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ON CARDINAL SPLINE SMOOTHING.(U)

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ON CARDINAL SPLINE SMOOTHING

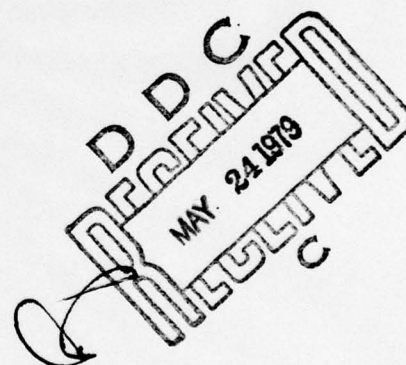
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MATHEMATICS RESEARCH CENTER

ON CARDINAL SPLINE SMOOTHING

I. J. Schoenberg

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ABSTRACT

The paper has two parts. In part I it describes the old results on the problem of smoothing a given bi-infinite sequence of equidistant data ([4], [5], [1], [2]). It includes in §3 the main results on the cardinal spline interpolation of equidistant data of power growth. All this is meant as background for the new developments in Part II.

In Part II we present a method of smoothing a sequence of equidistant data $\{y_v\}$ ($-\infty < v < \infty$). It is based on the ideas of E. T. Whittaker and may be described as follows. In terms of the central B-spline $M_{2m}(x)$, defined by (7) of Part I, §3, we define the cosine polynomial $\phi_{2m}(u)$ by (26) of Part I, §3. Given a positive smoothing parameter ϵ , we define the sequence of "weights" $\{\omega_v(\epsilon)\}$ by the Fourier expansion

$$\frac{1}{\phi_{2m}(u) + \epsilon(2 \sin \frac{u}{2})^{2m}} = \sum_{-\infty}^{\infty} \omega_v(\epsilon) e^{ivu}.$$

Let $\sum_{-\infty}^{\infty} |y_v| < \infty$. It is then shown that the cardinal spline

$$S(x; \epsilon) = \sum_{-\infty}^{\infty} c_j M_{2m}(x-j),$$

where

$$c_j = \sum_{v=-\infty}^{\infty} y_v \omega_{j-v}(\epsilon),$$

is the solution of the following minimum problem: Among all functions $f(x)$ such that

$$f^{(m)}(\epsilon) \in L_2(\mathbb{R})$$

to find $f(x)$ such that the functional

$$J(f) \equiv \epsilon \int_{-\infty}^{\infty} (f^{(m)}(x))^2 dx + \sum_{-\infty}^{\infty} (f(v) - y_v)^2 \text{ is to be minimal.}$$

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Key Words: Spline functions, Smoothing

Work Unit No. 6 - Spline Functions and Approximation Theory

SIGNIFICANCE AND EXPLANATION

Let $\{y_v\}$ ($-\infty < v < \infty$) be a given bi-infinite sequence of data that are in need of smoothing, i.e. replacement by a supposedly smoother sequence $\tilde{y}_v$. Let m be a natural number and ϵ a positive smoothing parameter. It is here shown how to construct a so-called cardinal spline function $S(x; \epsilon)$ having the following properties.

1. $S(x; \epsilon)$ reduces to a polynomial of degree $\leq 2m-1$ in each of the intervals $(i, i+1)$, for all integers i .
2. $S(x; \epsilon)$ has $2m-2$ continuous derivations for all real x .
3. If there exists a polynomial $P(x)$ of degree $\leq 2m-1$ so that $y_v = P(v)$ for all integers v , then $S(v; \epsilon) = y_v$ for all v .
4. If $\epsilon = 0$, then $S(v; 0) = y_v$, for all v , provided that the y_v do not grow too fast with $|v|$, in fact $y_v = O(|v|^\gamma)$ for some $\gamma > 0$.
5. If $\epsilon > 0$, then the smoothed sequence $\{\tilde{y}_v\}$ is given by

$$\tilde{y}_v = S(v; \epsilon) \text{ for all } v.$$

The spline $S(x; v)$ is obtained by an optimal compromise (depending on ϵ) between two simultaneously irreconcilable situations:

- (i) That $S(x; \epsilon)$ should reduce to a polynomial of degree $\leq m-1$,
- (ii) That $S(v; \epsilon) = y_v$ should hold for all v .

If we increase the smoothing parameter ϵ , then we place more emphasis on the condition (i), while allowing $\{S(v; \epsilon)\}$ to differ more from $\{y_v\}$.

In a future publication the author hopes to discuss the application of cardinal spline interpolation to the problem of approximating the Hilbert transform of a function $f(x)$ defined by the sequence of equidistant ordinates $\{f(vh)\}$ ($-\infty < v < \infty, h > 0$). If this sequence is subject to experimental errors, it would seem advisable to perform on the sequence $\{f(vh)\}$ a preliminary smoothing operation by the method of the present paper.

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ON CARDINAL SPLINE SMOOTHING

By I. J. Schoenberg

Introduction. Let me thank Professor Illo Galligani for asking me to visit Rome in May of 1977, where I gave a short 10-hours course on Cardinal Spline Interpolation at his Istituto per le applicazioni del Calcolo "Mauro Picone". He also asked me to prepare the above lectures for publication by his Institute. However, I noticed in Rome the strong current interest in the problem of smoothing of data prescribed at irregularly placed points in the plane. For this reason, and with Professor Galligani's approval, I decided to change the subject of this publication.

