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ADAPTIVE ACCELERATION OF SSOR FOR SOLVING LARGE LINEAR SYSTEMS.(U)
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ADAPTIVE ACCELERATION OF SSOR FOR SOLVING LARGE LINEAR SYSTEMS

Vitalius Benokraitis

March 1979

US ARMY ARMAMENT RESEARCH AND DEVELOPMENT COMMAND
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20. ABSTRACT (*Continue on reverse side if necessary and identify by block number*)

Symmetric successive overrelaxation (SSOR) for solving large, sparse systems of linear equations involves the estimation of a parameter ω. An adaptive procedure is outlined for improving the estimates for ω and the spectral radius $S(S_\omega)$ of the iteration matrix S_ω. These estimates are then used in the SSOR method with Chebyshev acceleration. The objective is to achieve convergence in only a few more iterations than would be required if the best possible values

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20. ABSTRACT (continued)

omega   Sigma omega

of ω and $S(S_{\omega})$ were used from the outset. The method is applied to obtain finite difference solutions of a number of generalized Dirichlet problems. In certain cases, the number of iterations required using the adaptive procedure increases like $h^{-\frac{1}{2}}$, where h is the mesh size.

h to the $-\frac{1}{2}$ power

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I. INTRODUCTION

We shall be concerned with iteratively determining the N-vector u of a large, sparse linear system

(1) $Au = b$

where A is a real, symmetric, positive definite matrix of order N and b is a given N-vector. Such systems arise in the finite difference solution of elliptic boundary value problems. Particularly, we shall develop an adaptive scheme based on the symmetric SOR (SSOR) iterative method with Chebyshev acceleration. Related work which has recently appeared includes Axelsson[1], Hayes and Young,[2] and Young.[3,4,5,6]

II. BASIC METHOD

By defining

$$B = I-D^{-1}A = L + U$$

$$c = D^{-1}b$$

where D = diag (A) and L and U are strictly lower and upper triangular matrices, respectively, we may replace the system (1) by

$$u = Bu + c.$$

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[1] Axelsson, O. (1972), "A Generalized SSOR Method," BIT 13, 443-467.

[2] Hayes, L. J. and D. M. Young (1977), "The Accelerated SSOR Method for Solving Large Linear Systems: Preliminary Report," Report CNA-123, Center for Numerical Analysis, The University of Texas, Austin, Texas.

[3] Young, D. M. (1974a), "On the Accelerated SSOR Method for Solving Elliptic Boundary Value Problems," in Lecture Notes in Mathematics, A. Dold and B. Eckmann (eds.), Vol. 363, Conference on the Numerical Solution of Differential Equations, Dundee 1973, (G. A. Watson, ed.), Springer-Verlag, New York, 195-206.

[4] Young, D. M. (1974b), "Solution of Linear Systems of Equations," in Numerical Solutions of Partial Differential Equations, J.G. Gram (ed.), D. Reidel Pub. Col., Holland, 35-54 (Proceedings of Conference "Advanced Study Institute on Numerical Solution of Partial Differential Equations," Kjeller, Norway, August 20-24, 1973).

[5] Young, D. M. (1974c), "Stopping Criteria and Adaptive Parameter Determination for Iterative Methods for Solving Large Linear Systems," Proceedings of the Gatlinburg VI Symposium on Numerical Algebra, Munchen, Germany, December 15-22, 1974.

[6] Young, D. M. (1977), "On the Accelerated SSOR Method for Solving Large Linear Systems," Advances in Mathematics 23, 215-271.

The SSOR method (Sheldon)[7] is then defined by forming a single SSOR iteration from a forward SOR iteration followed by a backward SOR iteration; that is, for n = 0, 1, 2, ... we set

$$(2) \quad \begin{cases} u^{(n+\frac{1}{2})} = \omega(Lu^{(n+\frac{1}{2})} + Uu^{(n)} + c) + (1-\omega)u^{(n)} \\ u^{(n+1)} = \omega(Lu^{(n+\frac{1}{2})} + Uu^{(n+1)} + c) + (1-\omega)u^{(n+\frac{1}{2})} \end{cases}$$

where $u^{(0)}$ is an arbitrary initial approximation to the solution u, and ω is a real relaxation parameter such that $0<\omega<2$.

