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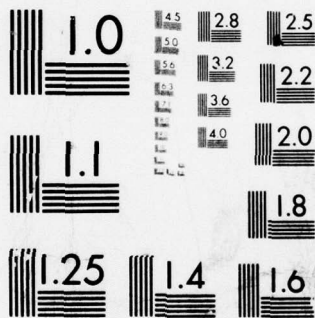
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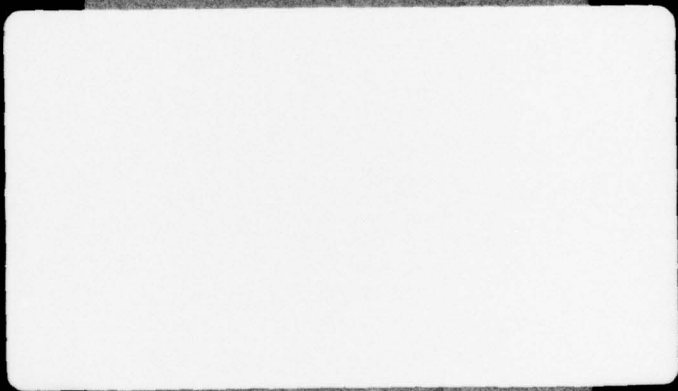
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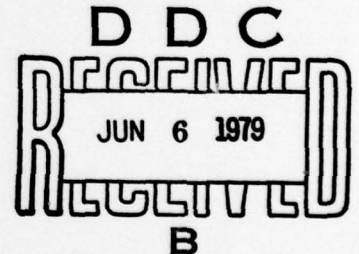
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ABSTRACT

Let U and V be independent random variables, and let $W = UV$. This paper concerns the distribution of U , given that V and W are distributed according to the gamma distributions. It is shown that U is distributed according to a beta distribution if the distributions of V and W are central gamma and that the distribution of U is degenerate at $u = 1$ if the distributions of V and W are non-central gamma. The given result is applied to determine the distribution of U when V and W are normally distributed.

Key words: Central and non-central gamma distributions.

AMS Classification: 62E10

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1. Main results. Let

$$g_m(x) = \frac{x^{m-1}}{\Gamma(m)} e^{-x}, \quad x > 0$$

denote the gamma density function with m degrees of freedom. The non-central gamma distribution $G_{m,\delta}$ with m degrees of freedom and non-centrality parameter equal to $\delta (> 0)$ is given by the density function

$$\begin{aligned} (1.1) \quad g_{m,\delta}(x) &= e^{-\delta} \sum_{r=0}^{\infty} g_{m+r}(x) \frac{\delta^r}{r!} \\ &= e^{-\delta-x} \left(\frac{x}{\delta}\right)^{\frac{m-1}{2}} I_{m-1}(2\sqrt{x\delta}) \end{aligned}$$

where $I_m(x)$ denotes the modified Bessel function with parameter m .

Let U and V be independent random variables and let

$$(1.2) \quad W = UV.$$

Suppose that the distribution of V is $G_{m,\delta}$ and the marginal distribution of W is $G_{m',\delta'}$. The main result of this paper concerns the distribution of U . Clearly U is positive with probability 1. Moreover

$$(1.3) \quad P\{0 < U \leq 1\} = 1.$$

Otherwise, let $P\{U \geq 1 + \xi\} = \alpha$, where ξ and α are positive numbers. Let $c = (1 + \xi)^{-1}$. Then

$$\begin{aligned} (1.4) \quad Ee^{cW} &= Ee^{cUV} \\ &\geq \alpha Ee^V \end{aligned}$$

where E denotes expectation. The left hand side of (1.4) is finite, whereas the right hand side is infinite. Therefore (1.3) is true.

It is easy to show that $m' \leq m$ and $\delta' \leq \delta$. Let $E(U) = E(W)/E(V) = (m' + \delta')/(m + \delta) = \gamma$, say. The Laplace transform of the gamma distribution is given by

$$\int_0^{\infty} e^{-\lambda x} dG_{m, \delta}(x) = (1 + \lambda)^{-m} \exp\left(-\frac{\lambda \delta}{1 + \lambda}\right), \quad \lambda > -1.$$

Therefore, (1.2) yields

$$(1.5) \quad (1 + \lambda)^{-m'} \exp\left(-\delta' + \frac{\delta'}{1 + \lambda}\right) = E(1 + \lambda U)^{-m} \exp\left(-\delta + \frac{\delta}{1 + \lambda U}\right) \\ \geq (1 + \lambda \gamma)^{-m} \exp\left(-\delta + \frac{\delta}{1 + \lambda \gamma}\right)$$

by Jensen's inequality. Comparing the two sides of (1.5) for large values of λ we find that $m' \leq m$.

