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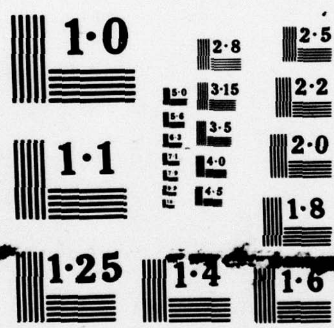

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(6) INSCRIBING AND CIRCUMSCRIBING CONVEX POLYHEDRA .

by

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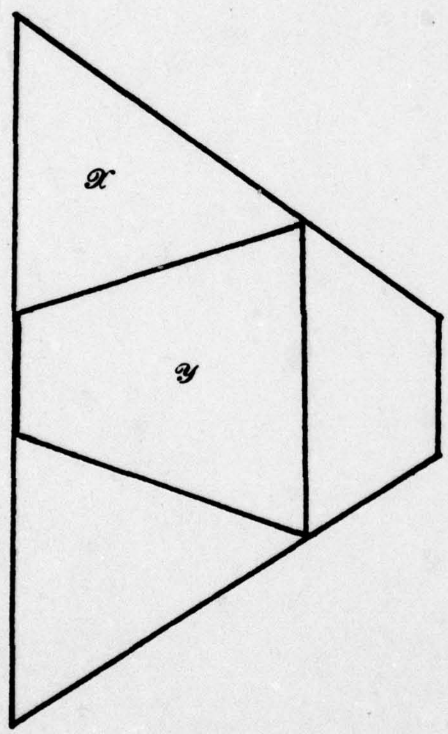
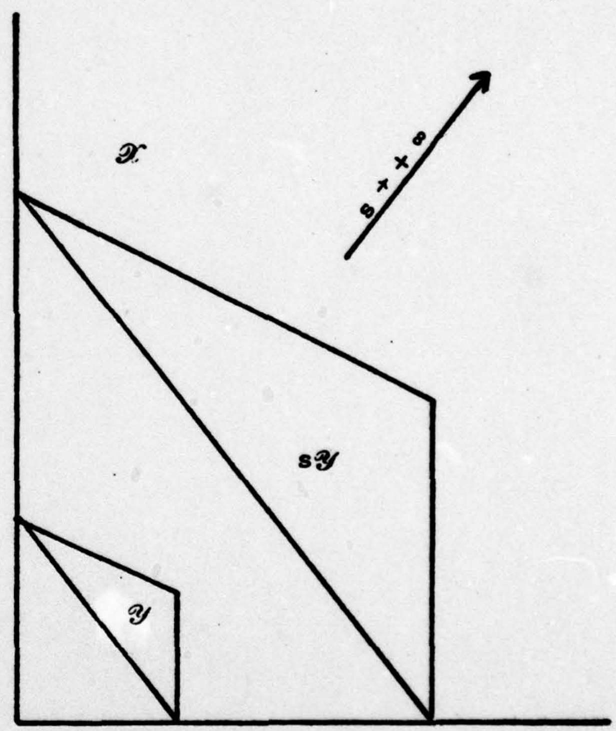
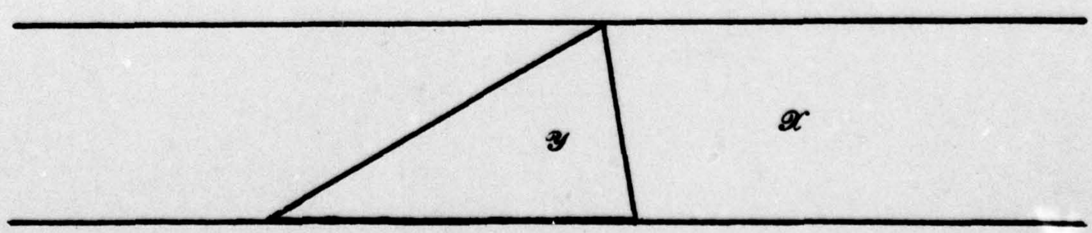
1. Introduction

Let $\mathscr{X}$ and $\mathscr{Y}$ be polyhedra; that is, closed polyhedral convex sets, bounded or not, in R^n. Our interest is in computing the smallest nonnegative scale $s\mathscr{X}$ of $\mathscr{X}$ for which some translate $s\mathscr{X} + t$ contains $\mathscr{Y}$, or equivalently, of computing the largest nonnegative scale $s\mathscr{Y}$ of $\mathscr{Y}$ for which some translate $s\mathscr{Y} + t$ is contained in $\mathscr{X}$.







