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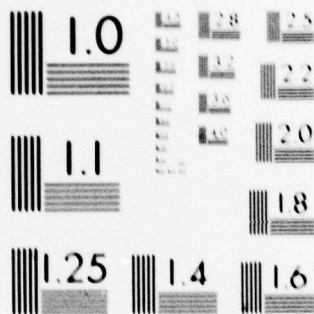
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ON THE COMPLEXITY OF

d-DIMENSIONAL VORONOI DIAGRAMS

by

Victor Klee

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13. ABSTRACT

For n points p_1^n, \dots, p_n^n of Euclidean d -space E^d , the associated Voronoi diagram $V(p_1^n, \dots, p_n^n)$ is a sequence (P_1^n, \dots, P_n^n) of convex polyhedra covering E^d , where P_i^n consists of all points of E^d that have p_i^n as a nearest point in the set $\{p_1^n, \dots, p_n^n\}$. Voronoi diagrams in E^2 have been of interest because of their use by Shamos and others in providing efficient algorithms for a number of computational problems. The efficiency depends on the fact that the diagram itself can be computed efficiently (in time $O(n \log n)$ when $d = 2$). The present paper deals with the complexity of Voronoi diagrams based on n points of E^d . It is shown, in particular, that if $M_0(d, n)$ is the maximum number of vertices that such a diagram may have, then both the limit inferior and the limit superior of the sequence $(\lceil d/2 \rceil ! M_0(d, n) / n^{\lceil d/2 \rceil})_{n=1, 2, \dots}$ are caught between 1 and 2 when d is even, and between $(\lceil d/2 \rceil e)^{-1}$ and 1 when d is odd.

complexity
diagram
facet
linear programming
neighborly
polyhedron
polytope
vertices
Voronoi diagram

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ON THE COMPLEXITY OF d-DIMENSIONAL VORONOI DIAGRAMS

VICTOR KLEE

Abstract Let $M_0(d,n)$ denote the maximum number of vertices of Voronoi diagrams based on n points of E^d . Then for each d , the cluster points of the sequence $(\lceil d/2 \rceil! M_0(d,n)/n^{\lceil d/2 \rceil})_{n=1,2,\dots}$ all lie in the interval $[1,2]$ when d is even and in $[\lceil d/2 \rceil e^{-1}, 1]$ when d is odd.

Introduction For n points p_1, \dots, p_n of Euclidean d -space E^d , the associated Voronoi diagram $V(p_1, \dots, p_n)$ is a sequence (P_1, \dots, P_n) of convex polyhedra covering E^d , where P_i consists of all points of E^d that have p_i as a nearest point in the set $\{p_1, \dots, p_n\}$. Thus

$$P_i = \{x \in E^d : \|x - p_i\| \leq \|x - p_j\| \text{ for all } j\} = \bigcap_{j \neq i} H_{ij},$$

where $H_{ij} = \{x \in E^d : \langle p_j - p_i, x \rangle \leq \frac{1}{2} (\|p_j\|^2 - \|p_i\|^2)\}$.

Note that H_{ij} is the closed halfspace which contains p_i and whose bounding hyperplane passes through the midpoint of the segment $[p_i, p_j]$ and is perpendicular to that segment.

For $0 \leq k < d$, let $\phi_k(p_1, \dots, p_n)$ denote the number of sets S such that S is a k -dimensional face of at least one of the polyhedra P_i . Then $\phi_k(p_1, \dots, p_n)$ is a natural measure of the complexity of the diagram, and the cases $k = 0$ and $k = d-1$ are of special interest. Let $M_k(d,n)$ denote the

maximum of $\phi_k(p_1, \dots, p_n)$ as (p_1, \dots, p_n) ranges over all n -tuples of distinct points of E^d . A routine application of Euler's theorem shows

$$M_0(2, n) = 2n - 5 \quad \text{and} \quad M_1(2, n) = 3n - 6 \quad \text{for all } n > 2.$$

