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A BIVARIATE FAILURE MODEL, (U)

JUL 76 T J TOSCH, P T HOLMES

N00014-71-A-0339-0002

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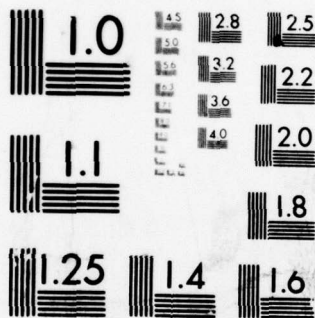
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(12) LEVEL II

6 A BIVARIATE FAILURE MODEL

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12 13 p.

DEPARTMENT OF MATHEMATICAL SCIENCES

CLEMSON UNIVERSITY

✓ 11 26 JUL 1976
TR 226

REPORT N 74, TR-226
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RESEARCH SUPPORTED BY THE OFFICE OF NAVAL RESEARCH
TASK NR 042-271 CONTRACT ~~N00014-71-A-0339-0002~~ ✓

15 N00014-75-C-0451

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ABSTRACT

A generalization of Freund's bivariate exponential model is discussed. Probabilistic properties of this model with minimal distribution assumptions are derived including the joint survival function and Laplace-Stieltjes transform.

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1. Introduction. A study is made of a two component failure system. Let the components be denoted by A and B and their lifetimes by S and T respectively. The general model to be discussed in this paper is a generalization of that of Freund [2] and arose from consideration of bivariate exponential distributions. In addition to Freund's there have been many such distributions suggested in the literature, including those suggested by Gumbel [3], Downton [1] and Hawkes [4]. Marshall and Olkin [5] have proposed the most widely referenced model. In their model, the $\Pr \{S > s + \Delta | S > s, T > s\} = \Pr \{S > s + \Delta | S > s\}$. This implies that, conditioned on the fact that component A is functioning at time s, the distribution of its residual lifetime is independent of whether component B has failed or not.

Freund [2] derives his distribution from the assumption that at the failure of one component the distribution of the residual lifetime of the other component is changed. Specifically, he assumes that the lifetime of A given that no failure of B occurs is distributed exponentially with parameter α , denoted $\exp(\alpha)$, and that the lifetime of B given no failure of A is distributed as $\exp(\beta)$. If B fails before A, then the residual lifetime of A is distributed as $\exp(\alpha')$. If A fails before B, then the residual lifetime of B is distributed as $\exp(\beta')$. This can be summarized as follows. Let X, Y, U, V be independent random variables such that $X \sim \exp(\alpha)$, $Y \sim \exp(\beta)$, $U \sim \exp(\alpha')$ and $V \sim \exp(\beta')$. Then

$$S = \begin{cases} X & \text{if } X \leq Y \\ Y+U & \text{if } X > Y \end{cases}, \quad T = \begin{cases} X+V & \text{if } X \leq Y \\ Y & \text{if } X > Y \end{cases} .$$

The model introduced in section 2 is a generalization of Freund's model with the distribution assumptions dropped. Section 3 includes derivations of expressions for the joint survival function and marginal distributions for this model. Letting $Y \sim \exp(\beta)$, an explicit expression for the joint Laplace-Stieltjes transform of the distribution is found in section 4. Using this expression, moments of the distribution are calculated. In section 5, two definitions of system life are defined and the Laplace-Stieltjes transform of the system life distribution is found in each case.

2. Model Definition. Label the two components of a system A and B. The lifetimes of the two components are dependent, in that the failure of one component affects the residual lifetime of the other. Formally, let A and B have lifetimes S and T respectively. Given random variables X, Y, U, V write

$$S = \begin{cases} X & \text{if } X \leq Y \\ Y+U & \text{if } X > Y \end{cases}, \quad T = \begin{cases} X+V & \text{if } X \leq Y \\ Y & \text{if } X > Y \end{cases} \quad (2.1)$$

to represent the two lifetimes. The following assumptions are made on X, Y, U, V:

- i) X, Y, U, V are mutually independent,
- ii) $\Pr\{X > 0\} = 1, \Pr\{Y > 0\} = 1,$
- iii) $\Pr\{U < 0\} = 0, \Pr\{V < 0\} = 0,$
- iv) X, Y are absolutely continuous.

(2.2)

Let $\Pr\{U = 0\}$ be denoted by p_U and $\Pr\{V = 0\}$ by p_V . The following is an example of the applicability of the model.

Let A and B be two generators that supply electricity to a hospital, each supplying a portion of the building. When one generator fails, the remaining generator must supply power to the entire hospital. This puts added strain on the generator and thus alters its residual life.

