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STABLE ADAPTIVE CONTROLLER DESIGN. PART II. PROOF OF STABILITY.(U)
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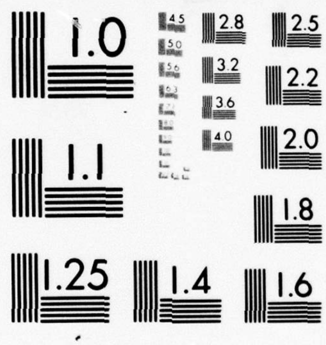

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STABLE ADAPTIVE CONTROLLER DESIGN.

PART II: PROOF OF STABILITY

Kumpati S. Narendra, Yuan-Hao Lin
and Lena S. Valavani

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Stable Adaptive Controller Design

Part II: Proof of Stability

Kumpati S. Narendra, Yuan-Hao Lin and Lena S. Valavani

1. Introduction: In this paper we present a proof of the stability in the large of an adaptive system which is a modification of that originally suggested by Monopoli [1] and later refined by Narendra and Valavani [2]. The proof of stability applies to both continuous and discrete systems.

In [2] the stability problem was clearly defined and a conjecture was made that the plant together with the controller would be stable if the augmented error is bounded. Recently, while applying the same approach to discrete systems, Lin and Narendra [3] suggested a new error model based on the third prototype discussed in [2] (Lemma 1) and on the earlier work of Kudva and Narendra [4]. This model contains an additional feedback term similar to that found in much of the adaptive observer literature [5,6] and is particularly useful in proving the stability in the large of discrete adaptive systems [7]. The successful resolution of the latter problem provides the motivation for using a similar model in the continuous case as well.

Ever since the publication of [1] there has been a great deal of interest in the problem of designing stable adaptive controllers. In [8] Feuer and Morse presented such a controller but its extreme complexity precludes its use in practical applications. Since then, several authors [9-13] have also attempted this problem for both discrete and continuous systems but their results are yet to be verified. In [9] the stability in the large of a simple discrete adaptive controller is considered and both discrete and continuous controllers are discussed in [10]. In [11] and [12] the continuous adaptive problem is treated when the input to the system is "sufficiently rich". A proof of stability for the continuous case is proposed in [13] for the adaptive control problem considered in [1] and [2] and also involves the additional feedback signal described earlier.

The problem discussed in this paper is essentially the same as that in [2]. The philosophy as well as the specific structure of the controller (except for the additional feedback term) and the adaptive laws are the same as in [2]. Hence, these aspects of the problem are discussed in section 2 only in a condensed form and the reader is referred to [2] for all details. The remaining sections are devoted to the proof of stability with which this paper is primarily concerned. The simple proof given in section 5 can be used, as mentioned earlier, for both continuous as well as discrete systems with only minor modifications. Hence, even though the emphasis is on the continuous case, comments are made in the paper to indicate how the proof could also be applied to the discrete case.

The augmented input to the reference model was first introduced in [1] in order to avoid differentiation. The concept of positive realness which is central to the development of stable adaptive laws led to the introduction of the operator* $P_L(\theta) = L(s)\theta(t)L^{-1}(s)$ in [2]. Much of the difficulty encountered in proving the stability of the adaptive system can be traced to the characteristics of $P_L(\theta)$ and the related operator $\bar{P}_L(\theta) = L^{-1}(s)\theta(t)L(s)$. Since $P_L(\theta) = \bar{P}_L(\theta) = \theta$, when θ is a constant, it was felt that the adaptive controller would be stable if $\theta(t)$ changes "slowly" with time. This intuitive assumption can now be justified by the stability analysis presented in this paper.

In section 2 the structure of the adaptive controller is discussed briefly and the principal stability question is restated in section 3. While state-variable equations are used wherever appropriate, the emphasis throughout the paper is on input-output descriptions. Section 4 is essentially self contained and all the mathematical concepts needed are developed here. Using these results the proof of the stability of the adaptive system is shown to follow directly in section 5.

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* Throughout the paper both differential equations and transfer functions are used in the arguments and, depending on the context, "s" is used as a differential operator or the Laplace transform variable.

2. Design of the Adaptive Controller:

A plant P to be controlled is completely represented by the input-output pair $\{u(t), y_p(t)\}$ and can be modeled by a linear time-invariant system

$$\begin{aligned} \dot{x}_p &= A_p x_p(t) + b_p u(t) \\ y_p &= h_p^T x_p(t) \end{aligned} \tag{1}$$

where A_p is an (nxn) matrix and h_p and b_p are n-vectors. The transfer function of the plant is $W_p(s)$ where

$$W_p(s) = h_p^T (sI-A_p)^{-1} b_p \triangleq \frac{K_p Z_p(s)}{R_p(s)} \tag{2}$$

with $W_p(s)$ strictly proper, $Z_p(s)$ a monic Hurwitz polynomial of degree $m(\leq n-1)$, $R_p(s)$ a monic polynomial of degree n and K_p a constant gain parameter. We further assume that only m,n and the sign of K_p are known for use in the design of an adaptive controller. $n^*(=n-m)$ is referred to as the relative degree of the plant.

A model M represents the behavior desired from the plant when it is augmented with a suitable controller. The model has a reference input r(t) which is uniformly bounded and an output $y_M(t)$. The transfer function of the model, denoted by $W_M(s)$ may be represented as

$$W_M(s) \triangleq K_M \frac{Z_M(s)}{R_M(s)} \tag{3}$$

where $Z_M(s)$ is a monic Hurwitz polynomial of degree $m_1 \leq m$, $R_M(s)$ is a monic Hurwitz polynomial of degree n and K_M is a constant. $W_M(s)$ is completely specified and the aim of the design is to generate suitable bounded inputs u(t) to the plant so that the deviation of the plant from the desired behavior as measured by an error signal $\epsilon_1(t)$ where

$$|\epsilon_1(t)| \triangleq |y_p(t) - y_M(t)| \tag{4}$$

tends to zero as $t \to \infty$.

Structure of the Controller ($n^* \leq 2$):

The basic structure of the adaptive controller used in the following sections is shown in Figure 1. The controller consists of a gain c_0 and two auxiliary signal generators F_1 and F_2. F_1 contains (n-1) parameters c_i $(i = 1,2,\ldots,n-1)$ and F_2 contains n parameters d_j $(j = 0,1,2,\ldots n-1)$. The 2n adjustable parameters are denoted by the elements of a parameter vector

$$\bar{\theta}^T(t) = [c_0(t),\theta^T(t)] \triangleq [c_0(t),c_1(t),\ldots c_{n-1}(t),d_0(t),d_1(t),\ldots d_{n-1}(t)] \quad (5)$$

F_1 and F_2 are described by the vector differential equations

$$\left.\begin{aligned} \dot{v}^{(1)} &= \Lambda v^{(1)} + b_f u \\ w^{(1)} &= c^T v^{(1)} \end{aligned}\right\} F_1$$
$$\left.\begin{aligned} \dot{v}^{(2)} &= \Lambda v^{(2)} + b_f y_p \\ w^{(2)} &= d_0 y_p + d^T v^{(2)} \end{aligned}\right\} F_2 \quad (6)$$

where Λ is an (n-1) x (n-1) stable matrix, $c^T = [c_1,c_2,\ldots,c_{n-1}]$ and $d^T = [d_1,d_2,\ldots,d_{n-1}]$.

Defining the vector of transfer functions V(s) as

$$V(s) \triangleq (sI-\Lambda)^{-1}b_f \quad (7)$$

$V_i(s)$, the i^{th} component of V(s), denotes the transfer function from u(t) to $v_i^{(1)}(t)$ or $y_p(t)$ to $v_i^{(2)}(t)$. For constant values of the control parameters the transfer functions of F_1 and F_2 are $W_1(s)$ and $W_2(s)$, where

$$W_1(s) = c^T V(s) \triangleq \frac{C(s)}{N(s)} \; ; \; W_2(s) = d^T V(s) + d_0 \triangleq \frac{D(s)}{N(s)} + d_0 \quad (8)$$

If (Λ,b_f) is in companion form, c_i and d_i $(i = 1,2,\ldots,n-1)$ represent the coefficients of s^{n-i-1} of the numerator polynomials of $W_1(s)$ and $W_2(s)$ respectively.

