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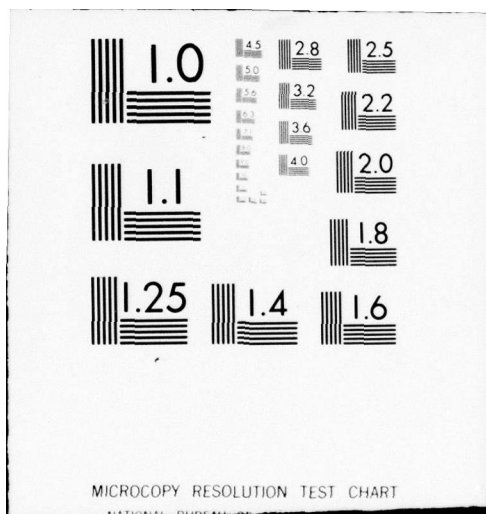
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POTENTIAL THEORY OF STEADY MOTION OF SHIPS,  
 PART 4: LOW-FROUDE-NUMBER APPROXIMATIONS<sup>†</sup>

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<sup>†</sup>This report is a revised and extended version of a previous study entitled "On the Theory of Steady Motion of Low-Froude-Number Displacement Ships" that was submitted and accepted for publication in the Journal of Ship Research in October 1977 and March 1978, respectively. Publication of this study however was interrupted due to its reliance upon a preliminary study entitled "Potential Theory of Steady Motion of Ships" that is still in the process of being reviewed for publication. An updated version of the latter study has been presented in two previous reports [1,2], which — together with the present report — provide a coherent and detailed exposition of the "generalized Neumann-Kelvin theory" of ship wave resistance.

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## ABSTRACT

The main object of the present Part 4 of the "generalized Neumann-Kelvin theory" of ship wave resistance is to present "low-Froude-number wave-resistance approximations" as well as "low-Froude-number slender-ship approximations", which supplement the basic "slender-ship approximations" obtained in Parts 2 and 3. First- and second-order approximations are given explicitly. In particular, the low-Froude-number wave-resistance approximations of Guevel et. al., Kayo, Baba, and Maruo are obtained in this study as first-order approximations in a sequence of low-Froude-number iterative approximations for solving the basic "generalized Neumann-Kelvin integral equation" established in Part 2 of the theory. New "first- and second-order low-Froude-number slender-ship wave-resistance approximations" are derived. In addition, a new integral equation for evaluating potential flow about a ship hull in the zero-Froude-number limit" is obtained. A method of solution of this integral equation, based on an iterative procedure leading to a sequence of "slender-ship approximations", is discussed in some detail.

## INTRODUCTION

In this study, the exposition of the "generalized Neumann-Kelvin theory" of ship wave resistance developed in [1] and [2] is pursued. Specifically, the object of the present Part 4 of the theory is to derive "low-Froude-number approximations", as well as "slender-ship low-Froude-number approximations", which supplement (and in a sense generalize) the basic "slender-ship iterative approximations given in [1] and [2]. The plan and main results of the study are briefly described below.

In the first section, the basic differential equations of the problem are listed, and the fundamental integral equation underlying the "generalized Neumann-Kelvin theory" developed in [1] and [2] is recalled. Section two is concerned with the analysis, and more particularly with the numerical evaluation, of the velocity potential,  $\phi$ , of the disturbance flow caused by the ship in the "zero-Froude-number limit". The well-known result that the "zero-Froude-number limit",  $\phi^0$ , of the disturbance potential  $\phi$  is identical to the "rigid-wall potential" — in which the free surface is replaced by a horizontal rigid wall coinciding with the undisturbed free surface — is first established (in two different ways, by starting from the differential and integral formulation of the problem). The main object of section two, however, is to present a new integral equation, namely equation (17), for the numerical evaluation of the "zero-Froude-number potential"  $\phi^0$ . Two alternative equivalent forms, given by equations (19) and (25), of a recurrence relation for solving the "zero-Froude-number integral equation" (17) iteratively are given. These equivalent recurrence formulas generate a sequence of "slender-ship zero-Froude-number approximations"  $\phi_k^0$ ,  $k \geq 0$ , where the "zeroth approximation"  $\phi_0^0$  is taken as zero, i.e.  $\phi_0^0 \equiv 0$ , which becomes exact in the limit of an extremely slender ship-hull form, and the "first approximation"  $\phi_1^0$  is the potential  $\phi_I^0$  defined by formula (18) and referred to as the "initial approximation". This "initial approximation" is examined in some detail. In particular, it is shown that the approximation  $\phi_I^0$  can be regarded as a generalization of the classical "thin-ship approximation", to which it reduces in the limit of a thin ship. It is also shown that in the case of a ship hull in the form of a half ellipsoid (say with beam/length and draft/length ratios equal to  $\beta$  and  $\delta$ , respectively) the initial potential  $\phi_I^0$  actually is proportional to the exact zero-Froude-number potential  $\phi^0$ , that is we have  $\phi^0 \equiv k(\beta, \delta)\phi_I^0$  where the multiplicative correction function  $k(\beta, \delta)$  is defined by formula (28a) and depicted in figure 2. For  $\beta = .2$  and  $\delta = .1$ , which can be regarded as typical values of the beam/length and draft/length ratios, we have  $k(\beta, \delta) \approx 1.06$ , so that the initial potential  $\phi_I^0$  given by formula (18) in fact provides a fairly accurate approximation to the zero-Froude-number potential of the disturbance

flow due to an ellipsoidal hull form of typical ship-like dimensions. To be sure, real ship hulls are not ellipsoidal in shape. Nonetheless, the initial potential  $\phi_I^0$  [the modified potential  $k(\beta, \delta)\phi_I^0$  may also be used] seems likely to provide a realistic approximation to the zero-Froude-number disturbance potential  $\phi^0$ .

In section three, "first-order low-Froude-number wave-resistance formulas" are obtained by approximating the near-field disturbance potential  $\phi$  in expression (31) for the Kochin free-wave spectrum function  $\Omega(t)$  by the "zero-Froude-number potential"  $\phi^0$ . The basic expression for the "first-order low-Froude-number approximation",  $\Omega_1^{\mathcal{L}F}(t)$ , to the Kochin spectrum function  $\Omega(t)$  is given by formula (37). An alternative expression for the first approximation  $\Omega_1^{\mathcal{L}F}$  is given by formula (40), which has been obtained from formula (37) by making use of a classical Green identity. This alternative expression for the first approximation  $\Omega_1^{\mathcal{L}F}$  then readily yields the low-Froude-number approximations obtained previously by Guevel, Vaussy, and Kobus [3], Kayo [4], Baba [5], and Maruo [6], which thus are obtained in this study as first-order approximations in a sequence of low-Froude-number iterative approximations (the second-order approximation in this sequence is presented in section five) for solving the basic "generalized Neumann-Kelvin integral equation" (6). If the slender-ship approximation  $k(\beta, \delta)\phi_I^0$  to the zero-Froude-number potential  $\phi^0$  is used, instead of  $\phi^0$ , as an approximation to the near-field disturbance potential  $\phi$  in expression (31) for the Kochin spectrum function  $\Omega(t)$ , the "first-order low-Froude-number slender-ship approximation"  $\Omega_1^{\mathcal{L}Fs\ell}$  may be obtained. This approximation is expressed by formula (45) or equivalently by formulas (46) and (47). The simplest (and thus perhaps the most attractive for practical purposes) of the "first-order low-Froude-number approximations" obtained in section three is the "linearized (that is, the nonlinear free-surface correction terms are neglected) first-order low-Froude-number slender-ship approximation",  $\Omega_{1\ell}^{\mathcal{L}Fs\ell}$ , given by formula (48). In fact, comparison of this formula with formula (32) for the basic "zeroth-order slender-ship approximation"  $\Omega_0$  shows that formula (48) for the approximation  $\Omega_{1\ell}^{\mathcal{L}Fs\ell}$  is not significantly more complex than formula (32) for  $\Omega_0$  from the computational point of view.

In section four, various (closely-related) "first-order low-Froude-number approximations"  $\phi_1^{\mathcal{L}F}$  and corresponding "first low-Froude-number slender-ship approximations"  $\phi_1^{\mathcal{L}Fs\ell}$  to the disturbance velocity potential  $\phi$  are obtained by approximating the unknown potential  $\phi$  in the "Neumann-Kelvin integral equation" (6) by the "zero-Froude-number potential"  $\phi^0$  and the corresponding slender-ship approximation  $k(\beta, \delta)\phi_I^0$ , respectively. This section thus parallels and supplements section three; specifically, the "first-order low-Froude-number approximations"  $\phi_1^{\mathcal{L}F}$  and  $\phi_1^{\mathcal{L}Fs\ell}$  to the disturbance velocity potential  $\phi$  correspond to the "first-order low-Froude-number approximations"  $\Omega_1^{\mathcal{L}F}$  and

$\Omega_1^{\ell F s \ell}$  to the Kochin free-wave spectrum function  $\Omega$  obtained in section three, and indeed the approximations  $\Omega_1^{\ell F}$  and  $\Omega_1^{\ell F s \ell}$  could alternatively have been derived from the analysis of the "far-field limit" of the approximations  $\phi_1^{\ell F}$  and  $\phi_1^{\ell F s \ell}$  to the disturbance potential.

In section five, "second-order low-Froude-number approximations"  $\phi_2^{\ell F}$  and  $\Omega_2^{\ell F}$  and corresponding "low-Froude-number slender-ship approximations"  $\phi_2^{\ell F s \ell}$  and  $\Omega_2^{\ell F s \ell}$  to the disturbance velocity potential  $\phi$  and the Kochin free-wave spectrum function  $\Omega$  are obtained by using the above-defined "first-order low-Froude-number approximations"  $\phi_1^{\ell F}$  and  $\phi_1^{\ell F s \ell}$  as approximations to the potential  $\phi$  in the "Neumann-Kelvin integral equation" (6). Of greatest interest for practical purposes may be the "second-order low-Froude-number slender-ship wave spectrum approximation"  $\Omega_2^{\ell F s \ell}$  defined by formula (64), which is based upon the "linearized first-order low-Froude-number slender-ship potential"  $\phi_{1\ell}^{\ell F s \ell}$  given by formula (58). While the second approximation  $\Omega_2^{\ell F s \ell}$  evidently is considerably more complex than the first approximation  $\Omega_{1\ell}^{\ell F s \ell}$  given by formula (48), it nonetheless appears to be well within present-day calculation capabilities. Indeed, the major computational task is that associated with the evaluation of the potential  $\phi_{1\ell}^{\ell F s \ell}$  defined by formula (58), which is not significantly more complex than the "initial potential"  $\phi_I$  given by formula (7) or than the classical Michell potential. Naturally, a significant simplification of formula (64) for the approximation  $\Omega_2^{\ell F s \ell}$  can be achieved if the hull-flux correction term  $q_{h1\ell}^{\ell F s \ell}$ , associated with sinkage and trim effects, and the nonlinear free-surface correction term  $q_{f1\ell}^{\ell F s \ell}$  are neglected.

In section six, it is briefly indicated how the low-Froude-number approximations  $\phi_1^{\ell F}$ ,  $\phi_1^{\ell F s \ell}$  and  $\phi_2^{\ell F}$ ,  $\phi_2^{\ell F s \ell}$  — which have been obtained in the previous sections as first and second approximations in a sequence of iterative approximations for solving the basic "generalized Neumann-Kelvin integral equation" (6) — can also be obtained by using two alternative (but equivalent and closely-related) approaches.

### 1. The generalized Neumann-Kelvin theory

The problem examined in this study is that of flow caused by a displacement ship in steady, rectilinear motion at the free surface of an otherwise calm sea, which is assumed to be of infinite depth and lateral extent. Water is supposed to be homogeneous and incompressible. Surface tension is neglected. Viscosity effects are ignored and irrotational flow is assumed. Effects of wavebreaking and spray formation at the ship bow are also neglected. (The ad-hoc corrections for effects of viscosity, spray, and wavebreaking which were included in the potential-flow theory developed in [1,2] will not be incorporated in this study for shortness.)

The flow is observed from a moving system of coordinates attached to the ship, so that the flow becomes independent of time. The  $z$  axis is chosen vertical, positive upwards, with the undisturbed free surface taken as the plane  $z = 0$ . The  $x$  axis is parallel to the direction of motion of the ship and positive towards the ship stern. Flow variables are rendered dimensionless with respect to some reference length  $L$ , the speed of the ship  $U$ , and the density of water  $\rho$ . The reference length  $L$  may be chosen as  $L = U^2/g$ , where  $g$  is the acceleration of gravity, as in [1,2], although it may be more convenient to take  $L$  as the length of the ship in the present study, which is specifically concerned with the steady motion of a ship at low Froude number. The parameter  $U/(gL)^{1/2}$  is denoted by the symbol  $F$ , which represents the usual Froude number if  $L$  is the length of the ship, while we simply have  $F \equiv 1$  if the reference length  $L$  is taken as  $U^2/g$  as in [1,2]. We thus define the dimensionless coordinates  $\vec{x} \equiv \vec{X}/L$  and velocity potential  $\phi \equiv \Phi/UL$ , where  $\vec{X}$  and  $\Phi$  are dimensional, while the dimensionless velocity is given by  $(\phi_x, \phi_y, \phi_z) \equiv (\Phi_x, \Phi_y, \Phi_z)/U$ .

The hydrodynamical problem amounts to determining the (dimensionless) velocity potential  $\phi$  of the disturbance flow caused by the ship. As it is explained in [1], this problem may be formulated in a "solution domain", (d) say, bounded by some arbitrary "fictitious hull" surface, (h) say, which may for instance (but need not) be taken as the wetted hull of the ship in position of rest, and the "undisturbed free surface", (f) say, defined as the portion of the plane  $z = 0$  located outside the intersection curve (c) of (h) with the plane  $z = 0$ . The problem then consists in solving the Laplace equation

$$\nabla^2 \phi = 0 \quad \text{in (d)} \quad , \quad (1)$$

subject to the "Kelvin boundary condition"

$$\phi_z + F^2 \phi_{xx} = -q_f \quad \text{on } (f), \quad (2)$$

the "radiation condition" of no waves upstream from the ship, and the "Neumann boundary condition"

$$\phi_n = -v - q_h \quad \text{on } (h), \quad (3)$$

where  $\phi_n \equiv \nabla\phi \cdot \vec{n}$  and  $v \equiv \vec{n} \cdot \vec{i}$ , with  $\vec{n}$  representing the unit normal vector to (h) directed towards the interior of the ship, and  $\vec{i}$  defined as the unit positive vector along the x axis.

