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CONVEX SPECTRAL FUNCTIONS

Shmuel Friedland

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CONVEX SPECTRAL FUNCTIONS

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ABSTRACT

In this paper we characterize all convex functionals defined on certain convex sets of hermitian matrices and which depend only on the eigenvalues of matrices. We extend these results to certain classes of non-negative matrices. This is done by formulating some new characterizations for the spectral radius of non-negative matrices, which are of independent interest.

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SIGNIFICANCE AND EXPLANATION

The following result is useful in connection with matrix applications: If $\lambda_1(A)$ denotes the largest eigenvalue of a hermitian matrix A , then

$$(1) \quad \lambda_1(A+B) \leq \lambda_1(A) + \lambda_1(B) ,$$

i.e., if A and B are hermitian matrices, the largest eigenvalue of $A+B$ is at most the sum of the largest eigenvalue of A and B . The quantity $\lambda_1(A)$ is a functional, i.e. a scalar depending on the matrix A . The above example suggests the following problem which is solved in this paper: Determine all functionals $\phi(A)$ depending only on the eigenvalues $\lambda_1, \dots, \lambda_n$ of A such that $\phi(A)$ is convex, i.e.

$$(2) \quad \phi(aA + (1-a)B) \leq a \phi(A) + (1-a)\phi(B), \quad 0 \leq a \leq 1$$

when A, B are hermitian. In economics and biology one very often deals with non-negative matrices. Denote by $\lambda_1(A)$ the spectral radius of a non-negative matrix $A \geq 0$, i.e. the largest non-negative eigenvalue of A . The fact that $\lambda_1(A) > 1$ or $\lambda_1(A) < 1$ plays a crucial role in the stability behaviour of the system. So any convexity results on $\lambda_1(A)$ are helpful to estimate $\lambda_1(A)$. Unfortunately (1) does not hold in general for A, B non-negative. In this paper we prove the validity of (1) for A, B non-negative if $B-A$ is a diagonal matrix. We extend this result for more special type of non-negative matrices. To derive these results we bring new characterizations of the spectral radius of non-negative matrices.

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CONVEX SPECTRAL FUNCTIONS

Shmuel Friedland

1. Introduction

Let A be an $n \times n$ matrix with complex entries. We arrange the eigenvalues of A in the following order

$$(1.1) \quad \operatorname{Re} \lambda_1(A) \geq \operatorname{Re} \lambda_2(A) \geq \dots \geq \operatorname{Re} \lambda_n(A) .$$

By H_n we denote the set of all $n \times n$ hermitian matrices. For $A \in H_n$ the classical maximal characterization states

$$(1.2) \quad \lambda_1(A) = \max_{(x,x)=1} (Ax, x) .$$

Thus $\lambda_1(A)$ is a convex functional on H_n . Ky Fan extended (1.2) [3]

$$(1.3) \quad \sum_{i=1}^k \lambda_i(A) = \max_{(x_i, x_j) = \delta_{ij}} \sum_{i=1}^k (Ax_i, x_i) .$$

In particular $\sum_{i=1}^k \lambda_i(A)$ is a convex functional on H_n . A function

$$(1.4) \quad \phi : A \rightarrow \mathbb{R} \quad (A \in H_n)$$

is called a spectral function if

$$(1.5) \quad \phi(A) = F(\lambda_1(A), \dots, \lambda_n(A)), \quad F : X \rightarrow \mathbb{R}, \quad X \subseteq \mathbb{R}_{\geq}^n .$$

Here $\mathbb{R}_{\geq}^n$ consists of all vectors $(x_1, \dots, x_n)$, $x_1 \geq x_2 \geq \dots \geq x_n$. In Section 2 of this paper we characterize all F for which ϕ is a convex functional on A . It turns out that F must be convex on X and F Schur's order preserving [11].

$$(1.6) \quad F(\alpha) \leq F(\beta) \quad \text{if} \quad \alpha = (\alpha_1, \dots, \alpha_n) < \beta = (\beta_1, \dots, \beta_n) ,$$

$$(1.7) \quad \sum_{i=1}^k \alpha_i \leq \sum_{i=1}^k \beta_i, \quad i = 1, \dots, n-1 ,$$

$$(1.8) \quad \sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i .$$

We also characterize all F such that ϕ is strictly convex. Let A be an $n \times n$ non-negative matrix. As usual denote by $r(A)$ the spectral radius of A . So $\lambda_1(A) = r(A)$. $r(A)$ is not a convex functional on non-negative matrices. For example consider

$$(1.9) \quad A = \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix}, \quad r(A) = \sqrt{\epsilon} .$$

Recently [1] Cohen proved that $r(A)$ is a convex function in i -th diagonal entry of A for any $1 \leq i \leq n$. We extend Cohen's result namely, we show that $r(A+D)$ is convex on D_n — the set of all $n \times n$ real diagonal matrices. In fact this result is a consequence of the Donsker-Varadhan characterization of $r(A)$ [2]. In Section 3 we bring more general characterizations of $r(A)$ by using a certain fundamental inequality for non-negative matrices established in [5]. This enables us to show that $\log r(e^D A)$ is also convex on D_n for a non-negative A . If A^{-1} happens to be an M -matrix then we have a stronger result. Namely, $r(DA)$ is convex on D_n^+ — the subset of non-negative matrices in D_n . This is done in Section 4.

In Section 5 we show how the results of Section 2 can be extended to the non-symmetric case by assuming that A is a totally positive matrix of order $j(TP_j)$. We shall state our results in case that A is a TP ($=TP_n$) matrix. That is all minors of A (of all orders) are non-negative. In that case we have

$$(1.10) \quad \lambda_1(e^D A) \geq \lambda_2(e^D A) \geq \dots \geq \lambda_n(e^D A) \geq 0, \quad D \in D_n .$$

If A is non-singular then the last inequality is strict. Let

$$(1.11) \quad \phi(D) = F(\log \lambda_1(e^D A), \dots, \log \lambda_n(e^D A)) .$$

Then ϕ is convex on $A \in D_n$ if and only if F is convex on X and Schur's order preserving.

We remark that the results in Section 2 hold for symmetric compact operators in Hilbert space. The results of Section 3-5 can be extended to appropriate integral operators, for example, as it was pointed out in [5].