Let

$$\phi(a, b; x) = 1 + \frac{a}{b}x + \frac{a(a+1)}{b(b+1)} \frac{x^2}{2!} + \dots$$

denote the confluent hypergeometric function. We have

$$EW^r = \frac{\Gamma(m' + r)}{\Gamma(m')} e^{-\delta'} \phi(m' + r, m'; \delta'), \quad r > -m'.$$

Similarly

$$EV^r = \frac{\Gamma(m + r)}{\Gamma(m)} e^{-\delta} \phi(m + r, m; \delta), \quad r > -m.$$

Therefore, (1.2) yields

$$\begin{aligned}
 (1.6) \quad EU^r &= \frac{\Gamma(m' + r)}{\Gamma(m')} e^{-\delta'} \phi(m'+r, m'; \delta') / \frac{\Gamma(m + r)}{\Gamma(m)} e^{-\delta} \phi(m+r, m; \delta) \\
 &= (\delta')^{\frac{3}{4} - \frac{m'}{2}} (\delta)^{\frac{m}{2} - \frac{3}{4}} r^{(m'-m)/2} \exp\left(\frac{\delta - \delta'}{2} - 2\sqrt{r}\right) \\
 &\quad (\sqrt{\delta} - \sqrt{\delta'}) (1 + O(r^{-1}))
 \end{aligned}$$

for large values of r . The asymptotic expression given above is derived from Formula 6.13.2 (12) of Erdelyi (1953). Since $EU^r \leq 1$ for $r \geq 0$, it follows from (1.6) that $\delta' \leq \delta$.

Let $H(u)$ denote the distribution function of U , and let $f_m^*(v) = (1+v)^{-m}$ and

$$f_m(v) = \int_0^1 (v+u)^{-m} dH(u), \quad v > 0.$$

Putting $v = \frac{1}{\lambda}$ in (1.5) we get

$$\begin{aligned}
 (1.7) \quad \frac{v^{m'-m}}{(1+v)^{m'}} \exp\left(-\frac{\delta'}{1+v}\right) &= e^{-\delta} \int_0^1 (v+u)^{-m} \exp\left(\frac{\delta v}{v+u}\right) dH(u) \\
 &= e^{-\delta} \sum_{r=0}^{\infty} \frac{(-\delta v)^r \Gamma(m)}{r! \Gamma(m+r)} f_m^{(r)}(v) \\
 &= \Gamma(m) e^{-\delta} H_{m-1}(2\sqrt{v\delta D}) f_m(v)
 \end{aligned}$$

where $D = \frac{d}{dv}$ denotes the derivative operator with respect to v , and

$$H_m(x) = \sum_{r=0}^{\infty} \frac{(-x^2/4)^r}{\Gamma(m+r+1)r!}.$$

Note that $J_m(x) = \left(\frac{x}{2}\right)^m H_m(x)$ represents a Bessel function.

Therefore

$$(1.8) \quad \left(x \frac{d^2}{dx^2} + (2m - 1) \frac{d}{dx} + x\right) H_{m-1}(x) = 0.$$

Writing the left hand side of (1.7) in the same form as the right hand side we get

$$(1.9) \quad \Gamma(m') v^{m'-m} e^{-\delta'} H_{m'-1}(2\sqrt{\theta\delta'D}) f_{m'}^*(v) = \\ \Gamma(m) e^{-\delta} H_{m-1}(2\sqrt{\theta\delta D}) f_m(v)$$

where $\theta = v$.

First let $m' = m$. A transformation of (1.8) gives

$$(1.10) \quad \left(4x \frac{d^2}{dx^2} + 4m \frac{d}{dx} + c^2\right) H_{m-1}(c\sqrt{x}) = 0$$

where c is a constant. An application of the differential equation (1.10) to both sides of (1.9) with $c = 2\sqrt{\theta\delta}$ gives

$$4\theta(\delta - \delta') e^{-\delta'} H_{m-1}(2\sqrt{\theta\delta'D}) f_m^*(v) = 0$$

or

$$(\delta - \delta') f_m^*(v) \exp\left(-\frac{\delta'}{1+v}\right) = 0.$$

Therefore, $\delta = \delta'$. Hence, $P\{U = 1\} = 1$.

