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A PROPOSAL FOR RESEARCH ON STOCHASTIC METHODS AND PROBLEMS IN A--ETC(U)

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AFOSR-77-3286

UNCLASSIFIED

AFOSR-TR-79-0603

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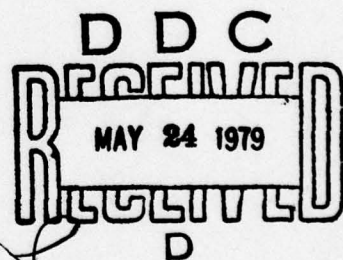
**STOCHASTIC METHODS AND  
PROBLEMS IN APPLIED MATHEMATICS  
(AFOSR 77-3286)**

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|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------|---------------------------------------------------------------------------------------------------------|--|
| 1. REPORT NUMBER<br><b>AFOSR-TR-79-0603</b>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        | 2. GOVT ACCESSION NO. | 3. RECIPIENT'S CATALOG NUMBER                                                                           |  |
| 4. TITLE (and Subtitle)<br><b>A Proposal for Research on<br/>STOCHASTIC METHODS AND PROBLEMS IN APPLIED<br/>MATHEMATICS</b>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        |                       | 5. TYPE OF REPORT & PERIOD COVERED<br><b>9 Final interim rept's</b>                                     |  |
| 6. AUTHOR(s)<br><b>10 Charles J. Holland</b>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |                       | 6. PERFORMING ORG. REPORT NUMBER                                                                        |  |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS<br>Purdue University<br>Department of Mathematics<br>West Lafayette, IN 47907                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          |                       | 8. CONTRACT OR GRANT NUMBER(s)<br><b>15 AFOSR-77-3286</b>                                               |  |
| 11. CONTROLLING OFFICE NAME AND ADDRESS<br>Air Force Office of Scientific Research/NM<br>Bolling AFB, Washington, DC 20332                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                         |                       | 10. PROGRAM ELEMENT, PROJECT, TASK<br>AREA & WORK UNIT NUMBERS<br>61102F <b>16</b> 2304/A1 <b>17</b> A1 |  |
| 14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)<br><b>12 17 p.</b>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     |                       | 11. REPORT DATE<br><b>11 1979</b>                                                                       |  |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |                       | 13. NUMBER OF PAGES<br>16                                                                               |  |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |                       | 15. SECURITY CLASS. (of this report)<br><br>UNCLASSIFIED                                                |  |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |                       | 15a. DECLASSIFICATION/DOWNGRADING<br>SCHEDULE                                                           |  |
| 16. DISTRIBUTION STATEMENT (of this Report)<br><br>Approved for public release; distribution unlimited                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             |                       |                                                                                                         |  |
| 17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                         |                       |                                                                                                         |  |
| 18. SUPPLEMENTARY NOTES                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                            |                       |                                                                                                         |  |
| 19. KEY WORDS (Continue on reverse side if necessary and identify by block number)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 |                       |                                                                                                         |  |
| 20. ABSTRACT (Continue on reverse side if necessary and identify by block number)<br>The determination of the limiting long time behavior of the system using a fixed control was answered in the case of a stochastic system perturbed by a small additive noise term where the control is such that the corresponding deterministic system possesses a stable limit cycle. A new characterization was given of the principal eigenvalue for second-order linear elliptic partial differential equations, not necessarily self-adjoint, with both natural and Dirichlet boundary conditions. A new alternative numerical method was given for calculating both the principal eigenvalue and corresponding eigenvector in the case of natural boundary conditions. |                       |                                                                                                         |  |

# FINAL REPORT

This is a final report of the work completed under AFOSR 77-3286. In Section I, the proposed research is reviewed and in Section II the work completed under the grant is discussed. Finally, in Section III work in progress under the support of AFOSR 77-3286B is discussed.

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# I. Research Proposal.

## Problems in Optimal Nonlinear Filtering

In this part of the research we seek a computationally convenient technique for solving the filtering and prediction problems for a class of nonlinear stochastic differential equations subject to partial observations at discrete time points. The applications of the technique described below to trajectory estimation are apparent.

Let there be given  $q$  functions  $f_1, f_2, \dots, f_q$  which are known to us. Suppose for some  $i \in \{1, 2, \dots, q\}$ , unknown to us, the state of the process is evolving according to the vector stochastic differential equations

$$dx = f_i(t, x)dt + g(t, x)dw, \quad x(0) = x.$$

At the discrete times  $t_1, \dots, t_p$ , which are known in advance, an intermediate observer (usually a machine) receives noise corrupted observations of the stochastic process  $x(t)$ . These observations  $y(t)$  satisfy the stochastic differential equations

$$dy = H(t, x, y)dt + \tilde{g}(t, x, y)dw, \quad y(0) = 0.$$

Suppose that  $y(t) \in R^m$  for some  $m$ . Assume there exists known pairwise disjoint sets  $B_i$ ,  $i = 1, \dots, m$ ,  $\cup B_i = R^m$ . Then at each time  $t_j$ ,  $j = 1, 2, \dots, p$ , we receive from the intermediate observer only the information as to which of the events  $y(t_j) \in B_i$  has occurred.

