha} & B_1 \end{array}$$

where φ is given, i_2 is inclusion and α exists via the universality of (E_1, p_1, B_1).

Now, suppose $\operatorname{cat}_{B_2} X_2^* = k < \infty$. There, X_2^* admits a closed cover $K_1^*, \ldots, K_k^*$ of sets contractible in B_2 to a point. If we set $A_\ell^* = \bar\varphi^{-1}(K_\ell^*)$, we have a closed cover $\{A_1^*, \ldots, A_k^*\}$ of X_1^* and

$$\bar\alpha \circ \bar i_2 \circ (\bar\varphi | A^*) \sim \text{constant} \quad (\text{in } B_1).$$

However, since (E_1, p_1, B_1) is universal, we have

$$\bar\alpha \circ \bar i_2 \circ \bar\varphi \sim \bar i_1$$

where $i_1 : X_1 \to E_1$ is inclusion. Thus, each A_ℓ^* is contractible to a point in B_1 and

$$\operatorname{cat}_{B_1} X_1^* \le \operatorname{cat}_{B_2} X_2^*\ .$$

3. Category vs Genus

3.1 **Theorem.** Let E denote a contractible, paracompact free G-space and let Σ denote the closed invariant subspaces of E. Then if $B = E/G$ and $A^* = A/G$, we have

$$\mathrm{cat}_B A^* = \text{G-genus } A, \quad A \in \Sigma .$$

Proof. a) We show first that $\mathrm{cat}_B A^* \leq$ G-genus A. Suppose that G-genus $A = k < \infty$. Then, we have an equivariant map

$$f : A \to Y = \overbrace{G \circ \cdots \circ G}^{k} \subset E_G .$$

But then, using Proposition 2.10 and Corollary 2.7

$$\mathrm{cat}_B A^* \leq \mathrm{cat}_{B_G} Y^* \leq \mathrm{cat}_{Y^*} Y^* \leq k .$$

Thus,

$$\mathrm{cat}_B A^* \leq \text{G-genus } A .$$

b) Now, suppose $\mathrm{cat}_B A^* = k < \infty$. Then,

$$A^* = A_1^* \cup \cdots \cup A_k^*$$

where each A_ℓ^* is closed and contractible in B. Now, since G-genus is subadditive (Proposition 2.8) we have

$$\text{G-genus } A \leq \sum_{\ell=1}^{k} \text{G-genus } A_\ell$$

where $A_\ell = p_A^{-1}(A_\ell^*)$, $p_A : A \to A/G = A^*$ the natural projection. Since each A_ℓ^* is contractible to a point in B, the bundle (A, p_A, A^*) is a trivial G-bundle and hence we have an equivariant map

$$f_\ell : A_\ell \to G$$

so that G-genus $A_\ell = 1$, $\ell = 1,\ldots,k$. This proves that

$$\text{G-genus } A \leq k = \mathrm{cat}_B A^*$$

and the proof is complete.

There are some noteworthy examples:

3.2 Let $\mathcal{B}$ denote an infinite dimensional Banach space over the reals $\mathbb{R}$. Let $G = \mathbb{Z}_2 = \{-1,1\}$ act on $\mathcal{B}$ by scalar multiplication and let Σ denote the closed invariant subsets of $E = \mathcal{B}-0$. Define the real genus of $A \in \Sigma$ by

$$\text{genus}_{\mathbb{R}} A = \mathbb{Z}_2\text{-genus } A .$$

Then,

$$\text{genus}_{\mathbb{R}} A = \text{cat}_B A^*$$

where $B = E/\mathbb{Z}_2$, $A^* = A/\mathbb{Z}_2$. As we have already observed, $\text{genus}_{\mathbb{R}} A = k < \infty$ is equivalent to saying that there is an equivariant (odd) map $f : A \to \mathbb{R}^k - 0$ and k is minimal with this property, so that $\text{genus}_{\mathbb{R}}$ is ordinary genus in the sense of Krasnoselskii [2].

3.3 Let $\mathcal{B}$ denote an infinite dimensional Banach space over the complex numbers $\mathbb{C}$. Let $G = S^1$, the complex numbers of norm 1. Then G acts freely on $E = \mathcal{B}-0$, again by scalar multiplications. Let Σ denote the closed invariant subsets of E and define the complex genus of $A \in \Sigma$ by

$$\text{genus}_{\mathbb{C}} A = S^1\text{-genus } A$$

then,

$$\text{genus}_{\mathbb{C}} A = \text{cat}_B A^*$$

where $B = E/S^1$, $A^* = A/S^1$. We also mention here that $\text{genus}_{\mathbb{C}} A = k < \infty$ is equivalent to saying that there is an equivariant map $f\colon A \to \mathbb{C}^k - 0$ and k is minimal with this property.