Elimination of $u^{(n+\frac{1}{2})}$ in (2) gives

$$u^{(n+1)} = S_\omega u^{(n)} + k_\omega$$

where

$$S_\omega = \mathcal{U}_\omega \mathcal{L}_\omega$$

$$= \left\{(I-\omega U)^{-1} \ (\omega L + (1-\omega)I)\right\} \left\{(I-\omega L)^{-1} (\omega U+(1-\omega)I)\right\}$$

$$k_\omega = \omega(2-\omega) \ (I-\omega U)^{-1} \ (I-\omega L)^{-1} c$$

Note that $\mathcal{L}_\omega$ corresponds to the familiar SOR iteration matrix. The backward SOR operator $\mathcal{U}_\omega$ is defined analogously.

If storage for an extra N-vector is provided, the work required for one SSOR iteration may be reduced to about the work necessary for a single SOR iteration. The work-saving technique is due to Niethammer[8] and

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[7] Sheldon, J. W. (1955), "On the numerical Solution of Elliptic Difference Equations," Math. Tables Aids Comput. 9, 101-112.

[8] Niethammer, W. (1964), "Relaxation bei Komplexen Matrizen," Math. Zeitsch. 86, 34-40.

is described in Benokraitis[9,10] and Young.[6] The method has been rediscovered by Conrad and Wallach.[11]

The SSOR method converges if $\bar{S}(S_\omega)$, the spectral radius of the iteration matrix S_ω, is less than 1, which holds if $0<\omega<2$ and A is positive definite. The rate of convergence is governed by the ordering of the equations and by the parameter ω. Assuming the natural ordering, Young[3,4,6] has shown that for a certain discrete generalized Dirichlet problem one can choose a "good" ω depending on bounds for the eigenvalues of B and LU so that the SSOR method converges with the same order-of-magnitude as the SOR method. For a finite difference discretization with mesh size h, the number of iterations required for both methods increases like h^{-1}.

Therefore, even by employing Niethammer's work-saving scheme, there is little justification for using SSOR. However, the SSOR method can be accelerated by an order-of-magnitude by means of Chebyshev semi-iteration since the eigenvalues of the matrix S_ω are real and nonnegative. (Chebyshev semi-iteration was first studied by Varga[12] and Golub and Varga.)[13] This approach is precluded for SOR with optimum $\omega = \omega_b$ since many of the eigenvalues of L_{ω_b} are complex. (See Varga[12] and Young[14].) Also, there is no improvement when semi-iteration is applied to SOR with $1<\omega<\omega_b$. (See Kincaid[15].) For accelerating the Gauss-Seidel method (SOR with $\omega = 1$), see Sheldon[16] and Young.[14]

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[9] Benokraitis, V.J. (1974), "On the Adaptive Acceleration of Symmetric Successive Overrelaxation," Doctoral Thesis, University of Texas, Austin.

[10] Benokraitis, V.J. (1976), "An Improved Iterative Method for Optimizing Symmetric Successive Overrelaxation," in ARO Report 76-3, Proceedings of the 1976 Army Numerical Analysis and Computers Conference, 133-140.

[11] Conrad, V. and Y. Wallach (1977), "A Faster SSOR Algorithm," Numer. Math. 27, 371-372.

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[13] Golub, G.H. and R.S. Varga (1961), "Chebyshev Semi-iterative Methods, Successive Overrelaxation Iterative Methods, and Second-order Richardson Iterative Methods," Numer. Math., Parts I and II, 3, 147-168.

[14] Young, D.M. (1971), Iterative Solution of Large Linear Systems, Academic Press, New York.

[15] Kincaid, D.R. (1974), "On Complex Second-degree Iterative Methods," SIAM J. Numer. Anal. 11, 211-218.

