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SEPARABLY-INFINITE PROGAMS.(U)

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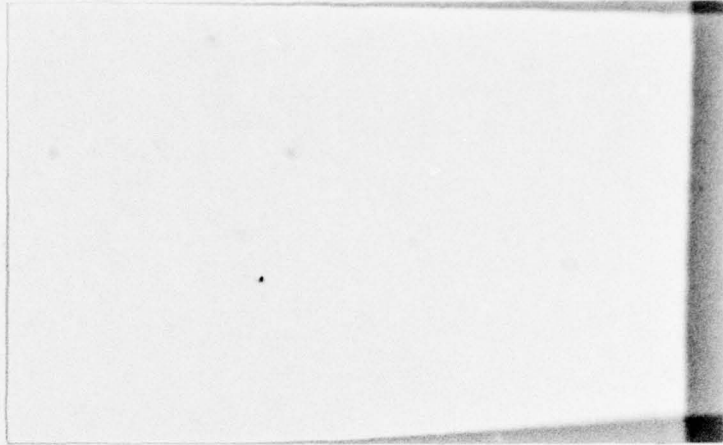
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by

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ABSTRACT

Separably-infinite Programs

by

A. Charnes, P. R. Gribik and K. O. Kortanek

Separably-infinite programs are a class of linear infinite programs that are related to semi-infinite programs and which have applications in economics and statistics. These programs have an infinite number of variables and an infinite number of constraints. However, only a finite number of variables appear in an infinite number of constraints, and only a finite number of constraints have an infinite number of variables. Duality in this class of programs is studied and used to develop a system of nonlinear equations satisfied by optimal solutions of the primal and dual programs. This nonlinear system has uses in numerical techniques for solving separably-infinite programs.

Key Words: Semi-infinite Programming, Separably-infinite Programming, Generalized Finite Sequence Space, Moment Cone, Duality, Nonlinear System.

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## 1. Introduction

In some recent applications of mathematical programming in statistics and economics, a new class of linear programs can be defined. These formulations have an infinite number of variables and an infinite number of constraints. As such, they are in the class of infinite programs [3]; however, they possess additional structure. Only a finite number of variables appear in an infinite number of constraints and only a finite number of constraints have an infinite number of variables. These programs are closely related to semi-infinite programs [1] which can be considered to be a subclass of this formulation.

This class of programs is of use in the study of the optimal experimental design problem. In this problem, one wishes to allocate measurement resources to points in a given region so that the measurements obtained can be used to make "good" estimates of some unknown constants in a linear regression problem. In [5, Section 6], this allocation problem is formulated as a linear program in which the Fisher information matrix related to the allocation must satisfy an infinite number of linear inequalities. The information matrix depends upon the allocation via a finite number of linear equalities. If the points at which measurements are possible are infinite in number, the allocation has an infinite number of variables. Thus this linear program is of the class we will examine.

There are also applications of this class of programs in economic equilibrium problems.

## 2. An Expansion of a Semi-infinite Program

In order to introduce the programs we study in this paper we first consider a semi-infinite program.

Program I: Let  $S$  be a set in  $\mathbb{R}^k$  and let  $u(\cdot):S \rightarrow \mathbb{R}^n$ ,  $u_{n+1}(\cdot):S \rightarrow \mathbb{R}$  and  $c \in \mathbb{R}^n$ .

$$\text{Find } V_I = \inf c^T x$$

from among  $x \in \mathbb{R}^n$  which satisfy  $u^T(t)x \geq u_{n+1}(t)$  for all  $t \in S$ . //

The dual we give for Program I uses the concept of a generalized finite sequence space [1]. Denote by  $R^{(S)}$  the set of all  $\lambda(\cdot):S \rightarrow \mathbb{R}$  which assume non-zero values at finitely many points only; that is all  $\lambda(\cdot)$  with finite support. The vector space  $R^{(S)}$  is termed a generalized finite sequence space. If the set  $S$  contains an infinite number of points,  $R^{(S)}$  is obviously infinite dimensional.

The following program is a dual for Program I.

Program II

$$\text{Find } V_{II} = \sup \sum_{t \in S} u_{n+1}(t) \lambda(t)$$

from among  $\lambda(\cdot) \in R^{(S)}$  which satisfy

$$\sum_{t \in S} u(t) \lambda(t) = c$$

$$\lambda(\cdot) \geq 0. //$$

The duality of Programs I and II has been studied extensively, [1][2][8].