Our first problem is to determine for each function  $f_i$ ,

the probability that the function  $f_i$  is being used given available information.

Our second prediction problem is the following: Given our information at times  $t = t_1, t_2, \dots, t_p$ , and the function  $f_i$ , determine the best prediction in mean square of some function  $h(x(T))$  of the process,  $T > t_p$ .

Let us discuss problem 2 which we have recently solved theoretically. For each possible information set, we must solve a coupled set of second order partial differential equations. Since the boundary conditions are of Cauchy type, they can be solved numerically. There are  $m \times p$  information sets, each requiring the solving of the coupled set of partial differential equations. Although these problems appear complex, they have the important practical advantage that they can be completely precomputed and do not need to be solved in real-time.

In this research we intend to consider the problems discussed above looking at both the theoretical solution and effective computational methods that can be developed from the theoretical solutions.

#### APPROXIMATION TECHNIQUES IN STOCHASTIC CONTROL PROBLEMS

In this research we continue our investigation of approximation techniques for a wide class of discrete and continuous time stochastic control problems. This research centers on small noise problems in the continuous case and on nonclassical stationary problems in the discrete case. Emphasis is placed on the development and theoretical justification of techniques

which yield computationally tractable algorithms that answer the following:

- (1) approximations to the optimal cost and the cost of using a particular control.
- (2) approximations to the optimal control.
- (3) evaluation of the relative performance of two controls.
- (4) estimates for the deterioration in system performance due to the failure to observe certain system components.

#### SMALL NOISE PROBLEMS IN CONTINUOUS TIME

##### Outline:

One general problem has been to solve "approximately" the stochastic control problem in terms of quantities computable from the solution to the corresponding deterministic control problem when the noise entering the system equation is a "small" parameter. The general procedure is to theoretically establish expansions of the optimal cost and control in powers of the noise coefficient. The expansions then suggest appropriate forms for suboptimal controls and numerical techniques for determining them. The theoretical development of the expansions involves the interplay between probability and partial differential equations. See [4] for a general review of the current state in this area.

The theoretical portion of this approach for the completely observable problem was treated successfully by Fleming [3]; the numerical algorithms suggested by the theorems were developed and employed on some two dimensional examples by this author in [5]. A partial solution to the open loop control problem was

developed by the author in [6] and [8]. In this research we intend to extend the results of the open loop control problem and derive similar results for the sampled-data problem.

Finally, we intend to investigate the stationary small noise control problem in the case of both complete and partial observations. Under certain assumptions we have computed an expansion of the cost of using a fixed control in [7]. The expansions in this case are nice in that the coefficients of the expansion satisfy algebraic equations and hence can be computed easily. In this research we seek expansions of the optimal cost and control under suitable assumptions. Further we intend to investigate approximation techniques suggested by our already established result.

#### Details:

Many problems of continuous time stochastic control can be formulated as follow. Let  $x(t) = (x_1(t), x_2(t), \dots, x_n(t)) \in \mathbb{R}^n$  denote the state of the system at time  $t$ , and suppose that the states evolve according to the stochastic differential equations

$$dx(t) = f(t, x(t), u(t))dt + b(t, x(t))dw(t), \quad t \geq t_0, \quad (1)$$

with initial condition  $x(t_0) = x_0 \in B$ , where  $x_0$  is a constant and  $B$  is defined below. In (1)  $w$  is a normalized  $n$  dimensional brownian motion process and  $u(t)$ , which takes values in a compact set  $K \subset \mathbb{R}^k$ , is the control applied at time  $t$ . As is customary dependence on (1) and other equations on the point  $\omega$  in the sample space  $\Omega$  is omitted.

Finite time control problems. Let  $\tau$  denote the first time  $t \geq t_0$  such that  $(t, x(t)) \notin Q$  where  $Q = (t_0, T) \times B$  and  $B$  is an open subset of  $R^n$  such that its boundary  $\partial B$  is a smooth manifold with compact closure. Then the optimal control problem is to choose  $u \in \mathcal{U}$ , the class of admissible controls, so as to minimize the quantity

$$E \int_{t_0}^{\tau} L(t, x(t), u(t)) dt \quad (2)$$

where  $E$  denotes expected value.

