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A STRAIGHTFORWARD GENERALIZATION OF DILIBERTO AND STRAUS' ALGOR--ETC(U)
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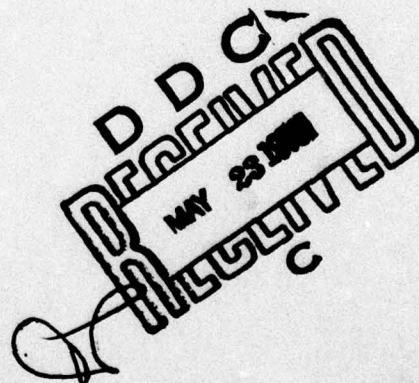
A STRAIGHTFORWARD GENERALIZATION
OF DILIBERTO AND STRAUS' ALGORITHM
DOES NOT WORK

Nira Richter-Dyn

Mathematics Research Center
University of Wisconsin-Madison
610 Walnut Street
Madison, Wisconsin 53706

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ABSTRACT

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An algorithm for best approximating in the sup-norm a function $f \in C[0,1]^2$ by functions from tensor-product spaces of the form $\pi_k \otimes C[0,1] \otimes C[0,1] \otimes \pi_l$, is considered. For the case $k = l = 0$ the Diliberto and Straus algorithm is known to converge. A straight-forward generalization of this algorithm to general k, l is formulated, and an example is constructed demonstrating that this algorithm does not converge for $k^2 + l^2 > 0$.

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SIGNIFICANCE AND EXPLANATION

Often it is desirable to approximate a given function as closely as possible by a member of a class of functions that are simpler to evaluate.

For a general continuous function of two variables $f(x,y)$ a best approximating function of the simpler form $h(y) + g(x)$ can be computed by the algorithm of Diliberto and Straus. Since such an approximation can be quite far from the approximated function, a better approximation of the form $\sum_{i=0}^k h_i(y)x^i + \sum_{j=0}^l g_j(x)y^j$ is considered. One way to try to construct such an approximation is to generalize the Diliberto and Straus algorithm to this more general setting. The generalized algorithm is simple in the sense that only one-dimensional best approximations by polynomials have to be computed. In this note it is shown by a simple example, that this "natural" generalization cannot be expected to converge, and therefore other methods should be developed.

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A STRAIGHTFORWARD GENERALIZATION
OF DILIBERTO AND STRAUS' ALGORITHM DOES NOT WORK

Nira Richter-Dyn

The algorithm of Diliberto and Straus for approximating a bivariate function by a sum of univariate ones proposed in 1951 [1], has been recently investigated in several works [2], [3], [4], where convergence and various properties of the algorithm are studied.

The algorithm, designed for computing the best approximation to $f \in C[0,1]^2$ in the sup-norm from the space

$$(1) \quad M = \{ \phi \mid \phi(x,y) \in C[0,1]^2, \phi(x,y) = h(y) + g(x) \},$$

is of the following form:

$$\begin{aligned} f_0(x,y) &= f(x,y) \\ f_{2n+1}(x,y) &= f_{2n}(x,y) - \frac{1}{2} \left[\max_{0 \leq \xi \leq 1} f_{2n}(\xi,y) + \min_{0 \leq \xi \leq 1} f_{2n}(\xi,y) \right], \\ (2) \quad n &= 0, 1, \dots, \\ f_{2n+2}(x,y) &= f_{2n+1}(x,y) - \frac{1}{2} \left[\max_{0 \leq \eta \leq 1} f_{2n+1}(x,\eta) + \min_{0 \leq \eta \leq 1} f_{2n+1}(x,\eta) \right], \\ n &= 0, 1, \dots \end{aligned}$$

It is proved in [1], [3], [4] that $\lim_{n \rightarrow \infty} \|f_n\| = \inf_{\phi \in M} \|f - \phi\|$, although the rate of convergence might be extremely slow [2]. Algorithm (2) can be interpreted as a sequence of repeated applications of the operator of one dimensional best approximation by constants to $f(x,y)$, regarded alternately as a function of x and as a function of y . More specifically, let J_x be the operator associating with $f(x,y) \in C[0,1]^2$ the function $(J_x f)(y) \in C[0,1]$, with $(J_x f)(y_0)$ the constant of best approximation to $f(x,y_0)$ in the sup-norm on $[0,1]$, and let J_y be defined similarly with the roles of x,y interchanged. Then (2) can be rewritten as

$$(3) \quad f_0 = f, f_{2n+1} = f_{2n} - J_x f_{2n}, f_{2n+2} = f_{2n+1} - J_y f_{2n+1}, \quad n=0,1,2,\dots$$