In the present form they describe the methods, with some changes to be mentioned below, whereby I solved the numerical problem assigned to me at the Ballistics Research Laboratories in Aberdeen, Maryland, during the second World War (see [4]). The problem was to smooth very extended equidistant tables of drag functions (or drag coefficients) by approximating them by very smooth functions that were easily computable with their first and second derivatives.

The first question to be answered was this: When may a "moving average" operation legitimately be called a smoothing formula? An answer is given in Section 1 of Part I. That it is a reasonable one is shown in Section 2 of Part I. These ideas were later greatly generalized by Fritz John in his important work on parabolic differential equations (see [2]). The connection is briefly mentioned in Section 2 of Part I.

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The second essential ingredient of our results of Part II is the process of cardinal spline interpolation (see [9]). The required results are described in Section 3 of Part I.

The changes from my war-time approach are developed completely in Part II. This is the new contribution of the paper, and it arises if we adapt the idea of E. T. Whittaker (see [10] and [7]) to the problem of smoothing a bi-infinite sequence of equidistant data. The result is the cardinal spline smoothing process.

Part I. Preliminaries

1. What is a smoothing formula? The several smoothing (or graduation)

methods discussed in [10] are all of the "moving average" type. By this we mean that we are given a sequence of real and symmetric weights

$$(1) \quad (a_n), \quad a_n = a_{-n} \quad (-\infty < n < \infty), \quad \sum_{n=-\infty}^{\infty} a_n = 1,$$

which we apply to the data (x_n) to produce their smoothed version (y_n) , by the formula

$$(2) \quad y_n = \sum_{v=-\infty}^{\infty} a_{n-v} x_v, \quad (n = 0, \pm 1, \dots).$$

In other words, we convolve the sequences (a_n) and (x_n) , an operation symbolically described by

$$(3) \quad (y_n) = (a_n) * (x_n).$$

By deriving a formula (2) according to a definite idea, such as a least squares formula, or Whittaker's formulae (see [10, Chapter XI]), we may reasonably expect that it will transform a given sequence (x_n) into a smoother sequence (y_n) . This point becomes doubtful, however, in case the formula (2)

was otherwise obtained, e.g., by some approximating procedure that is not strictly interpolatory. An example of such a procedure will be our cardinal spline smoothing of Part II. A criterion for (2) to be regarded as a smoothing formula was proposed by the author in [4, pp. 50-54] and proceeds as follows.

To begin with, we assume the Laurent series

$$(4) \quad f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad (r < |z| < r^{-1}),$$

to converge in some ring containing the unit circle $|z| = 1$. Setting $z = e^{iu}$,

we call the even function

$$(5) \quad \phi(u) = f(e^{iu}) = a_0 + 2a_1 \cos u + 2a_2 \cos 2u + \dots$$

the characteristic function of the formula (2). The regularity of $f(z)$ on $|z| = 1$, implies that $\phi(u)$ is regular in a certain strip $|\operatorname{Im} u| < \alpha$, whence the valid Taylor expansion

$$(6) \quad \phi(u) = \sum_{n=0}^{\infty} b_n u^n \quad (|u| < \alpha).$$

This expansion allows us to express easily an important property of (2): its reproductive power, or degree of exactness.

We say that (2) has the degree of exactness $2m-1$, where m is a natural number, provided that (2) reproduces exactly, i.e., $y_n = x_n$ for all n , a sequence (x_n) , if this sequence represents the values $P(n)$ of the polynomial of degree $2m-1$, but does not have this property for any higher degree. That the degree of exactness is always odd, follows from the symmetry condition (1). In terms of (6) we have the following easily established proposition:

The formula (2) has the degree of exactness $2m-1$, if and only if the expansion (6) is of the form

$$(7) \quad \phi(u) = 1 - \lambda u^{2m} + \dots \quad (\lambda \neq 0).$$

An example. Let (1) be the sequence

$$a_0 = 5/8, \quad a_1 = 1/4, \quad a_2 = -1/16, \quad a_n = 0 \quad \text{if } n > 2,$$

when the convolution (2) assumes the form

$$(8) \quad y_n = (-x_{n-2} + 4x_{n-1} + 10x_n + 4x_{n+1} - x_{n+2})/16.$$

From (5) we obtain

$$(9) \quad \phi(u) = (5 + 4 \cos u - \cos 2u)/8 = 1 - \frac{1}{16} u^4 + \dots$$

and (7) is seen to hold with $m = 2$, $\lambda = 1/16$. It follows that the formula (8) has the degree of exactness 3.

May (8) be regarded as a smoothing formula? The general criterion is described by the following.

Definition. We say that (2) is a smoothing formula, provided that its characteristic function $\phi(u)$ satisfies

$$(10) \quad -1 < \phi(u) < 1 \quad \text{if} \quad 0 < u < 2\pi.$$

This condition evidently implies that the coefficient λ , appearing in (7), is positive. The characteristic function (9) is easily seen to satisfy (10), and so (6) is a smoothing formula according to our definition of the term.

The criterion (10) raises the following kind of question: Do the numerous smoothing formulae as given in [10] satisfy our criterion (10)? See Greville's paper [1] where some of these questions are answered affirmatively.

In [4, pp. 50-54] good arguments in support of the criterion (10) are presented. Even more convincing reasons were given in [5, Part I]. In the near section we reproduce these arguments as presented in [6, pp. 200-204].