2. Convex functions on the spectrum of hermitian matrices

Let A be an $n \times n$ hermitian matrix. We can view A as a self adjoint operator on $\mathbb{C}^n$ endowed with the standard inner product

$$(2.1) \quad (x, y) = y^* x, \quad x, y \in \mathbb{C}^n.$$

Since the eigenvalues of A are real we arrange them in the decreasing order

$$(2.2) \quad \lambda_1(A) \geq \dots \geq \lambda_n(A).$$

Denote by $\xi_1, \dots, \xi_n$ the corresponding set of orthonormal eigen-vectors of A

$$(2.3) \quad A\xi_i = \lambda_i(A)\xi_i, \quad (\xi_i, \xi_j) = \delta_{ij}, \quad i, j = 1, \dots, n.$$

Let H_n denote the set of all $n \times n$ hermitian matrices. Since $\lambda_1(A)$ has the maximal characterization

$$\lambda_1(A) = \max_{(x, x)=1} (Ax, x),$$

$\lambda_1(A)$ is a convex function on H_n . More generally we have [4]

Theorem 2.1. Let $\{\alpha_i\}^n$ be a decreasing sequence of real numbers

$$(2.4) \quad \alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n.$$

Then for any A belonging to H_n

$$(2.5) \quad \sum_{i=1}^n \alpha_i \lambda_i(A) = \max_{(x_i, x_j) = \delta_{ij}, i, j=1, \dots, n} \sum_{i=1}^n \alpha_i (Ax_i, x_i).$$

Assume that the equality sign holds for some $x_1, \dots, x_n$. Let

$$(2.6) \quad \alpha_1 = \dots = \alpha_{i_1} > \alpha_{i_1+1} = \dots = \alpha_{i_2} > \dots > \alpha_{i_{r-1}+1} = \dots = \alpha_{i_r} = \alpha_n, (i_0=0).$$

Then there exists an orthonormal eigensystem of A such that the following subspaces coincide

$$(2.7) \quad [\xi_{i_j+1}, \dots, \xi_{i_{j+1}}] = [x_{i_j+1}, \dots, x_{i_{j+1}}], \quad j = 0, \dots, r-1.$$

The characterization (2.7) in the case that $\alpha_1 = \dots = \alpha_i = 1, \alpha_{i+1} = \dots = \alpha_n = 0$ was established by Fan [3].

In particular

$$(2.8) \quad \phi(A) = \sum_{i=1}^n \alpha_i \lambda_i(A)$$

is a convex functional on H_n if (2.4) is satisfied. That is

$$(2.9) \quad \phi(cA + (1-c)B) \leq c\phi(A) + (1-c)\phi(B), \quad A, B \in H_n, \quad 0 \leq c \leq 1.$$

We now are ready to state the problem which we solve in this section. A function

$$(2.10) \quad \phi : A \rightarrow \mathbb{R}, \quad A \in H_n$$

is called a spectral function if

$$(2.11) \quad \phi(A) = F(\lambda_1(A), \dots, \lambda_n(A)).$$

That is ϕ is defined on the spectrum of A . Our problem is to characterize all convex spectral functions on H_n . To answer this problem we introduce some notation and definitions.

Let $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ be two vectors satisfying (2.4). According to [7, Sec. 2.18] α is majorized by β , which is denoted by $\alpha < \beta$, if

$$(2.12) \quad \sum_{i=1}^k \alpha_i \leq \sum_{i=1}^k \beta_i, \quad k = 1, \dots, n-1,$$

$$(2.13) \quad \sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i.$$

Denote

$$(2.14) \quad \lambda(A) = (\lambda_1(A), \dots, \lambda_n(A)).$$

From Theorem 2.1 we obtain

Lemma 2.1. Let $A, B \in H_n$. Then

$$(2.15) \quad \lambda(A+B) < \lambda(A) + \lambda(B).$$

Moreover,

$$(2.16) \quad \lambda(A+B) = \lambda(A) + \lambda(B)$$

if and only A and B have a common eigenvector system

$$(2.17) \quad A\xi_i = \lambda_i(A)\xi_i, B\xi_i = \lambda_i(B)\xi_i, (\xi_i, \xi_j) = \delta_{ij}, i, j = 1, \dots, n.$$

Proof. Let

$$(2.18) \quad (A+B)\xi_i = \lambda_i(A+B)\xi_i, (\xi_i, \xi_j) = \delta_{ij}, i, j = 1, \dots, n.$$

So for any $\alpha = (\alpha_1, \dots, \alpha_n)$ which satisfies (2.4) we get

$$(2.19) \quad \sum_{i=1}^n \alpha_i \lambda_i(A+B) = \sum_{i=1}^n \alpha_i ((A+B)\xi_i, \xi_i) \leq \\ \sum_{i=1}^n \alpha_i \lambda_i(A) + \sum_{i=1}^n \alpha_i \lambda_i(B).$$

This establishes (2.15). Suppose that (2.16) holds. Then we must have

$$(2.20) \quad \sum_{i=1}^n \alpha_i \lambda_i(A) = \sum_{i=1}^n \alpha_i (A\xi_i, \xi_i), \sum_{i=1}^n \alpha_i \lambda_i(B) = \sum_{i=1}^n \alpha_i (B\xi_i, \xi_i).$$

for any $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n$. Choose $\alpha_i = n-i$. Then the equalities (2.7) imply (2.17). This conclusion is in fact is stated in Theorem 3.1 in [4].

By $\mathbb{R}_{\geq}^n$ denote the following subset of $\mathbb{R}^n$

$$(2.21) \quad \mathbb{R}_{\geq}^n = \{x | x = (x_1, \dots, x_n), x_1 \geq x_2 \geq \dots \geq x_n\}.$$

Clearly

$$(2.22) \quad \lambda : H_n \rightarrow \mathbb{R}_{\geq}^n \quad (\lambda(A) = (\lambda_1(A), \dots, \lambda_n(A)))$$

Let

$$(2.23) \quad \lambda(A) = X.$$

Thus the function F in terms of which ϕ is constructed satisfies $F : X \rightarrow \mathbb{R}$.