The class  $\mathcal{U}$  is chosen to reflect the amount of information available to the controller. In this part of the research we consider three special cases:

Open loop case. No observations of  $x(t)$  are made for  $t > t_0$ ; hence  $\mathcal{U}$  is selected to be the set of Borel measurable functions on  $[t_0, T]$ .

Complete observations. Here the controller observes the process at each time  $t$  and therefore  $\mathcal{U}$  is chosen to be  $\{u | u(t) = Y(t, x(t)) \text{ where } Y \text{ is a measurable function on } [t_0, T] \times R^n\}$ .

Sampled-data case. Suppose that the process is observed at times  $t_0, t_1, \dots, t_p$ ,  $t_0 < t_1 < \dots < t_p = T$ . Let  $b_k = (x(t_0), \dots, x(t_k))$  and let  $Y_k$  be a measurable function on  $[t_0, T] \times R^{nk}$ . Then  $\mathcal{U}$  is chosen to be

$\{u | u(t) = Y_k(t, b_k) \text{ for some } Y_k \text{ defined above, if}$

$t_{k-1} \leq t < t_k, \quad k = 0, 1, \dots, p-1\}$ .

Define  $\phi^\varepsilon(t_0, x_0)$ ,  $\psi^\varepsilon(t_0, x_0)$ ,  $\chi^\varepsilon(t_0, x_0)$ , and  $\phi^0(t_0, x_0)$  to be the optimal cost for the point  $(t_0, x_0)$  in the completely observable, open loop, sampled-data, and deterministic control problems, respectively, with the superscript  $\varepsilon$  indicating that  $b(t, x) = (2\varepsilon)^{1/\alpha}$  is used in (1). Assume that  $(t_0, x_0) \in N$  where  $N$  is a region of strong regularity. A region of strong regularity, see [3], p. 480, guarantees that the optimal deterministic control and trajectory have certain regularity properties. Then Fleming showed in [3] that

$$\phi^\varepsilon(t_0, x_0) = \phi^0(t_0, x_0) + \varepsilon\theta_1 + o(\varepsilon) \quad (3)$$

and this author showed in [6] in case  $B = \mathbb{R}^n$  that

$$\psi^\varepsilon(t_0, x_0) = \phi^0(t_0, x_0) + \varepsilon\theta_2 + o(\varepsilon) \quad (4)$$

where  $\theta_1$  and  $\theta_2$  can be determined from a knowledge of the solution of the corresponding deterministic control problem.

In this research the author seeks to extend the result (4) when  $B$  merely satisfies the earlier hypothesis and also to establish the analogous result

$$\chi^\varepsilon(t_0, x_0) = \phi^0(t_0, x_0) + \varepsilon\theta_3 + o(\varepsilon) \quad (5)$$

in the sampled-data problem.

Let  $Y^0(t, x)$  denote the optimal feedback control in the completely observable case, then Fleming [3] showed that

$$Y^\varepsilon(t, x) = Y^0(t, x) + \varepsilon W(t, x) + o(\varepsilon) \quad (6)$$

uniformly in  $(t, x)$  on compact subsets of those regions in  $N$  where  $Y^0$  is a  $C^\infty$  function. We seek similar expansions of the optimal control in the open loop and sampled-data cases. Under assumptions which included that each open loop control generate a Gaussian process, we established in [8]  $C^\infty$  expansions of the optimal control in powers of  $\epsilon$ .

The numerical methods for calculating the quantities  $\theta_1$  and  $W(t, x)$  in (3) and (6) has been discussed in a paper [5] by the author; the author has also developed under certain assumptions a numerical method for finding "best" controls of the form  $u^0 + \epsilon z$  for some function  $z$  in the open loop case. In case an expansion of the open loop control is established, then the function  $z$  must be the coefficient of  $\epsilon$ .

Stationary control problems. Let  $\mathcal{U}$  denote the class of autonomous feedback controls for which the autonomous version of (1) has a unique ergodic measure. See Kushner [11], Wonham [14], and Zakai [16] for a discussion of this assumption. The ergodic measure depends on both  $u \in \mathcal{U}$  and  $\epsilon$  will be denoted by  $\mu(u, \epsilon, \cdot)$ . The optimization problem is to choose  $u \in \mathcal{U}$  so that

$$J(u, \epsilon) = \int_{R^n} L(x, u(x)) d\mu(u, \epsilon, x) \quad (7)$$

is minimized. Denote the minimum cost by  $J^*(\epsilon)$  and the optimal control by  $u^\epsilon$ . Under certain assumptions we have shown [7] that for each  $u$ ,

$$J(u, \epsilon) = \sum_{k=0}^n \epsilon^k \theta^k + o(\epsilon^n) \quad (8)$$

for some constants  $\theta_1, \theta_2, \dots, \theta_n$  which satisfy algebraic equations. The integer  $n$  depends upon the smoothness of  $f, L$ , and  $u$ .