2. The behavior of the iterates of a moving average formula: The problem of De Forest. A conclusive argument in support of the necessity of our condition (1.10) is furnished by the solution of the following problem first stated and attacked by Erastus L. De Forest (1834-1885). If we subject the given sequence (x_k) n times in succession to the same transformation (1.2), we obtain a linear transformation

$$(1) \quad y_v^{(n)} = \sum_{k=-\infty}^{\infty} a_{v-k}^{(n)} x_k,$$

which is the n -fold iterate of (1.2). What is the asymptotic behavior of the

coefficients of (1) as $n \rightarrow \infty$? This question was answered by De Forest and by G. B. Dantzig (for references see [5]) for the case when all coefficients of (1.2) are non-negative, hence necessarily that $m = 1$ in (1.7). A general solution is as follows.

Let (1.2) be such that (1.1), (1.10), and (1.7), are satisfied, hence that $\lambda > 0$. Let

$$(2) \quad G_m(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda v} \cos vx \, dv,$$

which is the normal frequency function

$$(3) \quad G_1(x) = \frac{1}{2\lambda} e^{-\frac{x^2}{4\lambda}}.$$

If $m = 1$, otherwise ($m = 2, 3, \dots$) $G_m(x)$ is an entire function having infinitely many zeros, all real.

The coefficients of (1) satisfy the asymptotic relations

$$(4) \quad a_v^{(n)} = (\lambda n)^{-\frac{1}{2m}} G_m\left(v(\lambda n)^{-\frac{1}{2m}}\right) + o\left(n^{-\frac{1}{2m}}\right) \quad \text{as } n \rightarrow \infty,$$

where the "little o " symbol holds uniformly for all integers v .

For a proof see [5, Part I], where it is also shown by examples that (4) no longer holds if the equality sign is allowed in (1.10), and that the coefficient $a_0^{(n)}$ diverges exponentially to $+\infty$, as $n = 2k$ tends to infinity through even values, if the inequalities (1.10) are reversed anywhere in the interval $0 < u < 2\pi$.

The following discussion, while not directly related to our subject of smoothing, will show the connection of the asymptotic relation (4) with the wider field of parabolic differential equations.

Observe that (2) implies that

$$(5) \quad U(x, t) = t^{-\frac{1}{2m}} G\left(x t^{-\frac{1}{2m}}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-tv^2} + i\pi v \, dv, \quad (t > 0).$$

The function under the integral sign is immediately seen to satisfy for all v , the differential equation

$$(6) \quad \frac{\partial u}{\partial t} = (-1)^{m+1} \frac{\partial^{2m+1} u}{\partial x^{2m+1}} \quad (x \in \mathbb{R}, \quad t > 0),$$

which reduces to the familiar heat equation if $m = 1$. It follows that also $U(x, t)$, defined by (5), is a solution of (6) in the upper half-plane $t > 0$. On the other hand, applying to (2) Fourier's inversion formula and setting $v = 0$, we find that

$$\int_{-\infty}^{\infty} G(x) \, dx = 1.$$

These remarks imply the following: If $f(x)$ is continuous and $o(|x|^{-2})$, say, as $|x| \rightarrow \infty$, then

$$(7) \quad u(x, t) = t^{-\frac{1}{2m}} \int_{-\infty}^{\infty} G_m\left\{(x-v)t^{-\frac{1}{2m}}\right\} f(v) \, dv, \quad (t > 0),$$

is a solution of the differential equation (6) satisfying the boundary condition

$$(8) \quad \lim_{t \rightarrow 0+} u(x, t) = f(x).$$

This particular solution $u(x, t)$ may now also be approximated by the following numerical procedure: Draw in the (x, t) -plane the rectangular lattice of points $(v\Delta x, n\Delta t)$ ($v = 0, \pm 1, \dots; n = 0, 1, 2, \dots$).

Define on it a lattice function $u_{v,n}$ by starting with

$$u_{v,0} = f(v\Delta x),$$

and computing the values along each horizontal line from those on the line below it, by means of the transformation (1.2). This evidently amounts to iterating (1.2), and after n steps we obtain

$$(10) \quad u_{v,n} = \sum_{\mu=0}^{\infty} a_{\mu}^{(n)} f(\mu\Delta x).$$

For any given x and $t > 0$, (10) will go over into (8) if we do the following: We first connect the mesh-sizes Δx and Δt by the relation

$$(11) \quad \Delta t = \lambda(\Delta x)^{2m}.$$

If the integers v and n are such that

$$v\Delta x \rightarrow x, \quad \text{and} \quad n\Delta t \rightarrow t \quad \text{as} \quad \Delta x \rightarrow 0,$$

then

$$u_{v,n} \rightarrow u(x, t).$$

This follows readily from (10) and (8), in view of the asymptotic relation (4): (10) differs from a Cauchy-Riemann sum for the integral (8), by a quantity that tends to zero due to the uniformity in v of the error term of (4).

It is interesting to note that it does not matter which formula (1.2) we use in this construction, as long as it is of the degree of exactness $2m-1$, i.e., it satisfies (1.7), and above all that it satisfies the stability condition (1.10), the term "stability" meaning here stability on iteration. For the general theory of F. John, of which the equation (6) is a special example, see [3].

In this section we dealt exclusively with formulae (1.2) which satisfy the symmetry relation. In [2] T. M. E. Greville dealt with the more difficult case of unsymmetric formulae.