The "free-surface flux"  $q_f$  in the free-surface boundary condition (2) accounts for both the nonlinear terms in the free-surface condition and the fact that the position of the free surface, defined by  $z = -F^2(\phi_x + |\nabla\phi|^2/2)$ , generally differs from the plane  $z = 0$  (the undisturbed free surface) where the free-surface condition is enforced for mathematical simplicity. The nonlinear free-surface correction flux  $q_f$  is given by

$$q_f = \left[ \phi_z + F^2 \left\{ \phi_{xx} + (|\nabla\phi|^2)_x + \frac{1}{2} \nabla\phi \cdot \nabla |\nabla\phi|^2 \right\} \right]_{z = -F^2(\phi_x + \frac{1}{2} |\nabla\phi|^2)} - \left[ \phi_z + F^2 \phi_{xx} \right]_{z=0}. \quad (4)$$

The "hull flux"  $q_h$  in the hull boundary condition (3) is associated with the fact that the fictitious hull surface (h) where the hull condition is enforced may differ from the actual ship hull surface, (H) say. The hull flux  $q_h$  is given by

$$q_h = (\vec{i} + \nabla\phi)_H \cdot \vec{N} - (\vec{i} + \nabla\phi)_h \cdot \vec{n} \equiv (v + \phi_n)_H - (v + \phi_n)_h, \quad (5)$$

where  $\vec{i}$  is the unit positive vector along the x axis ( $\vec{i}$  represents the unit oncoming uniform stream equivalent to the ship speed in the present dimensionless steady flow formulation, and  $\vec{i} + \nabla\phi$  is the total velocity field) as defined above,  $\vec{N}$  and  $\vec{n}$  are the unit inward normal vectors to (H) and (h), respectively, and the notation  $( )_H$  and  $( )_h$  means that the expression between the parentheses is to be evaluated on (H) or (h), respectively. We evidently have  $q_h \equiv 0$  if (h) is chosen to coincide with (H), or rather the portion of (H) below the plane  $z = 0$ . In the present study, it is convenient to select (h) as the wetted hull of the ship in position of rest,  $(H_0)$  say, so that the fictitious hull (h)  $\equiv (H_0)$  and the real hull (H) do not differ much (sinkage and trim are not very significant at low Froude number), and  $q_h$  may be expected to be small.

It is shown in [1] that the foregoing "generalized Neumann-Kelvin problem" can be formulated in "integral form" given by the integral equation (2.21), which here becomes

$$\phi_* = \phi_*^I + \int_h (\phi - \phi_*) G_n da + F^2 \oint_c [G(\sigma\phi_s + \tau\phi_t) - (\phi - \phi_*) G_x] \mu ds + \int_f Gq_f dx dy + \int_h Gq_h da - F^2 \oint_c Gq_h \nu \mu ds, \quad (6)$$

where the significance of the previously-undefined symbols will now be explained. The symbol  $\phi_*$  is meant for  $\phi(\vec{x}_*)$  where  $\vec{x}_*$  is an arbitrary point in the solution domain (d) including its boundary (h) + (f) + (c), while  $\phi$  is meant for  $\phi(\vec{x})$  where  $\vec{x}$  represents the "point of integration" (integration variable) in the above integrals; the point  $\vec{x}_*$  thus is the "field point" where the potential is being evaluated, while  $\vec{x}$  represents the "dummy" variable of integration. The potential  $\phi_*^I \equiv \phi_I(\vec{x}_*)$  is the "initial potential" given by

$$\phi_I(\vec{x}_*) = \int_h G(\vec{x}_*, \vec{x}) v(\vec{x}) da(\vec{x}) - F^2 \oint_c G(\vec{x}_*, \vec{x}) v^2(s) \mu(s) ds. \quad (7)$$

The function  $G \equiv G(\vec{x}_*, \vec{x})$  in equations (6) and (7) is the fundamental solution (Green function) appropriate for the problem, that is, the function  $G(\vec{x}_*, \vec{x})$  represents the linearized velocity potential for the disturbance flow caused at point  $\vec{x}_*(x_*, y_*, z_* < 0)$  by a unit "outflow" at point  $\vec{x}(x, y, z < 0)$ , associated with a submerged source if  $z < 0$  or a flux across the free surface if  $z = 0$ , in an oncoming uniform stream along the positive x axis. For the purpose of the present study a convenient expression for the Green function G is

$$4\pi G(\vec{x}_*, \vec{x}) = -\frac{1}{r} - \frac{1}{r'} + \frac{1}{F^2} \frac{2}{\pi} \int_{-1}^1 \text{Im}[\exp(\zeta) E_1(\zeta) - \frac{1}{\zeta}] dt + H(x') \frac{-4}{F^2} \int_{-\infty}^{\infty} \exp\left(\frac{z'}{F^2} \tau^2\right) \sin\left(\frac{x'}{F^2} \tau + \frac{y'}{F^2} t\tau\right) dt, \quad (8)$$

where  $r \equiv |\vec{x}_* - \vec{x}|$  is the distance between the points  $\vec{x}_*$  and  $\vec{x}$ ,  $\vec{x}' \equiv (x', y', z') \equiv (x_* - x, y_* - y, z_* + z)$  is the vector joining the mirror image of the point  $\vec{x}$  with respect to the plane  $z = 0$  to the "field point"  $\vec{x}_*$ ,  $r' \equiv |\vec{x}'|$ ,  $\zeta \equiv F^{-2}(z' \sqrt{1-t^2} + y't + i|x'|) \sqrt{1-t^2}$ ,  $E_1(\zeta)$  is the exponential integral defined here as in Abramowitz and Stegun [7],  $H(x')$  is the Heaviside step function defined as  $H(x') = 0$  for  $x' < 0$ , that is upstream from the point  $\vec{x}$ , and  $H(x') = 1$  for  $x' > 0$ , and  $\tau \equiv \sqrt{1+t^2}$ . Expression (8) for the Green function G may readily be obtained from equations (1), (2), and (6)<sup>†</sup> in Noblesse [8]. The above expression for the fundamental solution

<sup>†</sup>Here, we used the relation  $F^2 \pi/r' = - \int_{-1}^1 \text{Im}(1/\zeta) dt$ , which may easily be verified.

G will actually be used in the following section for the purpose of investigating the "low-Froude-number limit" of the integral equation (6).

In equations (6) and (7), the symbol  $da$  in the surface integrals over (h) represents the differential element of area of (h), while  $ds$  in the line integrals around the "waterline" (c) represents the differential element of arc length of (c). In the line integrals around (c), we have  $v \equiv v(s) \equiv \vec{n}(s) \cdot \vec{i}$  where  $\vec{n}(s)$  is the unit inward normal vector to (h) at point  $s$  of (c), while  $\mu$  is defined as  $\mu \equiv \mu(s) \equiv \vec{n}'(s) \cdot \vec{i}$ , where  $\vec{n}'$  is the unit inward normal vector to (c) in the plane  $z = 0$ . In the (fairly common) case when the surface (h) intersects the plane  $z = 0$  orthogonally, we have  $\vec{n}(s) \equiv \vec{n}'(s)$  and  $v(s) \equiv \mu(s)$ . In the first line integral in equation (6), the symbols  $\sigma$  and  $\tau$  are defined as  $\sigma \equiv \vec{s} \cdot \vec{i}$  and  $\tau \equiv \vec{t} \cdot \vec{i}$ , where  $\vec{s} \equiv \vec{s}(s)$  is the unit tangent vector, at point  $s$ , to the "waterline" (c) oriented in the counterclockwise direction in the  $(x,y)$  plane, and  $\vec{t} \equiv \vec{t}(s)$  is the unit tangent vector to (h), mutually orthogonal to  $\vec{s}(s)$  and the normal  $\vec{n}(s)$  to (h) at point  $s$  of (c), and pointing downwards. In this line integral, the notation  $\phi_s \equiv \partial\phi(s,t,n)/\partial s$ ,  $\phi_t \equiv \partial\phi(s,t,n)/\partial t$ , and  $G_x \equiv \partial G(\vec{x}_*, \vec{x})/\partial x$  was used for shortness. The usual notation  $G_n \equiv \nabla G(\vec{x}_*, \vec{x}) \cdot \vec{n}(\vec{x})$  was also used in the first surface integral in equation (6). It will finally be noted that the axes  $x, y$ , and  $z$ , the "fictitious hull surface" (h), the undisturbed free surface (f), the "waterline" (c), the elements of area  $da$  and of arc length  $ds$ , and the unit vectors,  $\vec{i}$ ,  $\vec{n}$ ,  $\vec{n}'$ ,  $\vec{s}$ , and  $\vec{t}$ , are shown in figure 1.

A method of solution of the integral equation (6), based on an iterative procedure starting with the initial approximation  $\phi_I(\vec{x}_*)$  given by formula (7), is discussed in some detail in [1] and [2]. In particular, the recurrence relation defining the  $k$ th approximation  $\phi^{(k)}$  in the iterative sequence of approximations  $\phi^{(1)} \equiv \phi_I$ ,  $\phi^{(2)}$ ,  $\phi^{(3)}$ , ... is given by equation (30b) in [2]<sup>†</sup>. The formulas defining the iterative approximations  $R_k$  to the wave resistance  $R$  associated with the above-mentioned approximations  $\phi^{(k)}$  to the disturbance velocity potential  $\phi$  may also be found in [2]. The main object of the present study is to present an alternative iterative scheme for solving the integral equation (6) based upon the assumption that the value of the Froude number is fairly small, which is in fact the case for a large class of commercial ships. Before we proceed to derive these alternative low-Froude-number iterative approximations  $\phi_k^{\mathcal{L}F}$  to  $\phi$ , and the corresponding approximations  $R_k^{\mathcal{L}F}$  to  $R$ , we begin by investigating the limit of the disturbance velocity potential  $\phi$  as the Froude number vanishes.

<sup>†</sup>In fact, the recurrence relation (30b) in [2] is valid for  $k \geq 0$ , as it is indeed indicated in [2], so that we may actually regard the zeroth approximation  $\phi^{(0)} \equiv 0$  as the initial approximation in the iterative sequence of approximations  $\phi^{(k)}$ ,  $k \geq 0$ , which thus is defined by  $\phi^{(0)} \equiv 0$ ,  $\phi^{(1)} \equiv \phi_I$ ,  $\phi^{(2)}$ , ... .

## 2. The zero-Froude-number limit

In the differential formulation of the problem of steady motion of a ship stated by equations (1) through (5) the Froude number  $F \equiv U/(gL)^{1/2}$  — where  $L$  here is the length of the ship — appears explicitly only in the free surface boundary condition defined by equations (2) and (4). This free-surface condition may be expressed in the form

$$\phi_z = - \tilde{q}_f \quad , \quad (9)$$

where  $\tilde{q}_f$  is defined as

$$\tilde{q}_f = \left[ \phi_z + F^2 \{ \phi_{xx} + (|\nabla\phi|^2)_x + \frac{1}{2} \nabla\phi \cdot \nabla |\nabla\phi|^2 \} \right]_{z=0} = - F^2 \left( \phi_x + \frac{1}{2} |\nabla\phi|^2 \right) - [\phi_z]_{z=0} \quad . \quad (9a)$$

Equation (9a) shows that we have  $\tilde{q}_f \rightarrow 0$  as  $F \rightarrow 0$ , so that the free-surface boundary condition simply becomes the "rigid-wall condition"  $\phi_z = 0$  in the "zero-Froude-number limit"  $F = 0$ . Physically, as the Froude number vanishes, the force due to gravity becomes so much larger than inertia forces that deformations of the free surface are greatly inhibited, and the free surface indeed appears as a rigid wall, at least on the length scale of the ship.

In the zero-Froude-number limit the disturbance velocity potential  $\phi$  becomes the "zero-Froude-number potential",  $\phi^0$  say, which thus verifies the Laplace equation

$$\nabla^2 \phi^0 = 0 \quad \text{in (d)} \quad , \quad (10)$$

the "rigid-wall (Neumann) condition"

$$\phi_z^0 = 0 \quad \text{on (f)} \quad , \quad (11)$$

and the "Neumann boundary condition"

$$\phi_n^0 = -v \equiv -\vec{n} \cdot \vec{i} \quad \text{on (h)} \quad , \quad (12)$$

where (h) here is taken as the wetted hull of the ship in position of rest, i.e. (h)  $\equiv (H_0)^\dagger$ . The "zero-Froude-number potential"  $\phi^0$  may readily be seen to be identical to the disturbance velocity potential of the flow past the "double ship", consisting of  $(H_0)$

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<sup>†</sup>Sinkage and trim vanish as  $F \rightarrow 0$ , so that we have  $q_h = 0$ .

and its mirror image with respect to the plane  $z = 0$ , in an unbounded fluid (i.e. without free surface), and the potential  $\phi^0$  is indeed often referred to as the "double-ship potential".

It is also interesting to examine the zero-Froude-number limit of the integral equation (6), for this provides an alternative proof of the fact that  $\phi \rightarrow \phi^0$  as  $F \rightarrow 0$ . For this purpose, we begin by examining the zero-Froude-number limit of the fundamental solution (Green function)  $G(\vec{x}_*, \vec{x})$  associated with the Kelvin free-surface condition  $G_z + F^2 G_{xx} = 0$  on  $z = 0$ , and given by formula (8). It can be shown [by using the asymptotic expansion of the exponential integral  $E_1(\zeta)$  for  $|\zeta| \gg 1$ ] that the first integral in equation (8), which represents a nonoscillatory near-field disturbance (as it is shown in [8]), is  $O(F^4)$  as  $F \rightarrow 0$ , while the last integral, which is associated with free-surface gravity waves, is  $O(F^\alpha e^{-1/F^2})$  as  $F \rightarrow 0$ , with  $\alpha = 1, 2$ , or  $2/3$  depending on whether  $\vec{x}_*$  is inside, outside, or exactly on the boundary of the famous "Kelvin V-shaped sector" trailing downstream from the point outflow  $\vec{x}$ . We then have

$$G(\vec{x}_*, \vec{x}) = G^0(\vec{x}_*, \vec{x}) + O(F^2) + H(x') O(F^\beta e^{-1/F^2}) \text{ as } F \rightarrow 0^+, \quad (13)$$

where  $-4/3 \leq \beta \equiv \alpha - 2 \leq 0$ , and  $G^0(\vec{x}_*, \vec{x})$  is the "zero-Froude-number Green function" defined by

$$4\pi G^0(\vec{x}_*, \vec{x}) = -1/r - 1/r' \equiv -(1 + r/r')/r \quad . \quad (14)$$

The function  $G^0(\vec{x}_*, \vec{x})$  defined by equation (14) clearly is the Green function associated with the Neumann condition  $G_z = 0$  on  $z = 0$ , and thus is the Green function appropriate for the zero-Froude-number problem stated by equations (10) through (12). More precisely, it may be verified that the fundamental solution  $G^0(\vec{x}_*, \vec{x})$  verifies the equations

$$\left\{ \begin{array}{l} \nabla^2 G^0 = \delta(x - x_*) \delta(y - y_*) \delta(z - z_*) \text{ in } z < 0 \\ G_z^0 = 0 \text{ on } z = 0 \end{array} \right\} \text{ if } z_* < 0, \quad (15a)$$

$$(15b)$$

$$\left\{ \begin{array}{l} \nabla^2 G^0 = 0 \text{ in } z < 0 \\ G_z^0 = -\delta(x - x_*) \delta(y - y_*) \text{ on } z = 0 \end{array} \right\} \text{ if } z_* = 0, \quad (16a)$$

$$(16b)$$

where  $\delta(\ )$  is the usual Dirac " $\delta$  function".

<sup>†</sup>It ought however to be noted that  $r'$  is assumed to be  $O(1)$  as  $F \rightarrow 0$  in the asymptotic approximation (13), which is therefore not valid for  $r' = O(F^2)$ ; in other words, this "low-Froude-number approximation" is not uniformly valid with respect to  $\vec{x}'$ .

