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APEX SINGULARITIES FOR CORNER CRACKS UNDER OPENING, SLIDING, AN--ETC(U)

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MODES

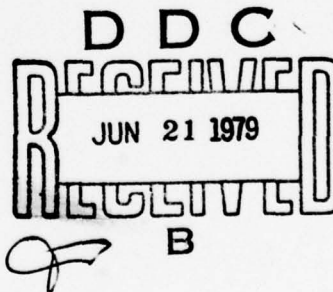
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March 1979

(Received February 5, 1979)



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APEX SINGULARITIES FOR CORNER CRACKS UNDER
OPENING, SLIDING, AND TEARING MODES

B. Noble, M. A. Hussain[†] and S. L. Pu[†]

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ABSTRACT

Power singularities at the apex of a flat, wedge-shaped, angular-sector crack in a three-dimensional solid are studied. Using Boussinesq stress functions of elasticity theory, the problem is reduced to an eigenvalue problem of a dual series. The stress singularity is found to be stronger or weaker than one half, depending upon whether the apex angle is greater or less than 180° . This tends to accelerate or retard the crack growth at the apex until the crack front straightens out.

AMS (MOS) Subject Classifications - 73.35, 73.65

Key Words - Stress intensity factor, Corner singularity,
Three-dimensional crack, Dual series

Work Unit Number 3 - Applications of Mathematics

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SIGNIFICANCE AND EXPLANATION

Crack propagation and fracture in metals depends on the intensity of the stresses in the vicinity of a crack. The classical two-dimensional theory of cracks indicates that close to the crack the stresses vary as the inverse of the square root of the distance from the tip of the crack. The constant of proportionality is known as the stress intensity factor. There have been extensive investigations of the magnitude of this factor for various geometrical configurations in two dimensions, since its size determines whether the crack will spread, and, if so, how fast it will spread.

Due to the difficulty of the problem, it is only in the last year or two that serious efforts have been made to determine stress intensity factors for three-dimensional situations, and some of these investigations have certainly been wrong. Part of the difficulty lies in the fact that the dependence of the stresses on the distance from the tip of the crack is much more complicated than the inverse square root behavior observed in two dimensions. This report examines the nature of the stress singularity at the apex of a flat wedge-shaped angular-sector crack in a three-dimensional infinite solid. The results indicate that the crack growth at the apex will be such as to straighten out the crack front.

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APEX SINGULARITIES FOR CORNER CRACKS UNDER
OPENING, SLIDING, AND TEARING MODES

B. Noble, M. A. Hussain[†] and S. L. Pu[†]

INTRODUCTION

In this paper, the stress singularities at the apex of a flat, wedge-shaped, angular sector crack in a three-dimensional solid are studied. The crack is subjected to an opening, sliding, or tearing mode loading condition. These modes, together with their two-dimensional idealizations as commonly seen in the literature, are shown in Figure 1. The two sets of figures become identical when the apex angle approaches 180° .

For the two-dimensional case, it is known that the stress singularity at the crack tip is of the order of one half for all of the modes. However, when the two crack fronts meet at a sharp corner, as shown in Figure 1, the stress singularity is no longer of the order of one half but depends mainly on the included angle as well as the mode the crack is subjected to, namely mode I, II, or III. A crack subjected to an arbitrary load can be decomposed into these three basic modes.

As far as singularities are concerned, it is of interest to note the equivalence of crack and punch problems. In such a case, the angle of the base of the punch and that of the cracked area are exponents of each other and in the loading process the punch is held stationary. A mode I crack problem can be reduced to a problem in potential theory, but the other modes cannot.