In this research we seek expansions of  $J^*(\epsilon)$  and  $u^\epsilon$  in powers of  $\epsilon$ . We will also examine the utility of implementing a truncated expansion of (8) as an approximation to the cost in order to determine suboptimal controls. Our present research seems to indicate that this is a good technique when the class of controls is restricted to those having prescribed forms with unknown parameters.

The stationary control problem has been previously studied by Wonham and Cashman [15] who used statistical linearization as an approximation tool.

#### STATIONARY DISCRETE TIME CONTROL

Discrete time stochastic control problems have been thoroughly considered, however, most prior work considers the case in which the controller remembers all previously obtained information. This has been called the classical information pattern by Witsenhausen [13]. We intend to investigate the stationary control case where the controller has only partial observations of the system state and no memory. This emphasis is motivated by systems in which it is too difficult or expensive to observe all system components, and systems in which it is difficult to implement controls using past information.

Under certain reasonable assumptions we have succeeded in reducing the optimization problem to a problem in nonlinear

programming [9]. We have also constructed examples to show that one can do better by using randomized controls than by only using nonrandomized controls of the current observed data. In problems of this type with complete observations it is known that the optimization problem can be treated as a problem in linear programming [12] and that the controller cannot do better by considering randomized controls [1].

In this research the author intends to investigate the computational complexity of the nonlinear programming algorithm and to seek conditions on the partially observable problem which guarantee that the optimal control can be chosen to be non-randomized. This latter question is important both for designing computational algorithms and system design.

The importance of such problems was first brought to our attention by Dr. Raymond Rishel.

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## II. Work Completed Under the Grant.

- (1) Stochastically perturbed limit cycles, J. Applied Probability 15, 311-320 (1978).

An important question in the stability and control of stochastic systems is the determination of the limiting long time behavior of the system using a fixed control. This work answered that question in the case of a stochastic system perturbed by a small additive noise term where the control is such that the corresponding deterministic system possesses a stable limit cycle.

It is shown that in the limit of large time the stochastic system is near the limit cycle. This is a stability result. Moreover, one can compute approximately at which portions of the limit cycle one is most likely to be found. Further various stationary average can be computed.

These results will be of use in designing approximate controls for stationary stochastic control systems. For a detailed discussion of these results, see the completed above paper.

- (2) A Minimum Principle for the Principal Eigenvalue for Second-Order Linear Elliptic Equations With Natural Boundary Conditions, Communications in Pure and Applied Mathematics 31(1978) 509-519.

This work gives a new characterization of the principal eigenvalue for second-order linear elliptic partial differential equations, not necessarily self-adjoint, with both natural and Dirichlet boundary conditions, and also gives a new alternative numerical method for calculating both the principal eigenvalue and corresponding eigenvector in the case of natural boundary conditions. The principal eigenvalue, if appropriate sign changes are made,

determines the stability of equilibrium solutions to certain second order nonlinear partial differential equations. The corresponding eigenvector enables one to determine the first approximation of the solution of the nonlinear equation to variations of the initial condition from the equilibrium solution. These nonlinear equations are important in the applications. For these reasons it is important to have these characterizations of the principal eigenvalue and eigenvector.

Our method converts the determination of the eigenvalue and eigenvector to determining the solution of a stationary stochastic control problem. This latter problem is solved and from it a numerical scheme arises naturally. This method appears to have applications in solving other problems.

### III. Work in Progress.

In the recent work in (2) above, we were able to derive a new characterization of the principal eigenvalue for second order linear elliptic partial differential equations, not necessarily self-adjoint, with both natural and Dirichlet boundary conditions, and also give a new alternative numerical method for calculating both the principal eigenvalue and corresponding eigenvector in the case of natural boundary conditions.

We shall use the above results to determine the asymptotic behavior of the principal eigenvalue for some singularly perturbed eigenvalue problems as a small nuisance parameter tends to zero. The principal eigenvalue is the optimal value for a singularly perturbed stationary stochastic control problem. We are thus able to determine the asymptotic behavior of the optimal value of certain stationary stochastic control problems.

Publications Resulting From  
AFOSR 77-3286

1. Stochastically perturbed limit cycles, J. Applied Probability 15, 311-320 (1978).
2. A Minimum principle for the principal eigenvalue for second-order linear elliptic equations with natural boundary conditions, Communications in Pure and Applied Mathematics, 31(1978) 509-519.
3. (In preparation) The principal eigenvalue for linear second order elliptic equations with natural boundary conditions.