By substituting equation (13) into equations (6) and (7), and by using the facts that the "free-surface flux"  $q_f$  and the "hull flux"  $q_h$  are  $O(F^2)$  as  $F \rightarrow 0$ , it may be seen that, in the "low-Froude-number limit", the integral equation (6) becomes

$$\phi_* = \int_h G^0 v da + \int_h (\phi - \phi_*) G_n^0 da + O(F^2) + O(e^{-1/F^2}) \text{ as } F \rightarrow 0,$$

where the  $O(F^2)$  term is associated with a "near-field disturbance" while the  $O(e^{-1/F^2})$  term corresponds to the free-surface gravity waves, which thus are "exponentially small" at low-Froude-number. The zero-Froude-number limit  $\phi^0$  of the disturbance velocity potential  $\phi$  thus is the solution of the "zero-Froude-number integral equation"

$$\phi^0(\vec{x}_*) = \phi_I^0(\vec{x}_*) + \int_h [\phi^0(\vec{x}) - \phi^0(\vec{x}_*)] G_n^0(\vec{x}_*, \vec{x}) da(\vec{x}), \quad (17)$$

where the "zero-Froude-number initial potential"  $\phi_I^0(\vec{x}_*)$  is defined as

$$\phi_I^0(\vec{x}_*) = \int_h G^0(\vec{x}_*, \vec{x}) v(\vec{x}) da(\vec{x}). \quad (18)$$

It may be verified that the integral equation (17), which was obtained above as the zero-Froude-number limit of the integral equation (6), is in fact identical to the integral equation that can be directly derived from equations (10) through (12) and equations (14) through (16) by applying a classical Green identity to the zero-Froude-number potential  $\phi^0$  and Green function  $G^0$  (in the manner shown in Part 2 of [1]), thereby establishing the consistency of the zero-Froude-number limits of the differential and integral formulations of the problem of steady motion of a ship stated by equations (1) through (5) and equations (6) through (8), respectively.

A straightforward method of solution of the integral equation (17) consists in using an iterative procedure based on the recurrence relation

$$\phi_{k+1}^0(\vec{x}_*) = \phi_I^0(\vec{x}_*) + \int_h [\phi_k^0(\vec{x}) - \phi_k^0(\vec{x}_*)] G_n^0(\vec{x}_*, \vec{x}) da(\vec{x}), \quad k \geq 0, \quad (19)$$

which immediately follows from equation (17), and beginning with the "zereth approximation"  $\phi_0^0(\vec{x}_*) \equiv 0$ . This "zereth approximation"  $\phi_0^0$  becomes exact in the limiting case of a "knife-blade-like" or "needle-like" ship-hull form, which suggests that the above iterative procedure corresponds to a "thin-, flat-, or slender-ship perturbation method of solution". In particular, the "first approximation"  $\phi_1^0$  in the sequence of iterative approximations  $\phi_k^0$ ,  $k \geq 0$ , defined by the recurrence relation (19) and the "zereth approximation"  $\phi_0^0 \equiv 0$  is given by  $\phi_1^0 \equiv \phi_I^0$ , so that the "initial potential"  $\phi_I^0$  defined

by formula (18) may thus be regarded as a first-order zero-Froude-number slender- or thin-, or flat- ship approximation.

As a matter of fact, the initial potential  $\phi_I^0$  in some respects appears to generalize the classical first-order thin- and slender-body perturbation approximations, as it will now be shown. For convenience, we begin by expressing the initial potential  $\phi_I^0$  given by formula (18) in the equivalent form

$$\phi_I^0(\vec{x}_*) = \frac{-1}{4\pi} \int_{h'} \frac{\vec{n}(\vec{x}) \cdot \vec{i} da(\vec{x})}{[(x_*-x)^2 + (y_*-y)^2 + (z_*-z)^2]^{1/2}}, \quad (20)$$

where equation (14) was used, and ( $h'$ ) represents the "double-hull surface" consisting of the surface ( $h$ ) and its mirror image with respect to the plane  $z = 0$ . With the coordinates  $\vec{x}$  and  $\vec{x}_*$  made dimensionless in terms of a reference length taken as the length of the ship, the dimensionless length of the double hull ( $h'$ ) is of course equal to unity, so that in the case of a thin ship form we have  $|y| \ll 1$ , and expression (20) may be approximated as

$$\phi_I^0(\vec{x}_*) \sim \frac{-1}{4\pi} \int_{h'} \frac{\vec{n}(\vec{x}) \cdot \vec{i} da(\vec{x})}{[(x_*-x)^2 + y_*^2 + (z_*-z)^2]^{1/2}}, \quad (21)$$

for  $|y_*| = 0(1)$ , that is in the "outer (thin-ship) limit" in the usual language of the "method of matched asymptotic expansions" (see for instance, Van Dyke [9]). If the equation of ( $h'$ ) is expressed by means of the two equations  $y = b^+(x,z)$  and  $y = -b^-(x,z)$  corresponding to the two sides of the surface ( $h'$ ), the surface integral (21) may be transformed into the following double integral over the projection, ( $h'_y$ ) say, of ( $h'$ ) onto the plane  $y = 0$

$$\phi_I^0(\vec{x}_*) \sim \frac{-1}{4\pi} \iint_{h'_y} \frac{t_x(x,z) dx dz}{[(x_*-x)^2 + y_*^2 + (z_*-z)^2]^{1/2}}, \quad (22)$$

where  $t(x,z) \equiv b^+(x,z) + b^-(x,z)$  is the "thickness" of the hull. In the usual case of a ship with port and starboard symmetry, we have  $b^-(x,z) = b^+(x,z)$ , so that  $t(x,z) = 2b(x,z)$ , and expression (22) becomes identical to the classical first-order perturbation approximation for a thin body. This well-known thin-ship (body) perturbation approximation may thus be seen to correspond to the thin-ship limit of the initial potential  $\phi_I^0$  defined by formula (18), in the same way, of course, that the classical Michell thin-ship approximation corresponds to the thin-ship limit of the initial potential  $\phi_I$  given by formula (7), as it was shown in [1]. Indeed, the recurrence relation (19) proposed for solving the zero-Froude-number integral equation (17) is essentially

identical to the recurrence relation (2.28a) in [1] related to the integral equation (6); furthermore, both of these recurrence relations are associated with the use of the approximations  $\phi^0 \equiv 0$  and  $\phi \equiv 0$  as "zeroth approximations", which clearly corresponds to a perturbation method of solution appropriate for thin- or slender-ship forms, as it was noted previously.

In the (usual) case of ship forms which are not only thin but also flat, i.e. slender, we have  $|z| \ll 1$  as well as  $|y| \ll 1$ , and expression (22) for the initial potential  $\phi_I^0$  may be further approximated as

$$\phi_I^0(\vec{x}_*) \sim \frac{-1}{4\pi} \int_0^1 \frac{\lambda(x) dx}{[(x_* - x)^2 + y_*^2 + z_*^2]^{1/2}} \quad (23)$$

where  $\lambda(x)$  is given by

$$\lambda(x) = \int_{-d(x)}^{d(x)} t_x(x, z) dz, \quad (23a)$$

with  $d(x)$  representing the local (at section  $x$ ) draft of the ship hull ( $h$ ). For usual ship forms we have  $\lambda(x) \equiv da(x)/dx$ , where  $a(x)$  is the cross-sectional area of the double-ship form ( $h^*$ ). The approximation (23), which was obtained above as the "slender-ship outer limit" [based on the assumption  $(y^2 + z^2)^{1/2} \ll 1$  with  $(y_*^2 + z_*^2)^{1/2} = O(1)$ ] of expression (20) for the initial potential  $\phi_I^0$ , may thus be seen to be identical to the "outer solution" in the well-known first-order slender-ship approximation based on the use of the method of matched asymptotic expansions.

It may also be interesting to examine the "slender-ship (body) inner limit" of the surface integral (20) defining the initial potential  $\phi_I^0$ . For this purpose, it is convenient to formally introduce the "inner variables"  $\eta$ ,  $\zeta$  and  $\eta_*$ ,  $\zeta_*$  related to the "outer variables"  $y$ ,  $z$  and  $y_*$ ,  $z_*$  by means of the equations  $y = \epsilon\eta$ ,  $z = \epsilon\zeta$ , and  $y_* = \epsilon\eta_*$ ,  $z_* = \epsilon\zeta_*$ , where  $\epsilon$  is the slenderness parameter attached to the ship ( $\epsilon$  may be taken as the ratio of the largest transversal dimension of the ship to its length). In terms of these inner variables, formula (20) may be expressed in the form

$$\phi_I^0(x_*, \eta_*, \zeta_*) = \frac{-1}{4\pi} \int_0^1 dx \oint_{\sigma} \frac{\vec{n}(x, \epsilon\eta, \epsilon\zeta) \cdot \vec{i} [1 - (\vec{n} \cdot \vec{i})^2]^{-1/2} \epsilon d\sigma}{[(x_* - x)^2 + \epsilon^2 \{(\eta_* - \eta)^2 + (\zeta_* - \zeta)^2\}]^{1/2}},$$

where  $(\sigma)$  is the intersection curve of the double-hull surface ( $h^*$ ) with any plane  $x = \text{constant}$ , and  $\epsilon d\sigma$  is the differential element of arc length of  $(\sigma)$ . By performing the change of variable  $x = x_* - \epsilon\xi$ , we may obtain

$$\phi_I^0(x_*, \eta_*, \zeta_*) = \frac{-1}{4\pi} \int_{x_*-1}^{x_*/\epsilon} d\xi \oint_{\sigma} \frac{\vec{n}(x_* - \epsilon\xi, \epsilon\eta, \epsilon\zeta) \cdot \vec{i} [1 - (\vec{n} \cdot \vec{i})^2]^{-1/2}}{[\xi^2 + (\eta_* - \eta)^2 + (\zeta_* - \zeta)^2]^{1/2}} \epsilon d\sigma \quad .$$

In the limit  $\epsilon \rightarrow 0$  — and for  $\epsilon^\alpha \leq x_* \leq 1 - \epsilon^\alpha$  with  $0 < \alpha < 1$ , that is away from the ship ends (bow and stern)  $x_* = 0$  and  $x_* = 1$  — we may then obtain

$$\phi_I^0(x_*, \eta_*, \zeta_*) \sim \frac{-1}{4\pi} \oint_{\sigma} \epsilon d\sigma \frac{\vec{n}(x_*, \epsilon\eta, \epsilon\zeta) \cdot \vec{i}}{[1 - (\vec{n} \cdot \vec{i})^2]^{1/2}} \int_{-\infty}^{\infty} \frac{d\xi}{[\xi^2 + (\eta_* - \eta)^2 + (\zeta_* - \zeta)^2]^{1/2}} \quad ,$$

where the longitudinal slope  $\vec{n} \cdot \vec{i}$  clearly was assumed to be continuous. The inner integral in the above expression is divergent; however, proper interpretation of this integral yields

$$\phi_I^0(x_*, \eta_*, \zeta_*) \sim \frac{1}{2\pi} \oint_{\sigma} \ln \sqrt{(\eta_* - \eta)^2 + (\zeta_* - \zeta)^2} \frac{\vec{n}(x_*, \epsilon\eta, \epsilon\zeta) \cdot \vec{i}}{[1 - (\vec{n} \cdot \vec{i})^2]^{1/2}} \epsilon d\sigma \quad . \quad (24)$$

This then shows that in the "inner region"  $(y_*^2 + z_*^2)^{1/2} = o(\epsilon)$  — and away from the ship ends  $x_* = 0$  and  $x_* = 1$  — the initial potential  $\phi_I^0$  is locally two dimensional, with  $x_*$  appearing as a parameter, i.e. we have  $\phi_I^0(\eta_*, \zeta_*; x_*)$ , which is a well-known, and indeed intuitively self-evident property of flows about slender bodies. More precisely, formula (24) expresses the inner approximation to the initial potential  $\phi_I^0(\eta_*, \zeta_*; x_*)$  as the potential due to a distribution of two-dimensional sources, of strength  $(\vec{n} \cdot \vec{i})/[1 - (\vec{n} \cdot \vec{i})^2]^{1/2}$ , along the intersection curve ( $\sigma$ ) of the double-hull surface ( $h^*$ ) with the plane  $x = x_*$ . This inner approximation to the initial potential  $\phi_I^0$  thus is clearly not identical to the inner solution in the classical slender-body perturbation approximation, which is also quasi two-dimensional of course, but satisfies the condition of no flow across the "framelines" ( $\sigma$ ) exactly, while this condition is only approximately verified by the initial potential  $\phi_I^0$ . In any case, we must evidently keep in mind that — in the present "longitudinal" motion — the most significant part of the ship hull surface in fact are its ends (bow and stern), where flow deceleration and acceleration mainly take place, rather than the approximately uniform "middle-body" section to which the above inner approximation is restricted.

It may be interesting to incidentally indicate here an alternative form of the recurrence relation (19). This alternative form can be obtained by noting that the  $k$ th approximation  $\phi_k^0$  on the right side of equation (19) verifies the following identity

$$\phi_k^0(\vec{x}_*) = - \int_h G^0(\vec{x}_*, \vec{x}) \phi_{kn}^0(\vec{x}) da(\vec{x}) + \int_h [\phi_k^0(\vec{x}) - \phi_k^0(\vec{x}_*)] G_n^0(\vec{x}_*, \vec{x}) da(\vec{x}) \quad ,$$

which can be established by applying a classical Green identity to the potential  $\phi_k^0(\vec{x})$  and the Green function  $G^0(\vec{x}_*, \vec{x})^\dagger$ . Use of the above identity and of equation (18) in equation (19) then yields the following alternative recurrence relation

$$\left[ \phi_{k+1}^0(\vec{x}_*) = \phi_k^0(\vec{x}_*) + \int_h G^0(\vec{x}_*, \vec{x}) [\nu(\vec{x}) + \phi_{kn}^0(\vec{x})] da(\vec{x}) \right], \quad k \geq 0, \quad (25)$$

which must also be associated with the zeroth approximation  $\phi_0^0(\vec{x}_*) \equiv 0$ . While the recurrence relation (19) is actually better suited than the (equivalent) alternative recurrence relation (25) for practical purposes of numerical calculations, the latter relation readily lends itself to an appealing physical interpretation, namely one keeps distributing sources of strength  $\nu + \phi_{kn}^0 \equiv (\vec{i} + \nabla\phi_k^0) \cdot \vec{n}$  equal to the fluid flux across (h) associated with each successive approximation  $\phi_k^0$  until the zero-flux ( $\nu + \phi_n^0 = 0$ ) hull boundary condition (12) is verified within the desired accuracy, which then provides a physical insight into the iterative scheme associated with the recurrence relation (19); in short, this iterative scheme thus might be interpreted as a successive "leakage stopping" scheme.