Here it is proved that

$$(1) \quad M_{d-1}(d, n) = \binom{n}{2} \quad \text{for } d \geq 3, \text{ all } n,$$

$$(2) \quad 1 \leq \liminf_{n \rightarrow \infty} \frac{M_0(d, n)}{n^r/r!} \leq \limsup_{n \rightarrow \infty} \frac{M_0(d, n)}{n^r/r!} \leq 2 \quad \text{for even } d = 2r,$$

$$(3) \quad \frac{1}{re} < \liminf_{n \rightarrow \infty} \frac{M_0(d, n)}{n^r/r!} \leq \limsup_{n \rightarrow \infty} \frac{M_0(d, n)}{n^r/r!} \leq 1 \quad \text{for odd } d = 2r - 1.$$

Our method can also be used to obtain inequalities for the other M_k 's

Theorems Not surprisingly, all our proofs are based on properties of neighborly polytopes. A d -polytope (that is, a bounded d -dimensional convex polyhedron) is said to be neighborly if each set of $\lfloor d/2 \rfloor$ of its vertices is the vertex-set of a face. This implies that for $1 \leq j \leq \lfloor d/2 \rfloor$, each set of j vertices is the vertex set of a $(j-1)$ -face. For discussions and constructions of neighborly polytopes, see Gale [4] and Grünbaum [5].

THEOREM 1 If $1 \leq j \leq \lfloor d/2 \rfloor$ then $M_{d-j+1}(d, n) = \binom{n}{j}$ for all n .

Proof. The cases in which $n \leq d + 1$ are left to the reader. With $n \geq d + 2$, let w_1, \dots, w_{n-1} be the vertices of a neighborly d -polytope

Q in E^d such that the origin is interior to Q . Then for $1 \leq j \leq \lfloor d/2 \rfloor$, each j facets ($(d-1)$ - faces) of the polar polytope

$$Q^0 = \{x \in E^d: \langle w_i, x \rangle \leq 1 \text{ for } 1 \leq i < n\}$$

intersect in a $(d-j)$ - face of Q^0 . For $1 \leq i < n$, let $p_i = 2w_i / \|w_i\|^2$. Then for each $x \in E^d$,

$$\langle w_i, x \rangle = 1 \iff \langle p_i, x \rangle = \frac{1}{2} \|p_i\|^2.$$

Let $p_n = 0$ and $(P_1, \dots, P_n) = V(p_1, \dots, p_n)$. Since the affine hulls of Q^0 's facets are the sets of the form $\{x: \langle w_i, x \rangle = 1\}$ for $1 \leq i \leq n$, it follows that $P_n = Q^0$ and for $1 \leq i < n$ the intersection $P_i \cap P_n$ is a facet F_i of P_n .

Let \underline{I} (resp. \underline{I}') consist of all j -sets $I \subset \{1, \dots, n\}$ such that $n \in I$ (resp. $n \notin I$), and for each $I \in \underline{I} \cup \underline{I}'$ let $G_I = \bigcap_{i \in I} P_i$. If $I \in \underline{I}$ then $G_I = \bigcap_{i \in I - \{n\}} F_i$, a $(d-j+1)$ - face of P_n . If $I \in \underline{I}'$ then $G_I \cap P_n$ is a $(d-j)$ - face of P_n and for each $i \in I$ is the intersection with P_n of a $(d-j+1)$ - face of P_i . Since different members of \underline{I} (resp. \underline{I}') give rise to distinct sets G_I (resp. $G_I \cap P_n$), the stated conclusion follows. \square

A polytope is simplicial if all its facets are simplices. It is known [4] that all neighborly d -polytopes are simplicial when d is even, and [5] that the number of facets of a simplicial neighborly d -polytope with n vertices is

$$\gamma(d, n) = \binom{n - \lfloor (d+1)/2 \rfloor}{n - d} + \binom{n - \lfloor (d+2)/2 \rfloor}{n - d}$$

McMullen [7] proved that $\gamma(d, n)$ is the maximum number of facets of d -polytopes with n vertices and hence, dually, of vertices of d -polytopes

with n facets.

A d -polyhedron is simple if it has at least one vertex and each of its vertices is incident to precisely d edges or, equivalently, to precisely d facets ($(d-1)$ - faces). A d -dimensional Voronoi diagram $V(p_1, \dots, p_n)$ is simple if it has at least one vertex and each vertex is incident to precisely $d + 1$ of the P_i 's; this implies that all the P_i 's are simple.