Denoting the principal signals of interest $r(t), v^{(1)}(t)$, $y_p(t)$ and $v^{(2)}(t)$ as the elements of a vector $\bar{\omega}(t)$

$$\bar{\omega}^T(t) \triangleq [r(t),\omega^T(t)] \triangleq [r(t),v^{(1)T}(t),y_p(t),v^{(2)T}(t)]^{\dagger} \tag{9}$$

the input to the plant, as shown in Figure 1 is

$$u(t) \triangleq \bar{\theta}^T(t)\bar{\omega}(t) - \alpha\varepsilon_1(t)\bar{\omega}^T(t)\Gamma\bar{\omega}(t) \qquad \Gamma = \Gamma^T > 0, \quad \alpha > 0 \tag{10}$$

The only difference between the controller structure described in equations (5-10) and that given in [2] is the existence of the second term in the right hand side of equation (10).

Following the results of [2] it can be shown that a constant control parameter vector $\bar{\theta}^*$ exists such that if $\bar{\theta}(t) \equiv \bar{\theta}^*$ the transfer function of the plant together with the controller matches that of the model exactly. If $\varepsilon(t)$ represents the state error between model and plant, the error differential equations are:

$$\begin{aligned} \dot{\varepsilon}(t) &= A_c\varepsilon(t) + b_c[\bar{\phi}^T(t)\bar{\omega}(t) - \alpha\varepsilon_1(t)\bar{\omega}^T(t)\Gamma\bar{\omega}(t)] \\ \varepsilon_1(t) &= h_c^T\varepsilon(t) \end{aligned} \tag{11}$$

where $W_e(s) \triangleq h_c^T(sI-A_c)^{-1}b_c = \frac{K_p}{K_M} W_M(s)$ is strictly positive real, and $\bar{\phi}(t) = \bar{\theta}(t) - \bar{\theta}^*$ is the parameter error vector [Ref. equations (18) and (19) in [2] and Figure 2]. The adaptive laws

$$\dot{\bar{\phi}}(t) = -\Gamma\varepsilon_1(t)\bar{\omega}(t) \tag{12}$$

assure the boundedness of $\bar{\phi}(t)$ and $\varepsilon(t)$ and that $\varepsilon_1(t) \to 0$ as $t \to \infty$ [Ref. section 3 for proof].†† Since the states of the model are bounded, this also assures the boundedness of the plant states.

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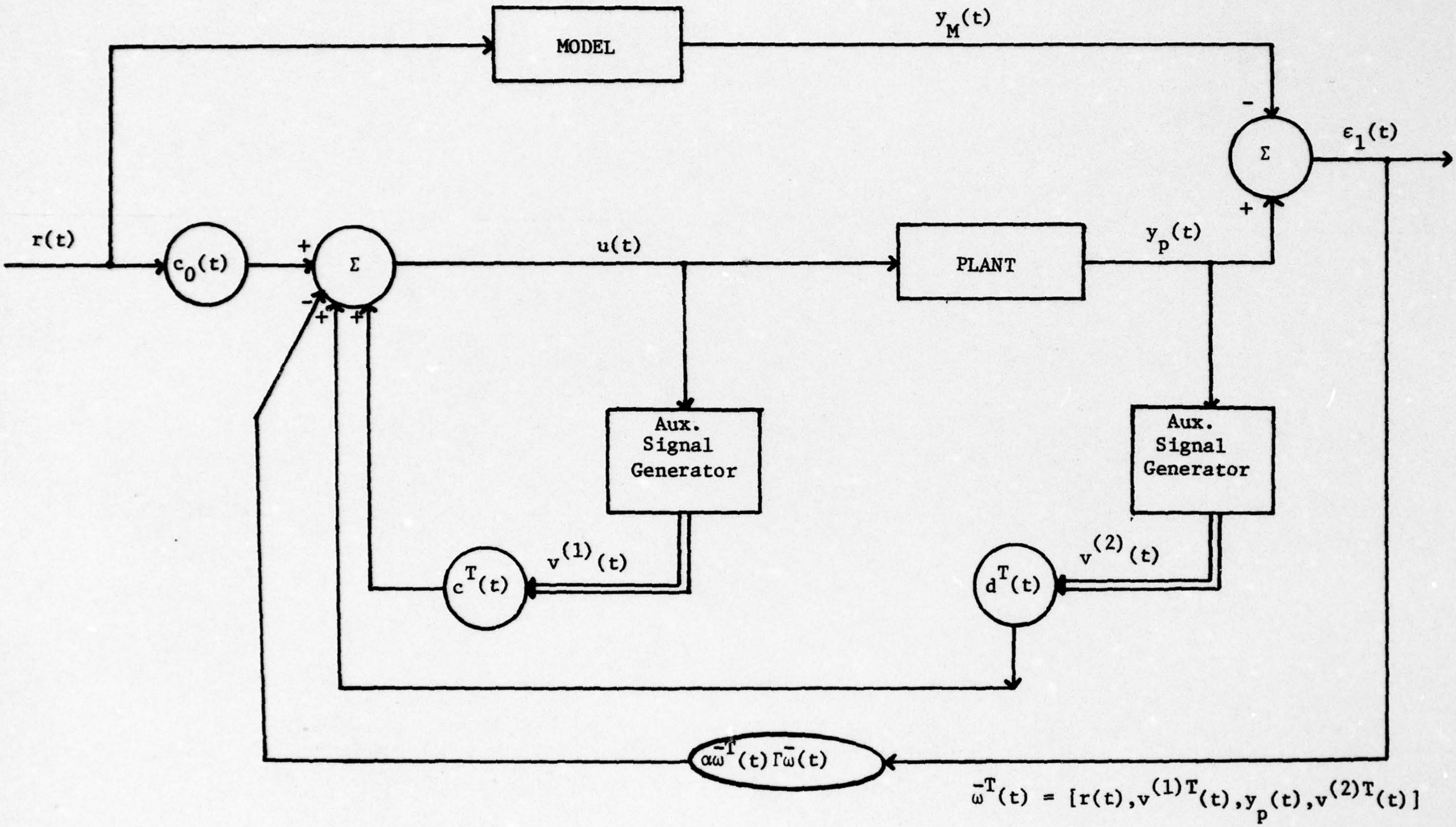
† In equations (5) and (9) the 2n dimensional vectors $\bar{\theta}(t)$ and $\bar{\omega}(t)$ are defined. In the main discussions in the paper, as explained later, only the 2n-1 dimensional vectors $\theta(t)$ and $\omega(t)$ are used.

†† $\varepsilon_1(t)$ is the error between plant and (unaugmented) model outputs. In section 3 $e_1(t)$ is the output error when the model output is augmented. The form of the differential equations (11) and (12), however, remains the same in the two cases.