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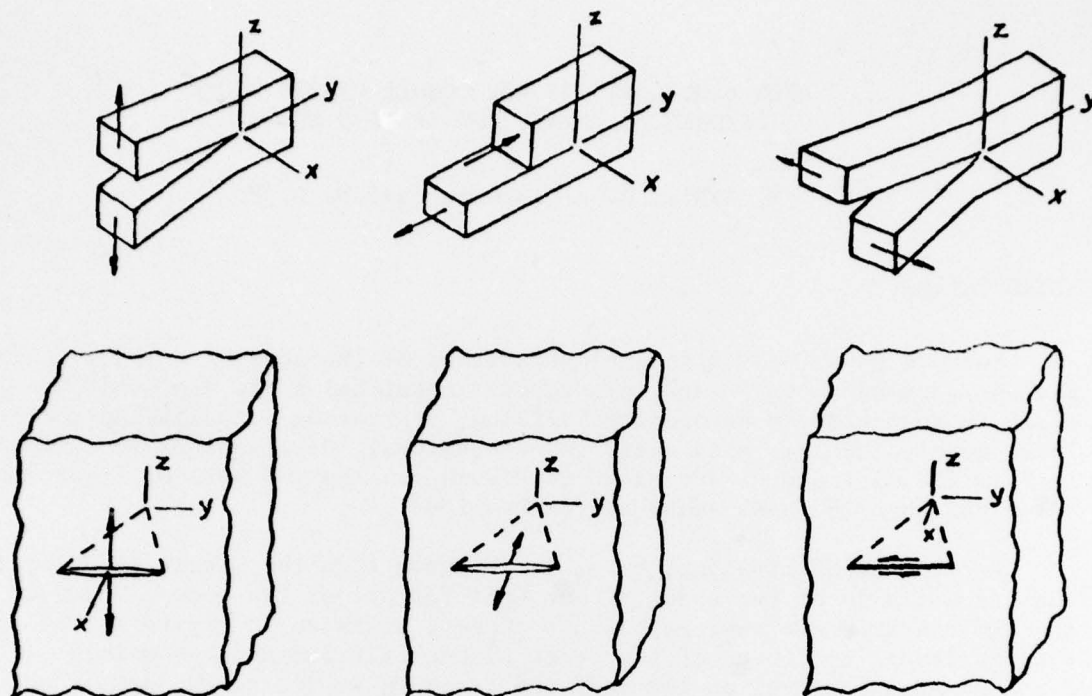


Fig. (1) - Three basic modes of two-dimensional, and the corresponding three-dimensional flat wedge-shaped, crack surface displacements.

The potential problem of charge distribution at the tip of a flat angular sector was solved by Noble (1959) [1]. Subsequently, due to a wide application of potential theory in the field of mathematical physics, a number of papers [2-7] appeared for the same problem. Even though the methods and approaches are different in these papers, the final results are the same as [1].

In this paper, we use Boussinesq potentials of the three-dimensional theory of elasticity. Assuming power law singularity and using separation of variables, the problem is reduced to a dual series relation for the case of mode I, and coupled dual series for modes II and III. Based on [1] a method is developed to solve these coupled dual series. The power, versatility, and generality of the method can be seen from the simplicity with which new results are obtained. It is shown how results of any desired degree of accuracy can be obtained from simple algebraic computations.

THE BOUSSINESQ SOLUTIONS IN SPHERICAL COORDINATES

In the absence of body forces, the equation of equilibrium for a homogeneous, isotropic, elastic body, in terms of displacement vector, is

$$\nabla^2 \bar{u} + (1-2\nu)^{-1} \nabla \nabla \cdot \bar{u} = 0 \quad (1)$$

Here ν is Poisson's ratio. According to Boussinesq, the general solution of Eq. (1) may be written as a superposition of three displacement fields, $\bar{u}_1$, $\bar{u}_2$, $\bar{u}_3$

$$2G\bar{u}_1 = \nabla \psi, \quad 2G\bar{u}_2 = 2\nabla x(\bar{k}\theta), \quad 2G\bar{u}_3 = \nabla(z\lambda) - 4(1-\nu)\lambda \bar{k} \quad (2)$$

where G is the shear modulus, and ψ , θ , λ are arbitrary harmonic functions. These will be referred to as basic solutions 1, 2, and 3, respectively.

We wish to investigate the order of power singularities at the apex. To this end, it is necessary to investigate the near field solution in spherical coordinates (r, θ, ϕ) , without implying that the solid under consideration is of a spherical shape.

We choose the following spherical harmonics

$$\psi_m = r^{\mu+1} p_{\mu+1}^m(\cos\theta) \begin{matrix} \cos \\ \sin \end{matrix} m\phi$$

$$\theta_m = r^{\mu+1} p_{\mu+1}^m(\cos\theta) \begin{matrix} \cos \\ \sin \end{matrix} m\phi$$

$$\lambda_m = r^\mu p_\mu^m(\cos\theta) \begin{matrix} \cos \\ \sin \end{matrix} m\phi$$