It is instructive to consider the simple particular case when the ship hull surface (h) is the "bottom half" of a triaxial ellipsoid with principal axes oriented parallel to the x, y, and z axes and corresponding principal dimensions 2a, 2b, and 2c, respectively, so that the length, beam, and draft of this simple ship hull form are 2a, 2b, and c, respectively. As it was noted previously, the velocity potential of the disturbance flow due to a hull form (h) in the zero-Froude-number limit is identical to the potential of the flow due to the "double hull" ( $\hat{h}$ ) consisting of (h) and its mirror image with respect to the plane  $z = 0$  in an unbounded fluid (i.e. without a free surface). In the particular case of a triaxial ellipsoid now being considered, an exact expression for the disturbance velocity potential is in fact known; this expression (see for instance Havelock [10]) is

$$\phi^0(\vec{x}_*) = \frac{-1}{2\pi(2-\alpha)} \frac{\partial}{\partial x_*} \int_{\hat{V}} \frac{dv(\vec{x})}{|\vec{x}_* - \vec{x}|}, \quad (26)$$

where  $\hat{V}$  represents the interior volume of the ellipsoid, i.e.  $\hat{V}$  is the volume bounded externally by the surface ( $\hat{h}$ ) of the ellipsoid,  $dv$  is the differential element of volume, and  $\alpha$  is a constant defined by the integral

<sup>†</sup>This expression may readily be verified by comparing it with the integral equation (17), in which one needs only replace  $\phi^0$  by  $\phi_k^0$  and, in formula (18),  $\nu$  by  $(-\phi_{kn}^0)$  as it may be seen from equation (12).

$$\alpha = abc \int_0^{\infty} (\lambda + a^2)^{-3/2} [(\lambda + b^2)(\lambda + c^2)]^{-1/2} d\lambda. \quad (26a)$$

Formula (26) may be expressed in the form

$$\phi^0(\vec{x}_*) = \frac{-1}{2\pi(2-\alpha)} \int_{\mathbb{V}} \frac{\partial}{\partial x_*} \left( \frac{1}{|\vec{x}_* - \vec{x}|} \right) dv(\vec{x}) = \frac{1}{2\pi(2-\alpha)} \int_{\mathbb{V}} \frac{\partial}{\partial x} \left( \frac{1}{|\vec{x}_* - \vec{x}|} \right) dv(\vec{x}).$$

The latter volume integral may finally be transformed into the surface integral

$$\phi^0(\vec{x}_*) = \frac{-1}{2\pi(2-\alpha)} \int_{h'} \frac{\vec{n}(\vec{x}) \cdot \vec{i}}{|\vec{x}_* - \vec{x}|} da(\vec{x}), \quad (27)$$

where  $\vec{n}(\vec{x})$  is the unit inward normal vector to  $(h')$  at point  $\vec{x}$ , in accordance with the definition introduced previously.

Comparison of expressions (27) and (20) then shows that the exact potential  $\phi^0$  in fact is proportional to the initial potential  $\phi_I^0$ , that is we have

$$\boxed{\phi^0(\vec{x}_*) = k \phi_I^0(\vec{x}_*)}, \quad (28)$$

where the constant of proportionality  $k$  is given by  $k = 1/(1 - \alpha/2)$ . By performing the change of variable  $\lambda = a^2 t$  in the integral (26a), and upon introducing the notation  $\beta \equiv b/a$  ( $\beta \equiv$  beam/length ratio) and  $\delta \equiv c/2a$  ( $\delta \equiv$  draft/length ratio), we may obtain

$$\boxed{k(\beta, \delta) = 1 / \left\{ 1 - \beta\delta \int_0^{\infty} (t+1)^{-3/2} [(t+\beta^2)(t+4\delta^2)]^{-1/2} dt \right\}}. \quad (28a)$$

It may be seen that  $k(\beta, \delta) \rightarrow 1$  if  $\beta \rightarrow 0$  (thin-ship limit), or if  $\delta \rightarrow 0$  (flat-ship limit), or if both  $\beta \rightarrow 0$  and  $\delta \rightarrow 0$  (slender-ship limit), in agreement with the fact that the initial potential  $\phi_I^0$  must become exact in the thin-, flat-, or slender-ship limits. Curves  $k(\beta; \delta)$  are represented in figure 2 for  $0 \leq \beta \leq 1$  and  $0 \leq \delta \leq 1/2$  (the ellipsoid becomes a sphere for  $\beta = 1, \delta = 1/2$ ). It may be seen that for  $\beta = .2$  and  $\delta = .1$ , which may be regarded as typical values of the beam/length and draft/length ratios, we have  $k \approx 1.06$ , so that the initial potential  $\phi_I^0$  given by formula (18) in fact provides a fairly accurate approximation to the velocity potential of the zero-Froude-number disturbance flow due to an ellipsoidal hull form of typical ship-like dimensions.

To be sure, real ship-hull forms are not ellipsoidal in shape. Nonetheless, the initial potential  $\phi_I^0$  seems likely to provide a realistic approximation to the zero-Froude-number disturbance potential  $\phi^0$ , and at least ought to provide a fairly good first approximation for solving the integral equation (17) iteratively on the basis of the recurrence relation (19). Furthermore, the remarkable result expressed by

equation (28) suggests that the "modified initial potential" given by  $k\phi_I^0$ , where  $k(\beta, \delta)$  is the function defined by formula (28a) and depicted in figure 2 — with  $\beta$  and  $\delta$  taken as the beam/length and draft/length ratios, respectively — may provide a slight improvement of the original initial potential  $\phi_I^0$  given by formula (18).

Equation (28) also suggests that a promising alternative approach to solving the integral equation (17) consists in expressing the solution  $\phi^0$  of this integral equation in the form  $\phi^0(\vec{x}_*) = k(\vec{x}_*)\phi_I^0(\vec{x}_*)$ . By substituting this relation into equation (17), and expressing the term  $\phi^0(\vec{x}) - \phi^0(\vec{x}_*) \equiv k(\vec{x})\phi_I^0(\vec{x}) - k(\vec{x}_*)\phi_I^0(\vec{x}_*)$  in the form  $k(\vec{x}_*)[\phi_I^0(\vec{x}) - \phi_I^0(\vec{x}_*)] + [k(\vec{x}) - k(\vec{x}_*)]\phi_I^0(\vec{x})$ , we may obtain

$$k(\vec{x}_*)\phi_I^0(\vec{x}_*) = \phi_I^0(\vec{x}_*) + k(\vec{x}_*) \int_h [\phi_I^0(\vec{x}) - \phi_I^0(\vec{x}_*)] G_n^0(\vec{x}_*, \vec{x}) da(\vec{x}) + \\ + \int_h [k(\vec{x}) - k(\vec{x}_*)] \phi_I^0(\vec{x}) G_n^0(\vec{x}_*, \vec{x}) da(\vec{x}) .$$

By multiplying both members of this equation by  $\phi_I^0(\vec{x}_*)$ , using the relations  $\phi^0(\vec{x}_*) = k(\vec{x}_*)\phi_I^0(\vec{x}_*)$  and  $\phi^0(\vec{x}) = k(\vec{x})\phi_I^0(\vec{x})$ , and rearranging, we may finally obtain the following alternative form of the integral equation (17)

$$\boxed{\phi^0(\vec{x}_*) = \bar{\phi}_I^0(\vec{x}_*) + \frac{\int_h [\phi^0(\vec{x})\phi_I^0(\vec{x}_*) - \phi^0(\vec{x}_*)\phi_I^0(\vec{x})] G_n^0(\vec{x}_*, \vec{x}) da(\vec{x})}{\phi_I^0(\vec{x}_*) - \int_h [\phi_I^0(\vec{x}) - \phi_I^0(\vec{x}_*)] G_n^0(\vec{x}_*, \vec{x}) da(\vec{x})}} , \quad (29)$$

where the potential  $\bar{\phi}_I^0(\vec{x}_*)$ , which corresponds to the initial potential associated with the modified integral equation (29), is given by

$$\boxed{\bar{\phi}_I^0(\vec{x}_*) = [\phi_I^0(\vec{x}_*)]^2 / \left\{ \phi_I^0(\vec{x}_*) - \int_h [\phi_I^0(\vec{x}) - \phi_I^0(\vec{x}_*)] G_n^0(\vec{x}_*, \vec{x}) da(\vec{x}) \right\}} . \quad (29a)$$

The above approximate expression for the zero-Froude-number potential  $\phi^0(\vec{x}_*)$  is exact in the case of a triaxial ellipsoid in translation parallel to its major axis, as it was shown previously, and thus seems likely to provide a fairly good approximation in the case of real ship-hull forms. It may be interesting to note that by performing a formal binomial expansion (and retaining only the first two terms in this expansion) of expression (29a) based on the assumption that the surface integral in the denominator is small compared with the potential  $\phi_I^0(\vec{x}_*)$ , we may obtain

$$\bar{\phi}_I^0(\vec{x}_*) \approx \phi_I^0(\vec{x}_*) + \int_h [\phi_I^0(\vec{x}) - \phi_I^0(\vec{x}_*)] G_n^0(\vec{x}_*, \vec{x}) da(\vec{x}) ,$$

which shows that  $\bar{\phi}_I^0(\vec{x}_*) \approx \phi_2^0(\vec{x}_*)$ , where  $\phi_2^0(\vec{x}_*)$  is the second approximation in the sequence of iterative approximations  $\phi_k^0(\vec{x}_*)$ ,  $k \geq 0$ , defined by the recurrence relation (19)

It may incidentally be noted that the above approach can obviously also be applied to the Neumann-Kelvin integral equation (6). Thus, by expressing the solution  $\phi(\vec{x}_*)$  of the "linearized" Neumann-Kelvin integral equation (6) — in which the nonlinear free-surface flux  $q_f$  and the hull flux  $q_h$  (associated with effects of sinkage and trim) are ignored — in the form  $\phi(\vec{x}_*) = k(\vec{x}_*)\phi_I(\vec{x}_*)$ , where  $\phi_I(\vec{x}_*)$  is the initial potential given by formula (7), we may obtain the following expression for the approximation  $\bar{\phi}_I^0(\vec{x}_*)$  corresponding to expression (29a) for the approximation  $\bar{\phi}_I^0(x_*)$

$$\left[ \bar{\phi}_I^0(\vec{x}_*) = (\phi_*^I)^2 / \left\{ \phi_*^I - \int_h (\phi^I - \phi_*^I) G_n da - F^2 \oint_c [G(\sigma\phi_s^I + \tau\phi_t^I) - (\phi^I - \phi_*^I) G_x] \mu ds \right\} \right],$$

where the notation  $\phi_*^I \equiv \phi_I^I(\vec{x}_*)$  and  $\phi^I \equiv \phi_I^I(\vec{x})$  was used for shortness. This approximation will be examined further in Part 5 of this study.

It may finally be noted that while the integral equation (17) and the modified integral equation (29) are valid for  $\vec{x}_*$  on the hull surface (h) and in the flow domain (d), and in principle may thus be used for determining  $\phi^0$  in (h) + (d), it may be more expedient in practice to use the integral equations (17) or (29) for determining  $\phi^0$  on (h), and then to obtain  $\phi^0$  in (d), strictly outside (h), by means of the well-known expression

$$\phi^0(\vec{x}_*) = \phi_I^0(\vec{x}_*) + \int_h \phi^0(\vec{x}) G_n^0(\vec{x}_*, \vec{x}) da(\vec{x}) \quad (30)$$

### 3. First-order low-Froude-number wave-resistance formulas

As it was explained in Part 3 of this study [2], the term  $\phi - \phi_*$  in the integrands of the first and second integrals on the right side of the integral equation (6) [which is valid for  $\vec{x}_*$  on the hull surface (h) + (c) and in the flow domain (d) + (f)] may be replaced by  $\phi$  in the far field of the ship, that is for  $\vec{x}_* \rightarrow \infty$ , since we have  $\phi_* \rightarrow 0$  as  $|\vec{x}_*| \rightarrow \infty$ , and therefore  $|\phi_*| \ll |\phi|$  for  $\vec{x}_*$  in the far field and  $\vec{x}$  in the near field. Furthermore, insofar as one is only interested in the surface waves trailing far downstream from the ship, the first three terms in expression (8) for the Green function  $G(\vec{x}_*, \vec{x})$ , which represent a nonoscillatory near-field disturbance, may be neglected. The disturbance velocity potential  $\phi(\vec{x}_*)$  far downstream from the ship may then be approximated as

$$\phi(\vec{x}_*) \sim F^2 \text{Re} \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\Omega(t)}{(1+t^2)^{3/2}} \exp \left\{ \frac{z_*}{F^2} (1+t^2) + i \left( \frac{x_*}{F^2} + \frac{y_*}{F^2} t \right) \sqrt{1+t^2} \right\} dt \text{ as } x_* \rightarrow \infty,$$

which expresses the potential in terms of a familiar superposition of elementary plane waves.

The function  $\Omega(t)$  in the above expression is the "Kochin free-wave spectrum function"<sup>†</sup> defined as

$$\Omega(t) = \Omega_0(t) + \int_h \phi E_n da + F^2 \oint_c [E(\sigma\phi_s + \tau\phi_t) - \phi E_x] \mu ds + \int_f E q_f dx dy + \int_h E q_h da - F^2 \oint_c E q_h \nu \mu ds, \quad (31)$$

where the function  $\Omega_0(t)$ , which is referred to as the "zerth approximation to the Kochin function"  $\Omega(t)$  since it corresponds to taking  $\phi \equiv 0$  as approximation to the disturbance potential  $\phi$  in the ship near field, is given by

$$\Omega_0(t) = \int_h E(\vec{x}; t) v(\vec{x}) da(\vec{x}) - F^2 \oint_c E(\vec{x}; t) v^2(s) \mu(s) ds, \quad (32)$$

with the function  $E(\vec{x}; t)$  defined as

$$E(\vec{x}; t) = \frac{(1+t^2)^{3/2}}{F^4} \exp \left\{ \frac{z}{F^2} (1+t^2) - i \left( \frac{x}{F^2} + \frac{y}{F^2} t \right) \sqrt{1+t^2} \right\}. \quad (33)$$

<sup>†</sup>It will be noted that there is here a slight difference in the definition of the Kochin function used in the present Part 4 and in Part 3 [2] of this study; specifically, the Kochin function  $\Omega(t)$  defined in this Part 4 is related to the Kochin function  $\Omega(\theta)$  of Part 3 by the equation  $\Omega(t) = \Omega(\theta) \sec^3 \theta$ .

The dimensionless wave resistance,  $R$  say, defined as  $R \equiv R' / \rho U^2 L^2$  where  $R'$  is dimensional ( $\rho$  is the density of water), can then be determined from the above-defined Kochin spectrum function  $\Omega(t)$  by means of the well-known "Havelock wave-resistance formula"

$$R = \frac{F^4}{2\pi} \int_{-\infty}^{\infty} |\Omega(t)|^2 (1+t^2)^{-5/2} dt \quad (34)$$

Further details on the derivation of the foregoing formulas may be found in Part 3 of this study [2].