[16] Sheldon, J.W. (1959), "On the Spectral Norms of Several Iterative Processes," J. Assoc. Comput. Mach. 5, 39-46.

III. ACCELERATED METHOD

The optimum semi-iterative method based on SSOR, denoted by SSOR-SI, is defined by

$$(3) \qquad u^{(n+1)} = \rho_{n+1}\left\{\bar{\rho}\ (S_\omega u^{(n)} + k_\omega) + (1-\bar{\rho})\ u^{(n)}\right\} + (1-\rho_{n+1})\ u^{(n-1)}$$

Here

$$\bar{\rho} = \frac{2}{2-S(S_\omega)}$$

$$\rho_1 = 1$$

$$\rho_2 = (1 - \sigma^2/2)^{-1}$$

$$\rho_{n+1} = (1 - \frac{\sigma^2 \rho_n}{4})^{-1} \quad , n = 2, 3, \ldots$$

where

$$\sigma = \frac{S(S_\omega)}{2-S(S_\omega)}$$

In order to apply the SSOR-SI method, we must determine the two parameters ω and $S(S_\omega)$. Some *a priori* methods for obtaining these parameters are discussed by Habetler and Wachspress,[17] Evans and Forrington,[18] Young[3,4,6] and Benokraitis.[9,10] Since finding these parameters may take as much work as solving the original problem, we are led to consider adaptive techniques which approximate the parameters and at the same time improve the solution of the linear system.

IV. FOUNDATION FOR ADAPTIVE METHOD

We begin by characterizing the eigenvalues of S_ω in terms of certain inner products. This result is due to Habetler and Wachspress.[17] See also Young.[6]

THEOREM 1. Let λ be an eigenvalue of S_ω where $0<\omega<2$ and let v be an associated eigenvector. Then λ may be represented by

[17] Habetler, G. J., and E. L. Wachspress (1961), "Symmetric Successive Overrelaxation in Solving Diffusion Difference Equations," Math. Comp. 15, 356-362.

[18] Evans, D. J., and C. V. D. Forrington (1963), "An Iterative Method for Optimizing Symmetric Successive Overrelaxation," Comput. J. 6, 271-273.

(4) $$\lambda = 1 - \omega(2-\omega)\frac{1-\alpha}{1-\omega\alpha+\omega^2\beta} = \phi(\omega,v)$$

where

(5) $$\begin{cases} \alpha = \dfrac{(v,DBv)}{(v,Dv)} \\ \\ \beta = \dfrac{(v,DLUv)}{(v,Dv)} \end{cases}$$

THEOREM 2. The representation $\phi(\omega,v)$ given by (4) for any vector $v \neq 0$ is a Rayleigh quotient with respect to the vector

$$w = (I-\omega\tilde{U})D^{\frac{1}{2}}v$$

and the symmetric matrix

$$\bar{S}_\omega = (I-\omega\tilde{U})D^{\frac{1}{2}}S_\omega D^{-\frac{1}{2}}(I-\omega\tilde{U})^{-1}$$

where

$$\tilde{U} = D^{\frac{1}{2}}UD^{-\frac{1}{2}} .$$

That is,

$$\phi(\omega,v) = \frac{(w,\bar{S}_\omega w)}{(w,w)} .$$

Furthermore, $\bar{S}_\omega$ is similar to S_ω and

(6) $$\phi(\omega,v) \leq S(\bar{S}_\omega) = S(S_\omega).$$

Proof: See Benokraitis (1974).[9]

We emphasize that (6) holds for any nonzero vector v, not just for eigenvectors of S_ω. However, the closer we approach a fundamental eigenvector, the closer we shall be able to determine $S(S_\omega)$ from $\phi(\omega,v)$ given by (4). Therefore, it would be fortunate if somehow we could determine the fundamental eigenvector without deviating from the path of improving the approximate solution of (1). A clue leading to the desired situation is contained in the following theorem (cf. Young (1974c)).[5]