Next, let $m > m'$. If $\delta = 0$ then $\delta' = 0$, since $\delta' \leq \delta$, as shown above. Then (1.6) reduces to

$$EU^r = \frac{\Gamma(m' + r)}{\Gamma(m')} / \frac{\Gamma(m + r)}{\Gamma(m)} \\ = \int_0^1 u^r dH^*(u)$$

where H^* denotes the beta distribution $\beta(u; m', m-m')$. Hence, $H(u) = H^*(u)$. Suppose that $\delta > 0$. Writing

$$e^{-\delta'} \sum_{r=0}^{\infty} \binom{m-m'+r-1}{r} \Gamma(m+r) H_{m+r-1}(2\sqrt{\theta\delta'D}) f_{m+r}^*(v)$$

for the left hand side of (1.7) and applying the differential equation (1.10) to both sides with $c = 2\sqrt{\theta\delta}$ we get

$$e^{-\delta'} \sum_{r=0}^{\infty} \binom{m-m'+r-1}{r} \Gamma(m+r) (4\theta(\delta-\delta') - 4r \frac{d}{dD}) H_{m+r-1}(2\sqrt{\theta\delta'D}) f_{m+r}^*(v) = 0$$

or

$$4\theta(\delta-\delta') \frac{v^{m'-m}}{(1+v)^{m'}} \exp\left(-\frac{\delta'}{1+v}\right) + 4\theta\delta' \sum_{r=0}^{\infty} r \binom{m-m'+r-1}{r} \sum_{s=0}^{\infty} \frac{(\theta\delta')^s}{s!} \frac{(1+v)^{m-r-s}}{m+r+s} = 0.$$

The above equation implies that $\delta = \delta' = 0$, contrary to the assumption that $\delta > 0$.

The foregoing results are summarized in the following theorem.

Theorem 1. Let $W = UV$ where U and V are independent random variables, and let V and W be distributed according to the gamma distributions $G_{m,\delta}$ and $G_{m',\delta'}$, respectively. Then $m' \leq m$, $\delta' \leq \delta$ and $P\{0 < U \leq 1\} = 1$. If $m' = m$ then $\delta' = \delta$ and $P\{U = 1\} = 1$. If $m' < m$ then $\delta' = \delta = 0$ and U is distributed according to the beta distribution $\beta(u; m', m-m')$.

Since the square of a normal random variable is distributed according to the gamma distribution with a scale factor we obtain the following corollary from the above theorem.

Corollary 1. Let $W = UV$ where U and V are independent random variables, and let V and W be normally distributed with unit variance and means equal to μ and μ' , respectively. Then $\mu^2 = \mu'^2$ and $P\{U^2 = 1\} = 1$. If $\mu = -(+)\mu' \neq 0$ then $P\{U = -(+)1\} = 1$.

An extension of Corollary 1 is given as follows: Let $\alpha_1, \dots, \alpha_p, Z_1, \dots, Z_p$ be independent random variables and let

$$Z = \sum_{i=1}^p \alpha_i Z_i. \quad \text{Let } EZ_i = \xi_i.$$

Corollary 2. If the random variables Z, Z_1, \dots, Z_p are normally distributed then $P\{\alpha_i = c_i\} = 1$ when $\xi_i \neq 0$ and $P\{\alpha_i^2 = c_i^2\} = 1$ when $\xi_i = 0$ for each $i = 1, \dots, p$ where c_1, \dots, c_p are certain constants.

Proof. Since Z is normally distributed it follows from the reproductive property of the normal distribution (see e.g., Lukacs and Laha (1964) Lemma 5.1.1) that $\alpha_i Z_i$ is normally distributed for each $i = 1, \dots, p$. The conclusion of the corollary follows from Corollary 1.

Let $\alpha_1, \dots, \alpha_p$ be m -component random vectors and let A denote the matrix whose i th column vector is α_i , $i = 1, \dots, p$. Let Z_1, \dots, Z_p be p independent normal random variables. Let

$EZ_i = \xi_i$, $Z = (Z_1, \dots, Z_p)'$ and $Y = AZ$. The random vector Y is distributed according to a multivariate normal distribution if and only if $\lambda' Y = \sum_{i=1}^p (\lambda' \alpha_i) Z_i$ is normally distributed for every non-null vector λ . The following result follows from Corollary 2.