For certain descriptions of $\mathscr{X}$ and $\mathscr{Y}$ we observe that this problem; namely, the circumscription program

$$\text{P1}\quad \begin{cases} z_1 = \underset{s,t}{\text{infimum:}}\ s \\ \text{subject to: } \mathscr{Y} \subseteq s\mathscr{X} + t \qquad s \geq 0 \end{cases}$$

or equivalently, the inscription program

$$\text{P2}\quad \begin{cases} z_2 = \underset{s,t}{\text{supremum:}}\ s \\ \text{subject to: } s\mathscr{Y} + t \subseteq \mathscr{X} \qquad s \geq 0 \end{cases}$$

is a linear program.

That an *n-sphere* of maximum radius in a polyhedron can be found by linear programming has been in the folklore for over a decade and has been used in a variety of applications. Perhaps our observations will be of use as well.

2. Preliminaries

To avoid trivialities we assume throughout that $\mathscr{X}$ and $\mathscr{Y}$ contain at least one and two points, respectively.

$\mathscr{X}$ and $\mathscr{Y}$ can be described as the intersection of finitely many hyperplanes

$$\mathscr{X} = \{x \in R^n : Ax \leq a\} \qquad \mathscr{Y} = \{x \in R^n : Bx \leq b\}$$

in which case we say that $\mathscr{X}$ and $\mathscr{Y}$ have representation H . Alternatively, $\mathscr{X}$ and $\mathscr{Y}$ can be described as a weighting of points and rays

$$\mathscr{X} = \{x \in R^n : x = X\lambda + U\omega,\ e\lambda = 1,\ \lambda \geq 0,\ \omega \geq 0\}$$

$$\mathscr{Y} = \{x \in R^n : x = Y\mu + V\pi,\ e\mu = 1,\ \mu \geq 0,\ \pi \geq 0\}$$

where $e = (1, 1, \ldots 1)$ in which case we say $\mathscr{X}$ and $\mathscr{Y}$ have representation W .

Given a representation one can systematically convert it to the other; however, we shall suppose, and typically rightly so, that the computational burden of this conversion is prohibitive.

Using the representation W we see that any polyhedron $\mathscr{X}$ can be expressed as $K + C$ where K is a compact polyhedron and C is the polyhedron of rays of $\mathscr{X}$. For any positive scale s we have $s\mathscr{X} = sK + sC = sK + C$, and, of course, for a zero scale s we have $s\mathscr{X} = \{0\}$. Thus the set $s\mathscr{X}$ is continuous in s for positive s, and is continuous in nonnegative s if and only if C contains only the origin.

Given $\mathscr{X}$ the cone of rays C is unique and we write $C = \text{cone}\,\mathscr{X}$. By $\text{tng}\,\mathscr{X}$ we mean the smallest subspace of R^n that contains a translate of $\mathscr{X}$.

The next lemmas describe the sense in which P1 and P2 are equivalent.