3. Cardinal spline interpolation (See [9, Lectures 1-4]). The problem of cardinal interpolation is to find solutions $f(x)$ of the interpolation problem

$$(1) \quad f(y) = y_v, \quad \text{for all integers } v,$$

where $\{y_v\}$ are the data. A formal solution is furnished by the series

$$(2) \quad f(x) = \sum_{-\infty}^{\infty} y_v \frac{\sin \pi(x-v)}{\pi(x-v)}$$

investigated in 1903 by de la Vallée Poussin, also later by E. T. Whittaker, who called it the cardinal series. The difficulty with (2) is the slow decay of the function $\frac{\sin \pi x}{(\pi x)}$ as $x \rightarrow \infty$. A much simpler solution of (1) is the piecewise linear interpolant $S_1(x)$ given by

$$(3) \quad S_1(x) = \sum_{-\infty}^{\infty} y_v M_1(x-v),$$

where $M_1(x)$ is the roof function defined by

$$M_1(x) = x+1 \text{ in } [-1,0], \quad M_1(x) = 1-x \text{ in } [0,1] \text{ and } M_1(x) = 0 \text{ if } |x| > 1.$$

The purpose of cardinal spline interpolation is to bridge the gap between the piecewise linear $S_1(x)$ defined by (3), and the cardinal series (2). It aims at retaining some of the sturdiness and simplicity of (3), at the same time capturing some of the smoothness and sophistication of (2).

Let m be a natural number, and let

$$(4) \quad \mathcal{S}_{2m-1} = \{S(x)\}$$

be the class of cardinal splines $S(x)$ of degree $2m-1$ defined by the two conditions:

$$(5) \quad S(x) \in C^{2m-1}(\mathbb{R}),$$

$$(6) \quad \text{The restriction of } S(x) \text{ to every unit interval } (v, v+1),$$

where v is an integer, is a polynomial of degree $\leq 2m-1$.

For $m=1$ we find $\mathcal{S}_1$ to be identical with the class (3) of continuous piecewise linear functions. Observe that $\mathcal{S}_{2m-1}$ contains the class of polynomials of degrees not exceeding $2m-1$.

The role of the roof-function $M_1(x)$, of (3), is taken over by the so-called

central B-spline $M_m(x)$: Writing $x_+ = \max(x, 0)$, it may be defined by*

$$(7) \quad M_m(x) = \frac{1}{(2m-1)!} \sum_{v=0}^{2m} (-1)^v \binom{2m}{v} (x+m-v)_+^{2m-1}.$$

Clearly $M_m(x) \in \mathcal{S}_{2m-1}$; we also find that $M_m(x) > 0$ in its support $-m < x < m$. The B-spline should be familiar in view of the fundamental identity

$$\sum_{v=-\infty}^{\infty} M_m(x-v) f(v) = \int_{-\infty}^{\infty} M_m(x) f'(v) dv,$$

which also shows that $\int_{-\infty}^{\infty} M_m(x) dx = 1$ if we choose $f(x) = x+m$.

The representation (3) also generalizes, and every $S(x) \in \mathcal{S}_{2m-1}$ admits a unique representation

$$(8) \quad S(x) = \sum_{v=-\infty}^{\infty} c_v M_m(x-v),$$

where the c_v are constants. This is the so-called standard representation.

The converse is clear: Every series (8) furnishes an element of $\mathcal{S}_{2m-1}$. We now try to solve the interpolation problem (1) by elements of $\mathcal{S}_{2m-1}$. In this direction there are two different kinds of results.

A. The data (y_v) are of power growth (See (8)). We say that the sequence (y_v) is of power growth, and write

$$(9) \quad (y_v) \in PG,$$

provided that

$$(10) \quad y_v = O(|v|^Y) \text{ as } v \rightarrow \pm\infty, \text{ for some } Y \geq 0.$$

Similarly, we write

$$(11) \quad f(x) \in PG,$$

* δ denotes the finite difference taken centrally.

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provided that $f(x) = O(|x|^n)$ as $x \rightarrow \pm \infty$, for some $n \geq 0$.

Below we exclude the trivial case when $n = 1$, since our problem is solved by (3) without any restriction on the (y_j) .

Theorem 1. If the sequence (y_j) is of power growth, then the interpolation problem

$$(12) \quad S(y) = y_j \quad \text{for all } j,$$

has a unique solution $S(x)$ such that

$$(13) \quad S(x) \in \mathcal{B}_{n-1} \cap PC.$$

The assumption (9) of Theorem 1 is a rough one; it admits, e.g., all bounded sequences (y_j) , with $y = 0$ in (10). The second assumption to which we now pass, is much more selective, and takes into account the finer structure of the sequence; in fact it admits only a narrow subclass of the sequences of PC. As usual, with stronger assumptions, stronger conclusions are possible: The interpolant $S(x)$ will exhibit an important extremum property.