MODEL
$y_M(t)$
Σ
$\varepsilon_1(t)$
$r(t)$
$c_0(t)$
Σ
$u(t)$
PLANT
$y_p(t)$
Aux. Signal Generator
Aux. Signal Generator
$v^{(1)}(t)$
$c^T(t)$
$v^{(2)}(t)$
$d^T(t)$
$\alpha\bar{\omega}^T(t)\Gamma\bar{\omega}(t)$
$\bar{\omega}^T(t) = [r(t), v^{(1)T}(t), y_p(t), v^{(2)T}(t)]$

Figure 1

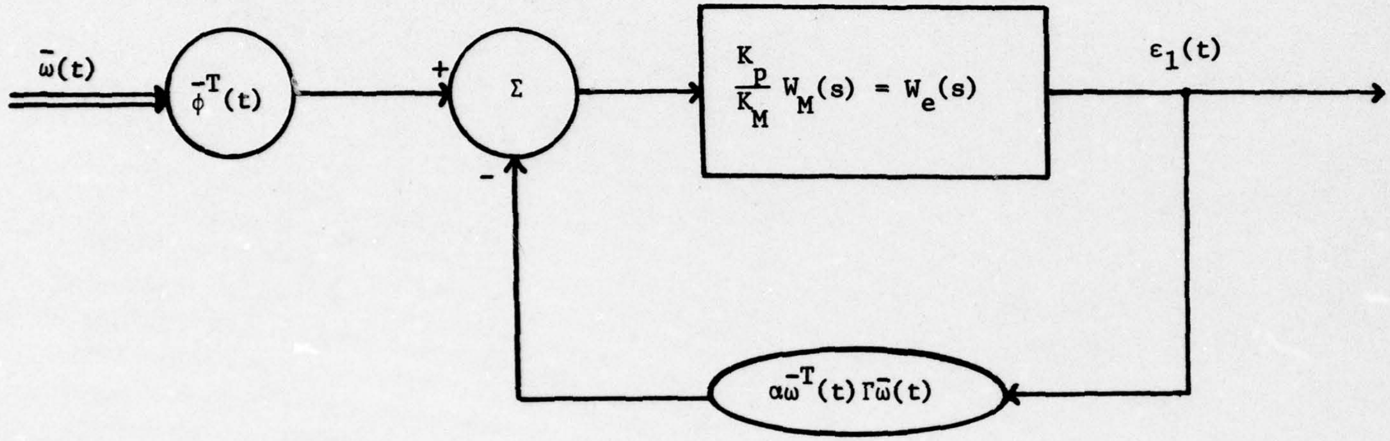


Figure 2

For a relative degree $n^* = 2$, the modification suggested in [2] can again be used in this case also to develop adaptive control laws of the form (12) [equation (22)in [2]]. The principal difficulty, therefore, arises when the relative degree n^* is greater than or equal to three. In such a case, an auxiliary signal corresponding to each control parameter has to be used as an input to the model as described in [2].

Structure of the Controller ($n^* \geq 3$):

If $n^* \geq 3$, with no loss of generality (Ref. [2]), let L(s) represent a Hurwitz polynomial in 's' such that $W_M(s)L(s)$ is a strictly positive real transfer function. If every parameter $\theta_i(t)$ in the controller in Figure 1 is replaced by an operator $P_L(\theta_i(t)) = L(s)\theta_i(t)L^{-1}(s)$, the same procedure as that outlined earlier can be used to derive the adaptive laws. In such a case, $\dot{\theta}(t) = -\Gamma\varepsilon_1(t)\zeta(t)$, where $L^{-1}(s)\omega(t) = \zeta(t)$. However, since the controller is to be differentiator free, such a procedure is not possible. Therefore, the following approach described in [2] is used. Corresponding to every feedback signal $\theta_i(t)\omega_i(t)$ to the plant, a signal $[\theta_i(t)-P_L(\theta_i(t))]\omega_i(t)$ is fed back into the model to preserve the form of the error differential equations (11) and hence the adaptive laws (12). This results in the structure of the adaptive controller described later in this section.

The complexity of the controller (as given in [2]) is also determined by the prior knowledge of the gain K_p of the plant. We therefore deal with the simpler case first, when K_p is known, and later consider the case when K_p is unknown. In section 5, for clarity of presentation, the proof of stability is also given separately for the two cases.

Case (i) (K_p known): With no loss of generality, we can assume here that $K_M = K_p = 1$. This implies that $c_0 = 1$ and hence only 2n-1 control parameters (the elements of $\theta(t)$) need to be adjusted.

If the input signal to the plant is

$$u(t) = \theta^T(t)\omega(t) + r(t) \tag{13}$$

the plant output can be expressed as

$$y_p(t) = W_M(s)[r(t) + \phi^T(t)\omega(t)] \tag{14}$$

The model output is augmented by the additional output $y_a(t)$ where

$$y_a(t) = W_M(s)L(s)[\{L^{-1}(s)\theta(t) - \theta(t)L^{-1}(s)\}^T\omega(t) + \alpha e_1(t)\zeta^T(t)\Gamma\zeta(t)] \tag{15}$$

and $L^{-1}(s)\omega(t) = \zeta(t)$ (Figure 3).

The total output of the model y(t) is the sum of the desired output $y_M(t) = W_M(s)r(t)$ and the augmented output $y_a(t)$; the augmented error $e_1(t)$

$$e_1(t) \triangleq y_p(t) - y(t) = \varepsilon_1(t) - y_a(t) \tag{16}$$

is given by

$$e_1(t) = W_M(s)L(s)[\phi^T(t)\zeta(t) - \alpha e_1(t)\zeta^T(t)\Gamma\zeta(t)] \tag{17}$$

which has the same form as (11) with $\bar{\omega}(t)$ and $\varepsilon_1(t)$ replaced by $\zeta(t)$ and $e_1(t)$ respectively.

The adaptive law for updating the control parameters is now

$$\dot{\phi}(t) = -\Gamma e_1(t)\zeta(t) \tag{18}$$

<u>Case (ii) (K_p unknown)</u>: While the error equations appear to be considerably more involved when K_p is unknown, they can be readily reduced to the same form as before by using a change of variables (as shown in [2]). The input to the plant is now given by

$$u(t) = \bar{\theta}^T(t)\bar{\omega}(t) \tag{19}$$

The main difficulty here is that K_p is different from K_M and hence an additional parameter $\psi_1(t)$ has to be introduced in series with the auxiliary input to the model to obtain the error equations in the form given in (17). For each signal $\theta_i(t)\omega_i(t)$ fed back into the plant a signal $L(s)\psi_1(t)[L^{-1}(s)\theta_i(t) - \theta_i(t)L^{-1}(s)]\omega_i(t)$ has to be fed back into the model (or $\psi_1(t)[L^{-1}(s)\theta_i(t) - \theta_i(t)L^{-1}(s)]\omega_i(t)$ into