<u>Lemma 1</u> (Principal Equivalence): If (s_1, t_1) is feasible or optimal for P1 and s_1 is positive, then $(s_2, t_2) = (1/s_1, -t_1/s_1)$ is feasible or optimal, respectively, for P2, and vice versa. ⊠

<u>Lemma 2</u> (Feasibility): P1 is feasible if and only if $\text{cone}\,\mathscr{Y} \subseteq \text{cone}\,\mathscr{X}$ and $\text{tng}\,\mathscr{Y} \subseteq \text{tng}\,\mathscr{X}$. P2 is always feasible. ⊠

<u>Lemma 3</u> (Attainment): The following are equivalent.

i) $0 < z_1 < +\infty$ ii) P1 has an optimum

iii) $0 < z_2 < +\infty$ iv) P2 has an optimum. ⊠

<u>Lemma 4</u> (Non-attainment): The following are equivalent:

i) $z_1 = 0$

ii) $z_2 = +\infty$

iii) $\mathscr{Y} + t \subseteq \text{cone}\,\mathscr{X}$ for some t $\boxtimes$

The possible discontinuity of $s\mathscr{X} + t$ at $s = 0$ accounts for the incomplete equivalence between P1 and P2 .

3. Results

In this section we formulate the circumscription and inscription problems P1 and P2 for three cases of representation; namely, HH, WW, and HW, where, for example, HW refers to $\mathscr{X}$ and $\mathscr{Y}$ having representations H and W, respectively.

In each case a more general problem is treated first. We generalize P1 and P2 to

$$\text{P3}\begin{cases}\displaystyle\inf_{\theta}\text{imum: } c\theta \\ \text{subject to: } \mathscr{Y} \subseteq \mathscr{X}(\theta) \qquad\qquad \theta \in \Theta\end{cases}$$

and

$$\text{P4}\begin{cases}\displaystyle\sup_{\theta}\text{remum: } c\theta \\ \text{subject to: } \mathscr{Y}(\theta) \subseteq \mathscr{X} \qquad\qquad \theta \in \Theta\end{cases}$$

where Θ is a polyhedron in R^k and $c\theta$ is a linear function of θ in Θ that measures some feature of the polyhedrons. We regain P1 and P2 from P3 and P4 by setting $\theta = (s,t)$, etc.

Case HH

Let $\mathscr{X}(\theta)$ be the set $\{x : A(\theta)x \leq a(\theta)\}$ where (A,a) is an affine function of θ in Θ and let $\mathscr{Y} = \{x : Bx \leq b\}$. The program P3 is seen to be, using an alternative theorem, the linear program

$$\text{P5}\begin{cases}\underset{\theta,\Lambda}{\text{minimize:}} & c\theta & \\ \text{subject to:} & \Lambda B = A(\theta) & \Lambda b \leq a(\theta) \\ & \Lambda \geq 0 & \theta \in \Theta\end{cases}$$

Observing that $s\mathscr{X} + t = \{x : Ax \leq sa + At\}$ for $s > 0$ and specializing P5 to solve P1 we obtain the linear program

$$\text{P6}\begin{cases}z_1 = \underset{s,t,\Lambda}{\text{minimize:}} & s & \\ \text{subject to:} & \Lambda B = A & \Lambda b \leq as + At \\ & \Lambda \geq 0 & s \geq 0\end{cases}$$

Case WW

Let $\mathscr{Y}(\theta)$ be the set $\{Y(\theta)\mu + V(\theta)\pi : e\mu = 1,\ \mu \geq 0,\ \pi \geq 0\}$ where (Y,V) is an affine function of θ in Θ and let $\mathscr{X}$ be the set $\{X\lambda + U\omega : e\lambda = 1,\ \lambda \geq 0,\ \omega \geq 0\}$. The program P4 is the linear program

$$\text{P7}\begin{cases}\underset{\theta,\Lambda,\Omega,\Pi}{\text{maximize:}} & c\theta & \\ \text{subject to:} & Y(\theta) = X\Lambda + U\Omega & e\Lambda = e \\ & V(\theta) = U\Pi & \theta \in \Theta \\ \Lambda \geq 0 & \Omega \geq 0 & \Pi \geq 0\end{cases}$$

Specializing P7 to solve P2 we obtain the linear program

$$
\text{P8}\left\{\begin{array}{lll}
z_2 = \underset{s,t,\Lambda,\Omega,\Pi}{\text{maximize:}} & s & \\
\text{subject to:} & t \circ e + sY = X\Lambda + U\Omega & e\Lambda = e \\
 & sV = U\Pi & s \geq 0 \\
 & \Lambda \geq 0 \quad \Omega \geq 0 & \Pi \geq 0
\end{array}\right.
$$

where $\circ$ denotes outer product. In solving P8 one first verifies that $V = U\Pi$ with $\Pi \geq 0$ has a solution, and then drops the constraints $sV = U\Pi$ and $\Pi \geq 0$.