Let D_n be the set of all $n \times n$ real diagonal matrices and D_n^1 the set of all diagonal matrices

$$(2.24) \quad D(\alpha) = \text{diag} \{\alpha_1, \dots, \alpha_n\}, \alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n.$$

Given $X \subseteq \mathbb{R}_{\geq}^n$ we require that A should be of the form

$$(2.25) \quad A = \lambda^{-1}(X).$$

Suppose $\beta \in X$. Then $D(\beta) \in A$. Thus the assumption that ϕ is convex on A implies in particular that ϕ is convex on $D_n^1 \cap A$. So we must have that F is convex on X which means also that X must be convex. Let $D(\beta) \in A$ and P be a permutation matrix $(\delta_{ij+1})_1^n$, $(n+1 \equiv 1)$. Then

$$(2.26) \quad \frac{1}{n} \sum_{i=1}^n P^i D(\beta) (P^T)^i = \left(\sum_{i=1}^n \beta_i / n \right) I.$$

Here by P^T we denote the transpose of P . Therefore if $\beta \in X$ then $\bar{\beta} = (b, \dots, b) \in X$ ($b = \sum_{i=1}^n \beta_i / n$). This in particular implies that

$$(2.27) \quad \text{if } \beta \in X, \alpha < \beta, \text{ then } \alpha \in X.$$

Definition 2.1. Let $X \subseteq \mathbb{R}_+^n$. The set X is called strongly convex if X is convex and the condition (2.27) is satisfied.

Theorem 2.2. Let X be a strongly convex set in $\mathbb{R}_+^n$ which contains at least one point α ,

$$(2.28) \quad \alpha_1 > \alpha_2 > \dots > \alpha_n.$$

Let $F : X \rightarrow \mathbb{R}$. Assume that $F \in C^{(1)}(X)$. Consider a spectral function $\phi : A \rightarrow \mathbb{R}$ ($A \subset H_n$) where ϕ and A are given by (2.11) and (2.25) accordingly. Then ϕ is convex on A if and only if F is convex on X and

$$(2.29) \quad \frac{\partial F}{\partial x_1}(\alpha) \geq \frac{\partial F}{\partial x_2}(\alpha) \geq \dots \geq \frac{\partial F}{\partial x_n}(\alpha)$$

for any $\alpha \in X$. Moreover, ϕ is strictly convex on A , i.e.

$$(2.30) \quad \phi(cA + (1-c)B) < c\phi(A) + (1-c)\phi(B), \quad A \neq B, \quad 0 < c < 1,$$

if and only if F is strictly convex on X and

$$(2.31) \quad \frac{\partial F}{\partial x_i}(\alpha) > \frac{\partial F}{\partial x_{i+1}}(\alpha) \quad \text{if } \alpha_i > \alpha_{i+1}.$$

To prove the theorem we need the following theorem of Ostrowski [11] (Theorems VII and VIII).

Theorem 2.3. Let X and F satisfy the assumptions of Theorem 2.2. Then F satisfies (2.29) if and only if

$$(2.32) \quad F(\alpha) \leq F(\beta) \quad \text{if} \quad \alpha < \beta.$$

Moreover

$$(2.33) \quad F(\alpha) < F(\beta) \quad \text{if} \quad \alpha < \beta \quad \text{and} \quad \alpha \neq \beta$$

if and only if the condition (2.31) holds.

Proof. Assume first that F is convex on X . So if $\lambda(A), \lambda(B) \in X$ then

$$(2.34) \quad F\left(\frac{\lambda(A)+\lambda(B)}{2}\right) \leq \frac{1}{2} (F(\lambda(A)) + F(\lambda(B))) .$$

According to Theorem 2.3, the assumption (2.29) implies

$$(2.35) \quad F\left(\frac{\lambda(A+B)}{2}\right) \leq F\left(\frac{\lambda(A)+\lambda(B)}{2}\right)$$

by the virtue of (2.15). This shows that ϕ is convex on A . Assume furthermore that F is strictly convex on X . So if $\lambda(A) \neq \lambda(B)$ the inequality sign holds in (2.34). This implies (2.30). Suppose that $\lambda(A) = \lambda(B)$ but $A \neq B$. According to Lemma 2.1 $\lambda(A+B) \neq (\lambda(A) + \lambda(B))$. So the additional assumption (2.31) yields the inequality sign in (2.35) according to Theorem 2.3. This manifests that ϕ is strictly convex on A . Assume now that ϕ is convex on A . In particular ϕ is convex on $D_n^1 \cap A$. This immediately implies that F is convex on X . Furthermore if ϕ is strictly convex then F is strictly convex. Let $\beta \in X$. So $D(\beta) \in A$. Assume that $\alpha < \beta$. Then $D(\alpha) \in A$. The classical result of [7, sec. 2.19] states that

$$(2.36) \quad G\beta = \alpha ,$$

where G is some doubly stochastic matrix. The Birkhoff theorem implies

$$(2.37) \quad G = \sum_{i=1}^k a_i P_i, \quad a_i > 0, \quad \sum_{i=1}^k a_i = 1$$

and P_i is a permutation matrix. So

$$(2.38) \quad D(\alpha) = \sum_{i=1}^k a_i P_i D(\beta) P_i^T .$$

So the convexity of ϕ implies

$$(2.39) \quad \phi(D(\alpha)) \leq \sum_{i=1}^k a_i \phi(P_i D(\beta) F_i^T) = \phi(D(\beta)) ,$$

which is equivalent to (2.32). Now (2.29) follows from Theorem 2.3. Assume furthermore that ϕ is strictly convex. Then we must have (2.33) which implies (2.31) according to Theorem 2.3. The proof of the theorem is concluded.