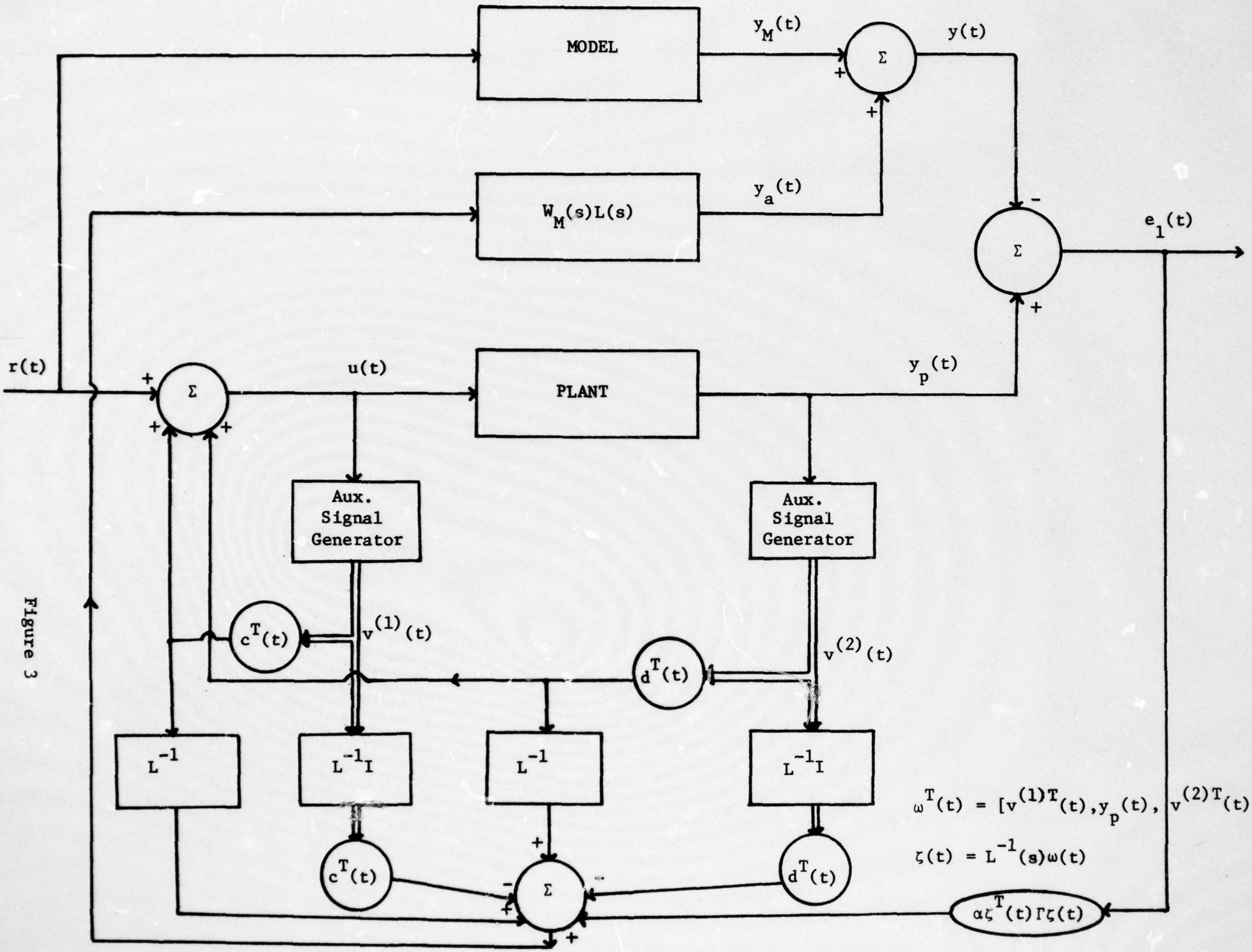

MODEL
$W_M(s)L(s)$
PLANT
Aux. Signal Generator
Aux. Signal Generator
r(t)
u(t)
$y_M(t)$
$y_a(t)$
y(t)
$y_p(t)$
$e_1(t)$
$v^{(1)}(t)$
$v^{(2)}(t)$
$c^T(t)$
$d^T(t)$
L^{-1}
$L^{-1}I$
$\alpha\zeta^T(t)\Gamma\zeta(t)$
$\omega^T(t) = [v^{(1)T}(t), y_p(t), v^{(2)T}(t)]$
$\zeta(t) = L^{-1}(s)\omega(t)$

Figure 3

$W_M(s)L(s)$ for practical realization). The 2n+1 parameters $\theta_i(t)$ $(i = 1,2,..,2n)$ and $\psi_1(t)$ have to be adaptively adjusted so that $\psi_1(t) \to K_p/K_M$ and $\theta_i(t) \to \theta_i^*$ as $t \to \infty$.

The augmented error $e_1(t)$ and the adaptive laws are now given by (Ref. [2])

$$e_1(t) = \frac{K_p}{K_M}[W_M(s)L(s)]\{[\bar{\phi}^T(t)\bar{\zeta}(t) + \psi(t)\bar{\xi}(t)] - \alpha_1 e_1(t)\bar{\zeta}^T(t)\Gamma\bar{\zeta}(t)\} \tag{20}$$

$$\begin{aligned} \dot{\bar{\phi}}(t) &= -\Gamma e_1(t)\bar{\zeta}(t) \\ \dot{\psi}(t) &= -\gamma e_1(t)\bar{\xi}(t) \end{aligned} \tag{21}$$

where

$$\bar{\zeta}(t) = L^{-1}(s)\bar{\omega}(t); \qquad \bar{\xi}(t) = [L^{-1}(s)\bar{\phi}(t) - \bar{\phi}(t)L^{-1}(s)]^T\bar{\omega}(t) \text{ and}$$

$$\psi(t) = 1 - \frac{K_M}{K_p}\psi_1(t) \tag{22}$$

The stability problem that arises is the same whether K_p is known or unknown and is stated in the next section.

3. Statement of the Stability Problem:

As shown in this section, when the adaptive laws (12), (18) or (19) are used (for the cases $n^* \leq 2$, $n^* \geq 3$(K_p known) and $n^* \geq 3$(K_p unknown) respectively) it is possible to demonstrate through the existence of a Lyapunov function that the state error vector $\varepsilon(t)$ or $e(t)$ as well as the parameter error vector $\phi(t)$ are bounded. When $n^* \leq 2$ (and no auxiliary input to the model is used) the output of the model $y_M(t)$ is bounded and hence the boundedness of the plant output can be concluded. This, in turn, results in $\varepsilon_1(t) \to 0$ as $t \to \infty$. However, when $n^* \geq 3$ we cannot assume that the model output $(y(t) = y_a(t) + y_M(t))$ is bounded and this leads to the principal stability question of interest in this paper.

If the input to the plant $u(t)$ is given by (13) (K_p known) or (19) (K_p unknown), the error model is defined by (17) or (20) and the corresponding adaptive laws are specified by (18) and (21), the problem of stability is to conclude the boundedness of the plant output $y_p(t)$ and hence all the relevant signals in the system. Demonstrating that this is indeed the case is the principal contribution of this paper. We now proceed to state the stability problem analytically for the case when K_p is known.

The Error Model (K_p known):

The differential equations describing the augmented error e(t) may be expressed as (Ref. [2])

$$\begin{aligned} \dot{e} &= A_c e(t) + b\nu(t) \\ e_1(t) &= h^T e(t) \\ \nu(t) &= \phi^T(t)\zeta(t) - \alpha\zeta^T(t)\Gamma\zeta(t)e_1(t) \qquad \alpha > 0 \end{aligned} \tag{23}$$

where

$$h^T(sI-A_c)^{-1}b = W_M(s)L(s) \text{ is SPR} \tag{24}$$

(compare with equation (17)). The corresponding adaptive laws are

$$\dot{\phi}(t) = -\Gamma e_1(t)\zeta(t). \tag{18}$$

For the case $n^* \leq 2$, the equations are the same with $\zeta(t)$ replaced by $\omega(t)$ and the boundedness of the plant output assures the boundedness of $\omega(t)$ in the corresponding error equations. A difficulty arises in this case since $y_p(t)$ and hence $\zeta(t)$ cannot be assumed to be bounded.

By the Kalman-Yacubovich lemma [14] it is known that if condition (24) is satisfied, a real matrix $P = P^T > 0$, a vector q, and an $\varepsilon > 0$ exist such that for any matrix $N = N^T > 0$

$$\begin{aligned} A_c^T P + PA_c &= -qq^T - \varepsilon N \\ Pb &= h \end{aligned} \tag{25}$$

are simultaneously satisfied. Defining a Lyapunov function for the set of differential equations (23), (18) as

$$V(e,\phi) \triangleq e^T(t)Pe(t) + \phi^T(t)\Gamma^{-1}\phi(t)$$

we obtain using (25)

$$\dot{V}(e,\phi) = -e^T(t)[qq^T + \varepsilon N]e(t) - 2\alpha[e_1(t)\zeta(t)]^T\Gamma[e_1(t)\zeta(t)]$$

$$\leq 0 \qquad \text{for } \alpha > 0 \tag{26}$$

It follows that the error model is uniformly stable and $e(t)$ (hence, $e_1(t)$) and $\phi(t)$ are uniformly bounded for any finite initial conditions. Further, since $V(t)$ is a non-increasing function which is bounded below, it converges to some limit V^*. [For $n^* \leq 2$ these conditions are adequate to assure that $e(t) \to 0$ as $t \to \infty$]. By (26) since $V(e,\phi)$ is uniformly bounded we have

$$e(t),\ e_1(t)\zeta(t) \in L^2 \tag{27}$$

Since $\dot{\phi}(t) = -\Gamma e_1(t)\zeta(t)$, (27) implies that

$$\dot{\phi}(t) \in L^2 \tag{28}$$

The importance of the additional feedback term $\alpha e_1(t)\zeta^T(t)\Gamma\zeta(t)$ in the error model is now apparent. Since $\dot{V}(e,\phi)$ contains the term $-e_1^2(t)\zeta^T(t)\Gamma\zeta(t)$, it is now possible to conclude that $\dot{\phi}(t) \in L^2$ which is central to the proof of stability.