The programs we consider have characteristics in common with both of the above programs.

Program P: Let  $S \subseteq R^k$ ,  $Q \subseteq R^l$  and let  $u(\cdot):S \rightarrow R^n$ ,  $u_{n+1}(\cdot):S \rightarrow R$   
 $v(\cdot):Q \rightarrow R^m$ ,  $v_{m+1}(\cdot):Q \rightarrow R$ ,  $c \in R^n$ ,  $b \in R^m$  and  $A \in R^{m \times n}$ .

$$\text{Find } V_P = \inf c^T x - \sum_{r \in Q} v_{m+1}(r) \eta(r)$$

from among  $x \in R^n$  and  $\eta(\cdot) \in R^{(Q)}$  which satisfy

$$u^T(t)x \geq u_{n+1}(t) \text{ for all } t \in S \quad (1a)$$

$$Ax + \sum_{r \in Q} v(r) \eta(r) = b \quad (1b)$$

$$\text{and} \quad \eta(\cdot) \geq 0. \quad // \quad (1c)$$

If the sets  $S$  and  $Q$  contain an infinite number of points, Program P has an infinite number of constraints and an infinite number of variables. Therefore, it is an infinite program. However, the constraints (1a) which are infinite in number contain only a finite number of variables, while the constraints (1b) which contain an infinite number of variables are finite in number. Because of this property, Program P is closely related to Programs I and II and we call it a separably-infinite program.

### 3. Duality in Separably-infinite Programming

Proceeding in a formal manner, the following program is obtained as the dual for Program P.

Program D:

$$\text{Find } V_D = \sup_{t \in S} \sum u_{n+1}(t) \lambda(t) - b^T y$$

from among  $\lambda(\cdot) \in R^S$  and  $y \in R^m$  which satisfy

$$\sum_{t \in S} u(t) \lambda(t) - A^T y = c \quad (2a)$$

$$v^T(r)y \geq v_{m+1}(r) \quad \text{for all } r \in Q \quad (2b)$$

$$\text{and } \lambda(\cdot) \geq 0. \quad // \quad (2c)$$

Program D is also separably infinite.

Lemma 1: Programs P and D are in duality in that for  $x, \eta(\cdot)$  feasible for Program P and  $y, \lambda(\cdot)$  feasible for Program D,

$$c^T x - \sum_{r \in Q} v_{m+1}(r) \eta(r) \geq \sum_{t \in S} u_{n+1}(t) \lambda(t) - b^T y.$$

Proof: Using (1a), (1b), and (2c),

$$\begin{aligned} & \sum_{t \in S} u_{n+1}(t) \lambda(t) - b^T y \\ & \leq \sum_{t \in S} (u^T(t)x) \lambda(t) - y^T(Ax + \sum_{r \in Q} v(r)\eta(r)) \\ & = \left( \sum_{t \in S} u(t) \lambda(t) - A^T y \right)^T x - \sum_{r \in Q} v^T(r)y \eta(r). \end{aligned}$$

Using (2a), (2b) and (1c), this last expression is less than or equal to

$$c^T x - \sum_{r \in Q} v_{m+1}(r) \eta(r). \quad //$$

Lemma 1 shows that Programs P and D satisfy the duality inequality  $V_D \leq V_P$ . Under additional assumptions, this is an equality. Rather than study the relations between Programs P and D in full generality, we will add assumptions that usually hold in practice and which permit us to develop relationships useful in computational techniques.

The following sets are convenient in the following developments:

$$K_S \equiv \{x \in \mathbb{R}^n \mid u^T(t)x \geq u_{n+1}(t) \text{ for all } t \in S\},$$

$$K_Q \equiv \{y \in \mathbb{R}^m \mid v^T(r)y \geq v_{m+1}(r) \text{ for all } r \in Q\},$$

$C_S \subseteq \mathbb{R}^{n+1}$  is the convex cone generated by

$$\left\{ \begin{pmatrix} u(t) \\ -u_{n+1}(t) \end{pmatrix} \mid t \in S \right\} \cup \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$

$C_Q \subseteq \mathbb{R}^{m+1}$  is the convex cone generated by

$$\left\{ \begin{pmatrix} v(r) \\ -v_{m+1}(r) \end{pmatrix} \mid r \in Q \right\} \cup \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

$C_S$  and  $C_Q$  are often called moment cones.