Corollary 3. Let $\alpha_1, \dots, \alpha_p$, Z be independent. If the distribution of Y is multivariate then $P\{\lambda' \alpha_i = c_i\} = 1$ when $\xi_i \neq 0$ and $P\{(\lambda' \alpha_i)^2 = c_i^2\}$ when $\xi_i = 0$, $i = 1, \dots, p$ for each non-null vector λ , where c_1, \dots, c_p are certain constants depending on λ .

Corollary 3 is related to the following result due to Kingman and Graybill (1970). Let Y_1, \dots, Y_p be independent and identically distributed random variables and let $A = (a_{ij})$ be a $p \times p$ random matrix which is orthogonal with probability 1 and $E(\sum_{j=1}^p a_{ij}) \neq 0$ for some i . Let $Y = (Y_1, \dots, Y_p)'$ and $Z = AY$.

Then the components of Z are independently and identically distributed according to the standard normal distribution if and only if the components of Y have the same distribution.

Theorem 2 below gives a characterization of the gamma distribution. Let

$$F(a, b; c; x) = \sum_{r=0}^{\infty} \frac{\Gamma(a+r) \Gamma(b+r) \Gamma(c)}{\Gamma(a) \Gamma(b) \Gamma(c+r)} \frac{x^r}{r!}$$

denote the hypergeometric function, and let

$$(1.11) \quad \phi(\lambda) = F(a, b; c; -\lambda)$$

$$= \frac{\Gamma(c)}{\Gamma(b) \Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1+t\lambda)^{-a} dt$$

$$a, b, c > 0, c > b.$$

It is seen that $\phi(0) = 1$ and that $\phi(\lambda)$ is a completely monotone function, that is, $(-1)^r \phi^{(r)}(\lambda) \geq 0$, $\lambda > 0$. Therefore, $\phi(\lambda)$ represents the Laplace transform of a probability distribution on $[0, \infty)$. A distribution on $[0, \infty)$, whose Laplace transform is given by (1.11) will be called inverse-hypergeometric. If $b = c$ then $\phi(\lambda) = (1 + \lambda)^{-a}$ is the Laplace transform of the gamma distribution with a degrees of freedom.

Theorem 2. Let $W = UV$ where U and V are independent random variables, and let U be distributed according to a beta distribution $\beta(u; p, q)$, say. Then W is distributed according to a gamma distribution with m degrees of freedom if and only if $m \leq p$ and the distribution V is inverse-hypergeometric, given by the Laplace transform $\phi(\lambda) = F(p+2; m; p; -\lambda)$.

Proof. Suppose that the Laplace transform of the distribution of V is given by $\phi(\lambda) = F(p+q; m; p; -\lambda)$. The r th moment of V is given by

$$(1.12) \quad EV^r = \frac{\Gamma(p+q+r) \Gamma(m+r) \Gamma(p)}{\Gamma(p+q) \Gamma(m) \Gamma(p+r)} .$$

Therefore,

$$(1.13) \quad \begin{aligned} EW^r &= EU^r EV^r \\ &= \frac{\Gamma(m+r)}{\Gamma(m)} . \end{aligned}$$

The right hand side of (1.13) represents the r th moment of the gamma distribution with m degrees of freedom. Therefore, the distribution of W is gamma.

Next, suppose that the distribution of W is gamma with m degrees of freedom. Considering the Laplace transform of the distribution of W , we have

$$\begin{aligned}
 (1.14) \quad (1 + \lambda)^{-m} &= Ee^{-\lambda W} \\
 &= Ee^{-\lambda UV} \\
 &= E \frac{\Gamma(p+q)}{\Gamma(p) \Gamma(q)} \int_0^1 u^{p-1} (1-u)^{q-1} e^{-\lambda uv} du \\
 &= E \phi(p, p+q; -\lambda V) \\
 &\geq \phi(p, p+q; -\lambda EV) \\
 &= \frac{\Gamma(p+q)}{\Gamma(q)} \lambda^{-p} (1 + o(\lambda^{-1})) \text{ as } \lambda \rightarrow \infty.
 \end{aligned}$$

It follows from (1.14) that $m \leq p$.

The r th moment of V is given by

$$\begin{aligned}
 EV^r &= EW^r / EU^r \\
 &= \frac{\Gamma(m+r)}{\Gamma(m)} \cdot \frac{\Gamma(p+q+r) \Gamma(p)}{\Gamma(p+q) \Gamma(p+r)}
 \end{aligned}$$

From (1.12) it follows that the Laplace transform of the distribution of V is given by $\phi(\lambda) = F(p+q; m; p; -\lambda)$. Hence the distribution of V is inverse-hypergeometric.

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