THEOREM 2 If p_1, \dots, p_n are distinct points of E^d such that the Voronoi diagram $V(p_1, \dots, p_n) = (P_1, \dots, P_n)$ is simple and u of the P_i 's are unbounded, then $\phi_0(p_1, \dots, p_n) \leq \gamma(d+1, n) + d - u$.

Proof. The assertion is obvious when $d = 2$, so we assume $d > 2$. A theorem of Davis [2] then guarantees the existence of a real-valued convex function f on E^d such that each P_i is a set X which is maximal with respect to there being an affine function on E_d that agrees with f on X . The epigraph $\{(x, \tau) : \tau \geq f(x)\}$ is a simple $(d+1)$ -polytope that has precisely n facets, u of which are unbounded. It then follows from an extension [6] of McMullen's theorem that the number of vertices of the epigraph, and hence of $V(p_1, \dots, p_n)$, is at most $\gamma(d+1, n) + d - u$. \square

THEOREM 3 If $n > d + 1$ then $\gamma(d, n-1) \leq M_0(d, n) < \gamma(d+1, n)$.

Proof. To establish the lower bound, carry out the construction of Theorem 1 with a neighborly polytope Q that is simplicial. Then Q has $\gamma(d, n-1)$ facets, so $\gamma(d, n-1)$ is also the number of vertices of the polar polytope $Q^0 = P_n$.

For the upper bound, note that whenever p_1, \dots, p_n are points of E^d (with $n > d$), they can be perturbed slightly so that the diagram $V(p_1, \dots, p_n)$

becomes simple and its number of vertices does not decrease. (A formal proof can be based on a semicontinuity theorem of [3].) Then use Theorem 2, noting that the number of unbounded P_i 's must exceed d . \square

Note that

$$\gamma(d,n) = \frac{n}{n-r} \binom{n-r}{r} \quad \text{for even } d = 2r$$

and
$$\gamma(d,n) = 2 \binom{n-r}{r-1} \quad \text{for odd } d = 2r-1.$$

Thus Theorem 3 yields the following corollary, which in turn implies (2).

COROLLARY 1 For even $d = 2r$ and for $n > d + 1$,

$$\frac{n-1}{n-1-r} \binom{n-1-r}{r} \leq M_0(d,n) < 2 \binom{n-1-r}{r}.$$

To establish (3) we use an idea of Preparata [8] in conjunction with some special neighborly polytopes.

THEOREM 4 If d is odd, $s > d$ and $t \geq 1$, then $M_0(d,s+t) \geq t\gamma(d-1,s-1)$.

Proof. Let $d = 2r+1$, and for each angle θ let

$$x(\theta) = (\sin \theta, \cos \theta, \sin 2\theta, \cos 2\theta, \dots, \sin r\theta, \cos r\theta) \in E^{2r}.$$

Let

$$C_r = \{x(\theta): 0 \leq \theta \leq 2\pi\},$$

a simple closed curve on the sphere in E^{2r} that is centered at 0 and has radius \sqrt{r} . This curve was studied by Caratheodory [1] and also by Gale

[4], who observed that the convex hull con X is a neighborly $(2r)$ -polytope for each finite set X of more than $2r$ points of C_r . Grünbaum [5] noted this is easily proved with the aid of Scott's identity [11] asserting that if $\delta(\theta_1, \dots, \theta_d)$ is the determinant of the matrix whose i^{th} row consists of a 1 followed by $x(\theta_i)$, then

$$\delta(\theta_1, \dots, \theta_d) = 2^{2r^2} \prod_{1 \leq i < j \leq d} \sin \frac{1}{2}(\theta_j - \theta_i).$$