THEOREM 3. The psudo-residual vector

(7) $$\delta^{(n)} = S_\omega u^{(n)} + k_\omega - u^{(n)}$$

where $u^{(n)}$ is the latest SSOR-SI iterate, satisfies

(8) $$\delta^{(n)} = P_n(S_\omega)\delta^{(0)} .$$

Here

$$P_n(S_\omega) = \frac{T_n\left(\frac{2S_\omega}{S(S_\omega)} - 1\right)}{T_n\left(\frac{2}{S(S_\omega)} - 1\right)}$$

where $T_n(x)$ is the n^{th} degree Chebyshev polynomial defined by the three-term recurrence relation

$$T_o(x) = 1, \; T_1(x) = x$$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \; n \geq 1 \; .$$

Proof: See Benokraitis (1974).[9]

We note that if $P_n(S_\omega)$ is replaced by $S_\omega{}^n$ then (8) reminds us of the power method for computing the dominant eigenvector. With this motivation, the next theorem comes as no surprise.

THEOREM 4. The pseudo-residual vector $\delta^{(n)}$ given by (7) converges in direction to the eigenvector associated with the eigenvalue $S(S_\omega)$ as n tends to infinity.

Proof: See Benokraitis (1974).[9] Also compare with Diamond (1971)[19] and Hageman (1972).[20]

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[19] Diamond, M.A. (1971), "An Economical Algorithm for the Solution of Finite Difference Equations," Report UIUC DCS-R-71-492, Department of Computer Science, University of Illinois at Urbana-Champaign, Urbana, Illinois.

[20] Hageman, L.A. (1972), "The Estimation of Acceleration Parameters for the Chebyshev Polynomial and the Successive Overrelaxation Iteration Methods," Report WAPD-TM-1038, Bettis Atomic Power Laboratory, Westinghouse Electric Corp., Pittsburgh, Pennsylvania.

By Theorem 4 it is possible, then, to determine the fundamental eigenvector with a little additional effort by computing the pseudo-residual vector. However, since

$$\delta^{(n)} = \underbrace{S_\omega u^{(n)} + k_\omega}_{\text{one SSOR iteration}} - \underbrace{u^{(n)}}_{\text{latest SSOR-SI iteration}} = \tilde{u}^{(n)} - u^{(n)}$$

and since

$$\tilde{u}^{(n)} = S_\omega u^{(n)} + k_\omega$$

must be computed as part of the next SSOR-SI iteration $u^{(n+1)}$ (see (3)), the pseudo-residual vector is essentially obtained as a byproduct of applying the SSOR-SI method.

V. ADAPTIVE METHOD

By using Theorems 1, 2, and 4 as a foundation we are able to present the basic structure for an adaptive procedure. For detailed descriptions of this method and several possible variations for the adaptive acceleration of SSOR, see Benokraitis (1974).[9]

We state the steps of the "algorithm" in outline form with a synopsis of the controlling theorem(s) in the heading. We use the word "algorithm" loosely, since admittedly much is left unspecified.

I. Theorem 2. For any $v \neq 0$, $\phi(\omega,v) \leq S(S_\omega)$.

1. Choose, $v_1, v_2 \neq 0$.
2. Observe

   a. $\phi_1 = \phi(\omega,v_1) \leq S(S_\omega)$

   b. $\phi_2 = \phi(\omega,v_2) \leq S(S_\omega)$

   c. $\Phi_1(\omega) = \max_{v_1,v_2} (\phi_1, \phi_2) \leq S(S_\omega)$

3. Minimize $\Phi_1(\omega)$ with respect to ω to obtain estimate ω_1.

4. Choose $\Phi_1(\omega_1)$ as $S_E(S_{\omega_1})$, an estimate of $S(S_{\omega_1})$. (The situation is depicted in Figure 1.)

II. Theorem 4. The pseudo-residual vector $\delta^{(n)}$ converges in direction to dominant eigenvector v.

1. Set i = 1

2. Iterate n times with SSOR-SI with parameters ω_i, $S_E(S_{\omega_i})$; test for convergence

3. Compute

$$\delta^{(n)} = (S_\omega n^{(n)} + k_\omega) - u^{(n)}$$

which approaches dominant eigenvector.