Suppose

$$(2.40) \quad A \subset H_m, m > n .$$

When we can define $\phi : A \rightarrow \mathbb{R}$ by (2.11). That is ϕ does not depend on $\lambda_{n+1}(A), \dots, \lambda_m(A)$, i.e. $\frac{\partial F}{\partial x_i} = 0$ for $i > n$. In that case Theorem 2.2 reads:

Corollary 2.1. Let the assumptions of Theorem 2.2 hold except that we have (2.40). Then ϕ is convex on A if and only if F is convex on X , the inequalities (2.29) hold and in addition

$$(2.41) \quad \frac{\partial F}{\partial x_n}(\alpha) \geq 0, \alpha \in X .$$

3. Some characterization of the spectral radius

Let A be an $n \times n$ non-negative matrix such that there exists two positive vectors u, v satisfying

$$(3.1) \quad Au = r(A)u, Av^T = r(A)v^T, u^T = (u_1, \dots, u_n) > 0, v^T = (v_1, \dots, v_n) > 0.$$

Assume the normalization

$$(3.2) \quad \sum_{i=1}^n u_i v_i = 1.$$

Let P_n be the set of probability vectors

$$(3.3) \quad P_n = \{\alpha \mid \alpha = (\alpha_1, \dots, \alpha_n), \alpha_i \geq 0, \sum_{i=1}^n \alpha_i = 1\}.$$

In [5, Sec. 3] it was manifested

Theorem 3.1. Let A be an $n \times n$ non-negative irreducible matrix having positive entries on the diagonal (or fully indecomposable, see Remark 3.3 in [5]). Then for any $\alpha \in P_n$, with positive entries ($\alpha_i > 0$). The function $f(x) = \sum_{i=1}^n \alpha_i \log \frac{(Ax)_i}{x_i}$ has a unique critical point $\xi = (\xi_1, \dots, \xi_n)$ in the interior point of P_n ($\xi_i > 0$) which must satisfy

$$(3.4) \quad \min_{x > 0} \sum_{i=1}^n \alpha_i \log \frac{(Ax)_i}{x_i} = \sum_{i=1}^n \alpha_i \log \frac{(A\xi)_i}{\xi_i}.$$

Thus, if α is chosen to be

$$(3.5) \quad \alpha = (u_1 v_1, \dots, u_n v_n),$$

where u and v satisfy (3.1) - (3.2) then

$$(3.6) \quad \sum_{i=1}^n u_i v_i \log \frac{(Ax)_i}{x_i} \geq \log r(A),$$

since $x = u$ is a critical point of $f(x)$.

From Theorem 3.1 we get

Theorem 3.2. Let A be an $n \times n$ non-negative matrix such that $r(A) > 0$. Then

$$(3.7) \quad \sup_{\alpha \in P_n} \inf_{x > 0} \sum_{i=1}^n \alpha_i \log \frac{(Ax)_i}{x_i} = \log r(A).$$

Suppose that there exists a positive vector u satisfying (3.1). Assume that

$$(3.8) \quad \inf_{x>0} \sum_{i=1}^n \alpha_i \log \frac{(Ax)_i}{x_i} = \log r(A) .$$

Then the vector v

$$(3.9) \quad v = (\alpha_1/u_1, \dots, \alpha_n/u_n)$$

fulfills (3.1). In particular if A is irreducible then α is unique and given by (3.5).

Proof: As the left-hand side of (3.7) is a continuous function of A it is enough to prove

(3.7) for A positive. Let $u > 0$ be the corresponding eigenvector of A . So

$$\inf_{x>0} \sum_{i=1}^n \alpha_i \log \frac{(Ax)_i}{x_i} \leq \sum_{i=1}^n \alpha_i \log \frac{(Au)_i}{u_i} = \log r(A)$$

for any α such that $\sum_{i=1}^n \alpha_i = 1$. Thus

$$\sup_{\alpha \in P_n} \inf_{x>0} \sum_{i=1}^n \alpha_i \log \frac{(Ax)_i}{x_i} \leq \log r(A) .$$

The above inequality together with (3.6) yields (3.7). Suppose that (3.8) holds. If $u > 0$

satisfies (3.1) then $x = u$ is a minimal point for $f(x) = \sum_{i=1}^n \alpha_i \log \frac{(Ax)_i}{x_i}$. So

$$0 = \frac{\partial f}{\partial x_j} \Big|_{x=u} = \sum_{i=1}^n \frac{\alpha_i a_{ij}}{(Ax)_i} - \frac{\alpha_j}{x_j} \Big|_{x=u} = r(A)^{-1} \sum_{i=1}^n \alpha_i u_i^{-1} a_{ij} - \alpha_j u_j^{-1} .$$

This shows that v given by (3.9) is a left eigenvector of A corresponding to $r(A)$. If A is irreducible, then u and v are unique up to a multiple of a positive scalar. Thus α is of the form (3.5) and since $\alpha \in P_n$, α unique. The proof of the theorem is completed.

We now bring an extended version of Theorem 3.2 which includes (3.7) and the Donsker-Varadhan characterization [2] as its special cases.

Theorem 3.3. Let $\Psi : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous convex function on $\mathbb{R}$. Define $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}$

$$(3.10) \quad \Phi(x) = \Psi(\log x) .$$

Let A be an $n \times n$ non-negative matrix such that $r(A) > 0$. Assume

$$(3.11) \quad \Psi'(\log r(A)) \geq 0 .$$

Then

$$(3.12) \quad \sup_{\alpha \in P_n} \inf_{x > 0} \sum_{i=1}^n \alpha_i \phi\left(\frac{(Ax)_i}{x_i}\right) = \phi(r(A)) .$$

Assume that the inequality sign holds in (3.11) and suppose that there exists a positive vector u satisfying (3.1). If

$$(3.13) \quad \inf_{x > 0} \sum_{i=1}^n \alpha_i \phi\left(\frac{(Ax)_i}{x_i}\right) = \phi(r(A)) ,$$

then the vector v (3.9) satisfies (3.1). In particular if A is irreducible then α is unique and given by (3.5).

Proof. Let $t_0 = \log r(A)$, $\Psi'(t_0) = e$. Then the convexity of Ψ implies

$$\Psi(t) \geq \Psi(t_0) + (t - t_0)\Psi'(t_0) .$$

So

$$(3.14) \quad \sum_{i=1}^n \alpha_i \phi\left(\frac{(Ax)_i}{x_i}\right) \geq (\phi(r(A)) - e \log r(A)) \\ + e \sum_{i=1}^n \alpha_i \log \frac{(Ax)_i}{x_i} , \quad \alpha \in P_n .$$

As $e \geq 0$ from Theorem 3.2 and the above inequality we get

$$(3.15) \quad \sup_{\alpha \in P_n} \inf_{x > 0} \sum_{i=1}^n \alpha_i \phi\left(\frac{(Ax)_i}{x_i}\right) \geq \phi(r(A)) .$$

Since ϕ is continuous we may assume that A is positive. By choosing $x = u$ the left-hand side of (3.15) we deduce an opposite inequality of (3.15). This establishes (3.12). In case the $e > 0$ we use the arguments of Theorem 3.2 to analyze the equality (3.12). End of proof.