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This formulation suggests a straightforward generalization of algorithm (3), namely best approximating $f(x,y)$ alternately in the x and y directions by polynomials of degree k and l respectively, in order to obtain a best approximation to $f(x,y)$ from the tensor-product space

$$(4) \quad M_{k,l} = \{ \phi(x,y) \mid \phi(x,y) \in C[0,1]^2, \phi(x,y) = \sum_{j=0}^k h_j(y)x^j + \sum_{j=0}^l g_j(x)y^j \} = \\ = \pi_k \otimes C[0,1] \otimes C[0,1] \otimes \pi_l.$$

(π_k denotes the space of all univariate polynomials of degree $\leq k$.) With this notation the subspace M in (1) is the tensor-product space $M_{0,0}$. The generalization of algorithm (3) to this more general setting is

$$(5) \quad f_0 = f, f_{2n+1} = f_{2n} - J_x^{(k)} f_{2n}, f_{2n+2} = f_{2n+1} - J_y^{(l)} f_{2n+1}, n=0,1,2,\dots,$$

where $(J_x^{(k)} f)(x, y_0) = \sum_{j=0}^k h_j(y_0)x^j$ is the polynomial of best approximation to $f(x, y_0)$ in the sup-norm on $[0,1]$ from π_k , and where $(J_y^{(l)} f)(x_0, y)$ is similarly defined.

In the following we present a simple example demonstrating that algorithm (5) for general k, l cannot be expected to converge to a best approximation to $f_0(x,y)$. We construct a function $f(x,y)$ such that $\|f\| > \inf_{\phi \in M_{0,1}} \|f - \phi\|$, while the functions $\{f_n\}$ generated from it by (5) with $k=0, l=1$ satisfy $\|f_n\| = \|f\|$ for all n .

Consider $f(x,y) \in C[0,1]^2$ subject to the following conditions:

$$(6) \quad \begin{aligned} f\left(\frac{1}{4}, \frac{j}{6}\right) &= (-1)^{i+j}, & j=2i, 2i+1, 2i+2, \quad i=0,1,2 \\ f\left(\frac{3}{4}, \frac{j}{6}\right) &= (-1)^{j+1}, & j=0,5,6 \\ f\left(1, \frac{2j+1}{6}\right) &= (-1)^j, & j=0,1,2 \\ |f(x,y)| &< 1 & \text{elsewhere in } [0,1]^2. \end{aligned}$$

As can be easily observed

$(J_x^{(0)} f)(x, \frac{1}{6}) = 0$, $i=0,1,\dots,6$ and $(J_y^{(1)} f)(\frac{1}{4}, y) = 0$, $i=0,1,2,3,4$, and both $f - J_x^{(0)} f$ and $f - J_y^{(1)} f$ satisfy (6). Thus algorithm (5) with $k=0, l=1$ generates a sequence $\{f_n\}$ of functions satisfying (6) whenever f_0 satisfies (6), and therefore $\|f_n\| = 1$ for all $n \geq 0$.

In order to verify that $\|f\| > \inf_{\phi \in M_{0,1}} \|f - \phi\|$, it is sufficient to show that there does not exist a bounded linear functional $\mu \in (C[0,1]^2)'$, $\mu \neq 0$, such that

$$(7) \quad \langle \phi, \mu \rangle = 0 \text{ for all } \phi \in M_{0,1},$$

$$(8) \quad \langle f, \mu \rangle = \|\mu\|.$$

Indeed any $\mu \neq 0$ with property (8) is necessarily of the form

$$(9) \quad \langle \zeta, \mu \rangle = \sum_{j=0}^r a_j \zeta(x_j, y_j), \quad \zeta \in C[0,1]^2, \text{ with } r > 0, a_j f(x_j, y_j) = |a_j|, j=0, \dots, r,$$

namely a linear combination of function values at extremal points of f . Moreover condition (7) implies that μ can be written as a linear combination of first differences in the x direction so as to vanish on all functions of the form $h(y)$, and as a linear combination of second order divided differences in the y direction, so as to vanish on all functions of the form $g_0(x) + g_1(x)y$.

These characteristics of μ are consistent with the special structure of the 15 extremal points of f , as given in (6), only if $r=14$ in (9). Then μ can be written as

$$(10) \quad \langle \zeta, \mu \rangle = \sum_{i=0}^4 c_i [1]_i \zeta,$$

where $[1]_i \zeta$ denotes the second order divided difference of $\zeta(\frac{i}{4}, y)$ at the extremal points of f with $x=\frac{i}{4}$. The sum (10) can be rewritten as a linear combination of first differences in the x direction only if $c_0, \dots, c_4$ satisfy the following system of linear equations:

$$c_0 = c_1 = c_2, \quad c_2 = \frac{c_3}{3}, \quad c_1 = \frac{c_4}{4}, \quad c_0 = \frac{c_3}{15}, \quad c_2 = \frac{2}{5}c_3 + \frac{c_4}{4},$$

which admits only the trivial solution.

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