The discussion of the previous section immediately suggests that a "first-order low-Froude-number approximation",  $\Omega_1^{\mathcal{L}F}(t)$  say, to the Kochin function  $\Omega(t)$  may be obtained by approximating the disturbance potential  $\phi$  in the ship near field, in formula (31), by the potential,  $\psi^0$  say, which may be taken as either the "exact zero-Froude-number potential"  $\phi^0$  given by the solution of the "zero-Froude-number integral equation" (17) or the "zero-Froude-number slender-ship approximation"  $k(\beta, \delta)\phi_I^0$  given by formulas (18) and (28a). [The approximation  $\bar{\phi}_I^0$  defined by formula (29a) will not be used here.] The first-low-Froude-number approximation  $\Omega_1^{\mathcal{L}F}(t)$  then is given by

$$\begin{aligned} \Omega_1^{\mathcal{L}F}(t) = & \Omega_0(t) + \int_h \psi^0 E_n da + F^2 \oint_c [E(\sigma\psi_s^0 + \tau\psi_t^0) - \psi^0 E_x] \mu ds + \\ & + \int_f E q_f^0 dx dy + \int_h E q_h^0 da - F^2 \oint_c E q_h^0 \nu \mu ds \quad , \end{aligned} \quad (35)$$

where the hull flux  $q_h^0$  is defined by formula (5), with  $\phi$  replaced by  $\psi^0$ , that is we have

$$q_h^0 = (\nu + \psi_n^0)_H - (\nu + \psi_n^0)_h \quad , \quad (36a)$$

while the nonlinear free-surface flux  $q_f^0$  is defined by formula (4), with  $\phi$  also replaced by  $\psi^0$  of course. By performing a Taylor series expansion of the right side of equation (4) about the plane  $z = 0$ , we may obtain

$$q_f^0 = F^2 \left\{ (|\nabla\phi|^2)_x + \frac{1}{2} \nabla\phi \cdot \nabla |\nabla\phi|^2 - (\phi_x + \frac{1}{2} |\nabla\phi|^2) \phi_{zz} \right\} + O(F^4) \quad ,$$

where the expression on the right side is to be evaluated at the plane  $z = 0$ . Replacement of  $\phi$  by  $\psi^0$ , and use of equations (38a,b) below, then yield

$$q_f^0 = F^2 \left\{ (|\nabla_2 \psi^0|^2)_x + \psi_x^0 \nabla_2^2 \psi^0 + \frac{1}{2} \left[ \nabla_2 \psi^0 \cdot \nabla_2 |\nabla_2 \psi^0|^2 + |\nabla_2 \psi^0|^2 \nabla_2^2 \psi^0 \right] \right\} + O(F^4) \quad , \quad (36b)$$

where  $\nabla_2$  is the two-dimensional differential operator  $(\partial_x, \partial_y)$ . By using formula (32) for the zeroth approximation  $\Omega_0(t)$  in expression (35), we may express the first low-Froude-number approximation  $\Omega_1^{\mathcal{L}F}(t)$  in the form

$$\left[ \begin{aligned} \Omega_1^{\mathcal{L}F}(t) = & \int_h [E(v + q_h^0) + \psi^0 E_n] da + \int_f Eq_f^0 dx dy \\ & - F^2 \int_c [E\{v(v + q_h^0) - (\sigma\psi_s^0 + \tau\psi_t^0)\} + \psi^0 E_x] \mu ds \end{aligned} \right]. \quad (37)$$

An interesting alternative expression for the above-defined approximation  $\Omega_1^{\mathcal{L}F}(t)$  may be obtained by applying a classical Green identity to the functions  $\psi^0(\vec{x})$  and  $E(\vec{x};t)$  in the domain (d), as it will now be shown. We begin by noticing that the zero-Froude-number potential  $\psi^0(\vec{x})$  verifies the following equations

$$\nabla^2 \psi^0 = 0 \quad \text{in } (d) \quad , \quad (38a)$$

$$\psi_z^0 = 0 \quad \text{on } (f) \quad , \quad (38b)$$

$$\psi_n^0 = -v + (v + \psi_n^0) \quad \text{on } (h) \quad , \quad (38c)$$

$$|\psi^0| \sim 1/|\vec{x}|^2 \quad \text{as } |\vec{x}| \rightarrow \infty \quad , \quad (38d)$$

where we actually have  $v + \psi_n^0 = 0$  if  $\psi^0$  is taken as the exact zero-Froude-number potential  $\phi^0$ , as it may be seen from equation (12). We also note that the function  $E(\vec{x};t)$  defined by formula (33) verifies the equations

$$\nabla^2 E = 0 \quad \text{in } z \leq 0 \quad , \quad (39a)$$

$$E_z + F^2 E_{xx} = 0 \quad \text{in } z \leq 0 \quad , \quad (39b)$$

$$E \sim F^{-4} (1 + t^2)^{3/2} \exp[z(1 + t^2)/F^2] \quad \text{as } z \rightarrow -\infty \quad , \quad (39c)$$

as it may readily be verified.

It follows from equations (38a,d) and equations (39a,c) that the following Green identity

$$\int_h (\psi^0 E_n - E \psi_n^0) da + \int_f (\psi^0 E_z - E \psi_z^0) dx dy = 0$$

holds. Equations (38b,c) and equation (39b) then yield

$$\int_h \psi^0 E_n da + \int_h E v da - \int_h E(v + \psi_n^0) da - F^2 \int_f \psi^0 E_{xx} dx dy = 0 .$$

By expressing the term  $\psi^0 E_{xx}$  in the last integral in the form  $\psi^0 E_{xx} = (\psi^0 E_x - E \psi_x^0)_x + E \psi_{xx}^0$ , and by using the well-known Green identity

$$\int_f (\psi^0 E_x - E \psi_x^0)_x dx dy = - \oint_c (\psi^0 E_x - E \psi_x^0) dy = \oint_c (\psi^0 E_x - E \psi_x^0) \mu ds ,$$

where we used the relation  $dy = - \mu ds$  along (c) [with  $\mu \equiv \vec{n} \cdot \vec{i}$  as it was defined previously in connection with equations (6) and (7)], we can obtain

$$\int_h E v da + \int_h \psi^0 E_n da + F^2 \oint_c (E \psi_x^0 - \psi^0 E_x) \mu ds = \int_h E(v + \psi_n^0) da + F^2 \int_f E \psi_{xx}^0 dx dy .$$

As it was shown in Parts 2 and 3 of this study [1,2] we have  $\psi_x^0 = \sigma \psi_s^0 + \tau \psi_t^0 + \nu \psi_n^0$ , which becomes  $\psi_x^0 = \sigma \psi_s^0 + \tau \psi_t^0 - \nu^2 + \nu(v + \psi_n^0)$  by virtue of equation (38c). We may then obtain

$$\begin{aligned} \int_h E v da - F^2 \oint_c E v^2 \mu ds + \int_h \psi^0 E_n da + F^2 \oint_c [E(\sigma \psi_s^0 + \tau \psi_t^0) - \psi^0 E_x] \mu ds = \\ F^2 \int_f E \psi_{xx}^0 dx dy + \int_h E(v + \psi_n^0) da - F^2 \oint_c E(v + \psi_n^0) \nu \mu ds . \end{aligned}$$

By substituting the above equation into expression (35), and using formulas (32) and (36a), we can finally obtain the following alternative expression for the first low-Froude-number approximation  $\Omega_1^{\mathcal{L}F}(t)$

$$\boxed{\Omega_1^{\mathcal{L}F}(t) = F^2 \int_f E(\psi_{xx}^0 + \bar{q}_f^0) dx dy + \int_h E(v + \psi_n^0)_H da - F^2 \oint_c E(v + \psi_n^0)_H \nu \mu ds} , \quad (40)$$

where  $\bar{q}_f^0$  is defined as  $\bar{q}_f^0 \equiv q_f^0 / F^2$ .

With (h) taken as the wetted hull of the ship in position of rest, and  $\psi^0$  chosen as the exact zero-Froude-number potential  $\phi^0$ , we have  $(v + \psi_n^0)_H = 0$  by virtue of equation (12), and expression (40) becomes

$$\Omega_1^{\mathcal{L}F}(t) = F^2 \int_f E(\phi_{xx}^0 + \bar{q}_f^0) dx dy . \quad (41)$$

If, furthermore, the nonlinear free-surface flux  $\bar{q}_f^0$  is neglected, we obtain the

Guevel-Kayo low-Froude-number approximation",  $\Omega_{GK}(t)$  say, given by

$$\boxed{\Omega_{GK}(t) = F^2 \int_f E(x,y,z=0;t) \phi_{xx}^0(x,y,z=0) dx dy} ; \quad (42)$$

indeed, the above approximation to the Kochin function  $\Omega(t)$  is essentially the same as the low-Froude-number approximation which was first obtained by Guevel, Vaussy and Kobus [ 3 ], and is precisely identical to the formula later derived by Kayo [ 4 ]. As it may be seen from equation (37), an alternative expression for the Guevel-Kayo approximation is

$$\boxed{\Omega_{GK}(t) = \int_h (E\nu + \phi^0 E_n) da - F^2 \oint_c \left[ E\{v^2 - (\sigma\phi_s^0 + \tau\phi_t^0)\} + \phi^0 E_x \right] \mu ds} , \quad (42a)$$

which may in fact be equally convenient as formula (40) for purposes of numerical calculations.

Approximation (41), in which the nonlinear free-surface flux  $\bar{q}_f^0$  is retained, may be shown to be identical to the "Baba-Maruo low-Froude-number approximation",  $\Omega_{BM}(t)$  say, which thus is given by

$$\boxed{\Omega_{BM}(t) = \Omega_{GK}(t) + \int_f E q_f^0 dx dy = F^2 \int_f E (\phi_{xx}^0 + \bar{q}_f^0) dx dy} , \quad (43a,b)$$

where  $q_f^0 \equiv F^2 \bar{q}_f^0$  is defined by formula (36b), with  $\psi^0$  replaced by  $\phi^0$ . Indeed, it can easily be verified that the term  $\phi_{xx}^0 + \bar{q}_f^0$  may be expressed in the form

$$\phi_{xx}^0 + \bar{q}_f^0 = \left[ (1 + \phi_x^0) (\phi_x^0 + \frac{1}{2} |\nabla_2 \phi^0|^2) \right]_x + \left[ \phi_y^0 (\phi_x^0 + \frac{1}{2} |\nabla_2 \phi^0|^2) \right]_y , \quad (43c)$$

which is identical to expression (11) in Baba [ 5 ] and expression (32) in Maruo [ 6 ].

If the potential  $\psi^0$  is taken as the approximation given by  $k(\beta, \delta) \phi_I^0$ , where  $k(\beta, \delta)$  is the function of the beam/length ratio  $\beta$  and draft/length ratio  $\delta$  defined by formula (28a) and  $\phi_I^0$  is the initial potential given by equation (18), an alternative form of expressions (37) and (40) for the first-order low-Froude-number approximation  $\Omega_1^{LF}(t)$  may be obtained by exploiting the fact that the potential  $\psi^0$  now is defined not only in the flow domain (d) but also in the "interior domain" ( $d_1$ ), i.e. in the domain bounded externally by the hull surface (h) and the portion, ( $f_1$ ) say, of the plane  $z = 0$  inside the "waterline" (c). In fact, it may easily be shown that the potential  $\psi^0$  verifies the equations

$$\nabla^2 \psi^0 = 0 \quad \text{in } (d_i) \quad , \quad (44a)$$

$$\psi_z^0 = 0 \quad \text{on } (f_i) \quad , \quad (44b)$$

$$\psi_n^0 = kv + k e \phi_{In}^0 \quad \text{on } (h) \quad , \quad (44c)$$

where the fact that the potential  $\psi^0 \equiv k\phi_I^0$  is associated with a distribution of sources of density  $kv$  on the surface  $(h)$  was used in equation (44c) [specifically, we have  $i\psi_n^0 - e\psi_n^0 = kv$ , where the superscripts  $i$  and  $e$  refer to the "interior" and "exterior" sides of  $(h)$ , respectively], and the superscript  $e$  in the symbol  $e\phi_{In}^0$  is meant to clearly indicate that the normal derivative  $\phi_{In}^0$  of the zero-Froude-number initial potential  $\phi_I^0$  is to be evaluated on the exterior side of the hull surface  $(h)$ .

By virtue of equations (39a) and (44a), the following Green identity

$$\int_h (\psi^0 E_n^0 - E\psi_n^0) da = \int_{f_i} (\psi^0 E_z^0 - E\psi_z^0) dx dy$$

holds. By using equations (44b), (39b), and (44c), we may obtain

$$\int_h \psi^0 E_n^0 da - k \int_h E v da - k \int_h E e \phi_{In}^0 da + F^2 \int_{f_i} \psi^0 E_{xx}^0 dx dy = 0 \quad .$$

By expressing the term  $\psi^0 E_{xx}^0$  in the form  $(\psi^0 E_x^0 - E\psi_x^0)_x + E\psi_{xx}^0$ , and using the Green identity

$$\int_{f_i} (\psi^0 E_x^0 - E\psi_x^0)_x dx dy = \oint_c (\psi^0 E_x^0 - E\psi_x^0) dy = - \oint_c (\psi^0 E_x^0 - E\psi_x^0) \mu ds \quad ,$$

where we used the relation  $dy = -\mu ds$  along  $(c)$ , we can obtain

$$\int_h \psi^0 E_n^0 da - k \int_h E v da - k \int_h E e \phi_{In}^0 da + F^2 \oint_c (E\psi_x^0 - \psi^0 E_x^0) \mu ds + F^2 \int_{f_i} E\psi_{xx}^0 dx dy = 0 \quad .$$

As it was noted previously, we have  $\psi_x^0 = \sigma\psi_s^0 + \tau\psi_t^0 + v\psi_n^0$ , which becomes  $\psi_x^0 = \sigma\psi_s^0 + \tau\psi_t^0 + kv^2 + k e \phi_{In}^0 v$  by virtue of equation (44c), so that we may obtain

$$\begin{aligned} \int_h \psi^0 E_n^0 da + F^2 \oint_c [E(\sigma\psi_s^0 + \tau\psi_t^0) - \psi^0 E_x^0] \mu ds &= k\Omega_0(t) - F^2 \int_{f_i} E\psi_{xx}^0 dx dy + \\ &+ k \int_h E e \phi_{In}^0 da - F^2 k \oint_c E e \phi_{In}^0 v \mu ds \quad , \end{aligned}$$

where formula (32) was used. By substituting the above equation into expression (35) for the first-order low-Froude-number approximation  $\Omega_1^{\mathcal{L}F}$ (t), and using equations (36a) and (32), we may then obtain the following alternative expression for the first-order low-Froude-number slender-ship approximation,  $\Omega_1^{\mathcal{L}Fs\mathcal{L}}$ (t), say

$$\Omega_1^{\mathcal{L}Fs\mathcal{L}}(t) = k\Omega_0(t) + \int_h E(v + \psi_n^0)_H da - F^2 \oint_c E(v + \psi_n^0)_H \nu \mu ds \\ - F^2 \int_{f_i} E\psi_{xx}^0 dx dy + \int_f Eq_f^0 dx dy .$$

With the potential  $\psi^0$  explicitly expressed as  $k(\beta, \delta)\phi_I^0$  we finally have

$$\boxed{\Omega_1^{\mathcal{L}Fs\mathcal{L}}(t) = k\Omega_0(t) + \int_h E(v + k\phi_n^0)_H da - F^2 \oint_c E(v + k\phi_n^0)_H \nu \mu ds \\ - F^2 k \int_{f_i} E\phi_{xx}^0 dx dy + F^2 k^2 \int_f Eq_f^0 dx dy} , \quad (45)$$

where the zero-Froude-number initial potential  $\phi_I^0$ , given by formula (18), was written in the form  $\phi_I^0$  for convenience, and the nonlinear free-surface flux  $q_f^0$  is defined as

$$q_f^0 = (|\nabla_2 \phi_I^0|^2)_x + \phi_x^0 \nabla_2^2 \phi_I^0 + \frac{1}{2} [\nabla_2 \phi_I^0 \cdot \nabla_2 |\nabla_2 \phi_I^0|^2 + |\nabla_2 \phi_I^0|^2 \nabla_2^2 \phi_I^0] , \quad (45a)$$

as it may be obtained from equation (36b). The first two terms on the right side of equation (45a) are  $O(|\nabla \phi_I^0|^2)$ , while the terms between brackets are  $O(|\nabla \phi_I^0|^3)$ , so that these "third-order" terms might be neglected for usual slender ship-hull forms.