Letting $\psi(x) = e^x$ in Theorem 3.3 we obtain the Donsker-Varadhan characterization [2].

Corollary 3.1. Let the assumptions of Theorem 3.2 hold. Then

$$(3.16) \quad \sup_{\alpha \in P_n} \inf_{x > 0} \sum_{i=1}^n \alpha_i \frac{(Ax)_i}{x_i} = r(A) .$$

Suppose that

$$(3.17) \quad \inf_{x>0} \sum_{i=1}^n \alpha_i \frac{(Ax)_i}{x_i} = r(A) .$$

If A has a positive eigenvector u then the conclusions of Theorem 3.2 apply.

Recall the classical characterization due to Wielandt [12]

$$(3.18) \quad \inf_{x>0} \max_{1 \leq i \leq n} \frac{(Ax)_i}{x_i} = r(A)$$

for any non-negative A . Assume that ϕ is an increasing function of x on $\mathbb{R}_+$. So

$$(3.19) \quad \begin{aligned} \inf_{x>0} \sup_{\alpha \in P_n} \sum_{i=1}^n \alpha_i \phi\left(\frac{(Ax)_i}{x_i}\right) &= \inf_{x>0} \phi\left(\max_{1 \leq i \leq n} \frac{(Ax)_i}{x_i}\right) \\ &= \phi\left(\inf_{x>0} \max_{1 \leq i \leq n} \frac{(Ax)_i}{x_i}\right) = \phi(r(A)) . \end{aligned}$$

Thus if ϕ is increasing and satisfies the assumptions of Theorem 3.3 then we can interchange $\sup$ with $\inf$ in (3.12). The characterization (3.19) is completely equivalent to the Wielandt characterization (3.18) while (3.12) seems to be a deeper characterization.

Let A be a non-negative and non-singular. Assume furthermore that A^{-1} is an M-matrix, i.e. the off-diagonal elements of A^{-1} are non-positive. Following [5] we bring another characterization of $r(A)$.

Theorem 3.4. Let A be a non-negative and non-singular matrix such that A^{-1} is an M-matrix. Then

$$(3.20) \quad \inf_{\alpha \in P_n} \sup_{x>0} \sum_{i=1}^n \alpha_i \frac{x_i}{(Ax)_i} = \frac{1}{r(A)} .$$

Assume that there exists a positive vector u satisfying (3.1) and suppose

$$(3.21) \quad \sup_{x>0} \sum_{i=1}^n \alpha_i \frac{x_i}{(Ax)_i} = \frac{1}{r(A)} .$$

Then v given by (3.9) satisfies (3.1). In particular if A is irreducible then α is unique and given by (3.5).

Proof. We have available the representation

$$(3.22) \quad A^{-1} = rI - B, \quad B \geq 0, \quad r > r(B)$$

and B is reducible if and only if A is reducible (e.g. [8, chap. 8]). Again, as in the proof of Theorem 3.2 one may assume that B is positive. By letting x to be equal to the positive eigenvector u of A we immediately deduce

$$(3.23) \quad \inf_{\alpha \in P_n, x > 0} \sum_{i=1}^n \alpha_i \frac{x_i}{(Ax)_i} \geq \frac{1}{r(A)} .$$

Let α be given by (3.5). Obviously for any $x > 0$ and $y = Ax$

$$(3.24) \quad \sum_{i=1}^n u_i v_i \frac{x_i}{(Ax)_i} = \sum_{i=1}^n u_i v_i \frac{(A^{-1}y)_i}{y_i} = r - \sum_{i=1}^n u_i v_i \frac{(By)_i}{y_i} .$$

From Corollary 3.1 it follows

$$(3.25) \quad \sum_{i=1}^n u_i v_i \frac{(By)_i}{y_i} \geq r(B) .$$

So

$$(3.26) \quad \sum_{i=1}^n u_i v_i \frac{x_i}{(Ax)_i} \leq r - r(B) = \frac{1}{r(A)}$$

and the equality sign holds if $x = u$. This establishes (3.20). The equality (3.21) is analyzed in the same way as in Theorem 3.2.

Remark 3.1. Theorem 3.4 does not hold for arbitrary non-negative matrices, take for example A to be a permutation matrix $P \neq I$. Therefore Theorem 3.4 is not a special case of Theorem 3.3.

4. Convexity properties of the spectral radius

Let A be an $n \times n$ non-negative matrix. Consider the matrix $A + D$, $D \in D_n$. Assume that the eigenvalues of $A + D$ arranged in the order

$$(4.1) \quad \operatorname{Re} \lambda_1(A) \geq \operatorname{Re} \lambda_2(A) \geq \dots \geq \operatorname{Re} \lambda_n(A) .$$

Let

$$(4.2) \quad \rho(D) = \lambda_1(A+D) .$$

We claim that $\rho(D)$ is real. If D is non-negative this fact is a consequence of the Perron-Frobenius theorem. For an arbitrary D consider $A + D + aI$

$$(4.3) \quad \lambda_k(A+D+aI) = \lambda_k(A+D) + a, \quad k = 1, \dots, n$$

for a big enough $A + D + aI \geq 0$ and (4.3) implies that $\rho(D)$ is real. Moreover by considering the matrix $B = A + D + aI$ and using the Donsker-Varadhan characterization for B we get the following characterization for $\rho(D)$

$$(4.4) \quad \rho(D) = \sup_{\alpha \in P_n} L_1(D, \alpha) .$$

Here $L_1(D, \alpha)$ is a linear functional on D_n

$$(4.5) \quad L_1(D, \alpha) = \sum_{i=1}^n \alpha_i d_i + \inf_{x > 0} \sum_{i=1}^n \alpha_i \frac{(Ax)_i}{x_i} ,$$

$$\alpha = (\alpha_1, \dots, \alpha_n), \quad D = \operatorname{diag}\{d_1, \dots, d_n\} .$$

It is a standard fact (4.4) and (4.5) imply the convexity of $\rho(D)$ on the set D_n . More precisely we have:

Theorem 4.1. Let A be a fixed $n \times n$ non-negative matrix. Assume that $\rho(D)$, $D \in D_n$, is given by (4.2). Then $\rho(D)$ is a real valued convex functional on D_n .

$$(4.6) \quad \rho((D_1+D_2)/2) \leq (\rho(D_1) + \rho(D_2))/2 .$$

Moreover if A is irreducible then the equality sign holds in (4.6) if and only if

$$(4.7) \quad D_2 - D_1 = aI$$

for some a .