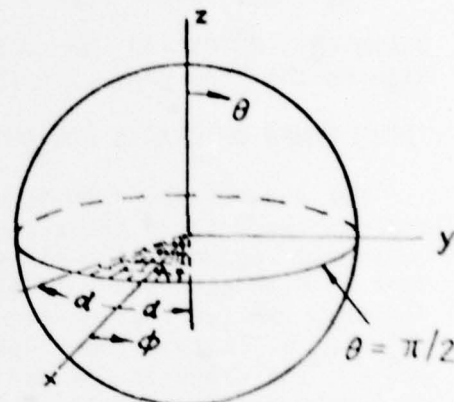


Fig. (2) - A flat, wedge-shaped crack.

where P_μ^m is the associated Legendre function of the first kind. Upon substitution ψ_m into the first basic solution of (2), we obtain a solution designated symbolically by $[1-\psi_m]$. Similarly we obtain solutions $[2-\theta_m]$ and $[3-\lambda_m]$. The selection of $\cos m\phi$ or $\sin m\phi$ in each stress function depends on the symmetry of the problem. Our objective is to find the value of μ in the open range of $(0,1)$. The final solution $[S]$ is the superposition of these solutions

$$[S] = \sum \{A_m[1-\psi_m] + B_m[2-\theta_m] + D_m[3-\lambda_m]\} \quad (3)$$

where, and thereafter, $\sum$ denotes the summation with respect to m for $m = 0, 1, \dots, \infty$.

On the plane $\theta = \pi/2$, the components of displacement and the pertinent components of stress for [S] are:

$$\begin{aligned}
 2Gu_r &= \sum [(\mu+1)A_m \pm 2mB_m] r^\mu p_{\mu+1}^m \cos m\phi \\
 2Gu_\theta &= \sum [-(m+\mu+1)A_m + (3-4\nu)D_m] r^\mu p_\mu^m \cos m\phi \\
 2Gu_\phi &= \sum [\mp mA_m - 2(\mu+1)B_m] r^\mu p_{\mu+1}^m \sin m\phi \\
 \sigma_\theta &= \sum [-(m+\mu+1)A_m + 2(1-\nu)D_m(-m+\mu+1)] r^{\mu-1} p_{\mu+1}^m \cos m\phi \\
 \tau_{\theta r} &= \sum [-\mu(m+\mu+1)A_m \mp m(m+\mu+1)B_m + (1-2\nu)\mu D_m] r^{\mu-1} p_\mu^m \cos m\phi \\
 \tau_{\theta\phi} &= \sum [\pm m(m+\mu+1)A_m + (m+\mu+1)B_m \mp (1-2\nu)mD_m] r^{\mu-1} p_\mu^m \sin m\phi
 \end{aligned} \tag{4}$$

where $p_\mu^m = p_\mu^m(0)$ and whenever there are two signs preceding a quantity, the sign on the top goes with the trigonometric function on the top and vice versa.

THREE MODES OF CRACKS AND DUAL SERIES RELATIONS

For a crack shown in Figure 2, the leading edges of the crack are $\phi = \pm \alpha$, and the crack is in the x-y plane ($\theta = \pi/2$). Let D^- and D^+ be the cracked and uncracked region of the plane $\theta = \pi/2$. Within the cracked region, the displacement is discontinuous. If the discontinuity is in the z-direction ($u_\theta^+ - u_\theta^- = \text{finite}$), the crack is under mode I; if the discontinuity is in the x-direction ($u_x^+ - u_x^- = \text{finite}$), the crack is defined to be under mode II; and if $u_y^+ - u_y^- = \text{finite}$, the crack is defined to be under mode III. Boundary conditions for various modes are tabulated below.

Boundary Conditions on $\theta = \pi/2$

	Non Mixed Conditions (in $D^- + D^+$)	Mixed Conditions	
		in D^-	in D^+
Mode I	$\tau_{\theta r} = \tau_{\theta\phi} = 0$	$\sigma_\theta = 0$	$u_\theta = 0$
Mode II	$\sigma_\theta = 0$	$\tau_{\theta r} = \tau_{\theta\phi} = 0$	$u_r = u_\phi = 0$
Mode III	$\sigma_\theta = 0$	$\tau_{\theta r} = \tau_{\theta\phi} = 0$	$u_r = u_\phi = 0$

(5)

(6)

(7)

For mode I, u_θ is even in ϕ . This leads to the use of the trigonometric functions at the top of (4). The boundary conditions of (6) and (7) are identical, but the symmetric properties are different for mode II and mode III. In the former case, u_r is even and u_ϕ is odd in ϕ while in the latter, the reverse is true. Hence the proper set of quantities should be selected in (4), for each case.