3. The Joint Survival and Marginal Distributions. Let $\bar{F}(s, t) = \Pr\{S > s, T > t\}$. We begin by deriving an expression for $\bar{F}(s, t)$ when $s < t$. First condition on the values of X and Y . Then $\bar{F}(s, t) = \int_0^\infty \int_0^\infty \Pr\{S > s, T > t \mid X=x, Y=y\} dF_X(x) dF_Y(y)$. Now partition the region $[0, \infty) \times [0, \infty)$ into 12 subregions based on the relative sizes of x, y, s, t . The only non-zero contributions are $\Pr\{V > t-x\}$ on the region $[s, t] \times [x, \infty)$ and 1 on the region $[t, \infty) \times [t, \infty)$. Therefore when $s < t$,

$$\begin{aligned} \bar{F}(s, t) &= \int_s^t \int_x^\infty \bar{F}_V(t-x) dF_Y(y) dF_X(x) + \int_t^\infty \int_t^\infty 1 dF_X(x) dF_Y(y) \\ &= \bar{F}_X(t) \bar{F}_Y(t) + \int_s^t \bar{F}_V(t-x) \bar{F}_Y(x) dF_X(x). \end{aligned}$$

The case $s > t$ follows similarly and the case $s = t$ is trivial.

We have therefore proved

$$\text{Theorem 1. } F(s,t) = \begin{cases} F_X(t) F_Y(t) + \int_s^t F_V(t-x) F_Y(x) dF_X(x) & \text{if } s < t \\ F_X(s) F_Y(s) & \text{if } s = t \\ F_X(s) F_Y(s) + \int_t^s F_U(s-y) F_X(y) dF_Y(y) & \text{if } s > t . \end{cases}$$

We can obtain the marginal distributions using $F_S(s) = F(s,0)$.

Corollary 2.

$$\begin{aligned} \text{i) } F_S(s) &= F_X(s) F_Y(s) + \int_0^s F_U(s-y) F_X(y) dF_Y(y), \\ \text{ii) } F_T(t) &= F_X(t) F_Y(t) + \int_0^t F_V(t-x) F_Y(x) dF_X(x) . \end{aligned}$$

From the model definition we have the following representation for the variable S ,

$$S = \min(X,Y) + U \cdot I_{\{X > Y\}} , \quad (3.1)$$

where $I_{\{X > Y\}}$ is the indicator function for the set $\{X > Y\}$. A similar representation exists for T . With this representation we see that in general

$$\begin{aligned} E(S) &= E(\min(X,Y)) + E(U) \Pr\{X > Y\}, \\ E(T) &= E(\min(X,Y)) + E(V) \Pr\{Y > X\}. \end{aligned} \quad (3.2)$$

4. The Joint Laplace-Stieltjes Transform. The joint Laplace-Stieltjes (L-S) transform of (S,T) is defined to be

$\int_0^\infty \int_0^\infty e^{-as-bt} F(ds,dt)$, where $F(ds,dt)$ is the measure determined by the survival function $\bar{F}(s,t)$. We begin with a representation of $F(ds,dt)$.

$$\text{Lemma. } F(ds,dt) = \begin{cases} \bar{F}_Y(s) dF_V(t-s) dF_X(s) & \text{if } s < t \\ \bar{F}_Y(s) p_V dF_X(s) + \bar{F}_X(s) p_U dF_Y(s) & \text{if } s = t \\ \bar{F}_X(t) dF_U(s-t) dF_Y(t) & \text{if } s > t \end{cases}$$

Proof for the case $s < t$: We must investigate how A can fail at time s and B fail at time t . If $s < t$, then $S < T$, and thus $X < Y$, so that $S = X = s$ and $T = X + V = s + V = t$. Therefore $X = s$, $Y > s$ and $V = t - s$. Since the variables are independent the result follows. QED

Using this lemma, the joint L - S transform of (S,T) can be written as

$$\begin{aligned} f^*(a,b) = & \int_0^\infty \int_{s+}^\infty e^{-as-bt} \bar{F}_Y(s) dF_V(t-s) dF_X(s) + \\ & p_V \int_0^\infty e^{-(a+b)s} \bar{F}_Y(s) dF_X(s) + \\ & p_U \int_0^\infty e^{-(a+b)s} \bar{F}_X(s) dF_Y(s) + \\ & \int_0^\infty \int_{t+}^\infty e^{-as-at} \bar{F}_X(t) dF_V(s-t) dF_Y(t) \end{aligned} \quad (4.1)$$

This expression can be evaluated piece by piece. We have

$$\begin{aligned}
& \int_0^{\infty} \int_{s+}^{\infty} e^{-as-bt} \bar{F}_Y(s) dF_V(t-s) dF_X(s) \\
&= \int_0^{\infty} e^{-as} \bar{F}_Y(s) \int_{s+}^{\infty} e^{-bt} dF_V(t-s) dF_X(s) \\
&= \int_0^{\infty} e^{-as} \bar{F}_X(s) e^{-bs} \int_{0+}^{\infty} e^{-bw} dF_V(w) dF_X(s) \\
&= (f_V^*(b) - p_V) \int_0^{\infty} e^{-(a+b)s} \bar{F}_Y(s) dF_X(s) .
\end{aligned}$$

Here $f_V^*(b)$ is the L - S transform of F_V evaluated at b .