For $1 \leq i \leq d$, let $\alpha_i = 2\pi(i-1)/d$ and $w_i = x(\alpha_i)$. From neighborliness and a remark of Gale [4], and also from Scott's identity, it follows that the convex hull of the w_i 's is a $(2r)$ -simplex. Since $\sum_1^d w_i = 0$, the origin is interior to the simplex. For the given $s > d$, let w_{d+1}, \dots, w_s be distinct points of $C_r - \{w_1, \dots, w_d\}$. For $1 \leq i < s$, let $p_i = 2w_i / \|w_i\|^2$, so that $\|p_i\| = 2/\sqrt{r}$, and let $p_s = 0$. With $(P_1, \dots, P_s) = V(p_1, \dots, p_s)$, P_s is the polar of the neighborly $(2r)$ -polytope con $\{w_1, \dots, w_{s-1}\}$ and hence P_s has $\gamma(d-1, s-1)$ vertices. Let $q_i = (\sqrt{r}/2) p_i$ for $1 \leq i \leq s$, so that $q_s = 0$, $\|q_i\| = 1$ for $1 \leq i < s$, and the polytope

$$K = \{x \in E^{2r} : \|x\| \leq \|x - q_i\| \text{ for } 1 \leq i < s\}$$

is equal to $(\sqrt{r}/2)P_s$.

Now let E^{2r} be embedded in E^d as a hyperplane through the origin, having a line Rz with $\|z\| = 1$ as orthogonal supplement. For $1 \leq i \leq t$ let $q_{s+i} = 2iz$. Let (Q_1, \dots, Q_{s+t}) denote the Voronoi diagram $V(q_1, \dots, q_{s+t})$. We prove $M_0(d, s+t) \geq t\gamma(d-1, s-1)$ by showing for $1 \leq i \leq t$ that Q_{s+i} has $\gamma(d-1, s-1)$ vertices in the hyperplane

$$J_i = E^{2r} + (2i-1)Z.$$

All points of J_i are equidistant from q_{s+i-1} and q_{s+i} , and are closer to these than to any other point of the set $\{q_s, \dots, q_{s+t}\}$. The point $(2i-1)Z$ is closer to q_{s+i-1} and to q_{s+i} than to any other point of the set $\{q_1, \dots, q_{s+t}\}$. Thus J_i contains a facet F of Q_{s+i} , and in fact

$$F = \bigcap_{1 \leq k < s} (H_k \cap J_i)$$

where

$$H_k = \{x \in E^d : \|x - q_{s+i}\| \leq \|x - q_k\|\}.$$

To see that F has the same number of vertices as K , note that there is a point $-\mu z$ such that F is the intersection with J_i of the convex cone formed by all rays that issue from $-\mu z$ and pass through points of K . The vertices of F are the intersections of J_i with the edges of the cone, and these in turn correspond to vertices of K . The existence of $-\mu z$, which can be deduced from the lemma below, depends on all the points q_1, \dots, q_{s-1} having the same norm, and that was the reason for the special choice of neighborly polytopes in this construction. (Thus having $\|q_1\| = \dots = \|q_{s-1}\|$ appears to be essential here, though having these norms = 1 is merely a computational convenience.)

Lemma. Suppose E is a hyperplane through the origin in a Euclidean space, having a line Rz with $\|z\| = 1$ as orthogonal supplement. Suppose $q \in E$ with $\|q\| = 1$, and suppose $0 < \beta < \eta \leq 2\beta$. Let

$$\psi = \frac{\eta(2\beta - \eta) + 1}{2} \quad \text{and} \quad \mu = \frac{\beta}{2\psi - 1} = \frac{\beta}{\eta(2\beta - \eta)}.$$

Then for each point x of the hyperplane $E + \beta z$, the following two conditions are equivalent:

(i) $\|x - nz\| \leq \|x - q\|$;

(ii) if x' is the point at which the segment $[-\mu z, x]$ intersects E , then $\|x'\| \leq \|x' - q\|$.

To prove lemma, consider an arbitrary point $x \in E + \beta z$ — say $z = y + \beta z$ with $y \in E$. Consideration of similar triangles shows that $x' = \epsilon y$ with $\epsilon = \mu/(\mu + \beta) = 1/(2\psi)$. Using the facts that $\langle z, y \rangle = \langle z, q \rangle = 0$ and $\langle z, z \rangle = \langle q, q \rangle = 1$, both (i) and (ii) are seen to be equivalent to the inequality $\langle q, y \rangle \leq \psi$. That settles the lemma and completes the proof of Theorem 4. \square