2. The case when $\int_{-\infty}^{\infty} |L^n y|^2 dx < \infty$ (See [9, Lecture 6]). We introduce the class of sequences and functions as follows:

$$(14) \quad \tilde{L}_2^n = \{(y_j); \int_{-\infty}^{\infty} |L^n y|^2 dx < \infty\},$$

(15) $\tilde{L}_2^n = \{f(x); f, \dots, f^{(n-1)} \text{ are absolutely continuous, } f^{(n)}(x) \in L_1(\mathbb{R})\}$. Of course $\tilde{L}_2^0$ and $\tilde{L}_2^1$ are the familiar L_2 and L_1 , respectively. We may also describe $\tilde{L}_2^n$ as the class of sequences obtained from elements of L_2 by n successive summations. Similarly the elements of $\tilde{L}_2^n$ are obtained from those of L_2 by n successive integrations.

Theorem 2. If

$$(16) \quad (y_j) \in L_2^n,$$

then the interpolation problem

$$(17) \quad S(y) = y_j \quad \text{for all } j,$$

has a unique solution such that

$$(18) \quad S(x) \in \mathcal{B}_{n-1} \cap L_2^n.$$

This solution $S(x)$ has the following extremum property: If $f(x)$ is an arbitrary function such that

$$(19) \quad f(x) \in L_2^n$$

and

$$(20) \quad f(y) = y_j \quad \text{for all } j,$$

then

$$(21) \quad \int_{-\infty}^{\infty} |f^{(n)}(x)|^2 dx \geq \int_{-\infty}^{\infty} |S^{(n)}(x)|^2 dx,$$

unless $f(x) = S(x)$ for all real x .

In words: If $(y_j) \in L_2^n$, then the spline interpolant $S(x)$ minimizes the integral

$$(22) \quad I(f) = \int_{-\infty}^{\infty} |f^{(n)}(x)|^2 dx$$

among all sufficiently smooth interpolants of (y_j) .

If $y_j = P(y)$ for all j , where $P(x) \in \pi_{n-1}$, then $P(x) \in \mathcal{B}_{n-1} \cap L_2^n$, and therefore $S(x) = P(x)$ by the unicity of the solution in Theorem 2. However, here $I(S) = 0$. In the general case of $(y_j) \in L_2^n$ we may therefore say that $S(x)$ is among all interpolants of (y_j) , the one that "is most nearly" a polynomial of degree $\leq n-1$.

π_{n-1} is the class of polynomials of degree $n-1$ or less.

$$(28) \quad \frac{1}{\phi_m(u)} = \sum_{v=-\infty}^{\infty} a_v e^{ivu}$$

with real coefficients a_v , $a_{-v} = \overline{a_v}$, that decay exponentially. Let us find the standard representation

$$(29) \quad S(x) = \sum_j c_j M_{2m}(x-j)$$

of the solution of the interpolation problem (17), which requires that

$$(30) \quad \sum_j c_j M_{2m}(v-j) = \gamma_v \quad \text{for all } v.$$

Furthermore let

$$(31) \quad C(u) = \sum_j c_j e^{iju}, \quad \text{or } (c_j) \mapsto C(u),$$

be the as yet unknown generating function of the (c_j) . Since the convolution of two sequences has a generating function that is the product of the generating functions of the two sequences, we see by (24), (26), and (31), that the relations (30) are equivalent to the relation

$$(32) \quad C(u) \phi_{2m}(u) = T(u), \quad \text{or } C(u) = \frac{T(u)}{\phi_{2m}(u)}.$$

Now (28) shows that $(c_j) = (y_j) * (a_j)$ and therefore

$$(33) \quad c_v = \sum_j y_j a_{v-j} \quad \text{for all } v.$$

These are the coefficients of the interpolating spline (29).

Examples. 1. If $m = 1$, then $\phi_2(u) = 1$, hence $a_0 = 1$, $a_v = 0$ ($v \neq 0$), and we obtain $c_v = y_v$ for all v . If $m = 2$, we find (see [9, Lecture 4, Section 5]) that

$$a_v = \sqrt{3} \cdot \lambda^{|v|}, \quad \text{where } \lambda = -2 + \sqrt{3} = -.26795.$$

2. If we choose $y_v = \delta_v$, where $\delta_0 = 1$, $\delta_v = 0$ ($v \neq 0$), then (33) shows that $c_v = a_v$. Therefore the spline

If $y_v = p(v)$, where $p(x) \in \pi_{2m-1}$, but $p(x) \notin \pi_{2m-1}$, then again

$p(x) \in \pi_{2m-1} \cap \pi_0$, and so $S(x) = p(x)$ is the unique solution of Theorem 1.

Theorem 2 does not apply here because $(y_v) \notin J_1^m$. There is no interpolant $f(x)$ such that $I(f) < \infty$.

How do we actually construct the solutions $\psi(x)$ of these interpolation problems? To answer this question let us for the moment assume that

$$(23) \quad (y_v) \in J_1, \quad \text{hence a fortiori } (y_v) \in J_2.$$

This insures the continuity of the periodic function

$$(24) \quad T(u) = \sum_{v=-\infty}^{\infty} y_v e^{ivu}$$

which we call the generating function of the sequence (y_v) . Here and below we denote the relationship between a sequence and its generating function symbolically by writing

$$(25) \quad (y_v) \mapsto T(u).$$

We also require the generating function of the sequence $(M_{2m}(v))$, which is

$$(26) \quad \phi_{2m}(u) = \sum_{v=-(m-1)}^{m-1} M_{2m}(v) e^{ivu}.$$

This is a cosine polynomial of order $m-1$, because $M_{2m}(x) = 0$ if $|x| \geq m$. It is readily evaluated by (7), and we find that

$$\phi_2(u) = 1, \quad \phi_4(u) = \frac{1}{3}(2 + \cos u), \quad \phi_6(u) = \frac{1}{60}(33 + 26 \cos u + \cos 2u), \dots$$