<u>The Plant Feedback Loop</u>:

The plant together with the controller may be described by the differential equations

$$\begin{aligned}\dot{x}(t) &= A_c x(t) + b_m[\phi^T(t)\omega(t) + r(t)]\\ \omega(t) &= C_m x(t)\end{aligned} \tag{29}$$

or equivalently by a vector transfer function $W(s) \triangleq C_m(sI-A_c)^{-1}b_m = \begin{bmatrix} \frac{W_M(s)}{W_p(s)} V(s) \\ W_M(s) \\ W_M(s)V(s) \end{bmatrix}$

(where V(s) is defined in (7)) in the forward path and a gain vector $\phi(t)$ in the feedback path. Further, $\zeta(t) = L^{-1}(s)\omega(t)$ and the plant feedback loop together with the error model may be represented as shown in the Figure 4 .

The Stability Problem: is then to show that the plant feedback loop with $\phi(t)$ adjusted according to the adaptive law (18) and $\dot{\phi} \in L^2$ is stable in the large.

When K_p is unknown, the error equations (23) have to be replaced by those corresponding to (20) but the plant feedback loop remains essentially the same ($c_0(t)$ is also adjusted) (Figure 5). Hence, the nature of the stability problem is the same in both cases.

Stability Problem in the Discrete Case: The form of the difference equations describing the plant, the reference model and the error model is the same as that described in this section but the stability problem is considerably simplified by the fact that $\Delta\phi(k) \in \ell^2$ and hence $\Delta\phi(k) \to 0$ as $k \to \infty$. $V(e(k),\phi(k)) > 0$ and $\Delta V(e(k),\phi(k)) \leq 0$ also results in $e(k), e_1(k), e_1(k)\zeta(k)$ and $\nu(k) \to 0$ as $k \to \infty$. Hence, all the results described in section (5) carry over directly to the discrete case as well.

4. Mathematical Preliminaries:

The proof of stability of the adaptive control problem is discussed in section 5 and is based on the five lemmas given in this section. The definitions and related mathematical concepts needed for the proof of the lemmas are developed in this section.

Definition 1: L_e^∞ denotes the space of functions which are bounded for finite time, i.e.

$$L_e^\infty = \{f: R^+ \to R \,/ \sup_{t \geq \tau} |f(\tau)| < \infty \quad , \text{ for all } t \in R^+\}$$

Remark 1: Since the parameter error vector $\phi(t)$ is uniformly bounded all the signals in this paper can grow at most exponentially and, therefore, belong to this space.

Definition 2: Let $x(t), y(t) \in L_e^\infty$. Let $\beta(t)$ be a continuous function such that $\beta(t) \to 0$ as $t \to \infty$. If $y(t) = \beta(t)x(t)$, we denote

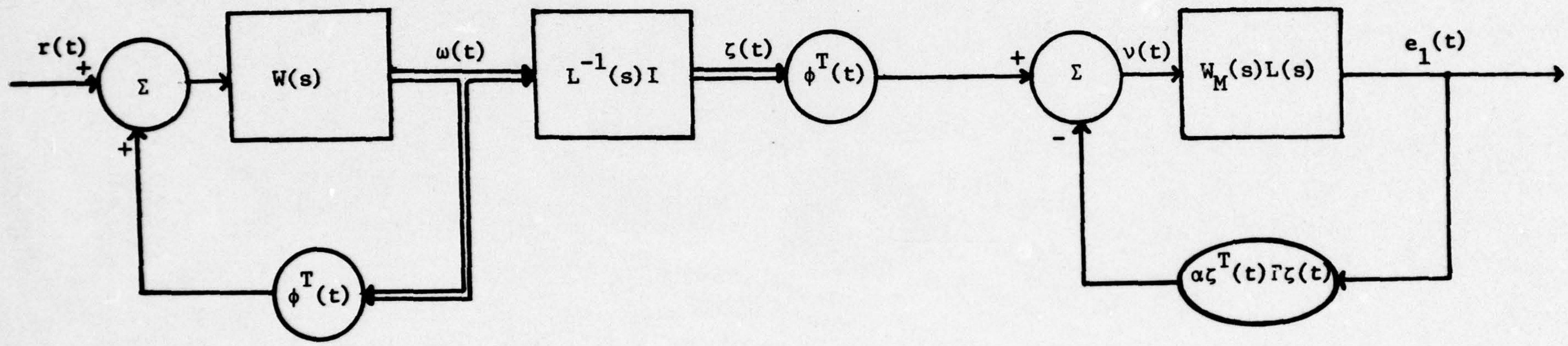


Figure 4

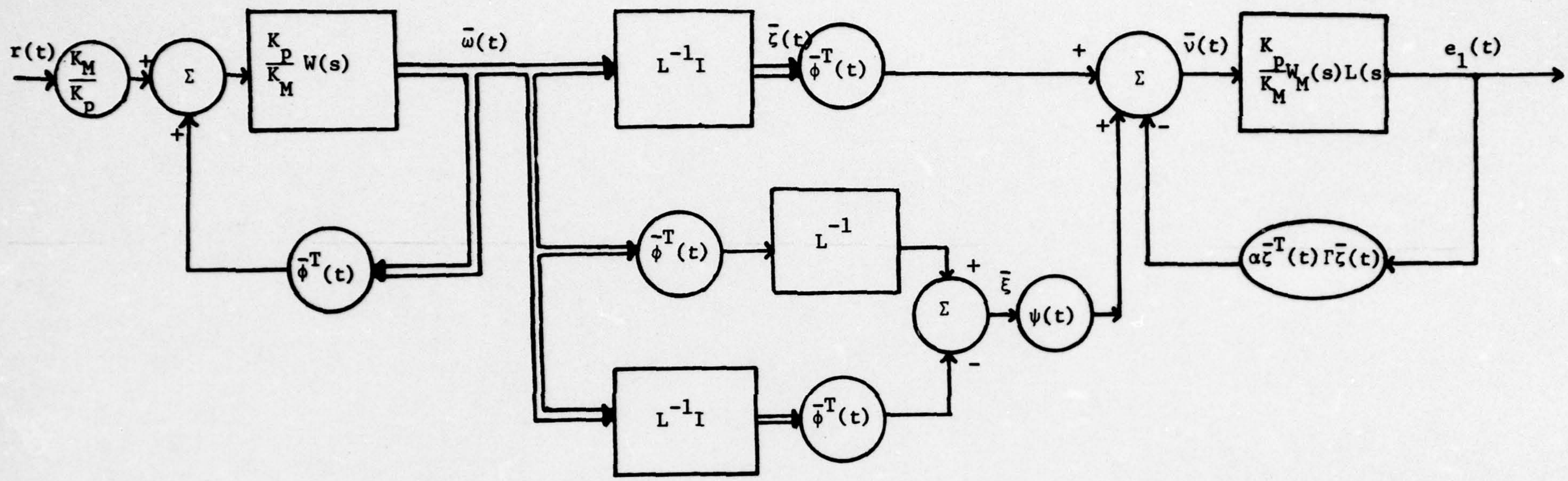

r(t)
K_M/K_P
Σ
K_P/K_M W(s)
ω̄(t)
L^-1 I
ζ̄(t)
φ̄^T(t)
Σ
ν̄(t)
K_P/K_M W_M(s)L(s
e_1(t)
φ̄^T(t)
φ̄^T(t)
L^-1
Σ
ξ̄
ψ(t)
αζ̄^T(t)Γζ̄(t)
L^-1 I
φ̄^T(t)

Figure 5

$$y(t) = o[x(t)]$$

Remark 2: (i) If $y(t)= o[y(t)]$, then $y(t) \to 0$ as $t \to \infty$.