By a similar calculation

$$\begin{aligned}
& \int_0^{\infty} \int_{t+}^{\infty} e^{-as-bt} \bar{F}_X(t) dF_V(s-t) dF_Y(t) \\
&= (f_U^*(a) - p_U) \int_0^{\infty} e^{-(a+b)s} \bar{F}_X(s) dF_Y(s) .
\end{aligned}$$

Combining these into (4.1) we get

Theorem 4.

$$f^*(a,b) = f_V^*(b) \int_0^{\infty} e^{-(a+b)s} \bar{F}_Y(s) dF_X(s) + f_U^*(a) \int_0^{\infty} e^{-(a+b)s} \bar{F}_X(s) dF_Y(s) .$$

These integrals cannot be evaluated in general, but can be for certain important special cases. In particular we have

Corollary 5.

If $Y \sim \exp(\beta)$ then

$$f^*(a,b) = f_V^*(b) f_X^*(a+b+\beta) + \frac{f_U^*(a)\beta}{a+b+\beta} [1 - f_X^*(a+b+\beta)] .$$

Corollary 6.

If $X \sim \exp(\alpha)$ and $Y \sim \exp(\beta)$, then

$$f^*(a,b) = \frac{1}{a+b+\alpha+\beta} [\alpha f_V^*(b) + \beta f_U^*(a)].$$

Using the expression in Corollary 6, moments of (S,T) can be found by evaluating the appropriate partial derivatives.

Theorem 7.

If $X \sim \exp(\alpha)$ and $Y \sim \exp(\beta)$, then

i) $E(S) = \frac{1}{\alpha+\beta} [1+\beta E(U)]$, $E(T) = \frac{1}{\alpha+\beta} [1+\alpha E(V)]$,

ii) $Var(S) = \frac{1}{(\alpha+\beta)^2} [1+\beta^2 Var(U) + \alpha\beta E(U^2)]$,

$$Var(T) = \frac{1}{(\alpha+\beta)^2} [1+\alpha^2 Var(V) + \alpha\beta E(V^2)] ,$$

iii) $Cov(S,T) = \frac{1}{(\alpha+\beta)^2} [1-\alpha\beta E(U) E(V)]$.

It can be shown that all values between +1 and -1 are possible for the correlation of S and T even in the special case considered in Theorem 7.

5. Time to System Failure. Failure in a series system and a parallel system will be discussed. Label the system life L_1 for the series system and L_2 for the parallel system. We have $L_1 = \min(S,T)$ and $L_2 = \max(S,T)$.

Since $\min(S,T) = \min(X,Y)$, $\bar{F}_{L_1}(t) = \bar{F}_X(t) \bar{F}_Y(t)$ and the model does not really enter in. If X and Y are exponentially distributed then so is L_1 .

To work with L_2 , the random variable $D = L_2 - L_1$ will be considered. If we let $p = \Pr \{X > Y\}$ and $q = 1 - p$, then $D \sim pV + qU$. Since V and U are independent $f_D^*(a) = pq f_V^*(ap) f_U^*(aq)$. It follows from $L_2 = L_1 + D$ and the independence of L_1 and D that

$$f_{L_2}^*(a) = pq f_V^*(ap) f_U^*(aq) f_{L_1}^*(a).$$

6. Discussion. The question of practicality arises here as it should with any model formulation. Is the class of models sufficiently rich yet simple enough to obtain meaningful results? The proposed model is based upon the assumption that the residual life of one component is dependent upon whether or not the other component has failed. This seems to be a realistic assumption in many applications.

The first meaningful results one would like are the moments of (S, T) . For given distributions the Laplace-Stieltjes transform (Theorem 4) may be used to obtain the moments. This expression may be difficult to evaluate though. However, the much simpler expression in Corollaries 5 and 6 still come from rich classes of models. Next, given a sample can one obtain reasonable estimators for the parameters? In Freund's special case, simple expressions for the maximum likelihood estimates can be obtained.

Finally, does this model enable one to make decisions about how to control the system. When the components fail, can we arrive at optimal replacement policies? Here again, this is possible when using this model.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER N74	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A BIVARIATE FAILURE MODEL		5. TYPE OF REPORT & PERIOD COVERED Technical Report
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Thomas J. Tosch Paul T. Holmes		8. CONTRACT OR GRANT NUMBER(s) N00014-75-C-0451
9. PERFORMING ORGANIZATION NAME AND ADDRESS Clemson University Dept. of Mathematical Sciences Clemson, South Carolina 29631		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR 042-271
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Code 436 Arlington, Va. 22217		12. REPORT DATE 26 July 1976
		13. NUMBER OF PAGES 9
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Bivariate failure distribution, exponential distribution		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) A generalization of Freund's bivariate exponential model is discussed. Probabilistic properties of this model with minimal distribution assumptions are derived including the joint survival function and Laplace-Stieltjes transform.		

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