4. Check if parameters should be changed.

a. If $\phi(\omega_i, \delta^{(n)}) \leqslant S_E(S_{\omega_i})$ do not change parameters. Go to II.2

b. If $\phi(\omega_i, \delta^{(n)}) > S_E(S_{\omega_i})$ continue to step III to change parameters.

III. Theorems 1, 2, 4. As $\delta^{(n)}$ approaches dominant eigenvector, $\phi(\omega_i, \delta^{(n)})$ approaches $S(S_{\omega_i})$.

1. Set $v_{i+2} = \delta^{(n)}$

$$\phi_{i+2} = \phi(\omega, v_{i+2})$$

2. Observe

$$\Phi_{i+1}(\omega) = \max_{v_k, k=1, \ldots, i+2} (\phi_1, \phi_2, \ldots, \phi_{i+2}) \leqslant S(S_\omega)$$

3. Minimize $\Phi_{i+1}(\omega)$ with respect to ω to obtain next estimate ω_{i+1}.

4. Choose $\Phi_{i+1}(\omega_{i+1})$ as $S_E(S_{\omega_{i+1}})$, an estimate of $S(S_{\omega_{i+1}})$. Set i = i+1.

Go to II.2. Process is continued until convergence.

We briefly discuss how to choose n in step II.2. Here we make use of the average and asymptotic average rates of convergence for the SSOR-SI method (Young (1971)).[14] A strategy which produces acceptable results is to choose n so that the average rate of convergence after n iterations is 90% of the asymptotic average rate of convergence. The convergence rates are computed using the latest estimate of $S(\mathcal{S}_\omega)$. That is, n is chosen to be the least n which satisfies

$$-\frac{1}{n}\log\frac{2r^n}{1+r^{2n}} \geq .9\,(-\log r)$$

where

$$r = \frac{1-\sqrt{1-S_E(\mathcal{S}_\omega)}}{1+\sqrt{1-S_E(\mathcal{S}_\omega)}}\,.$$

A word about the additional work required in the adaptive algorithm is in order. Mainly, the added expense comes in changing the parameters. This involves the computation of α and β, two quotients of inner products in the formula for $\phi(\omega,v)$ given in (4)-(5). For problems of the type discussed in Section VI, to compute α and β requires approximately $28J^2$ arithmetic operations if the mesh size is $h = 1/J$. One SSOR-SI iteration requires approximately $39J^2$ operations. Therefore, four parameter changes are approximately equivalent to three SSOR-SI iterations in terms of work performed. Since no more than four parameter changes were required for the problems considered, the number of iterations for the adaptive algorithm should effectively be increased by about three iterations.

VI. NUMERICAL EXAMPLES

We present results for a sample of the generalized Dirichlet problems considered. The results are given in graphic form in Figures 2-4. In each figure, we give the differential equation, the region considered and the boundary values. We replace the differential equation by a 5-point symmetric difference equation (see Young (1977)).[6]

The number of iterations required for varying mesh sizes is recorded for optimum, adaptive, and estimated SSOR-SI parameters. In the adaptive case, the subscript on the number of iterations indicates the number of parameter changes required. The estimated parameters are the values of Young (1977)[6] which depend on bounds for the eigenvalues of B and LU. (For Problem 3, the results for the estimated parameters are not given since an excessive number of iterations are required.) The slopes s of the lines indicate that the number of iterations required for convergence increases like h^{-s}.

For smooth and some discontinuous coefficients (Problems 1 and 2), the number of iterations required behaves like $h^{-\frac{1}{2}}$, an order-of-magnitude better than SOR or SSOR. For cases involving higher discontinuity (Problem 3), the behavior is like $h^{-3/4}$.