(ii) If $|y(t)| = o[\sup_{t \geq \tau} |y(\tau)|]$, then $y(t) \to 0$ as $t \to \infty$.

(iii) If $|y(t)| = x(t) + o[\sup_{t \geq \tau} |y(\tau)|]$ and $x(t)$ is uniformly bounded then

$y(t)$ is also uniformly bounded.

Definition 3: Let $x(t), y(t) \in L_e^\infty$. If there exists a constant $M > 0$ such that $|y(t)| \leq M|x(t)|$, then we denote

$$y(t) = O[x(t)]$$

Remark 3: If the input to a linear exponentially stable system is $x(t) \in L_e^\infty$ and the corresponding output is $y(t)$, then

$$|y(t)| = O[\sup_{t \geq \tau} |x(\tau)|] \tag{30}$$

In particular, the input and output of an asymptotically stable linear time-invariant system satisfy (30).

Definition 4: Let $x(t), y(t) \in L_e^\infty$. If $y(t) = O[x(t)]$ and $x(t) = O[y(t)]$, then we say that $x(t)$ and $y(t)$ are equivalent and denote this by

$$x(t) \sim y(t)$$

Definition 5: Let $x(t), y(t) \in L_e^\infty$. If $\sup_{t \geq \tau} |y(\tau)| \sim \sup_{t \geq \tau} |x(\tau)|$, we say that $x(t)$ and $y(t)$ grow at the same rate.

Remark 4: (i) Two signals which are equivalent grow at the same rate but the converse is not true.

(ii) Two signals which are uniformly bounded grow at the same rate. Hence this definition is of interest mainly for unbounded signals.

(iii) If $x(t), y(t) \in L_e^\infty$, only one of the following three conditions can hold:

$$\sup_{t \geq \tau} |x(\tau)| \sim \sup_{t \geq \tau} |y(\tau)|, \quad \sup_{t \geq \tau} |y(\tau)| = o[\sup_{t \geq \tau} |x(\tau)|], \text{ or } \sup_{t \geq \tau} |x(\tau)| = o[\sup_{t \geq \tau} |y(\tau)|]$$

(iv) $\{\sup_{t\geq\tau}|x(\tau)| = o[\sup_{t\geq\tau}|y(\tau)|]$ or $\sup_{t\geq\tau}|x(\tau)| \sim \sup_{t\geq\tau}|y(\tau)|\}$

$$\Longleftrightarrow \quad \{\sup_{t\geq\tau}|x(\tau)| = O[\sup_{t\geq\tau}|y(\tau)|]\}.$$

(v) Let an n-dimensional vector $x(t) \in L_e$ be unbounded, then there exists at least a component $x_{i_0}(t)$ such that

$$\sup_{t\geq\tau}|x_{i_0}(\tau)| \sim \sup_{t\geq\tau}\|x(\tau)\|$$

In the following Lemmas 1-3, $x(t), y(t) \in L_e^\infty$ and H(s) is the rational transfer function of an asymptotically stable system and is strictly proper. All initial conditions are assumed to be identically zero.*

Lemma 1: Let x(t) and y(t) be the input and output respectively of a system with transfer function H(s). Then

(i) $x(t) \in L^p \Rightarrow y(t), \dot{y}(t) \in L^p \quad 1 \leq p \leq \infty$

(ii) $x(t) \in L^1$ or $L^2 \Rightarrow y(t) \to 0$ as $t \to \infty$

(iii) $x(t) \to 0$ as $t \to \infty \Rightarrow y(t) \to 0$ as $t \to \infty$

The proof of Lemma 1 can be found in any good text on linear systems (Ref. [15]).

Lemma 2: Let $x(t) \in L^2$ (or L^1) and $\rho(t) \in L_e^\infty$. If $x(t)\rho(t)$ is the input to H(s) and y(t) the corresponding output, then

$$y(t) = o[\sup_{t\geq\tau}|\rho(\tau)|]$$

Proof: Let h(t) be the impulse response of H(s). $h(t) \in L^1$.

$$y(t) = \int_0^t h(t-\tau)x(\tau)\rho(\tau)d\tau$$

$$|y(t)| \leq \sup_{t\geq\tau}|\rho(\tau)| \int_0^t |h(t-\tau)x(\tau)|d\tau$$

Since Lemma 1 is valid for any $x(t) \in L^2$ (or L^1), $\int_0^t |h(t-\tau)x(\tau)|d\tau \to 0$ as $t \to \infty$ or

$$|y(t)| = o[\sup_{t\geq\tau}|\rho(\tau)|]$$

* If the initial conditions are non-zero, the output in Lemma 2 contains an additional exponentially decaying term.

<u>Lemma 3</u>: In the feedback system shown in Figure 6

$$y(t) = H(s)[x(t)+\beta(t)y(t)]$$

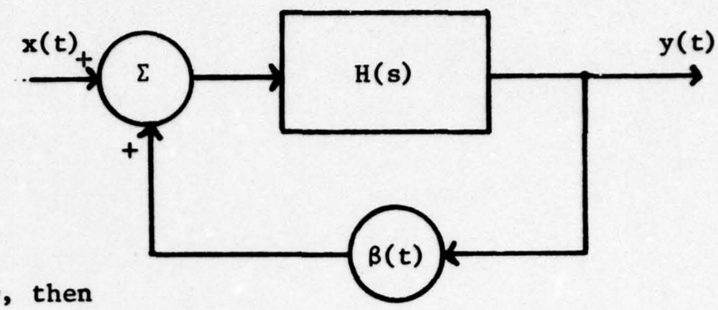


Figure 6

If $\beta(t) \in L^2$ or L^1 or $\beta(t) \to 0$ as $t \to \infty$, then

(i) $y(t) \to 0$ as $t \to \infty$, if $x(t) \to 0$ as $t \to \infty$

(ii) $y(t)$ is uniformly bounded, if $x(t)$ is uniformly bounded.

<u>Proof</u>: The proof is a direct consequence of Lemma 2, the linearity of H and Remark 2.

<u>Remark 5</u>: If in Lemma 3 $|x(t)| = o[\sup_{t \geq \tau}|y(\tau)|]$, the same results hold and $x(t), y(t) \to 0$.

<u>Remark 6</u>: In the discrete case $x(k) \in \ell^1$ or ℓ^2 implies that $x(k) \to 0$ as $k \to \infty$. With obvious modifications Lemmas 1-3 hold even if H(z) is a proper (and not strictly proper) discrete asymptotically stable transfer function.

While various generalizations of the following lemma can be stated, Lemma 4 is adequate for the proof of stability in section 5.