Case HW_1

We treat the case HW twice as HW_1 and HW_2 where we approach the circumscription/inscription problems through P3 and P4, respectively. Let $\mathscr{X}(\theta) = \{x : A(\theta)\, x \leq a(\theta)\}$ where (A,a) is an affine function of θ in Θ and let $\mathscr{Y} = \{Y\mu + V\pi : e\mu = 1,\ \mu \geq 0,\ \pi \geq 0\}$. The program P3 is the linear program

$$
\text{P9}\left\{\begin{array}{ll}
\underset{\theta}{\text{minimize:}} & c\theta \\
\text{subject to:} & A(\theta)Y \leq a(\theta)\circ e \\
 & A(\theta)V \leq 0 \qquad \theta \in \Theta
\end{array}\right.
$$

Specializing P9 to solve P1 we obtain the linear program

$$\text{P10}\begin{cases} z_1 = \underset{s,t}{\text{minimize:}} & s \\ \text{subject to:} & AY \leq s a \circ e + A(t \circ e) \\ & AV \leq 0 \qquad s \geq 0 \end{cases}$$

In solving P10 one verifies $AV \leq 0$ and then drops the constraints $AV \leq 0$.

Case HW_2

Let $\mathcal{Y}(\theta)$ be the set $\{Y(\theta)\mu + V(\theta)\pi : e\mu = 1, \mu \geq 0, \pi \geq 0\}$ where (Y,V) is an affine function of θ in Θ and $\mathcal{X}$ is the set $\{x : Ax \leq a\}$. Then the program P4 is the linear program

$$\text{P11}\begin{cases} \underset{\theta}{\text{maximize:}} & c\theta \\ \text{subject to:} & AY(\theta) \leq a \circ e \\ & AV(\theta) \leq 0 \qquad \theta \in \Theta \end{cases}$$

Specializing P11 to solve P2 we obtain the linear program

$$\text{P12}\begin{cases} z_2 = \underset{s,t}{\text{maximize:}} & s \\ \text{subject to:} & A(sY + t \circ e) \leq a \\ & sAV \leq 0 \qquad s \geq 0 \end{cases}$$

In solving P12 one verifies $AV \leq 0$ and then drops the constraints $sAV \leq 0$. Observe that P11 remains a linear program if a also is an affine function of θ in Θ .

4. Related Problems

The forgoing raises the question as to whether the following problems can be solved as linear programs.

a) Case WH

b) Finding the largest $\mathcal{Y}$ in $\mathcal{X}$ or smallest $\mathcal{X}$ containing $\mathcal{Y}$ where rotations as well as scales and translations are permitted.

c) Finding the largest n-sphere $\mathcal{Y}$ in $\mathcal{X}$ where $\mathcal{X}$ has representation W .

d) Finding the smallest n-sphere $\mathcal{X}$ containing $\mathcal{Y}$ where $\mathcal{Y}$ has representation H or W .

We suspect, but have no comprehensive proofs, that none of these problems can be formulated as a linear program. Observe in a) that for fixed (X,U,B,b) Case WH can be formulated as a linear program by converting to one of the other cases; in b) the set of optimal $\mathcal{Y}$ may not be a convex set; and in d) the n-sphere may have an irrational radius given rational data.

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Let $\mathcal{X}$ and $\mathcal{Y}$ be closed polyhedral convex sets, bounded or not, in R^n. For certain representations of $\mathcal{X}$ and $\mathcal{Y}$ it is shown that the task of finding the smallest scale of $\mathcal{X}$ for which some translate contains $\mathcal{Y}$ can be resolved with linear programming.