COROLLARY 2 For odd $d = 2r-1$ and for $n > d+1$,

$$\frac{n-r-1}{r+1} \frac{nr-r-1}{nr-r^2+1} \binom{\lceil nr/(r+1) \rceil - r}{r-1} < M_0(d, n) < \frac{n}{n-r} \binom{n-r}{r}.$$

Proof. Use Theorem 3 for the upper bound. For the lower bound, apply Theorem 4 with $s = \lceil nr/(r+1) \rceil$ and $t = \lfloor n/(r+1) \rfloor$, obtaining

$$M_0(d, n) \geq t \gamma(2r-2, s-1) = t \frac{s-1}{s-r} \binom{s-r}{r-1}$$

and hence the stated lower bound. From the latter it follows that

$$\liminf_{n \rightarrow \infty} M_0(d, n) \geq \frac{1}{r} \left(\frac{r}{r+1} \right)^r n^r > \frac{1}{re} n^r$$

thus settling (3). \square

Comments For applications of Voronoi diagrams to problems of packing and covering in E^d , and for references to the earlier literature, see Rogers

[10]. In recent years, Voronoi diagrams in E^2 have been of interest because of their use by Shamos [12] and Shamos and Hoey [13] in providing efficient algorithms for a number of computational problems. For n points p_1, \dots, p_n of E^2 , the diagram (P_1, \dots, P_n) can be computed in time $O(n \log n)$ each P_i being output as its sequence of successive vertices. The same computation in E^d would in worst cases require time $\Omega(n^{\lceil d/2 \rceil})$ because of the possible number of vertices. However, it is unknown whether, in time bounded by some polynomial in d and n , one can compute the facets of the P_i 's. For input $p_1, \dots, p_n \in E^d$, the output would consist of n subsets I_1, \dots, I_n of $\{1, \dots, n\}$ such that $i \in I_j$ if and only if the hyperplane $\{x: \|x - p_i\| = \|x - p_j\|\}$ contains a facet of P_j . By results of Reiss and Dobkin [9], this can be accomplished in polynomial time if and only if linear programming problems with d variables and n constraints can be solved in polynomial time.

REFERENCES

- [1] C. Carathéodory, Über den Variabilitätsbereich der Fourier'schen Konstanten von positiven Harmonischen Funktionen. Rend. Circ. Matem. Palermo 32 (1911) 193-217. ...
- [2] C. Davis, The set of non-linearity of a convex piecewise-linear function. Scripta Math. 24 (1959) 219-228.
- [3] H. G. Eggleston, B. Grünbaum and V. Klee, Some semicontinuity theorems for convex polytopes and cell-complexes. Comment. Math. Helv. 39(1964) 165-188.
- [4] Gale, Neighborly and cyclic polytopes. Convexity (V. Klee, ed),

- Amer. Math. Soc. Proc. Symp. Pure Math. 7(1963) 225-232.
- [5] B. Grünbaum, Convex Polytopes. Interscience-Wiley, London, 1967.
- [6] V. Klee, Polytope pairs and their relationship to linear programming. Acta Math. 133 (1974) 1-25.
- [7] P. McMullen, The maximum number of faces of a convex polytope. Mathematika 17 (1970) 179-184.
- [8] F. P. Preparata, A nearest-point Voronoi polyhedron for n -points may have $O(n^2)$ vertices. Steps onto Computational Geometry (F. P. Preparata, ed.), Tech. Rep., Coordinated Science Lab., University of Illinois, Urbana, 1977, pp. 23-24.
- [9] S. P. Reiss and D. P. Dobkin, The complexity of linear programming. Tech. Rep. No. 69, Yale Univ. Comp. Sci. Dept.
- [10] C. A. Rogers, Packing and Covering. Cambridge Tracts in Math. and Math. Physics, No. 54, Cambridge Univ. Press, 1964.
- [11] R. F. Scott, Note on a theorem of Prof. Cayley's. Messenger Math. 8 (1879) 155-157.
- [12] M. I. Shamos, Computational Geometry. Ph.D. thesis, Yale Univ. Comp. Sci. Dept., 1975.
- [13] M. I. Shamos and D. Hoey, Closest-point problems. Proc. 16th Ann. IEEE Symp. Foundations Comp. Sci. (1975) 151-162.