<u>Lemma 4</u>: In the plant feedback loop (Figure 3) defined by equations (1) and (6), the input to the plant and the states of F_1 and F_2 can not grow faster than the output of the plant i.e.

$$u(t), v_i^{(1)}(t), v_i^{(2)}(t) = O[\sup_{t \geq \tau}|y_p(\tau)|] \qquad i = 1,2,\ldots,n-1$$

<u>Proof</u>: Since $v_i^{(1)}(t)$ and $v_i^{(2)}(t)$ are the states of the asymptotically stable systems F_1 and F_2 with inputs $u(t)$ and $y_p(t)$ respectively, we have $v_i^{(2)}(t) = O[\sup_{t \geq \tau}|y_p(\tau)|]$. Hence it suffices to show that

$$u(t) = O[\sup_{t \geq \tau}|y_p(\tau)|]$$

We show in what follows that

$$y_p(t) = o[\sup_{t \geq \tau}|u(\tau)|] \tag{31}$$

leads to a contradiction. By (31) we have

$$v_i^{(2)}(t) = o[\sup_{t \geq \tau} |u(\tau)|] \qquad (32)$$

Consider the feedback loop F_1 defined by

$$\dot{v}^{(1)} = \Lambda v^{(1)} + b_f u \qquad (6)$$

and

$$u(t) = \theta^T(t)\omega(t) + r(t) \qquad (13)$$

Since $r(t)$ is uniformly bounded and $y_p(t)$ and $v^{(2)}(t)$ satisfy (31) and (32), if $u(t)$ is unbounded, $v^{(1)}(t)$ is also unbounded and by Remark 4 there is at least one variable $v_{i_0}^{(1)}(t)$, $1 \leq i_0 \leq n-1$ which grows at the same rate as the input $u(t)$ or

$$\sup_{t \geq \tau} |v_{i_0}^{(1)}(\tau)| \sim \sup_{t \geq \tau} |u(\tau)| \qquad (33)$$

Since F_1 is a vector differential equation with bounded coefficients

$$|\dot{v}_{i_0}^{(1)}(t)| = O[\sup_{t \geq \tau} |v_{i_0}^{(1)}(\tau)|] \qquad (34)$$

Since $W_p(s)$ is a transfer function with zeros in the open left half plane and

$$v_{i_0}^{(2)}(t) = W_p(s) v_{i_0}^{(1)}(t) \qquad (35)$$

it follows from (34) that

$$\sup_{t \geq \tau} |v_{i_0}^{(1)}(\tau)| = O \ [\sup_{t \geq \tau} |v_{i_0}^{(2)}(\tau)|] \qquad (36)$$

From (33) and (36)

$$\sup_{t \geq \tau} |u(\tau)| = O[\sup_{t \geq \tau} |v_{i_0}^{(2)}(\tau)|]$$

which is a contradiction to (32). Hence

$$u(t) = O[\sup_{t \geq \tau} |y_p(\tau)|]$$

which proves the lemma.

Remark 7: Lemma 4 is needed since we cannot directly conclude that $u(t) = O[\sup_{t \geq \tau} |y_p(\tau)|]$. In the error model, though $e_1(t)$ is uniformly bounded, we cannot conclude that $\nu(t)$ is uniformly bounded for the same reason. However, in discrete systems it follows directly that $\nu(k) \to 0$ as $k \to \infty$.

Lemma 5 is central to the proof of stability in section 5 and relates the operator $\bar{P}_L(\phi) = L^{-1}(s)\phi(t)L(s)$ to the pure gain $\phi(t)$ [Ref. equation (10) in [2]].

Let L(s) be a Hurwitz polynomial in 's' of degree r. With no loss of generality* assume

$$L(s) = \prod_{i=1}^{r} (s+a_i) \tag{37}$$

where a_i is real. Defining $A_i(s)$ as

$$A_i(s) \triangleq L(s) / \prod_{j=1}^{i} (s+a_i),\ i = 1,2,\ldots r-1;\quad A_0(s) = L(s)$$

if $\zeta(t)$ is a function which is r-times differentiable, then

$$[L^{-1}(s)\phi(t)L(s)]\zeta(t) = [\phi(t) - \sum_{i=0}^{r-1} A_i^{-1}(s)\dot{\phi}(t)A_{i+1}(s)]\zeta(t) + \varepsilon(t) \tag{38}$$

where $\varepsilon(t)$ is an exponentially decaying term due to the initial conditions. Figure 7 shows the two equivalent forms in (38) and the corresponding signals.

Remark 8: For the discrete case, if L(z) is a polynomial with all its zeros within the unit circle

$$[L^{-1}(z)\phi(k)L(z)]\zeta(k) = [\phi(k) - \sum_{i=0}^{r-1} A_i^{-1}(z)\Delta\phi(k)zA_{i+1}(z)]\zeta(k) + \varepsilon(k) \tag{39}$$

where $\phi(k+1) - \phi(k) = \Delta\phi(k)$ and $\varepsilon(k) \to 0$ geometrically in k.

Lemma 5: If L(s) is defined as in (37) and $L(s)\zeta(t) = \omega(t)$ with $\omega(t) \in L_e^\infty$ and $\dot{\phi}(t) \in L^2 \cap L_e^\infty$ then

$$[L^{-1}(s)\phi(t)L(s)]\zeta(t) = \phi(t)\zeta(t) + o[\sup_{t \geq \tau} |\omega(\tau)|]$$

---

* If L(s) contains complex conjugate zeros, the form of the expansion of the operator $L^{-1}\phi L$ remains the same but the coefficients of the polynomials have to be modified. $A_i(s)$ is, however, stable for all $i = 0,1,\ldots r-1$, e.g., if $L(s) = s^2+2as + b$ and has complex zeros $L^{-1}\phi L = \phi - \frac{s+a}{s^2+2as+b}\dot{\phi} - \frac{1}{s^2+2as+b}\dot{\phi}(s+a)$.