The following assumptions are made throughout the remainder of the paper.

- (A1) Either the set  $K_S$  is non-empty and bounded, or  $u_{n+1}(\cdot) = 0$ .  
 (A2) The set  $K_Q$  is non-empty and bounded.  
 (A3) The cone  $C_S$  is closed.

Lemma 2: Let  $g \in \mathbb{R}^n$  and  $h \in \mathbb{R}$ . Then  $\begin{pmatrix} g \\ h \end{pmatrix} \in C_S$  if and only if

$$x^T g \geq h \text{ for all } x \in K_S.$$

Proof: Consider the polar cone of  $C_S$

$$C_S^* = \left\{ \begin{pmatrix} w \\ w_{n+1} \end{pmatrix} \mid w \in \mathbb{R}^n, w_{n+1} \in \mathbb{R} \mid w^T d + w_{n+1} e \geq 0 \text{ for all } \begin{pmatrix} d \\ e \end{pmatrix} \in C_S \right\}.$$

Since  $C_S$  is assumed to be closed, we have

$$(C_S^*)^* = C_S \text{ ([9] Theorem 14.1 or [10] Lemma 2.7.6).}$$

Thus  $\begin{pmatrix} g \\ -h \end{pmatrix} \in C_S$  if and only if

$$w^T g - w_{n+1} h \geq 0 \text{ for all } \begin{pmatrix} w \\ w_{n+1} \end{pmatrix} \in C_S^*.$$

Now  $C_S$  is generated by

$$\left\{ \begin{pmatrix} u(t) \\ -u_{n+1}(t) \end{pmatrix} \mid t \in S \right\} \cup \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

Thus

$$C_S^* = \left\{ \begin{pmatrix} w \\ w_{n+1} \end{pmatrix} \mid w^T u(t) - w_{n+1} u_{n+1}(t) \geq 0 \text{ for all } t \in S \text{ and } w^T 0 + w_{n+1} 1 \geq 0 \right\}.$$

Therefore, we have that  $\begin{pmatrix} g \\ -h \end{pmatrix} \in \mathcal{C}_S$  if and only if for  $w \in \mathbb{R}^n$  and  $w_{n+1} \in \mathbb{R}$ ,

$$w^T g - w_{n+1} h \geq 0 \quad (3a)$$

whenever

$$w^T u(t) - w_{n+1} u_{n+1}(t) \geq 0 \quad \text{for all } t \in S \quad (3b)$$

$$w_{n+1} \geq 0 \quad (3c)$$

Case 1:  $u_{n+1}(\cdot) \neq 0$ .

Let  $\bar{w} \in \mathbb{R}^n$  and  $\bar{w}_{n+1} \in \mathbb{R}$  satisfy (3b) with  $w_{n+1} = 0$ . That is

$$\bar{w}^T u(t) \geq 0 \quad \text{for all } t \in S.$$

If  $\bar{w} \neq 0$ , it is a direction of infinity [10] (or recession direction [9]) of the set  $K_S$  which contradicts the assumption that  $K_S$  is bounded if  $u_{n+1}(\cdot) \neq 0$ . Therefore, we may divide (3a) and (3b) by  $w_{n+1}$  to obtain the desired result.

Case 2:  $u_{n+1}(\cdot) = 0$ .

In this case (3a), (3b) and (3c) become

$$w^T g - w_{n+1} h \geq 0$$

whenever

$$w^T u(t) \geq 0 \quad \text{for all } t \in S$$

$$w_{n+1} \geq 0.$$

These conditions are equivalent to

$$w^T g \geq 0 \quad (4a)$$

$$-h \geq 0 \quad (4b)$$

whenever

$$w^T u(t) \geq 0 \text{ for all } t \in S. \quad (4c)$$

If (4a), (4b) and (4c) hold, then obviously

$$w^T g - h \geq 0 \quad (5a)$$

whenever

$$w^T u(t) \geq 0 \quad (5b)$$

Now suppose that (5a) and (5b) are satisfied by  $\bar{w}$ . Then for any  $\alpha > 0$ ,  $\alpha \bar{w}$  satisfies (5b) and so

$$\alpha \bar{w}^T g - h \geq 0 \text{ for all } \alpha > 0$$

or

$$\bar{w}^T g \geq 0.$$

If we set  $w = 0$ , we get

$$-h \geq 0.$$

Thus (5a) and (5b) imply (4a), (4b) and (4c). //

Lemma 2 is used in showing that a duality equality exists between Programs P and D under our assumptions.