It also has the property that

$$(27) \quad 0 < \phi_{2m}(\pi) \leq \phi_{2m}(u) \leq \phi_{2m}(0) = 1 \quad \text{for all } u.$$

It follows that its reciprocal has an expansion

$$(34) \quad L_{2m-1}(x) = \sum_{\nu} w_{\nu} M_{2m}(x - \nu).$$

is the solution of the interpolation problem

$$(35) \quad L_{2m-1}(\nu) = \delta_{\nu}, \quad \text{for all } \nu.$$

The function (34) is the fundamental function of the process, and the solution $S(x)$ of the general problem (17) is given by

$$(36) \quad S(x) = \sum_{\nu=-\infty}^{\infty} y_{\nu} L_{2m-1}(x - \nu).$$

This cardinal interpolation formula bridges the gap between the linear interpolant (3) and the cardinal series (2). In fact, notice that if $m = 1$ then (36) reduces to (3), while we have

$$(37) \quad \lim_{m \rightarrow \infty} S_{2m-1}(x) = \frac{\sin \pi x}{\pi x}.$$

Also every derivative $S_{2m-1}^{(k)}(x)$ converges to the corresponding derivative of the right side of (37), uniformly for all real x .

In our discussion we have assumed that (23) holds. However, the relations (33), (29), and (36) are valid for both Theorems 1 and 2, under their respective assumptions.

1. Statement of the problem. We assume now that

$$(1) \quad (y_{\nu}) \in L_1, \text{ hence } \sum |y_{\nu}| < \infty,$$

and restrict ourselves to real-valued data and functions. We also recall the definitions (3.14) and (3.15) of Part I, of the classes L_1^m and L_2^m . In view of the inclusion relations

$$(2) \quad L_1 \subset L_2 \subset L_2^1 \subset \dots \subset L_2^k \subset L_2^{k+1} \subset \dots$$

(See [9, p. 104]), we observe that (1) implies that (y_{ν}) satisfies the assumptions of Theorem 2 for all m .

The problem. We are given m and a smoothing parameter $\epsilon > 0$. Among all functions

$$(3) \quad f(x) \in L_1^m, \text{ hence } f^{(m)}(x) \in L_1(R),$$

we wish to find the solution of the problem

$$(4) \quad J(f) = \epsilon \cdot \int_{-\infty}^{\infty} (f^{(m)}(x))^2 dx + \int_{-\infty}^{\infty} (f(\nu) - y_{\nu})^2 d\nu = \text{minimum}.$$

Lemma 1. In solving the minimum problem (4) we may restrict the choice of admissible functions $f(x)$ to the elements of

$$(5) \quad S_{2m-1} \cap L_2^m.$$

Proof: If $f(x)$ is such that $J(f) < \infty$, then $(f(\nu) - y_{\nu}) \in L_1$. We apply Theorem 2 to the sequence $(f(\nu))$, and let

$$(6) \quad s(x) \in S_{2m-1} \cap L_2^m$$

be such that $s(\nu) = f(\nu)$ for all ν . But then

$$\begin{aligned} J(s) &= \epsilon \int_{-\infty}^{\infty} (s^{(m)})^2 dx + \int_{-\infty}^{\infty} (s(u) - y_u)^2 \\ &= \epsilon \int_{-\infty}^{\infty} (s^{(m)})^2 dx + \int_{-\infty}^{\infty} (f(u) - y_u)^2, \end{aligned}$$

and so

$$J(s) \leq \epsilon \int_{-\infty}^{\infty} (f^{(m)})^2 dx + \int_{-\infty}^{\infty} (f(u) - y_u)^2 = J(f).$$

In view of the extremum property of $s(x)$ as expressed by (3.21) of Theorem 2. Therefore, for any $f(x)$, the spline $s(x)$ that interpolates $f(x)$, produces a value $J(s) \leq J(f)$.

Let us therefore find the solution

$$(7) \quad S(x) = \sum_j c_j M_{2m}(x-j)$$

of the minimum problem

$$(8) \quad J(S) = \epsilon \int_{-\infty}^{\infty} (S^{(m)})^2 dx + \int_{-\infty}^{\infty} (S(u) - y_u)^2 = \text{minimum}.$$

Here we need another

Lemma 2. If (7) satisfies $S(x) \in L_2(K)$, hence also $(c_j) \in L_2$, then

$$(9) \quad \int_{-\infty}^{\infty} (S^{(m)}(x))^2 dx = \sum_{j,v} y_{j-v} c_j c_v.$$

where

$$(10) \quad (y_v) \mapsto \left(2 \sin \frac{u}{2} \right)^2 \phi_{2m}(u).$$

Proof. From (7) we find that

$$\begin{aligned} (11) \quad \int_{-\infty}^{\infty} (S^{(m)}(x))^2 dx &= \int \left\{ \sum_{j,k} c_j c_k M_{2m}^{(m)}(x-j) M_{2m}^{(m)}(x-k) \right\} dx \\ &= \sum_{j,k} y_{j-k} c_j c_k. \end{aligned}$$

where (Y_r) is the even sequence defined by

$$(12) \quad Y_r = \int_{-\infty}^{\infty} M^{(m)}(x) M^{(m)}(x-r) dx,$$

where, to simplify notations we dropped the subscript $2m$ of $M_{2m}(x)$. Integrations by parts show that

$$(13) \quad Y_r = (-1)^{m-1} \int_{-\infty}^{\infty} M'(x) M^{(2m-1)}(x-r) dx.$$