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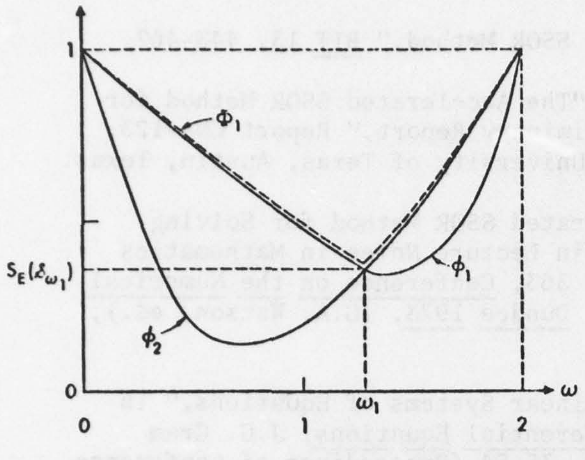

Fig. 1. Determination of First Approximation of ω and $S(S_\omega)$

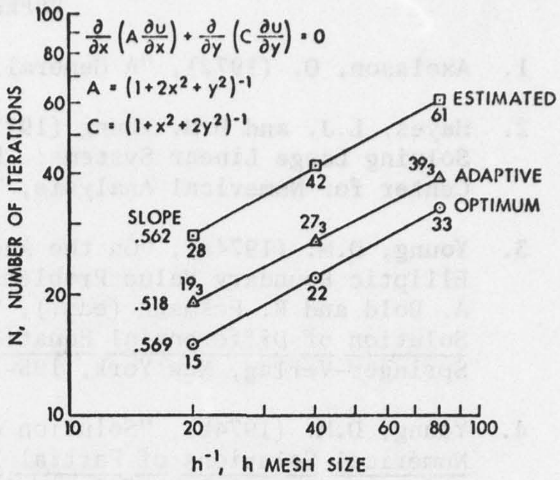

Fig. 2. Prob. 1. Unit Square w/Zero Boundary Values, Except Unity on Side Y = 0

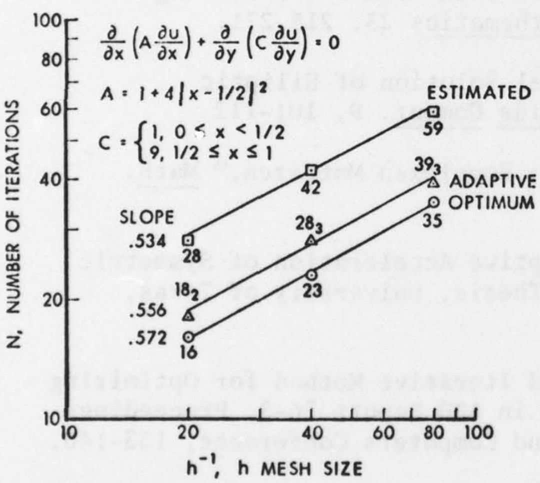

Fig. 3. Prob. 2. Unit Square w/Zero Boundary Values, Except Unity on Side Y = 0

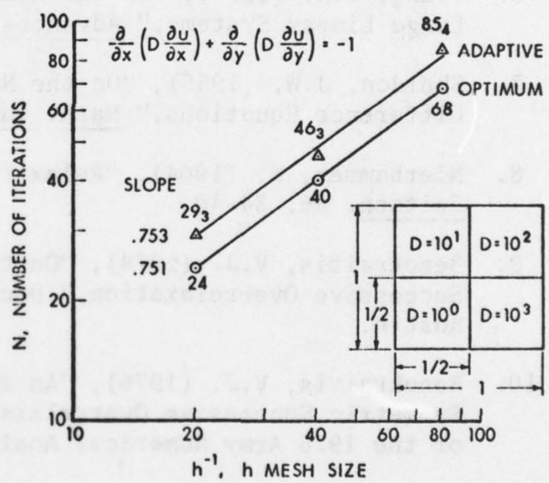

Fig. 4. Prob. 3. Unit Square w/Zero Boundary Values

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20. Hageman, L.A. (1972), "The Estimation of Acceleration Parameters for the Chebyshev Polynomial and the Successive Overrelaxation Iteration Methods," Report WAPD-TM-1038, Bettis Atomic Power Laboratory, Westinghouse Electric Corp., Pittsburgh, Pennsylvania.

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