$$\eta(t) = L^{-1}(s)\phi(t)L(s)\zeta(t)$$

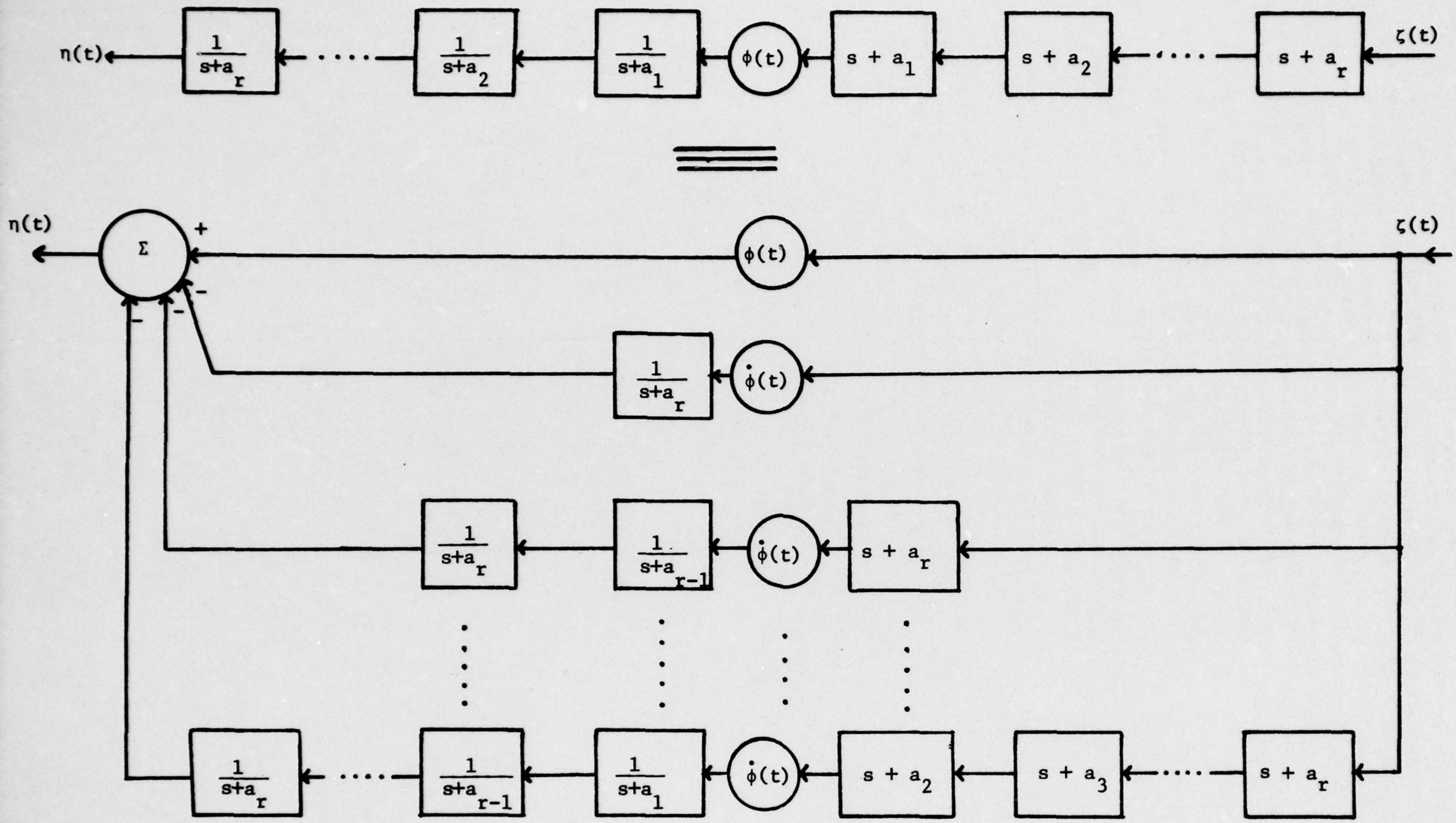


Figure 7

Proof: The proof follows directly by using the expansion (38), Lemma 2 and Remark 3 since

$$A_i(s)\zeta(t) = A_i(s)L^{-1}(s)\omega(t) = O[\sup_{t\geq\tau}|\omega(\tau)|]$$

5. Proof of Stability:

The proof of stability of the plant feedback loop follows directly from Lemmas 4 and 5 in section 4.

The motivation for the proof may be described qualitatively as follows: Assuming that $\omega(t)$ and hence $\zeta(t)$ are unbounded, Lemma 4 assures us that they cannot grow faster than the plant output $y_p(t)$. Using Lemma 5, $y_p(t)$ can be shown to be the sum of three signals, $W_M(s)r(t)$, $e_1(t)$, and $y_\varepsilon(t)$ where the first two are bounded and the third term by Lemmas 2,3 and 4 is $o[\sup_{t\geq\tau}|y_p(\tau)|]$. Hence, we have a contradiction and $y_p(t)$ is uniformly bounded. This, in turn, implies that $\omega(t)$, $\zeta(t)$ and $x_p(t)$ are uniformly bounded.

Proof:

Case (i) $n^* \geq 3$ and $K_p = K_M = 1$: (Figure 4)

By Lemma 4, if we can show that $y_p(t)$ is uniformly bounded, then all signals in the system are uniformly bounded. From (14)

$$y_p(t) = W_M(s)r(t) + W_M(s)\phi^T(t)\omega(t) \tag{40}$$

where $W_M(s)r(t)$ is a uniformly bounded signal.

The second term in the right hand side of (40) can be expressed as

$$W_M(s)\phi^T(t)\omega(t) = [W_M(s)L(s)][L^{-1}(s)\phi^T(t)L(s)]\zeta(t)$$

By Lemma 4 and Lemma 5

$$W_M(s)\phi^T(t)\omega(t) = [W_M(s)L(s)]\{\phi^T(t)\zeta(t) + o[\sup_{t\geq\tau}|y_p(\tau)|]\}$$

Further, since by (23)

$$\phi^T(t)\zeta(t) = \nu(t) - \alpha\dot{\phi}^T(t)\zeta(t)$$

where $[W_M(s)L(s)]v(t) = e_1(t)$

and $[W_M(s)L(s)](\alpha\dot{\phi}^T(t)\zeta(t)) = o[\sup_{t\geq\tau}|y_p(\tau)|]$ by Lemma 3,4 and $\dot{\phi}(t) \in L^2$

Hence, we have

$$y_p(t) = W_M(s)r(t) + e_1(t) + o[\sup_{t\geq\tau}|y_p(\tau)|]$$

It follows that $y_p(t)$ is uniformly bounded and the plant feedback loop is stable in the large.

Case (ii) $n^* \geq 3$ and K_p is unknown: (Figure 5)

In this case,

$$y_p(t) = [\frac{K_p}{K_M} W_M(s)][\frac{K_M}{K_p} r(t) + \bar{\phi}^T(t)\bar{\omega}(t)]$$

$$= W_M(s)r(t) + [\frac{K_p}{K_M} W_M(s)L(s)][L^{-1}(s)\bar{\phi}^T(t)L(s)]\bar{\zeta}(t)$$

By the same arguments as case (i)

$$\frac{K_p}{K_M} W_M(s)\bar{\phi}^T(t)\bar{\omega}(t) = [\frac{K_p}{K_M} W_M(s)L(s)]\{\bar{\phi}^T(t)\bar{\zeta}(t) + o[\sup_{t\geq\tau}|y_p(\tau)|]\}$$

From Figure 5

$$\bar{\phi}^T(t)\bar{\zeta}(t) = \bar{\nu}(t) - \alpha\dot{\bar{\phi}}^T(t)\bar{\zeta}(t) - \psi(t)\bar{\xi}(t)$$

But $\bar{\xi}(t) = [L^{-1}(s)\bar{\phi}^T(t) - \bar{\phi}^T(t)L^{-1}(s)]\bar{\omega}(t)$

$$= L^{-1}(s)\bar{\phi}^T(t)L(s)\bar{\zeta}(t) - \bar{\phi}^T(t)\bar{\zeta}(t)$$

$$= o[\sup_{t\geq\tau}|y_p(\tau)|]$$

Since $\psi(t)$ is uniformly bounded, again we conclude that $y_p(t)$ is uniformly bounded.

Case (iii) discrete system:

For the discrete system, with $\nu(t)$ replaced by $\nu(k) \to 0$ and $\Delta\phi(k) \to 0$, by eq. (39) stability follows trivially.

We have established that $\omega(t)$ is uniformly bounded, hence the input to the error model $\zeta(t)$ is also uniformly bounded. It follows by (26) that $e(t)$ is uniformly bounded, hence the second derivative of the Lyapunov function $\ddot{V}(t)$ is also uniformly bounded, which assures that $\dot{V}(t)$ is uniformly continuous. Hence

$$\lim_{t\to\infty} \dot{V}(t) = \lim_{t\to\infty} e(t) = \lim_{t\to\infty} \dot{\phi}(t) = 0$$

We also notice that the augmented input signal to the model $[L^{-1}(s)\phi^T(t)-\phi^T(t)L^{-1}(s)]\omega(t)$ and $\alpha e_1(t)\zeta^T(t)\Gamma\zeta(t)$ also tend to zero as $t \to \infty$. Hence, the error

$$\varepsilon_1(t) = y_p(t) - y_M(t) \to 0 \quad \text{as } t \to \infty.$$

5. Conclusion: The paper presents a complete and unified proof of the stability in the large of both continuous and discrete adaptive control systems. Except for an additional term, the controller structure is identical to that given in [2]. When the specified model transfer function $W_M(s)$ is not positive real, an operator $P_L(\theta) = L(s)\theta(t)L^{-1}(s)$ was used in [2] to generate a stable adaptive law for a parameter $\theta(t)$. Almost all the stability questions that arise in the adaptive control problem can be traced to this operator as well as a related operator $\bar{P}_L(\theta) = L^{-1}(s)\theta(t)L(s)$ and the extent to which these approximate a pure gain $\theta(t)$. Intuition suggests that when $\theta(t)$ is constrained to vary "slowly" in some sense (e.g. $\dot{\theta}(t) \to 0$ or $\theta(t) \in L^2$) the operator $\bar{P}_L(\theta)$ would approximate $\theta(t)$. The proof given in section 5 may be considered to be a mathematical justification of this statement. In this sense, it is based on our intuition regarding the behavior of the adaptive loop and the augmented inputs which originally led to the schemes suggested in [1] and [2].

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