**Theorem 1:** If Program P is feasible and finite valued, then Program D is feasible and  $V_P = V_D$ . Moreover, Program D assumes its supremum as a maximum.

**Proof:** Program P can be written in the following form:

Find  $V_P = \inf z$

from among  $x \in \mathbb{R}^n$ ,  $\eta(\cdot) \in \mathcal{C}_Q$ ,  $z \in \mathbb{R}$ ,  $w \in \mathbb{R}$  which satisfy

$$u^T(t)x \geq u_{n+1}(t) \quad \text{for all } t \in S$$

$$\begin{pmatrix} Ax \\ c^T x \end{pmatrix} + \sum_{r \in Q} \begin{pmatrix} v(r) \\ -v_{m+1}(r) \end{pmatrix} \eta(r) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w = \begin{pmatrix} b \\ z \end{pmatrix}$$

and  $\eta(\cdot) \geq 0$ ,  $w \geq 0$ . //

This is equivalent to the following form.

Find  $V_P = \inf z$

from among  $x \in \mathbb{R}^n$ ,  $z \in \mathbb{R}$  which satisfy

$$x \in K_S$$

$$\begin{pmatrix} b - Ax \\ z - c^T x \end{pmatrix} \in \mathcal{C}_Q. //$$

Let us define the set  $\bar{K} \subseteq \mathbb{R}^{n+1}$

$$\bar{K} = \left\{ \begin{pmatrix} b - Ax \\ z - c^T x \end{pmatrix} \mid x \in K_S \text{ and } z < V_P \right\}.$$

Since  $K_S$  is convex,  $\bar{K}$  is also convex. Now  $V_P$  is the value of Program P. Thus there cannot be an  $\bar{x} \in K_S$  and  $\bar{z} < V_P$  such that

$$\begin{pmatrix} b - A\bar{x} \\ \bar{z} - c^T \bar{x} \end{pmatrix} \in \mathcal{C}_Q.$$

Hence

$$\bar{K} \cap \mathcal{C}_Q = \emptyset.$$

Since  $\bar{K}$  and  $\mathcal{C}_Q$  are disjoint, non-empty convex sets, we can find a hyperplane that separates them ([10] Theorem 3.3.9 or [9] Theorem 11.3).

That is, there exist  $y \in \mathbb{R}^m$  and  $y_{m+1} \in \mathbb{R}$ , not both zero, such that

$$y^T d + y_{m+1} d_{m+1} \geq 0 \quad \text{for all} \quad \begin{pmatrix} d \\ d_{m+1} \end{pmatrix} \in \mathcal{C}_Q \quad (6)$$

and

$$y^T (b - Ax) + y_{m+1} (z - c^T x) \leq 0 \\ \text{for all } x \in K_S \text{ and } z < v_P. \quad (7)$$

Now  $\begin{pmatrix} v(r) \\ -v_{m+1}(r) \end{pmatrix} \in \mathcal{C}_Q$  for all  $r \in Q$  and  $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathcal{C}_Q$ . Thus (6) implies that

$$y^T v(r) - y_{m+1} v_{m+1}(r) \geq 0 \quad \text{for all } r \in Q \\ y_{m+1} \geq 0 \quad (8)$$

If  $y_{m+1} = 0$ ,  $y^T v(r) \geq 0$  for all  $r \in Q$  which implies that  $y$  is a direction of infinity of  $K_Q$ , which contradicts the assumption that  $K_Q$  is bounded.