Another consequence of Theorem 3.1 is the following result which asserts the independence of L - S category on the ambient Banach space.

3.4 Corollary. If $\mathcal{B}_i$, $i = 1,2$ are real (complex) Banach spaces (not necessarily infinite dimensional) and $A_i \subset \mathcal{B}_i - 0$ are closed invariant subsets admitting an equivariant homeomorphism $\varphi\colon A_1 \to A_2$, then

$$\text{cat}_{B_1} A_1^* = \text{cat}_{B_2} A_1^*$$

where $B_i = \mathcal{B}_i - 0)/\mathbb{Z}_2(S^1)$.

Proof. If both Banach spaces are infinite then

$$\mathrm{cat}_{B_1} A_1^* = G\text{-genus } A_1 = G\text{-genus } A_2 = \mathrm{cat}_{B_2} A_2^* .$$

To complete the proof it suffices to prove the following lemma.

3.5 Lemma. Let $\mathcal{B}$ denote an infinite dimensional Banach space over $\mathbb{R}$ or $\mathbb{C}$ and let L denote a finite dimensional subspace. Let A denote a closed invariant set in $L - 0$. If $C = (L - 0)/G$, $B = (\mathcal{B} - 0)/G$, $A^* = A/G$, where $G = \mathbb{Z}_2$ or S^1, then

$$\mathrm{cat}_C A^* = \mathrm{cat}_B A^* .$$

Proof. We consider only the real case. We may identify L with $\mathbb{R}^n$ and if $\mathbb{Z}_2$-genus $A = k$, then $k \leq n$ and we have a diagram of bundle maps

$$\begin{array}{ccccccc}
A & \xrightarrow{\varphi} & S^{k-1} & \xrightarrow{i} & S^{n-1} & \xrightarrow{j} & S^n \\
q\downarrow & & \downarrow p_{k-1} & & \downarrow p_{n-1} & & \downarrow p_n \\
A^* & \xrightarrow{\bar{\varphi}} & \mathbb{R}P^{k-1} & \xrightarrow{\bar{i}} & \mathbb{R}P^{n-1} & \xrightarrow{\bar{j}} & \mathbb{R}P^n
\end{array}$$

where φ is the equivariant map obtained from the fact that $\mathbb{Z}_2$-genus $A = k$ and i is inclusion. $\mathbb{R}P^{k-1}$ is the union of k contractible closed sets, $K_1^*, \ldots, K_k^*$ and hence if we set $A_\ell^* = \bar{\varphi}^{-1}(K_\ell^*)$, we have that each map

$$\bar{i} \circ (\bar{\varphi}|A_\ell^*) \sim \text{constant} \quad (\text{in } \mathbb{R}P^{n-1}).$$

We may assume without loss that $A_\ell = q^{-1}(A_\ell^*) \subseteq S^{n-1}$ and is a finite subcomplex of dimension $\leq n-1$. Since $(S^n, P_n, \mathbb{R}P^n)$ is n-universal [11]

$$j^* \circ \bar{i} \circ \bar{\varphi}|A_\ell^* \sim j_\ell : A_\ell^* \subset \mathbb{R}P^n .$$

Thus, A_ℓ^* is contractible in $\mathbb{R}P^n$. This forces A_ℓ^* to be a proper subset of $\mathbb{R}P^{n-1}$ and hence A_ℓ^* is deformable in $\mathbb{R}P^{n-1}$ to $\mathbb{R}P^{n-2}$. Repeating the above argument then forces A_ℓ^* to be contractible in $\mathbb{R}P^{n-1}$ and so

$$\mathrm{cat}_C A^* \leq k = \mathbb{Z}_2\text{-genus } A = \mathrm{cat}_B A^* .$$

Since the inequality $\mathrm{cat}_B A^* \leq \mathrm{cat}_C A^*$ is obvious the lemma follows and the proof of Corollary 3.4 is complete.

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The concept of genus of an invariant, closed set A in a paracompact free G-space E is introduced for any compact Lie group G and the general result that G-genus A = $\text{cat}_B A^*$ is proven where $B = E/G$, $A^* = E/G$ and cat is short for Ljusternik-Schnirelman category. As a special case, the concept of genus (Krasnoselskii) coincides with the notion of category (Ljusternik-Schnirelman) as employed in a real or complex Banach space.

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