A. Mode I

Using Eq. (4) and the non-mixed conditions of (5), we have

$$B_m = 0 \quad \text{and} \quad A_m = (m+\mu+1)^{-1}(1-2\nu)D_m \quad (8)$$

The mixed boundary conditions of (5) and using (8) and (4), yield

$$\begin{aligned} \sum b_m \cos m\phi &= 0 & 0 < \phi < \alpha \\ \sum q_m b_m \cos m\phi &= 0 & \alpha < \phi < \pi \end{aligned} \quad (9)$$

where

$$b_m = (-m+\mu+1)D_m P_{\mu+1}^m, \quad q_m = (-m+\mu+1)^{-1} P_{\mu}^m / P_{\mu+1}^m \quad (10)$$

B. Mode II

For the homogeneous condition of σ_θ in (6) using (4) the coefficients A and D must satisfy

$$A_m = (m+\mu+1)^{-1} 2(1-\nu)D_m$$

This relation and the mixed conditions of (6) yield the following coupled dual series:

$$\sum E_m \cos m\phi = 0 \quad 0 < \phi < \alpha \quad (11)$$

$$\sum (R_m E_m + S_m F_m) \cos m\phi = 0 \quad \alpha < \phi < \pi$$

$$\sum_1 F_m \sin m\phi = 0 \quad 0 < \phi < \alpha \quad (12)$$

$$\sum_1 (U_m F_m + T_m E_m) \sin m\phi = 0 \quad \alpha < \phi < \pi$$

where $\sum_1$ denotes the summation with respect to m for $m = 1, 2, \dots, \infty$ and

$$E_m = (m+\mu+1)[\mu A_m + 2m(1-\nu)R_m]P_{\mu}^m, \quad F_m = (m+\mu+1)[mA_m + 2\mu(1-\nu)B_m]P_{\mu}^m \quad (13)$$

$$R_m = [m^2 - \mu(\mu+1)(1-\nu)]V_m, \quad S_m = m(1-\nu-\nu\mu)V_m \quad (14)$$

$$T_m = m(1+\mu\nu)V_m, \quad U_m = [(1-\nu)m^2 - \mu(\mu+1)]V_m \quad (15)$$

in (14) and (15) V_m stands for $P_{\mu+1}^m / P_{\mu}^m / [(m+\mu+1)(m^2 - \mu^2)]$.

C. Mode III

Similar to the preceding case, we have

$$\sum_1 E_m \sin m\phi = 0 \quad 0 < \phi < \alpha \quad (16)$$

$$\sum_1 (R_m E_m + S_m F_m) \sin m\phi = 0 \quad \alpha < \phi < \pi$$

$$\begin{aligned}\sum F_m \cos m\phi &= 0 & 0 \leq \phi < \alpha \\ \sum (U_m F_m + T_m E_m) \cos m\phi &= 0 & \alpha < \phi < \pi\end{aligned}\quad (17)$$

SOLUTIONS OF SINE AND COSINE SERIES

For the solution of (9), (11), and (12), (16) and (17), we need the exact solutions of certain dual sine and cosine series.

A. Sine Series

Consider a dual series of the form

$$\begin{aligned}\sum_1 a_n \sin nx &= 0 & x_0 < x < \pi \\ \sum_1 \frac{1}{n} a_n \sin nx &= f(x) & 0 \leq x < x_0\end{aligned}\quad (18)$$

A solution is given by [8]

$$g(t) = -\frac{1}{\pi} \sec \frac{t}{2} \frac{d}{dt} \int_0^c \frac{\sin u G(u) du}{t \sqrt{\cos t - \cos u}} \quad (19)$$

$$\text{with } G(u) = \cot \frac{u}{2} \frac{d}{du} \int_0^u \frac{\sin(x/2) f(x) dx}{\sqrt{(\cos x - \cos u)}} \quad (20)$$

Where

$$g(x) = \sum_1 a_n \sin nx, \text{ for } 0 < x < x_0, \text{ and } a_m = \frac{2}{\pi} \int_0^{x_0} \frac{g(t) \sin mt dt}{x}$$

For $f(x) = C_1 \sin x$, we obtain, using the abbreviation $\beta = \cos x_0$,

$$g(x) = \frac{C_1}{\sqrt{2}} \sin \frac{x}{2} (2 \cos x + 1 - \beta) (\cos x - \beta)^{-1/2} \quad (21)$$

$$a_m = \frac{1}{2} (\beta P_m - \beta P_{m-1} - P_{m+1} + P_{m-2}), \quad P_m = P_m(\beta) \quad (22)$$