Thus  $y_{m+1} > 0$ . Dividing (7) and (8) by  $y_{m+1}$ , we obtain

$$y^T v(r) \geq v_{m+1}(r) \quad \text{for all } r \in Q \quad (9)$$

and

$$y^T (b - Ax) + (z - c^T x) \leq 0 \quad \text{for all } x \in K_S \text{ and } z < v_P.$$

This last inequality can be written as

$$x^T (A^T y + c) \geq b^T y + z \quad \text{for all } x \in K_S \text{ and } z < v_P.$$

By Lemma 2, this is equivalent to

$$\begin{pmatrix} A^T y + c \\ -b^T y - z \end{pmatrix} \in \mathcal{C}_S \quad \text{for all } z < v_P.$$

Since  $C_S$  is closed,

$$\begin{pmatrix} A^T y + c \\ -b^T y - v_P \end{pmatrix} \in C_S.$$

Thus there exist a  $\lambda(\cdot) \in R^{(S)}$  and  $w \in R$  which satisfy

$$A^T y + c = \sum_{t \in S} u(t) \lambda(t) \quad (10)$$

$$-b^T y - v_P = \sum_{t \in S} -u_{n+1}(t) \lambda(t) + w \quad (11)$$

$$\lambda(\cdot) \geq 0 \text{ and } w \geq 0. \quad (12)$$

Combining (9), (10) and (12), we have that  $y, \lambda(\cdot)$  is feasible for Program D. Using Lemma 1, (11) and (12), we have

$$v_P \leq \sum_{t \in S} u_{n+1}(t) \lambda(t) - b^T y \leq v_D \leq v_P. \quad //$$

Remark 1: Let us make the following assumptions.

- (A1\*) Either the set  $K_Q$  is non-empty and bounded or  $v_{m+1}(\cdot) = 0$ .
- (A2\*) The set  $K_S$  is non-empty and bounded.
- (A3\*) The cone  $C_Q$  is closed.

Under these assumptions, if Program D is feasible and finite valued, then Program P is feasible, attains its value and  $v_P = v_D$ .

Remark 2: Let us assume that the sets  $K_S$  and  $K_Q$  are non-empty and bounded. Also, assume that  $C_Q$  and  $C_S$  are closed. Then, if either program is feasible and finite valued, the other program is feasible and has the same value. Moreover, both programs attain their values.

The next lemma shows that we do not have to consider  $\lambda(\cdot) \in \mathbb{R}^{(S)}$  with arbitrarily large support.

**Lemma 3:** Suppose that Program D assumes its value. Then there exists a pair  $y^* \in \mathbb{R}^m$ ,  $\lambda^*(\cdot) \in \mathbb{R}^{(S)}$  with  $\lambda^*(\cdot)$  having at most  $n$  points in its support at which Program D assumes its value.

**Proof:** Suppose that Program D assumes its value at  $\hat{y}$ ,  $\hat{\lambda}(\cdot)$ . Then

$$\begin{pmatrix} A^T \hat{y} + c \\ -b^T \hat{y} - z \end{pmatrix} \in \mathcal{C}_S \quad \text{for all } z \leq v_D, \quad (13a)$$

and

$$\begin{pmatrix} A^T \hat{y} + c \\ -b^T \hat{y} - z \end{pmatrix} \notin \mathcal{C}_S \quad \text{for } z > v_D. \quad (13b)$$

From Caratheodory's theorem for cones, we have that there exist  $t_1, \dots, t_{n+1}$  in  $S$  and  $\hat{p}_1, \dots, \hat{p}_{n+1}$  in  $R$  which satisfy

$$b^T \hat{y} + v_D = \sum_{i=1}^{n+1} \hat{p}_i u_{n+1}(t_i) \quad (14)$$

$$A^T \hat{y} + c = \sum_{i=1}^{n+1} \hat{p}_i u(t_i) \quad (15)$$

$$\hat{p}_i \geq 0 \quad \text{for } i=1, \dots, n+1 \quad (16)$$

Consider the program

$$\max \sum_{i=1}^{n+1} \hat{p}_i u_{n+1}(t_i)$$

from among  $\hat{p}_1, \dots, \hat{p}_{n+1}$  in  $R$  which satisfy

$$\sum_{i=1}^{n+1} \hat{p}_i u(t_i) = A^T \hat{y} + c$$

$$\hat{p}_i \geq 0 \quad \text{for } i = 1, \dots, n+1. \quad //$$

Relations (13a) through (16) show that the value of this program is  $b^T y + V_D$ , and the theory of linear programming shows that this value can be achieved with at most  $n$  of the variables positive. //

Remark 3: If we assume that Program P attains its value, we can show in the same fashion that it assumes its value at a pair  $x^* \in \mathbb{R}^n$ ,  $\eta^*(\cdot) \in \mathbb{R}^{(Q)}$  such that  $\eta^*(\cdot)$  has at most  $m$  points in its support.