Proof. As we pointed out (4.6) is a consequence of (4.4). So it is enough to analyze the equality case. Let

$$(4.8) \quad A_1 = A + (D_1 + D_2)/2, \quad A_1 u = r_1 u, \quad A_1^T v = r_1 v, \quad r_1 = \rho((D_1 + D_2)/2) .$$

As A is irreducible we may assume that $u, v > 0$ and the normalization (3.2) holds. Let α be given by (3.5). So

$$(4.9) \quad L_1((D_1 + D_2)/2, \alpha) = \inf_{x > 0} \sum_{i=1}^n \alpha_i \frac{(A_1 x)_i}{x_i} = \sum_{i=1}^n \alpha_i \frac{(A_1 u)_i}{u_i} = r_1 .$$

If we apply the results of Section 3 in [5]

$$(4.10) \quad f(x, B) = \sum_{i=1}^n \alpha_i \frac{(Bx)_i}{x_i}$$

where $B + bI$ is irreducible matrix, for some positive b , then $f(x, B)$ has a unique critical point in the interior of P_n which must be the minimum point ($f(x) = +\infty$ on the boundary of P_n). The equality sign in (4.6) implies

$$(4.11) \quad L_1(D_1, \alpha) = \rho(D_1), \quad L_1(D_2, \alpha) = \rho(D_2) .$$

That is

$$(4.12) \quad f(x, A + D_1) \geq f(u, A + D_1) = \rho(D_1), \quad f(x, A + D_2) \geq f(u, A + D_2) = \rho(D_2) .$$

The uniqueness of the minimal point of $f(x, B)$ implies

$$(4.13) \quad (A + D_1)u = \rho(D_1)u, \quad (A + D_2)u = \rho(D_2)u .$$

As $u > 0$ (4.7) follows the above equality. The proof of the theorem is completed.

The inequality (4.6) extends Cohen's result [1]. Let A be a non-negative matrix such that $r(A) > 0$. Clearly, for any $D \in D_n$, $r(e^D A)$ is also positive. Define

$$(4.14) \quad R(D) = \log r(e^D A) .$$

According to Theorem 3.2,

$$(4.15) \quad R(D) = \sup_{\alpha \in P_n} L_2(D, \alpha) ,$$

where

$$(4.16) \quad L_2(D, \alpha) = \sum_{i=1}^n \alpha_i d_i + \inf_{x > 0} \sum_{i=1}^n \alpha_i \log \frac{(Ax)_i}{x_i}.$$

Combining (4.15) and (4.16) and using the uniqueness result stated in Theorem 3.1 as in the proof of Theorem 4.1 we deduce.

Theorem 4.2. Let A be a fixed $n \times n$ non-negative matrix having a positive spectral radius. Assume that $R(D)$ is given by (4.14). Then $R(D)$ is a convex functional on D_n .

$$(4.17) \quad R((D_1 + D_2)/2) \leq (R(D_1) + R(D_2))/2.$$

Moreover if A is irreducible and the diagonal entries of A are positive (or A is fully indecomposable) then the equality sign holds in (4.17) if and only if (4.7) holds for some a .

Assume that $A, B \in H_n$ and furthermore A is positive definite ($(Ax, x) > 0$ for $x \neq 0$).

BA is similar to $A^{1/2}BA^{1/2}$. This shows that $\lambda_1(BA)$ is a convex functional on H_n for a fixed positive definite A . If in addition A has non-negative entries then $\lambda_1(DA)$ is convex on D_n . This result does not apply in general for non-negative matrices. For example, take A to be a permutation matrix $P \neq I$. However, $\lambda_1(DA)$ is convex on D_n^+ - the set of $n \times n$ non-negative diagonal matrices if A^{-1} is an M-matrix.

Theorem 4.3. Let A^{-1} be an M-matrix. Then $r(DA)$ is a convex functional on D_n^+ .

$$(4.18) \quad r\left(\frac{(D_1 + D_2)}{2} A\right) \leq \frac{1}{2} (r(D_1 A) + r(D_2 A)).$$

Moreover if A is irreducible then the equality sign in (4.18) holds if and only if

$$(4.19) \quad D_2 = aD_1$$

for some positive a provided that D_1 or D_2 have positive diagonal elements.

Proof. Using the continuity argument we may assume that in the decomposition (3.22) B is positive (irreducible), i.e. A is positive (irreducible). Thus if all diagonal elements of $D_0 = \text{diag}\{d_1^0, \dots, d_n^0\}$ are positive then $D_0 A$ is positive (irreducible). According to the Perron-Frobenius theorem $r(D_0 A)$ is a simple root of the $\det(\lambda I - D_0 A) = 0$. By the implicit function theorem $r(DA)$ is an analytic function of D in the neighborhood of D_0 . Then the convexity of $r(DA)$ would follow if we show that

$$(4.20) \quad r(DA) \geq r(D_0A) + \sum_{i=1}^n (d_i - d_i^{(0)}) \left. \frac{\partial r(DA)}{\partial d_i} \right|_{D_0},$$

for any D_0 with positive diagonal elements. Let ξ, η be the eigenvectors corresponding to D_0A and $A^T D_0$

$$(4.21) \quad D_0A\xi = r(D_0A)\xi, \quad A^T D_0\eta = r(D_0A)\eta,$$

$$0 < \xi = (\xi_1, \dots, \xi_n), \quad 0 < \eta = (\eta_1, \dots, \eta_n), \quad \sum_{i=1}^n \xi_i \eta_i = 1.$$

It can be shown that

$$(4.22) \quad \left. \frac{\partial r(DA)}{\partial d_i} \right|_{D_0} = \eta^T \frac{\partial D}{\partial d_i} A\xi = r(D_0A) \frac{\eta_i \xi_i}{d_i^{(0)}} \quad i = 1, \dots, n.$$