For $f(x) = C_2 \sin 2x$, the corresponding results are

$$g(x) = -\sqrt{2} C_2 \sin \frac{x}{2} (\cos x - \beta)^{-1/2} (\beta + \beta^2 - 2 \cos x + 2 \beta \cos x - 4 \cos^2 x) \quad (23)$$

$$a_n = -\frac{1}{2} [-2P_{n-3} + 2\beta P_{n-2} + (-1 + \beta^2)P_{n-1} + (1 - \beta^2)P_n - 2\beta P_{n+1} + 2P_{n+2}] \quad (24)$$

Similarly, for $f(x) = C_3 \sin 3x$, we have

$$g(x) = \frac{3C_3 \sin(x/2)}{2\sqrt{2}(\cos x - \beta)} [16 \cos^3 x + 8 \cos^2 x (1 - \beta) - 2 \cos x (3 + 2\beta + \beta^2) - \beta^3 - \beta^2 + 3\beta - 1] \quad (25)$$

$$a_n = 3[2(P_{n-4} - P_{n+3}) - 2\beta(P_{n-3} - P_{n+2}) - (\beta^2 - 1)\{P_{n-2} + \beta(P_{n-1} + P_n) - P_{n+1}\}]/4 \quad (26)$$

B. Cosine Series

The cosine series under consideration have the form

$$\sum a_n \cos nx = 0 \quad x_0 < x < \pi \quad (27)$$

$$\rho_0 a_0 + \sum_{n=1}^{\infty} \frac{1}{n} a_n \cos nx = \sum_{n=1}^N \rho_n a_n \cos nx, \quad 0 < x < x_0$$

A method of solution is developed by Noble [1] for finding the lowest eigenvalue for mixed problems in potential theory. The method uses a stretching transformation due to Schwinger [9]. Let

$$\sum a_n \cos nx = h(x) \quad \text{for } 0 < x < x_0 \quad (28)$$

then

$$a_0 = \frac{1}{\pi} \int_0^{x_0} h(x) dx, \quad a_n = \frac{2}{\pi} \int_0^{x_0} h(x) \cos nx dx \quad (29)$$

Upon substitution from (29) into the second equation of (27), we have, for $0 < x < x_0$,

$$\rho_0 \frac{a_0}{\pi} \int_0^{x_0} h(u) du + \sum_{n=1}^{\infty} \frac{1}{n} \frac{2}{\pi} \int_0^{x_0} h(u) \cos nu \cos mx du = \sum_{n=1}^N \rho_n \frac{2}{\pi} \int_0^{x_0} h(u) \cos nu \cos mx du, \quad (30)$$

Using the stretching transformation

$$\cos u = t + s \cos \xi \quad (31)$$

where t and s , determined from $u = 0, x_0$ corresponding to $\xi = 0, \pi$ are

$$t = \cos^2(x_0/2), \quad s = \sin^2(x_0/2) \quad (32)$$

A similar relation is assumed between x and ζ . Using [10], we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n} \cos nu \cos mx = -\frac{1}{2} \ln \{2 |\cos u - \cos x|\} = -\ln s / 2 + \sum_{n=1}^{\infty} \frac{1}{n} \cos m \xi \cos m \zeta \quad (33)$$

Equation (30) becomes

$$(\rho_0 - \ln s) \frac{1}{\pi} \int_0^{\pi} h(u) \frac{du}{d\xi} d\xi + \sum_{n=1}^{\infty} \frac{1}{n} \frac{2}{\pi} \int_0^{\pi} h(u) \frac{du}{d\xi} \cos m \xi d\xi \cos m \zeta =$$

$$\sum_{n=1}^N \rho_n \frac{2}{\pi} \int_0^{\pi} h(u) \frac{du}{d\xi} \cos m \xi d\xi \cos m \zeta, \quad 0 < x < \pi \quad (34)$$

On the right hand side of (34), $\cos nu$ and $\cos mx$ can be transformed to $\cos m \xi$ and $\cos m \zeta$ in virtue of (31). Let

$$h(u) \frac{du}{d\xi} = \sum c_n \cos n \xi \quad 0 < \xi < \pi \quad (35)$$

then

$$\frac{1}{\pi} \int_0^{\pi} h(u) \frac{du}{d\xi} d\xi = c_0, \quad \frac{2}{\pi} \int_0^{\pi} h(u) \frac{du}{d\xi} \cos m\xi d\xi = c_m \quad (36)$$

Upon substitution from (36) into (34), we have a trigonometric equation in $\cos m\xi$. Equating the coefficients of like terms, a system of algebraic equations in c 's is obtained. For non-trivial solutions, the determinant of the set of equations must vanish which is used to find the eigenvalues.