The last duality relation that we consider in this paper is a complementary slackness result.

Theorem 2: Let  $x^*$ ,  $\eta^*(\cdot)$  be optimal for Program P and let  $y^*$ ,  $\lambda^*(\cdot)$  be optimal for Program D. Then

$$(u^T(t)x^* - u_{n+1}(t))\lambda^*(t) = 0 \text{ for all } t \in S$$

and

$$(v^T(r)y^* - v_{m+1}(r))\eta^*(r) = 0 \text{ for all } r \in Q.$$

Proof: From Theorem 1, we have that  $V_P = V_D$ , thus

$$c^T x^* - \sum_{r \in Q} v_{m+1}(r)\eta^*(r) = \sum_{t \in S} u_{n+1}(t)\lambda^*(t) - b^T y^*.$$

Using (1b) and (2a), this can be written as

$$\sum_{t \in S} \left[ u^T(t)x^* - u_{n+1}(t) \right] \lambda^*(t) = - \sum_{r \in Q} \left[ v^T(r)y^* - v_{m+1}(r) \right] \eta^*(r).$$

From (1a),

$$u^T(t)x^* - u_{n+1}(t) \geq 0 \text{ for all } t \in S;$$

while (1c) gives  $\lambda^*(t) \geq 0$  for all  $t \in S$ . Thus the left hand side of the equation is non-negative. From (2b),

$$v^T(r)y^* - v_{m+1}(r) \geq 0 \text{ for all } r \in Q;$$

while (2c) gives  $\eta^*(r) \geq 0$  for all  $r \in Q$ . Thus the right hand side of the equation is non-positive. Therefore, the only way the equation can be true is if everything is zero. //

#### 4. A Nonlinear System for Separably-infinite Programs

One of the most successful methods for solving semi-infinite programs consists roughly of the following steps:

- (i) Replace the region  $S$  of Program I by a subset containing a finite number of points and solve the resulting finite linear program and its dual.
- (ii) Use the results of (i) to find a starting point for Newton-Raphson iterations on a system of nonlinear equations obtained from duality relationships between Programs I and II [4], [6], [7].

In this section, the results of Section 3 are used in developing a nonlinear system for separably-infinite programs that is similar to that developed by Gustafson and Kortanek for semi-infinite programs. An additional assumption is imposed.

(A4) Denote by  $u_j(\cdot)$  the  $j$ -th component of  $u(\cdot)$  and by  $v_j(\cdot)$  the  $j$ -th component of  $v(\cdot)$ . Let  $u_1(\cdot), \dots, u_{n+1}(\cdot)$  be continuously differentiable on an open region containing  $S$ ; and let  $v_1(\cdot), \dots, v_{m+1}(\cdot)$  be continuously differentiable on an open region containing  $Q$ .

Lemma 4: Let  $x \in \mathbb{R}^n$ ,  $\eta(\cdot) \in \mathbb{R}^{(Q)}$  be feasible for Program P. If  $\hat{t}$  is in the interior of  $S$  and

$$u^T(\hat{t})x = u_{n+1}(\hat{t}),$$

then

$$\sum_{j=1}^n \nabla u_j(\hat{t})x_j = \nabla u_{n+1}(\hat{t}).$$

Proof: Since the pair  $x, \eta(\cdot)$  is feasible, the function

$$\theta(\cdot) = u^T(\cdot)x - u_{n+1}(\cdot)$$

is non-negative on  $S$ ; and, therefore,  $\hat{t}$  is a point at which the minimum is attained. Since  $\hat{t}$  is in the interior of  $S$ ,

$$\nabla \theta(\hat{t}) = 0. \quad //$$

Lemma 4 holds for Program D with appropriate changes of notation.