This can be done by bringing D_0A to the Jordan form and using the simplicity of $r(D_0A)$. See for example [10, II, §5.4]. Thus (4.20) is equivalent to

$$(4.23) \quad r(DA) \geq r(D_0A) \sum_{i=1}^n \frac{d_i}{d_i^{(0)}} \xi_i \eta_i.$$

This inequality was established in [5]. It follows directly from (3.26). Indeed suppose that D has positive diagonal elements and let

$$(4.24) \quad DAw = r(DA)w, \quad w = (w_1, \dots, w_n) > 0.$$

Then according to (3.26)

$$\begin{aligned} \frac{1}{r(D_0A)} &\geq \sum_{i=1}^n \xi_i \eta_i \frac{w_i}{(D_0Aw)_i} = \\ \sum_{i=1}^n \xi_i \eta_i \frac{d_i}{d_i^{(0)}} \frac{w_i}{(DAw)_i} &= \frac{1}{r(DA)} \sum_{i=1}^n \xi_i \eta_i \frac{d_i}{d_i^{(0)}}, \end{aligned}$$

which establishes (4.23) for D with positive diagonal. So (4.18) holds in the interior of D_n^+ . The continuity argument implies the validity of (4.18) on D_n^+ . Suppose that A is also irreducible. Then B in the decomposition (3.22) is also irreducible, since the inverse of block triangular matrix is also a block triangular one. As in the proof of Theorem 4.1 strict inequality holds in (4.23) unless D_0A and DA have the same positive eigenvector. So

$D = aD_0$ for some $a > 0$. This shows that we have strict inequality in (4.18) unless (4.19) holds provided that D_0 (which is either D_1 or D_2) have positive diagonal. The proof of the theorem is completed.

We conclude this section by pointing out that the convexity of $r(DA)$ on D_n^+ is a stronger result than the convexity of $\log r(e^Q A)$ on D_n . Indeed, let

$$(4.25) \quad D_0 = e^{Q_0}, \quad Q_0 = \{q_1^{(0)}, \dots, q_n^{(0)}\}, \quad q_i^{(0)} = \log d_i^{(0)}, \quad i = 1, \dots, n.$$

Suppose that $\log r(e^Q A)$ is convex at $Q = Q_0$. This means

$$(4.26) \quad \log r(e^Q A) \geq \log r(D_0 A) + r(D_0 A)^{-1} \sum_{i=1}^n \frac{\partial r(e^Q A)}{\partial q_i} \Big|_{Q=Q_0} (q_i - q_i^{(0)}), \quad Q = \text{diag}\{q_1, \dots, q_n\}.$$

As in the proof of Theorem 4.3

$$(4.27) \quad \frac{\partial r(e^Q A)}{\partial q_i} \Big|_{Q=Q_0} = \eta^T \frac{\partial e^Q}{\partial q_i} \Big|_{Q_0} A \xi = r(D_0 A) \eta_i \xi_i, \quad i = 1, \dots, n$$

where η, ξ given by (4.21).

Thus (4.26) is equivalent

$$(4.28) \quad r(e^Q A) \geq r(D_0 A) \prod_{i=1}^n \left(\frac{e^{q_i}}{d_i^{(0)}} \right)^{\xi_i \eta_i} = r(D_0 A) \prod_{i=1}^n \left(\frac{d_i}{d_i^{(0)}} \right)^{\xi_i \eta_i}, \quad q_i = \log d_i, \quad i = 1, \dots, n.$$

Using the relation between the arithmetic and the geometric means from (4.23) we get

$$(4.29) \quad r(e^Q A) \geq r(D_0 A) \sum_{i=1}^n \frac{e^{q_i}}{d_i^{(0)}} \xi_i \eta_i \geq r(D_0 A) \prod_{i=1}^n \left(\frac{e^{q_i}}{d_i^{(0)}} \right)^{\xi_i \eta_i}.$$

That is the convexity of $r(DA)$ at $D_0 \in D_n^+$ implies the convexity of $\log r(e^Q A)$ at $Q_0 = \log D_0$. This demonstrates that the convexity of $r(DA)$ on D_n^+ implies the convexity of $\log r(e^D A)$ on D_n . On the other hand if A is a permutation matrix $\neq I$ then $r(DP)$ is not convex on D_n^+ (for details see [5], Section 3).

5. Convex functions on the spectrum of totally positive matrices

A real valued $n \times n$ matrix is called a totally (strictly totally) positive matrix of order k if all minors of A of order less or equal to k are non-negative (positive). We denote these matrices by $TP_j(STP_j)$. For $j = n$ we call these matrices simply by $TP(STP)$. A matrix A is called oscillating if A is TP and some power of A is STP . It is known that a TP matrix is oscillating if and only if

$$(5.1) \quad a_{ii} > 0, a_{i(i-1)} > 0, a_{i(i+1)} > 0, i = 1, \dots, n, A = (a_{ij})_{i,j=1}^n \geq 0.$$

In that case A is totally indecomposable.

If A is TP_j then

$$(5.2) \quad \lambda_1(A) \geq \lambda_2(A) \geq \dots \geq \lambda_j(A) \geq |\lambda_k(A)|, k = j+1, \dots, n.$$

If A is STP_j then we have strict inequalities in (5.2). See [6] and [9] for proofs of these results and more properties of these matrices. Let A be TP_j . Define $\phi: A \rightarrow \mathbb{R} (A \in D_n)$ as follows

$$(5.3) \quad \phi(D) = F(\log \lambda_1(e^D A), \dots, \log \lambda_j(e^D A)).$$

As in Section 2 we were looking for necessary and sufficient conditions on F which imply that ϕ is a convex function on $A \in D$ for any A which is TP_j . It turns out that we have an analogous result to Theorem 2.2. To do so we need few notations and definitions. Let $\bar{\alpha} = (\alpha_1, \dots, \alpha_j)$ and $\bar{\beta} = (\beta_1, \dots, \beta_j)$ and $j < n$. We define $\bar{\alpha} \ll \bar{\beta}$ if (2.12) holds for $k = 1, \dots, j$. Thus if $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$ and $\alpha < \beta$ then $\bar{\alpha} \ll \bar{\beta}$. Conversely, if $\bar{\alpha} \ll \bar{\beta}$ we can extend $\bar{\alpha}$ to α and $\bar{\beta}$ to β such that $\alpha < \beta$. A set $\bar{X} \subseteq \mathbb{R}_{\geq}^j$ is called a super convex if $\bar{X}$ is convex and

$$(5.4) \quad \text{if } \bar{\beta} \in \bar{X}, \bar{\alpha} \ll \bar{\beta}, \text{ then } \bar{\alpha} \in \bar{X}.$$