The above "first low-Froude-number approximation"  $\Omega_1^{\mathcal{L}Fs\mathcal{L}}$  may be regarded as an approximation, specifically a low-Froude-number approximation, to the "first slender-ship approximation"  $\Omega_1$  defined by formula (22) in Part 3 of this study [2]. Indeed, comparison of formula (45) above and of formula (22) in Part 3 shows that, besides the fact that the multiplicative correction function  $k(\beta, \delta)$  was not introduced in expression (22) in Part 3 (i.e. we simply have  $k = 1$  in this formula), the present low-Froude-number approximation (45) essentially corresponds to approximating the initial potential  $\phi_I$ , given by formula (7), by the zero-Froude-number initial potential  $\phi_I^0$  given by formula (18). In addition, however, the above expression (45) contains a surface integral over the ship "waterplane" ( $f_i$ ) that has no counterpart in expression (22) in Part 3. The numerical simplification resulting from the substitution of the potential  $\phi_I$  given by formula (7) by the zero-Froude-number potential  $\phi_I^0$  given by formula (18), which mainly stems from the much-simpler form of the zero-Froude-number

Green function  $G^0$  given by formula (14) in comparison with the Kelvin Green function  $G$  given by formula (8), is significant, and indeed a main recommendation of the low-Froude-number slender-ship approximation (45) in comparison with the slender-ship approximation (22) of Part 3 resides in the fact that the approximation (45) does not require evaluation of the fairly complicated Kelvin Green function (14). The approximation  $\Omega_1^{\ell F s \ell}$  defined by formula (45) also appears to be somewhat simpler than the low-Froude-number approximations  $\Omega_{GK}$  and  $\Omega_{BM}$  defined by formulas (42,a) and (43a,b) in that these approximations, based on the exact zero-Froude-number potential  $\phi^0$ , evidently require the preliminary determination of the potential  $\phi^0$ , that is the preliminary solution of a zero-Froude-number integral equation (the method of Hess and Smith [11] is usually used for this purpose).

Expression (45) for the first low-Froude-number slender-ship approximation  $\Omega_1^{\ell F s \ell}(t)$  may be written in the form

$$\boxed{\Omega_1^{\ell F s \ell}(t) = \Omega_{1\ell}^{\ell F s \ell}(t) + F^2 k^2 \int_f E^0 q_f^I dx dy} \quad (46)$$

where the linearized first low-Froude-number slender-ship approximation  $\Omega_{1\ell}^{\ell F s \ell}(t)$  is given by

$$\boxed{\Omega_{1\ell}^{\ell F s \ell}(t) = \int_h E \lambda da - F^2 \oint_c E \lambda v \mu ds - F^2 k \int_{f_1} E^0 \phi_{xx}^I dx dy} \quad (47)$$

with the source density  $\lambda$  defined as

$$\lambda = kv + (v + k \phi_n^I)_{\text{H}} \quad (47a)$$

as it can be obtained by using formula (32). An interesting alternative expression for the approximation  $\Omega_{1\ell}^{\ell F s \ell}$  is

$$\boxed{\Omega_{1\ell}^{\ell F s \ell}(t) = \int_h (Ev + k \phi_I^0 E_n) da - F^2 \oint_c [E\{v^2 - k(\sigma \phi_s^I + \tau \phi_t^I) + k \phi_x^I E_x\} \mu ds} \quad (48)$$

which can readily be obtained by substituting  $k\phi_I^0$  for  $\phi^0$  into expression (42a). The above-defined approximation  $\Omega_{1\ell}^{\ell F s \ell}$  is the simplest of the low-Froude-number approximations obtained in this section. As a matter of fact, comparison of formula (32) for the zeroth approximation  $\Omega_0$  with formulas (48) and (47) for the approximation  $\Omega_{1\ell}^{\ell F s \ell}$  shows that, from the computational point of view, the latter approximation requires a relatively simple modification of the basic zeroth approximation  $\Omega_0(t)$ .

In summary, several (evidently related) low-Froude-number approximations to the Kochin free-wave spectrum function  $\Omega(t)$  have been obtained. The common characteristic feature of these various first-order low-Froude-number approximations resides in that they are based upon approximating the disturbance velocity potential  $\phi$  in the ship near field, in formula (31), by the "zero-Froude-number potential"  $\psi^0$ , which was taken as either the "exact zero-Froude number potential"  $\phi^0$ , given by the solution of the "zero-Froude-number integral equation" (17), or as the "zero-Froude-number slender-ship approximation"  $k(\beta, \delta)\phi_I^0$  defined by formulas (18) and (28a). While the first-order low-Froude-number approximations  $\Omega_1^{\ell F}$  and  $\Omega_1^{\ell F s \ell}$  are somewhat more "sophisticated" than the basic "zeroth approximation"  $\Omega_0$  given by formula (32), which is simply associated with the approximation  $\phi = 0$  in the ship near field, these approximations admittedly are still fairly "crude". More "refined" (second-order low-Froude-number) approximations ( $\Omega_2^{\ell F}$  and  $\Omega_2^{\ell F s \ell}$ ) can however be defined by using a more "realistic" approximation to the potential  $\phi$  in the ship near field, in formula (31), than the "wave-free" zero-Froude-number potential  $\psi^0$ . The analysis of section 2 readily suggests that natural approximations for  $\phi$  are the "first-order low-Froude-number approximations",  $\phi_1^{\ell F}$  and  $\phi_1^{\ell F s \ell}$  say, defined by evaluating the various unknown terms on the right side of the integral equation (6) on the basis of the zero-Froude-number potential  $\psi^0$  (taken as  $\phi^0$  or as  $k\phi_I^0$ ) as an approximation to  $\phi$ . These first low-Froude-number approximations to the potential  $\phi$  will now be determined.

4. First-order low-Froude-number approximations to the velocity potential

As it was noted above, the first-order low-Froude number approximation  $\phi_1^{\mathcal{L}F}$  to the potential  $\phi$  may be defined by approximating  $\phi$  in the unknown terms on the right side of the integral equation (6) by the zero-Froude-number potential  $\psi^0$ . We thus have

$$\begin{aligned} \phi_1^{\mathcal{L}F}(\vec{x}_*) &= \phi_I(\vec{x}_*) + \int_h (\psi^0 - \psi_*^0) G_n da + F^2 \oint_c [G(\sigma\psi_s^0 + \tau\psi_t^0) - (\psi^0 - \psi_*^0) G_x] \mu ds + \\ &+ \int_f Gq_f^0 dx dy + \int_h Gq_h^0 da - F^2 \oint_c Gq_h^0 \nu \mu ds, \end{aligned} \quad (49)$$

where the potential  $\phi_I(\vec{x}_*)$  is the initial potential defined by formula (7), and  $q_h^0$  and  $q_f^0$  are the hull flux and the nonlinear free-surface flux given by formulas (36a) and (36b), respectively. By using equation (7), expression (49) for the approximation  $\phi_1^{\mathcal{L}F}$  may be written in the form

$$\boxed{\begin{aligned} \phi_1^{\mathcal{L}F}(\vec{x}_*) &= \int_h [G(\nu + q_h^0) + (\psi^0 - \psi_*^0) G_n] da + \int_f Gq_f^0 dx dy \\ &- F^2 \oint_c [G\{\nu(\nu + q_h^0) - (\sigma\psi_s^0 + \tau\psi_t^0)\} + (\psi^0 - \psi_*^0) G_x] \mu ds \end{aligned}} \quad (50)$$

An interesting alternative expression for the first-order low-Froude-number approximation  $\phi_1^{\mathcal{L}F}$  can be obtained by exploiting the fact that the potential  $\psi^0$  verifies the equations

$$\nabla^2 \psi^0 = 0 \quad \text{in (d)}, \quad (51a)$$

$$\psi_z^0 + F^2 \psi_{xx}^0 = F^2 \psi_{xx}^0 \quad \text{on (f)}, \quad (51b)$$

$$\psi_n^0 = -[\nu - (\nu + \psi_n^0)_h] \quad \text{on (h)}, \quad (51c)$$

as it can readily be seen from equations (38a,b,c), so that  $\psi^0$  must satisfy the following identity<sup>†</sup>

$$\begin{aligned} \psi_*^0 &= \int_h G[\nu - (\nu + \psi_n^0)_h] da - F^2 \oint_c G[\nu - (\nu + \psi_n^0)_h] \nu \mu ds - F^2 \int_f G\psi_{xx}^0 dx dy + \\ &+ \int_h (\psi^0 - \psi_*^0) G_n da + F^2 \oint_c [G(\sigma\psi_s^0 + \tau\psi_t^0) - (\psi^0 - \psi_*^0) G_x] \mu ds, \end{aligned} \quad (51d)$$

<sup>†</sup>Equation (51d) may be written down at once by comparing equations (51b,c), (2), and (3), which show that one simply needs to replace the terms  $\phi$ ,  $q_f$ , and  $(\nu + q_h)$  by the terms  $\psi^0$ ,  $-F^2 \psi_{xx}^0$ , and  $[\nu - (\nu + \psi_n^0)_h]$ , respectively, in the integral equation (6).

where we used the notation  $\psi_*^0 \equiv \psi^0(\vec{x}_*)$  and  $\psi^0 \equiv \psi^0(\vec{x})$ . By using equations (51d), (7), and (36a) into equation (49), we may obtain the following alternative expression for the approximation  $\phi_1^{\mathcal{L}F}$

$$\boxed{\phi_1^{\mathcal{L}F}(\vec{x}_*) = \psi^0(\vec{x}_*) + F^2 \int_f G(\psi_{xx}^0 + \bar{q}_f^0) dx dy + \int_h G(v + \psi_n^0)_{,H} da - F^2 \oint_c G(v + \psi_n^0)_{,H} v \mu ds}, \quad (52)$$

where  $\bar{q}_f^0 \equiv q_f^0/F^2$  as it was defined previously.

If the fictitious hull (h) is taken as the wetted hull of the ship in position of rest [so that we have  $q_h^0 = 0$  in expression (50), by virtue of equation (36a)], the potential  $\psi^0$  is taken as the exact zero-Froude-number potential  $\phi^0$  [so that, by virtue of equation (12), we have  $(v + \psi_n^0)_{,H} = 0$  in expression (52)], and the nonlinear free-surface flux  $q_f^0 \equiv F^2 \bar{q}_f^0$  is neglected, the potential  $\phi_1^{\mathcal{L}F}$  becomes the "Guevel-Kayo potential",  $\phi_{GK}$  say, given by the alternative expressions

$$\boxed{\phi_{GK}(\vec{x}_*) = \phi^0(\vec{x}_*) + F^2 \int_f G(\vec{x}_*, x, y, 0) \phi_{xx}^0(x, y, 0) dx dy}, \quad (53)$$

$$\boxed{\phi_{GK}(\vec{x}_*) = \int_h [Gv + (\phi^0 - \phi_*^0) G_n] da - F^2 \oint_c [G\{v^2 - (\sigma\phi_s^0 + \tau\phi_t^0)\} + (\phi^0 - \phi_*^0) G_x] \mu ds}, \quad (53a)$$

corresponding to expressions (52) and (50), respectively. If the nonlinear free-surface flux  $q_f^0$  is retained, the potential  $\phi_1^{\mathcal{L}F}$  becomes the "Baba-Maruo potential",  $\phi_{BM}$  say, given by

$$\boxed{\phi_{BM}(\vec{x}_*) = \phi_{GK}(\vec{x}_*) + \int_f G q_f^0 dx dy = \phi^0(\vec{x}_*) + F^2 \int_f G(\phi_{xx}^0 + \bar{q}_f^0) dx dy}. \quad (53b,c)$$

If the zero-Froude-number potential  $\psi^0$  is taken as the slender-ship approximation  $k(\beta, \delta)\phi_I^0$ , where  $k(\beta, \delta)$  is the function of the beam/length ratio  $\beta$  and draft/length ratio  $\delta$  defined by equation (28a) and  $\phi_I^0$  is the potential given by formula (18), an alternative form of expressions (50) and (52) for the first low-Froude-number potential  $\phi_1^{\mathcal{L}F}$  can be obtained by exploiting the fact that the potential  $\psi^0 \equiv k\phi_I^0$  now is defined not only in the flow domain (d) but also in the interior domain ( $d_1$ ) bounded by the hull surface (h) and the "waterplane" ( $f_1$ ). More precisely, the potential  $\psi^0$  verifies the following equations

$$\nabla^2 \psi^0 = 0 \quad \text{in } (d_1), \quad (54a)$$

$$\psi_z^0 + F^2 \psi_{xx}^0 = F^2 k^0 \phi_{xx}^I \quad \text{on } (f_i) \quad , \quad (54b)$$

$$\psi_n^0 = kv + k^e \phi_{In}^0 \quad \text{on } (h) \quad , \quad (54c)$$

as it may be obtained from equations (44a,b,c). It can then be shown that  $\psi^0$  must satisfy the following identity

$$\int_h (\psi^0 - \psi_*^0) G_n da + F^2 \oint_c [G(\sigma\psi_s^0 + \tau\psi_t^0) - (\psi^0 - \psi_*^0) G_x] \mu ds =$$

$$\int_h G(kv + k^e \phi_{In}^0) da - F^2 \oint_c G(kv + k^e \phi_{In}^0) \nu \mu ds - F^2 k \int_{f_i} G^0 \phi_{xx}^I dx dy \quad . \quad (54d)^\dagger$$

By substituting equation (54d) into equation (49), and using formulas (7) and (36a), we may then obtain the following expression for the "first-order low-Froude-number slender-ship approximation",  $\phi_1^{\ell Fsl}$  say,

$$\boxed{\phi_1^{\ell Fsl}(\vec{x}_*) = k\phi_I^0(\vec{x}_*) + \int_h G(\nu + k^0 \phi_n^I) da - F^2 \oint_c G(\nu + k^0 \phi_n^I) \nu \mu ds - F^2 k \int_{f_i} G^0 \phi_{xx}^I dx dy + F^2 k^2 \int_f G^0 q_f^I dx dy} \quad , \quad (55)$$

where the zero-Froude-number initial potential  $\phi_I^0$ , given by formula (18), was written in the form  $\phi^I$  for convenience, and the nonlinear free-surface flux  $q_f^I$  is given by formula (45a). Equation (55) may be written in the form

$$\boxed{\phi_1^{\ell Fsl}(\vec{x}_*) = \phi_{1\ell}^{\ell Fsl}(\vec{x}_*) + F^2 k^2 \int_f G^0 q_f^I dx dy} \quad , \quad (56)$$

where the linearized low-Froude-number slender-ship potential  $\phi_{1\ell}^{\ell Fsl}$  can [by using formula (7)] be expressed as

$$\boxed{\phi_{1\ell}^{\ell Fsl}(\vec{x}_*) = \int_h G\lambda da - F^2 \oint_c G\lambda \nu \mu ds - F^2 k \int_{f_i} G^0 \phi_{xx}^I dx dy} \quad , \quad (57)$$

with the source density  $\lambda$  given by formula (47a). An interesting alternative expression for the approximation  $\phi_{1\ell}^{\ell Fsl}$  is

<sup>†</sup>The identity (54d) can be established by applying the approach used in Part 2 of this study [1], with straightforward modifications accounting for the fact that equations (54a,b,c) correspond to an interior (rather than exterior) problem.

$$\phi_{1\ell}^{\lambda F s \ell}(\vec{x}_*) = \int_h [Gv + k(\phi_I^0 - \phi_*^0)G_n] da - F^2 \int_c [G\{v^2 - k(\sigma \phi_s^0 + \tau \phi_t^0)\} + k(\phi_I^0 - \phi_*^0)G_x] \mu ds, \quad (58)$$

which can be obtained by substituting  $k \phi_I^0 \equiv k \phi_I^0$  for  $\phi^0$  into expression (53a).