Let us assume that Program P attains its value. By Remark 3, we have that there exists a pair  $x^* \in \mathbb{R}^n$ ,  $\eta^*(\cdot) \in \mathbb{R}^{(Q)}$  at which Program P attains its value such that  $\eta^*(\cdot)$  has at most  $m$  points in its support. Let  $\{r_1, \dots, r_q\}$  be the support of  $\eta^*(\cdot)$  and let  $\eta_i^* = \eta^*(r_i)$ .

By Theorem 1 and Lemma 3, there exists a pair  $y^* \in \mathbb{R}^m$ ,  $\lambda^*(\cdot) \in \mathbb{R}^{(S)}$  at which Program D attains its value such that  $\lambda^*(\cdot)$  has at most  $n$  points in its support. Let  $\{t_1, \dots, t_p\}$  be the support of  $\lambda^*(\cdot)$  and let  $\lambda_i^* = \lambda^*(t_i)$ .

Because of the feasibility of these pairs and Theorem 2, they must satisfy the following equations.

$$\sum_{j=1}^n u_j(t_i) x_j^* - u_{n+1}(t_i) = 0 \quad \text{for } i = 1, \dots, p \quad (17a)$$

$$\sum_{j=1}^m v_j(r_i) y_j^* - v_{m+1}(r_i) = 0 \quad \text{for } i=1, \dots, q \quad (17b)$$

$$Ax^* + \sum_{i=1}^q v(r_i) \eta_i^* = b \quad (17c)$$

$$-A^T y^* + \sum_{i=1}^p u(t_i) \lambda_i^* = c \quad (17d)$$

If  $t_1, \dots, t_p$  are in the interior of  $S$  and if  $r_1, \dots, r_q$  are in the interior of  $Q$ , Lemma 4 yields the following equations.

$$\sum_{j=1}^n \nabla_{u_j}(t_i) x_j^* - \nabla_{u_{n+1}}(t_i) = 0 \quad \text{for } i=1, \dots, p \quad (17e)$$

$$\sum_{j=1}^m \nabla_{v_j}(r_i) y_j^* - \nabla_{v_{m+1}}(r_i) = 0 \quad \text{for } i=1, \dots, q. \quad (17f)$$

Once  $p$  and  $q$  are set, the unknowns in system (17) are  $x^* \in \mathbb{R}^n$ ,  $y^* \in \mathbb{R}^m$ ,  $t_i \in \mathbb{R}^k$  and  $\lambda_i^* \in \mathbb{R}$  for  $i=1, \dots, p$ ,  $r_i \in \mathbb{R}^l$  and  $\eta_i^* \in \mathbb{R}$  for  $i=1, \dots, q$ . System (17) consists of  $p + q + m + n + kp + ql$  nonlinear equations in the same number of variables that are satisfied at the optima. System (17) and its uses in numerical techniques are analogous to the corresponding system for semi-infinite programs and its uses.

Remark 4: In developing (17e), we assumed that  $t_i$  for  $i=1, \dots, p$  are in the interior of  $S$ . If some  $t_i$  is a boundary point of  $S$ , then some independent vectors  $h_1, h_2, \dots, h_f$  ( $f < k$ ) may possibly be found such that for a  $\delta > 0$ ,

$$t_i + \epsilon h_s \in S \quad \text{for } -\delta \leq \epsilon \leq \delta.$$

We have that

$$g_s(\epsilon) \equiv u^T(t_i + \epsilon h_s) x^* - u_{n+1}(t_i + \epsilon h_s)$$

is non-negative for  $-\delta \leq \epsilon \leq \delta$  with  $g_s(0) = 0$ . Thus

$$\sum_{j=1}^n h_s^T \nabla_{u_j}(t_i) x_j^* - h_s^T \nabla_{u_{n+1}}(t_i) = 0$$

for  $s = 1, 2, \dots, f$ . These equations should be included in (17e) in place of the vector equation for  $t_i$ . The same result holds for an  $r_i$  which is a boundary point of  $Q$ .

## 5. Conclusion

In this paper, we introduced a class of infinite programs that have applications in economics and statistics. The duality of these programs was examined under assumptions that are usually satisfied by practical problems and permit us to develop duality relationships that are useful in developing numerical techniques. The development of these techniques will be explored in future papers. Also, the duality in separably-infinite programming will be studied in more generality.

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