Observe that $M^{(2m-1)}(x)$ is a step function assuming in consecutive unit intervals the values

$$(14) \quad \dots, 0, 0, 1, -\binom{2m-1}{1}, \binom{2m-1}{2}, \dots, -1, 0, 0, \dots$$

This sequence has the generating function

$$(15) \quad \sum_{j=0}^{2m-1} (-1)^j \binom{2m-1}{j} e^{iju} = (1 - e^{iu})^{2m-1},$$

except for a shift factor e^{iuk} which we disregard. Now (13) indicates that (Y_r) is the convolution of the sequence (14) with the sequence

$$\int_{-v+1}^{v+1} M'(x) dx = M(v+1) - M(v) \mapsto (e^{-iu} - 1) \phi_{2m}(u).$$

However, in (13) the sequence (Y_r) appears as a sum of the form $\sum_v a_v b_{v-r}$.

If we pass from (a_v) to the reversed sequence (a_{-v}) , we obtain a genuine convolution $\sum_v a_{-v} b_{v-r}$. Let us therefore reverse the first sequence (14). As we obtain the generating function of the reversed sequence by changing u into $-u$ in its original generating function, we find the generating function of (Y_r) to be (up to a shift factor e^{iuk}) the product

$$\begin{aligned}
& (-1)^{m-1} (1 - e^{-iu})^{2m-1} \cdot (e^{-iu} - 1) \phi_{2m}(u) \\
& = (-1)^m (1 - e^{-iu})^{2m} \phi_{2m}(u) \\
& = e^{-i2um} (-1)^m (e^{iu/2} - e^{-iu/2})^{2m} \phi_{2m}(u) \\
& = e^{-i2um} (2 \sin \frac{u}{2})^{2m} \phi_{2m}(u).
\end{aligned}$$

Since $\langle Y_r \rangle$ is an even sequence, its generating function must be even, and therefore

$$\langle Y_r \rangle \mapsto (2 \sin \frac{u}{2})^{2m} \phi_{2m}(u),$$

establishing (10).

2. Solution of the problem. From (8), (9), and (7) we find that

$$(1) \quad J(S) = e \sum_{j,v} Y_{j-v} c_j c_v + \sum_v \left\{ \sum_j c_j M(v-j) - Y_v \right\}^2.$$

Let us minimize this function of the (c_j) . To obtain the normal equations, we differentiate $J(S)$ obtaining

$$\frac{1}{2} \frac{\partial}{\partial c_k} J(S) = e \sum_j Y_{j-k} c_j + \sum_v \left\{ \sum_j c_j M(v-j) - Y_v \right\} M(v-k) = 0 \quad (k \in \mathbb{Z}).$$

If we sum within the double-sum only with respect to v , we obtain

$$(2) \quad \sum_v M_{2m}(v-j) M_{2m}(v-k) = \alpha_{j-k}$$

where

$$(3) \quad (\alpha_j) \mapsto (\phi_{2m}(u))^2.$$

The normal equations thus become

$$e \sum_j Y_{j-k} c_j + \sum_j \alpha_{j-k} c_j = \sum_v Y_v M_{2m}(v-k) \quad (k \in \mathbb{Z}),$$

or

$$(4) \quad \sum_j (\alpha_{j-k} + e Y_{j-k}) c_j = \sum_v Y_v M_{2m}(v-k) \quad (k \in \mathbb{Z}).$$

However, by (3) and (1.10) we find

$$(5) \quad (\alpha_j + e Y_j) \mapsto (\phi_{2m}(u))^2 + e (2 \sin \frac{u}{2})^{2m} \phi_{2m}(u).$$

From

$$(M_{2m}(v)) \mapsto \phi_{2m}(u),$$

and writing

$$(6) \quad (c_j) \mapsto C(u), \quad (Y_j) \mapsto T(u),$$

we find the normal equations (4) to be equivalent to the relation

$$\left\{ (\phi_{2m}(u))^2 + e (2 \sin \frac{u}{2})^{2m} \phi_{2m}(u) \right\} C(u) = T(u) \phi_{2m}(u),$$

whence

$$(7) \quad (c_j) \mapsto C(u) = \frac{T(u)}{\phi_{2m}(u) + e (2 \sin \frac{u}{2})^{2m}}.$$

This establishes

Theorem 3. In terms of the expansion

$$(8) \quad \frac{1}{\phi_{2m}(u) + e (2 \sin \frac{u}{2})^{2m}} = \sum_{v=-\infty}^{\infty} x_v(\epsilon) e^{i v u},$$

where the coefficients $x_v(\epsilon) = x_{-v}(\epsilon)$ decay exponentially, the coefficients

(c_j) of the solution

$$(9) \quad S(x) = \sum_j c_j M_{2m}(x-j)$$

of the minimum problem, are

$$(10) \quad c_v = \sum_{j=-\infty}^{\infty} y_j w_{v-j}(\epsilon).$$

We call the solution (9) the cardinal smoothing spline.