Comparison of expression (55) for the first-order low-Froude-number slender-ship approximation  $\phi_1^{\lambda F s \ell}$  and expression (2.27) in Part 2 of this study [1] for the second-order slender-ship approximation  $\phi_2$  shows that the potential  $\phi_1^{\lambda F s \ell}$  can be regarded as a low-Froude-number approximation to the potential  $\phi_2$ . Specifically, it may be seen that, besides the fact that the multiplicative correction function  $k(\beta, \delta)$  was not introduced in expression (2.27) for the approximation  $\phi_2$ , the potential  $\phi_1^{\lambda F s \ell}$  essentially corresponds to approximating the initial potential  $\phi_I$  given by formula (7) by the zero-Froude-number initial potential  $\phi_I^0$  given by formula (18), although an additional integral, over the ship "waterplane" ( $f_1$ ), appears in expression (55) for  $\phi_1^{\lambda F s \ell}$ . A main recommendation of the low-Froude-number approximation  $\phi_1^{\lambda F s \ell}$ , in comparison with the approximation  $\phi_2$ , obviously resides in the notable numerical simplification associated with the substitution of the potential  $\phi_I^0$  for the potential  $\phi_I$ . As a matter of fact, from the computational point of view, the approximation  $\phi_1^{\lambda F s \ell}$ , expressed in the form given by formula (58) say, corresponds to a relatively simple modification of the basic initial potential  $\phi_I$ , given by formula (7), associated with the approximation  $\phi \equiv 0$  in the unknown terms on the right side of the integral equation (6). Actually, the first low-Froude-number slender-ship approximation  $\phi_1^{\lambda F s \ell}$ , associated with the approximation  $\phi = k \phi_I^0$  in the unknown terms in the integral equation (6), may just as well be regarded as a low-Froude-number approximation of the second slender-ship approximation  $\phi_2$ , as it was just discussed, or as a generalization of the initial potential  $\phi_I$  ( $\equiv$  first-order slender-ship approximation  $\phi_1$ ), and it may indeed readily be verified that expressions (55) through (58) become expression (7) if  $k$  is replaced by zero. In conclusion to this section, it will be noted that expressions (50), (52), (53,a,b,c), (55), (56), (57) and (58) for the several (obviously related) first low-Froude-number approximations to the potential  $\phi$  clearly correspond to expressions (37), (40), (42,a), (43a,b), (45), (46), (47), and (48), respectively, for the first low-Froude-number approximations to the Kochin free-wave spectrum function  $\Omega$ , and indeed the latter approximations to the Kochin spectrum function could evidently have been derived from the approximations for the potential  $\phi$ , that is following the present section rather than preceding it.

### 5. Second-order low-Froude-number approximations

A second-order low-Froude-number approximation,  $\phi_2^{\mathcal{L}F}$  say, to the disturbance potential  $\phi$  may readily be defined by approximating the potential  $\phi$  on the right side of the integral equation (6) by the previously-determined first low-Froude-number approximation  $\phi_1^{\mathcal{L}F}$ . We may then obtain

$$\begin{aligned} \phi_2^{\mathcal{L}F}(\vec{x}_*) &= \phi_I(\vec{x}_*) + \int_h (\phi_1^{\mathcal{L}F} - \phi_{1*}^{\mathcal{L}F}) G_n da + F^2 \oint_c [G(\sigma\phi_{1s}^{\mathcal{L}F} + \tau\phi_{1t}^{\mathcal{L}F}) - (\phi_1^{\mathcal{L}F} - \phi_{1*}^{\mathcal{L}F}) G_x] \mu ds + \\ &+ \int_f Gq_{f1}^{\mathcal{L}F} dx dy + \int_h Gq_{h1}^{\mathcal{L}F} da - F^2 \oint_c Gq_{h1}^{\mathcal{L}F} \nu \mu ds, \end{aligned} \quad (59)$$

where the nonlinear free-surface flux  $q_{f1}^{\mathcal{L}F}$  and the hull flux  $q_{h1}^{\mathcal{L}F}$  are given by formulas (4) and (5), respectively, in which  $\phi$  is replaced by  $\phi_1^{\mathcal{L}F}$ . By using formula (7) for the initial potential  $\phi_I(\vec{x}_*)$ , expression (59) for  $\phi_2^{\mathcal{L}F}(\vec{x}_*)$  may be written in the form

$$\boxed{\begin{aligned} \phi_2^{\mathcal{L}F}(\vec{x}_*) &= \int_h [G(\nu + q_{h1}^{\mathcal{L}F}) + (\phi_1^{\mathcal{L}F} - \phi_{1*}^{\mathcal{L}F}) G_n] da + \int_f Gq_{f1}^{\mathcal{L}F} dx dy \\ &- F^2 \oint_c [G\{\nu + q_{h1}^{\mathcal{L}F}\} - (\sigma\phi_{1s}^{\mathcal{L}F} + \tau\phi_{1t}^{\mathcal{L}F})] + (\phi_1^{\mathcal{L}F} - \phi_{1*}^{\mathcal{L}F}) G_x \mu ds \end{aligned}} \quad (60)$$

An alternative expression for the above second-order low-Froude-number approximation  $\phi_2^{\mathcal{L}F}$  can be obtained by exploiting the fact that the first approximation  $\phi_1^{\mathcal{L}F}$  verifies the equations

$$\nabla^2 \phi_1^{\mathcal{L}F} = 0 \quad \text{in (d)}, \quad (61a)$$

$$\phi_{1z}^{\mathcal{L}F} + F^2 \phi_{1xx}^{\mathcal{L}F} = -q_f^0 \quad \text{on (f)}, \quad (61b)$$

$$\phi_{1n}^{\mathcal{L}F} = -[\nu - (\nu + \phi_{1n}^{\mathcal{L}F})_h] \quad \text{on (h)}, \quad (61c)$$

as it can readily be seen, so that we have the identity

$$\begin{aligned} \phi_{1*}^{\mathcal{L}F} &= \int_h G[\nu - (\nu + \phi_{1n}^{\mathcal{L}F})_h] da - F^2 \oint_c G[\nu - (\nu + \phi_{1n}^{\mathcal{L}F})_h] \nu \mu ds + \int_f Gq_f^0 dx dy + \\ &+ \int_h (\phi_1^{\mathcal{L}F} - \phi_{1*}^{\mathcal{L}F}) G_n da + F^2 \oint_c [G(\sigma\phi_{1s}^{\mathcal{L}F} + \tau\phi_{1t}^{\mathcal{L}F}) - (\phi_1^{\mathcal{L}F} - \phi_{1*}^{\mathcal{L}F}) G_x] \mu ds, \end{aligned} \quad (61d)$$

where  $\phi_{1*}^{\mathcal{L}F} \equiv \phi_1^{\mathcal{L}F}(\vec{x}_*)$  and  $\phi_{1n}^{\mathcal{L}F} \equiv \nabla\phi_1^{\mathcal{L}F} \cdot \vec{n}$ . By using equations (61d), (5), and (7) into

equation (59), we may then obtain the following alternative expression for the approximation  $\phi_2^{\mathcal{L}F}$

$$\boxed{\phi_2^{\mathcal{L}F}(\vec{x}_*) = \phi_1^{\mathcal{L}F}(\vec{x}_*) + \int_h G(v+\phi_{1n}^{\mathcal{L}F})_H da - F^2 \oint_c G(v+\phi_{1n}^{\mathcal{L}F})_H v \mu ds + \int_f G \pi_1^{\mathcal{L}F} dx dy} \quad , \quad (62)$$

where the free-surface flux  $\pi_1^{\mathcal{L}F}$  is defined as  $\pi_1^{\mathcal{L}F} = q_{f1}^{\mathcal{L}F} - q_f^0$ . It follows from equation (4), in which  $\phi$  is replaced by  $\phi_1^{\mathcal{L}F}$ , and equation (61b) that  $\pi_1^{\mathcal{L}F}$  is given by

$$\pi_1^{\mathcal{L}F} = \left[ \phi_{1z}^{\mathcal{L}F} + F^2 \left\{ \phi_{1xx}^{\mathcal{L}F} + (|\nabla \phi_1^{\mathcal{L}F}|^2)_x + \frac{1}{2} \nabla \phi_1^{\mathcal{L}F} \cdot \nabla |\nabla \phi_1^{\mathcal{L}F}|^2 \right\} \right]_z = -F^2 \left( \phi_{1x}^{\mathcal{L}F} + \frac{1}{2} |\nabla \phi_1^{\mathcal{L}F}|^2 \right) \quad . \quad (62a)$$

From the alternative expressions (60) and (62) for the second low-Froude-number approximation  $\phi_2^{\mathcal{L}F}$  to the potential  $\phi$ , we can easily derive the following alternative expressions for the second-order low-Froude-number approximation  $\Omega_2^{\mathcal{L}F}$  to the Kochin free-wave spectrum function  $\Omega(t)$

$$\boxed{\Omega_2^{\mathcal{L}F}(t) = \int_h [E(v + q_{h1}^{\mathcal{L}F}) + \phi_1^{\mathcal{L}F} E_n] da + \int_f E q_{f1}^{\mathcal{L}F} dx dy - F^2 \oint_c [E\{v(v + q_{h1}^{\mathcal{L}F}) - (\sigma \phi_{1s}^{\mathcal{L}F} + \tau \phi_{1t}^{\mathcal{L}F})\} + \phi_1^{\mathcal{L}F} E_x] \mu ds} \quad , \quad (63a)$$

$$\boxed{\Omega_2^{\mathcal{L}F}(t) = \Omega_1^{\mathcal{L}F}(t) + \int_h E(v+\phi_{1n}^{\mathcal{L}F})_H da - F^2 \oint_c E(v+\phi_{1n}^{\mathcal{L}F})_H v \mu ds + \int_f E \pi_1^{\mathcal{L}F} dx dy} \quad . \quad (63b)$$

Second-order low-Froude-number approximations  $R_2^{\mathcal{L}F}$  to the wave resistance  $R$  can then be readily defined by using the above second-order approximation  $\Omega_2^{\mathcal{L}F}$  to the Kochin free-wave spectrum function  $\Omega$  in the Havelock wave resistance formula (34).

Second-order low-Froude-number slender-ship approximations  $\phi_2^{\mathcal{L}Fs\ell}$ ,  $\Omega_2^{\mathcal{L}Fs\ell}$ , and  $R_2^{\mathcal{L}Fs\ell}$  to the potential  $\phi$ , the Kochin spectrum function  $\Omega$ , and the wave resistance  $R$ , respectively, can similarly be defined by merely replacing  $\phi_1^{\mathcal{L}F}$  by  $\phi_1^{\mathcal{L}Fs\ell}$  in expressions (60), (62), and (63a,b). Of main practical interest may be the second-order low-Froude-number slender-ship wave resistance approximation  $R_2^{\mathcal{L}Fs\ell}$  associated with the approximation  $\Omega_2^{\mathcal{L}Fs\ell}$  to the Kochin spectrum function  $\Omega$ , corresponding to the use of the linearized first-order slender-ship approximation  $\phi_{1\ell}^{\mathcal{L}Fs\ell}$  defined by formula (58) as an approximation to the potential  $\phi$  in formula (31) for  $\Omega$ . A slight change of notation in equation (63a) readily yields

$$\Omega_2^{\ell Fsl}(t) = \int_h [E(v + q_{h1l}^{\ell Fsl}) + \phi_{1l}^{\ell Fsl} E_n] da + \int_f E q_{f1l}^{\ell Fsl} dx dy - F^2 \oint_c [E(v(v + q_{h1l}^{\ell Fsl}) - (\sigma \phi_{1l,s}^{\ell Fsl} + \tau \phi_{1l,t}^{\ell Fsl})) + \phi_{1l}^{\ell Fsl} E_x] \mu ds, \quad (64)$$

where the notation  $\phi_{1l,s}^{\ell Fsl} \equiv \partial \phi_{1l}^{\ell Fsl} / \partial s$ , and  $\phi_{1l,t}^{\ell Fsl} \equiv \partial \phi_{1l}^{\ell Fsl} / \partial t$  was used, the hull flux  $q_{h1l}^{\ell Fsl}$  is given by equation (5) with  $\phi$  replaced by  $\phi_{1l}^{\ell Fsl}$ , that is we have

$$q_{h1l}^{\ell Fsl} = (v + \phi_{1l,n}^{\ell Fsl})_H - (v + \phi_{1l,n}^{\ell Fsl})_h, \quad (64a)$$

and the nonlinear free-surface flux  $q_{f1l}^{\ell Fsl}$  is defined by formula (4) in which  $\phi$  must evidently also be replaced by  $\phi_{1l}^{\ell Fsl}$ ; by performing a Taylor series expansion of the right side of equation (4) about the plane  $z = 0$ , and retaining only the terms that are  $O(F^2 |\nabla \phi|^2)$  as a first approximation, we may obtain

$$q_{f1l}^{\ell Fsl} \approx F^2 \left[ (|\nabla \phi_{1l}^{\ell Fsl}|^2)_x - \phi_{1l,x}^{\ell Fsl} \phi_{1l,zz}^{\ell Fsl} \right], \quad (64b)$$

where the expression on the right side is to be evaluated on (f), i.e. on the plane  $z = 0$ .

It is interesting to compare the above second-order low-Froude-number slender-ship approximation  $\Omega_2^{\ell Fsl}$  and the straightforward first-order slender-ship approximation  $\Omega_1$  given by equation (18) in Part 3 of this study [2]. In terms of the notation used in the present Part 4, this slender-ship approximation  $\Omega_1$  is given by the expression

$$\Omega_1(t) = \int_h [E(v + q_h^I) + \phi^I E_n] da + \int_f E q_f^I dx dy - F^2 \oint_c [E(v(v + q_h^I) - (\sigma \phi_s^I + \tau \phi_t^I)) + \phi^I E_x] \mu ds, \quad (65)$$

where  $\phi^I \equiv \phi_I$  is the initial potential defined by formula (7), the hull flux  $q_h^I$  is given by equation (5) with  $\phi$  replaced by  $\phi_I$ , that is we have

$$q_h^I = (v + \phi_n^I)_H - (v + \phi_n^I)_h, \quad (65a)$$

and the nonlinear free-surface flux is given by equation (3a) in [2], which becomes

$$q_f^I = F^2 \left[ (|\nabla \phi_I|^2)_x - \phi_x^I (\phi_z^I + F^2 \phi_{xx}^I)_z \right]. \quad (65b)$$

Expressions (64,a,b) and (65,a,b) clearly are identical, except for the fact that the initial potential  $\phi_I$  in expressions (65) for  $\Omega_1$  becomes the linearized first low-Froude-number slender-ship approximation  $\phi_{1l}^{\ell Fsl}$  in expressions (64) for  $\Omega_2^{\ell Fsl}$ . Any difference, notably from the computational point of view, between the approximations  $\Omega_2^{\ell Fsl}$  and  $\Omega_1$  must then stem from the potentials  $\phi_I$  and  $\phi_{1l}^{\ell Fsl}$ , which are given by formulas (7) and (58), respectively. As it was already noted in the previous section, the problem of evaluating the potential  $\phi_{1l}^{\ell Fsl}$  numerically is not significantly more difficult than that of evaluating the potential  $\phi_I$ , so that the wave-resistance approximations  $R_2^{\ell Fsl}$  and  $R_1$  associated with the approximations  $\Omega_2^{\ell Fsl}$  and  $\Omega_1$  are comparable from the point of view of required numerical calculations.