Clearly $\bar{X}$ is super convex in $\mathbb{R}_{\geq}^j$ if and only if it could be extended to $X \subseteq \mathbb{R}_{\geq}^n$ such that X is strongly convex in $\mathbb{R}_{\geq}^n$. Using the above arguments and Ostrowski's result (Theorem 2.3) we get

Lemma 5.1. Let $\bar{X}$ be a super convex set in $\mathbb{R}_{\geq}^j$. Let $F: \bar{X} \rightarrow \mathbb{R}$. Assume that $F \in C^{(1)}(\bar{X})$.

Then

$$(5.5) \quad F(\bar{\alpha}) \leq F(\bar{\beta}) \quad \text{if} \quad \bar{\alpha} < \bar{\beta}$$

if and only if

$$(5.6) \quad \frac{\partial F}{\partial x_1}(\bar{\alpha}) \geq \frac{\partial F}{\partial x_2}(\bar{\alpha}) \geq \dots \geq \frac{\partial F}{\partial x_j}(\bar{\alpha}) \geq 0$$

for any $\bar{\alpha} \in \bar{X}$. Moreover strict inequality in (5.5) holds for $\bar{\alpha} \neq \bar{\beta}$ if and only if

$$(5.7) \quad \frac{\partial F}{\partial x_i}(\bar{\alpha}) > \frac{\partial F}{\partial x_{i+1}}(\bar{\alpha}) \quad \text{if} \quad \alpha_i > \alpha_{i+1}, \quad \frac{\partial F}{\partial x_j}(\bar{\alpha}) > 0 \quad \text{if} \quad \alpha_j > 0.$$

Assume that A is TP_j . Denote

$$(5.8) \quad \lambda^{(j)}(A) = (\lambda_1(A), \dots, \lambda_j(A)), \quad \log \lambda^{(j)}(A) = (\log \lambda_1(A), \dots, \log \lambda_j(A)).$$

Theorem 5.1. Let A be an $n \times n$ non-singular TP_j matrix. If $j < n$ then

$$(5.9) \quad \log \lambda^{(j)}(e^{(D_1+D_2)/2} A) < \frac{1}{2} \log \lambda^{(j)}(e^{D_1} A) + \log \lambda^{(j)}(e^{D_2} A).$$

If $j = n$ then

$$(5.10) \quad \log \lambda(e^{(D_1+D_2)/2} A) < \frac{1}{2} [\log \lambda(e^{D_1} A) + \log \lambda(e^{D_2} A)].$$

If in addition A satisfies (5.1), or more generally A is totally indecomposable, then

$$(5.11) \quad \log \lambda^{(j)}(e^{(D_1+D_2)/2} A) = \frac{1}{2} [\log \lambda^{(j)}(e^{D_1} A) + \log \lambda^{(j)}(e^{D_2} A)]$$

for any $1 \leq j \leq n$ if and only if (4.7) is satisfied for some a .

Proof. Denote by $C_k(A)$ the k -th compound of A . Thus

$$(5.12) \quad C_k(e^D) = e^{\varphi_k(D)}$$

where φ_k is well defined map $\varphi_k : D_n \rightarrow D_{\binom{n}{k}}$. It is easy to see using the properties of the compound matrices that φ_k is a linear map. According to Theorem 4.2 $\log r(e^{D'} C_k(A))$ is convex on $D_{\binom{n}{k}}$ for $k = 1, \dots, j$. Note that the non-singularity of A implies that $r(C_k(A)) > 0$.

Thus $\log r(e^{\varphi_k(D)} C_k(A))$ is convex on D_n . Let

$$(5.13) \quad R_k(D) = \sum_{i=1}^k \log \lambda_i(e^D A) .$$

It is well known that

$$(5.14) \quad R_k(D) = \log r(C_k(e^D A)) .$$

Therefore $R_k(D)$ is convex on D_n for $k = 1, \dots, j$. This is equivalent to (5.9) for $j < n$.

For $j = n$, $R_n(D)$ is linear on D as

$$(5.15) \quad R_n(D) = \log \det(e^D A) = \sum_{i=1}^n d_i + \log \det(A) .$$

This verifies (5.10) if A is a TP matrix. Suppose that in addition A is totally indecomposable. According to Theorem 4.2 we have a strict inequality in (4.17) unless (4.7) holds. Thus (5.11) can be satisfied if only (4.7) holds. Trivially (4.7) implies (5.11). The proof of the theorem is completed.

Theorem 5.2. Let $\bar{X}$ be a super convex set in $R_{\geq}^j$ for $1 \leq j < n$ (a strongly convex containing a point $\alpha, \alpha_1 > \dots > \alpha_n$, if $j = n$). Let $F : \bar{X} \rightarrow \mathbb{R}$. Assume that $F \in C^{(1)}(\bar{X})$. Let A be a given $n \times n$ non-singular TP_j matrix. Consider a spectral function $\phi : A \rightarrow \mathbb{R}$, given by (5.3), where A is a convex set in D_n such that

$$(5.16) \quad \log \lambda^{(j)}(e^D A) \subseteq \bar{X}, D \in A .$$

Then, for all such A , ϕ is convex if and only if F is convex on $\bar{X}$ and satisfies (5.6) in case that $1 \leq j < n$. Moreover, if A is totally indecomposable then ϕ is strictly convex if and only if F is strictly convex and satisfies (5.7). In case $j = n$, ϕ is convex (strictly convex provided that A is totally indecomposable) if and only if F satisfies the assumptions of Theorem 2.2.

Proof. A proof of this theorem can be achieved by modifying in the obvious way the proof of Theorem 2.2. In fact, all the arguments of the proof of Theorem 2.2 carry over if one notices that the identity matrix is TP.

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