3. A few properties of the cardinal smoothing spline $S(x) = S(x; \epsilon)$.

A. We have assumed above that $\epsilon > 0$. However, if we set $\epsilon = 0$ in (2.8),

it becomes

$$(11) \quad \frac{1}{\phi_{2m}(u)} = \sum_v w_v(0) e^{i v u},$$

and a comparison with the expansion (3.28) of Part I, shows that $w_v(0) = w_v$

for all v . This shows that $S(x; 0) = S(x)$ reduces to the interpolating cardinal

spline $S(x)$ of Theorem 2.

3. What is the effect of the smoothing spline $S(x) = S(x; \epsilon)$ on the original

data $\{y_j\}$? This we answer by determining the "smoothed" sequence $\{S(y_j)\}$, to compare it with $\{y_j\}$. By (2.9) and (2.10) we find

$$S(y_j) = (c_j)^* (W_{2m}(y)),$$

and therefore, by (2.7),

$$(2) \quad (S(y_j)) \mapsto C(u) \phi_{2m}(u) = \frac{T(u)}{1 + \epsilon \frac{(2 \sin \frac{u}{2})^{2m}}{\phi_{2m}(u)}}.$$

In terms of the expansion

$$(3) \quad \frac{1}{1 + \epsilon \frac{(2 \sin \frac{u}{2})^{2m}}{\phi_{2m}(u)}} = \sum_v \sigma_v(\epsilon) e^{i v u}$$

(2) shows that the sequence $\{S(y_j; \epsilon)\}$ arises from the data $\{y_j\}$ by the smoothing formula

$$(4) \quad S(y_j; \epsilon) = \sum_j \sigma_{v-j}(\epsilon) y_j.$$

Observe that by (2.8) and (3) the coefficients $\sigma_v(\epsilon)$ are expressed in terms of $w_v(\epsilon)$ by $\sigma_v(\epsilon) = \sum_j W_{2m}(v-j) w_j(\epsilon)$.

Is (4) a smoothing formula according to our definition of Part I, Section 1? That it is one we see if we inspect its characteristic function

$$(5) \quad K(u; \epsilon) = \frac{1}{1 + \epsilon \frac{(2 \sin \frac{u}{2})^{2m}}{\phi_{2m}(u)}},$$

for it is evident that

$$(6) \quad 0 < K(u; \epsilon) < K(0; \epsilon) = 1 \quad \text{for } 0 < u < 2\pi.$$

C. The smoothing power of the formula (4) increases with increasing ϵ . In [4, Definition 2, p. 53] we gave good reasons for the following definition: Of two different smoothing formulae having the characteristic functions $\phi(u)$ and $\tilde{\phi}(u)$, we say that the second has greater smoothing power, provided that

$$(7) \quad |\tilde{\phi}(u)| \leq |\phi(u)| \quad \text{for all } u, \text{ excluding equality for all } u.$$

However, if $0 < \epsilon < \tilde{\epsilon}$, it is clear by (5) that

$$0 < K(u; \tilde{\epsilon}) < K(u; \epsilon) \quad \text{if } 0 < u < 2\pi,$$

and the criterion (7) is satisfied.

D. The degree of exactness of the smoothing formula (4) is $2m-1$. This follows from (1.7) of Part I, because (5) shows that we have the expansion in powers of u

$$(8) \quad K(u; \epsilon) = 1 - \epsilon u^{2m} + \dots$$

E. If we drop our assumption (1.1), and assume only that (y_j) is of power growth, then our construction of the smoothing spline $S(x) = S(x; \epsilon)$ by the formulae (2.8), (2.10), and (2.9), remains applicable. Of course, its earlier connection with the functional $J(S)$, of (1.8), no longer holds. In fact we will find that $J(S) = \infty$ for all splines S . Presumably, it is still true that our $S(x; \epsilon)$ minimizes $J(S)$, provided that (y_j) satisfies the condition

$$(y_j) \in L^{\infty}$$

of Theorem 2. However, this I was not able to establish.

In any case I recommend the cardinal smoothing spline $S(x; \epsilon)$, which represents the modification, found more than 30 years later, of my war-time approach to the problem of cardinal smoothing.

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Abstract continued.

described as follows. In terms of the central B-spline $M_{2m}(x)$, defined by (7) of Part I, §3, we define the cosine polynomial $\phi_{2m}(u)$ by (26) of Part I, §3. Given a positive smoothing parameter ϵ , we define the sequence of "weights" $\{\omega_v(\epsilon)\}$ by the Fourier expansion

$$\frac{1}{\phi_{2m}(u) + \epsilon(2 \sin \frac{u}{2})^{2m}} = \sum_{-\infty}^{\infty} \omega_v(\epsilon) e^{i v u}.$$

Let $\sum_{-\infty}^{\infty} |y_v| < \infty$. It is then shown that the cardinal spline

$$S(x; \epsilon) = \sum_{-\infty}^{\infty} c_j M_{2m}(x-j),$$

where

$$c_j = \sum_{v=-\infty}^{\infty} y_v \omega_{j-v}(\epsilon),$$

is the solution of the following minimum problem: Among all functions $f(x)$ such that

$$f^{(m)}(\epsilon) \in L_2(\mathbb{R})$$

to find $f(x)$ such that the functional

$$J(f) \equiv \epsilon \int_{-\infty}^{\infty} (f^{(m)}(x))^2 dx + \sum_{-\infty}^{\infty} (f(v) - y_v)^2 \quad \text{is to be minimal.}$$