6. Alternative derivations of the low-Froude-number approximations

The low-Froude-number approximations  $\phi_1^{\ell F}$ ,  $\phi_1^{\ell F s \ell}$  and  $\phi_2^{\ell F}$ ,  $\phi_2^{\ell F s \ell}$  were obtained in the previous sections as first and second approximations in a sequence of iterative approximations associated with the integral equation (6) and the "zerth approximation"  $\psi^0$ , which was taken as the "exact" zero-Froude-number potential  $\phi^0$  given by the solution of the zero-Froude-number integral equation (17) or as the zero-Froude-number slender-ship approximation  $k(\beta, \delta)\phi_I^0$  defined by formulas (18) and (28a). These low-Froude-number approximations can also be obtained by using two related alternative approaches based on formally expressing the potential  $\phi$  in the form

$$\phi = \psi^0 + \psi \quad (66)$$

either in equations (1) through (5) or in equation (6), that is in the differential or integral formulation of the problem. These alternative approaches will now be briefly examined.

By substituting equation (66) into equations (1) through (5), which must evidently be satisfied by the potential  $\phi \equiv \psi^0 + \psi$ , and using equations (38a,b,c) verified by the zero-Froude-number potential  $\psi^0$ , we may obtain the following equations to be satisfied by the "low-Froude-number potential"  $\psi$

$$\nabla^2 \psi = 0 \quad \text{in (d)} \quad , \quad (67a)$$

$$\psi_z + F^2 \psi_{xx} = - (F^2 \psi_{xx}^0 + q_f^0 + q_f') \quad \text{on (f)} \quad , \quad (67b)$$

$$\psi_n = - \left[ (\psi + \psi_n^0)_H + (\psi_n)_H - (\psi_n)_h \right] \quad \text{on (h)} \quad , \quad (67c)$$

where the nonlinear free-surface flux  $q_f^0 + q_f'$  is identical to  $q_f^0$  (that is we have  $q_f^0 + q_f' \equiv q_f^0$ ) given by formula (4) in which  $\phi$  is replaced by  $\psi^0 + \psi$ , with  $q_f^0$  defined as

$$q_f^0 = \left[ \psi_z^0 + F^2 \left\{ \psi_{xx}^0 + (|\nabla \psi^0|^2)_x + \frac{1}{2} \nabla \psi^0 \cdot \nabla |\nabla \psi^0|^2 \right\} \right]_{z=0} - F^2 \left[ \psi_{xx}^0 \right]_{z=0} \quad , \quad (67d)$$

and  $q_f'$  defined by the relation

$$q_f' = q_f - q_f^0 \quad . \quad (67e)$$

In addition to equations (67a,b,c), the usual radiation condition of no waves upstream from the ship must evidently be satisfied. It may be verified, by performing a Taylor series expansion of the right side of equation (67d), that the low-Froude-number approximation to the nonlinear free-surface flux  $q_f^0$  is given by equation (36b) obtained previously. It will also be noted that the expression on the right side of the hull boundary condition (67c) vanishes if the fictitious hull (h) is taken as the wetted hull of the ship in position of rest and effects of sinkage and trim are neglected, and if  $\psi^0$  is taken as the exact zero-Froude-number potential  $\phi^0$ , since we then have  $(v + \phi_n^0)_H = 0$  by virtue of equation (12).

The "generalized Neumann-Kelvin problem" defined by equations (67a,b,c) and the radiation condition may be stated in integral form, that is in the form of an integral equation, in the manner shown in Part 2 of this study [1]. Specifically, the following integral equation can be obtained for the low-Froude-number correction potential  $\psi$

$$\begin{aligned} \psi(\vec{x}_*) &= \int_h G \left[ (v + \psi_n^0)_H + (\psi_n)_h^H \right] da - F^2 \oint_c G \left[ (v + \psi_n^0)_H + (\psi_n)_h^H \right] v \mu ds + \\ &+ \int_f G (F^2 \psi_{xx}^0 + q_f^0 + q_f^1) dx dy + \int_h (\psi - \psi_*) G_n da + F^2 \oint_c \left[ G(\sigma \psi_s + \tau \psi_t) - (\psi - \psi_*) G_x \right] \mu ds, \end{aligned} \quad (68)^\dagger$$

where the notation  $(\psi_n)_h^H \equiv (\psi_n)_H - (\psi_n)_h$  was used for shortness, and  $\psi_*$  and  $\psi$  are meant for  $\psi(\vec{x}_*)$  and  $\psi(\vec{x})$ , respectively, with  $\vec{x}_*$  representing an arbitrary point in the domain (d) + (f) + (h) + (c) while  $\vec{x}$  is the "integration point", in accordance with previous practice. The integral equation (68) may be expressed in the form

$$\begin{aligned} \psi(\vec{x}_*) &= \psi_I(\vec{x}_*) + \int_h (\psi - \psi_*) G_n da + F^2 \oint_c \left[ G(\sigma \psi_s + \tau \psi_t) - (\psi - \psi_*) G_x \right] \mu ds + \\ &+ \int_h G (\psi_n)_h^H da - F^2 \oint_c G (\psi_n)_h^H v \mu ds + \int_f G q_f^1 dx dy, \end{aligned} \quad (69)$$

where the potential  $\psi_I(\vec{x}_*)$  is defined as

$$\psi_I(\vec{x}_*) = \int_f G (F^2 \psi_{xx}^0 + q_f^0) dx dy + \int_h G (v + \psi_n^0)_H da - F^2 \oint_c G (v + \psi_n^0)_H v \mu ds. \quad (69a)$$

<sup>†</sup>The integral equation (68) may readily be obtained by comparing equations (67b,c) and equations (2) and (3), which show that one simply needs to replace the terms  $q_f$  and  $(v + q_h)$  by the terms  $(F^2 \psi_{xx}^0 + q_f^0 + q_f^1)$  and  $[(v + \psi_n^0)_H + (\psi_n)_h^H]$ , respectively, in the integral equation (6).

The above potential  $\psi_I(\vec{x}_*)$ , which clearly regroups the various "known terms" (that is, the terms involving the presumably-known zero-Froude-number potential  $\psi^0$  alone) in the integral equation (68) provides a natural initial approximation [corresponding to the neglect of the various "unknown terms", i.e. the integrals involving the unknown potential  $\psi$ , in the integral equation (68)] for solving this integral equation iteratively, in the manner used previously in Part 2 of this study [1]. The potential  $\phi$  defined by substituting the initial approximation  $\psi_I$  given by formula (69a) into expression (66) may be seen to be identical to the approximation  $\phi^{\mathcal{L}F}$  obtained previously in equation (52). The second low-Froude-number approximation  $\phi_2^{\mathcal{L}F}$  could likewise be re-derived here as the second approximation in the sequence of iterative approximations associated with the integral equation (69).

The integral equation (69) for the low-Froude-number correction potential  $\psi$  was derived above from the "differential formulation" of the problem, specifically from equations (1) through (5), into which equation (66) was substituted. As one would expect, the integral equation (69) can also be obtained by substituting equation (66) directly into the "integral formulation" of the problem, that is the integral equation (6), and it may be interesting to briefly consider this alternative approach here. By substituting equation (66) into the integral equation (6), expressing the nonlinear free-surface flux  $q_f$  and the hull flux  $q_h$  in the forms  $q_f = q_f^0 + q_f'$ , as defined by equations (67d,e), and  $q_h = q_h^0 + (\psi_n^0)_h^H$ , with  $q_h^0 \equiv (\nu + \psi_n^0)_H - (\nu + \psi_n^0)_h$  as defined by equation (36a) and  $(\psi_n^0)_h^H \equiv (\psi_n^0)_H - (\psi_n^0)_h$  as defined previously, we may express the integral equation (6) in the form of equation (69), where the potential  $\psi_I$  here is defined as

$$\begin{aligned} \psi_I(\vec{x}_*) = & -\psi^0(\vec{x}_*) + \phi_I(\vec{x}_*) + \int_h (\psi^0 - \psi_n^0) G_n da + F^2 \oint_c \left[ G(\sigma\psi_s^0 + \tau\psi_t^0) - (\psi^0 - \psi_n^0) G_x \right] \mu ds + \\ & + \int_f Gq_f^0 dx dy + \int_h Gq_h^0 da - F^2 \oint_c Gq_h^0 \nu \mu ds . \end{aligned}$$

That the above expression for the potential  $\psi_I$  is actually identical to expression (69a) may be verified by using equations (51d), (7) and (36a).

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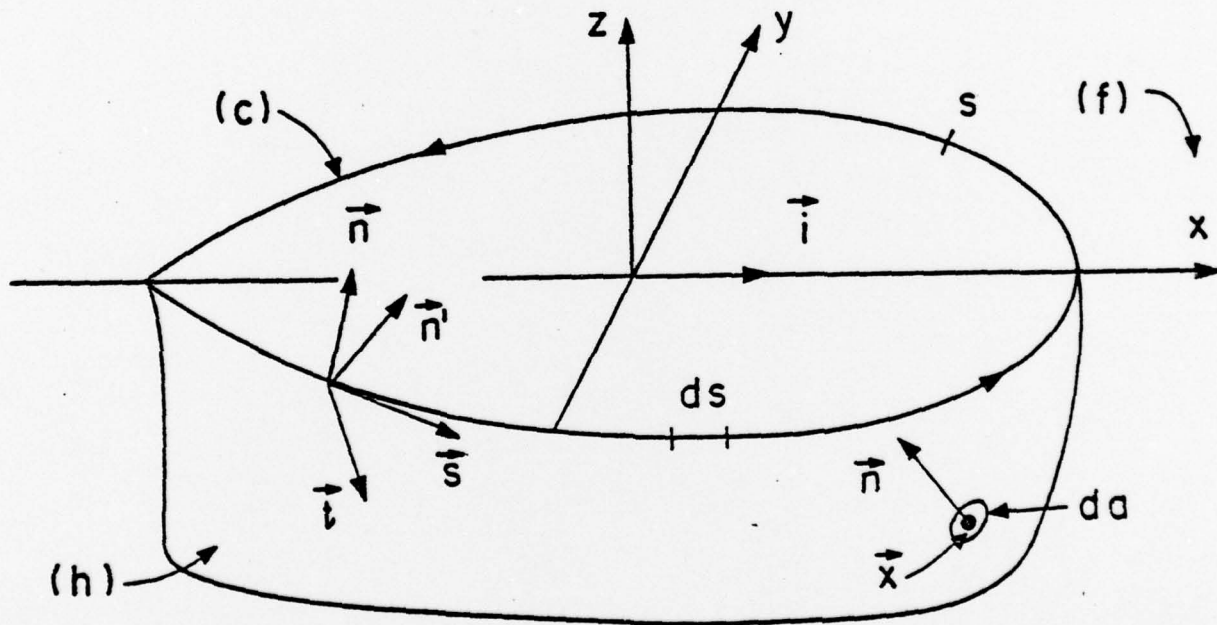


Figure 1: Definition Sketch

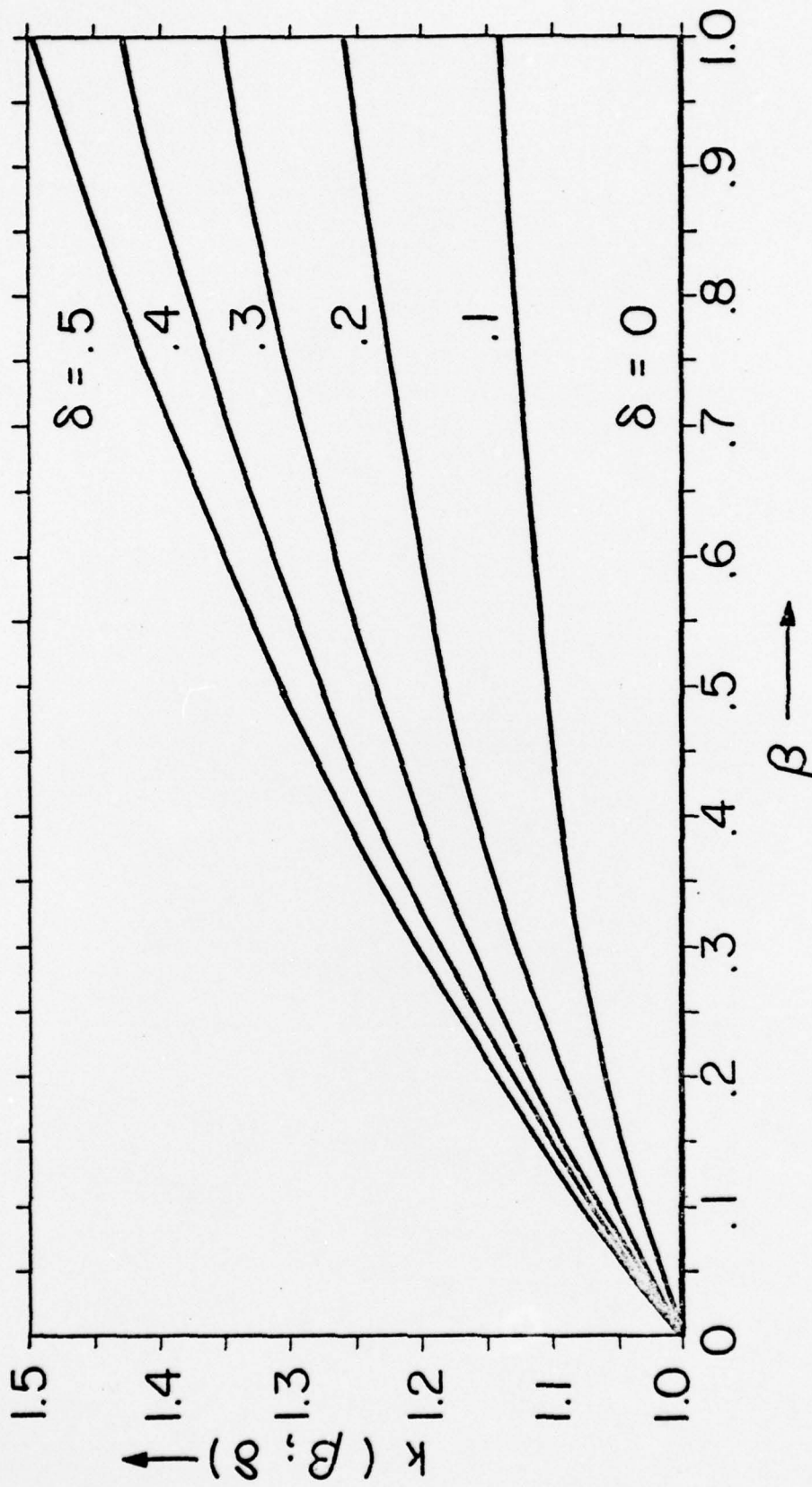


Figure 2: The correction constant  $k(\beta, \delta)$  as a function of the beam/length ratio  $\beta$  and for several values of the draft/length ratio  $\delta$ .

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