Specifically, if $\rho_n = 0$ for $n \geq 1$ in the last equation of (27), then $c_m = 0$ for $m \geq 1$ and for $c_0 \neq 0$, we must have

$$\rho_0 - \ell n s = 0 \quad (37)$$

For the case $\rho_1 \neq 0$ and $\rho_m = 0$, $m \geq 2$ then $c_m = 0$ for $m \geq 2$ and

$$\begin{bmatrix} -\rho_0 + \ell n s + 2t^2 \rho_1 & st \rho_1 \\ 2st \rho_1 & -1 + \rho_1 s^2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = 0 \quad (38)$$

Similarly, when $N = 2$ on the right hand side of (27), $c_m = 0$ for $m \geq 3$ and c_0, c_1, c_2 satisfy, using ℓ_1 for $s^2 + 2t^2 - 1$,

$$\begin{bmatrix} -\rho_0 + \ell n s + 2t^2 \rho_1 + 2\ell_1^2 \rho_2 & st \rho_1 + 4st \ell_1 \rho_2 & s^2 \ell_1 \rho_2 \\ 2st \rho_1 + 2(4st) \ell_1 \rho_2 & -1 + s^2 \rho_1 + (4st)^2 \rho_2 & s^2 (4st) \rho_2 \\ 2s^2 \ell_1 \rho_2 & s^2 (4st) \rho_2 & -\frac{1}{2} + s^4 \rho_2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = 0 \quad (39)$$

SOLUTIONS OF FLAT, WEDGE-SHAPED CRACKS

A. Mode I

The second equation of the dual cosine series (9) can be written as

$$q_0 b_0 + \sum_1 \frac{1}{m} b_m \cos m\phi = \sum_1 \left(\frac{1}{m} - q_m \right) b_m \cos m\phi \quad \alpha < \phi < \pi \quad (40)$$

Using the last equation, p. 63 of [11], we have

$$q_m = \frac{1}{2} \Gamma\left(\frac{m+\mu+1}{2}\right) \Gamma\left(\frac{m-\mu}{2}\right) / \Gamma\left(1 + \frac{m+\mu}{2}\right) \Gamma\left(\frac{m-\mu+1}{2}\right) \quad (41)$$

For a large m , $q_m \rightarrow 1/m$, [1], therefore the infinite series on the right hand side of equation (40) could be truncated.

In order to use the solution of cosine series presented in the preceding section, we make the following change of variables

$$\phi = \pi - \omega, \quad x_0 = \pi - \alpha, \quad \cos m\phi = (-1)^m \cos m\omega, \quad (-1)^m b_m = a_m \quad (42)$$

The dual cosine series (9) take the form

$$\begin{aligned} a_0 + \sum_1 a_m \cos m\omega &= 0 & x_0 < \omega < \pi \\ q_0 a_0 + \sum_1 \frac{1}{m} a_m \cos m\omega &= \sum_1 \left(\frac{1}{m} - q_m\right) a_m \cos m\omega & 0 < \omega < x_0 \end{aligned} \quad (43)$$

As a first approximation, we drop out the infinite series on the right hand side of (43). Using (37), we have a transcendental equation

$$\cos\left(\frac{\alpha}{2}\right) = \exp(q_0/2), \quad q_0 = -\frac{1}{2} \left[\Gamma\left(\frac{1+\mu}{2}\right) / \Gamma\left(1 + \frac{\mu}{2}\right) \right]^2 \quad (44)$$

from which μ can be determined for a given half angle α of the wedge. For the second approximation, only one term is taken in the series on the right hand side of (43). Using (38) the value of μ is found as follows:

$$\begin{vmatrix} q_0 - \ln s - 2t^2(1-q_1) & -st(1-q_1) \\ -2st(1-q_1) & 1-s^2(1-q_1) \end{vmatrix} = 0 \quad (45)$$

where s, t are given by (32), (42) and q_0, q_1 from (41).

Similarly, the third approximation can be carried out. The results of each approximation are shown in Figure 3, showing a remarkably fast convergence.

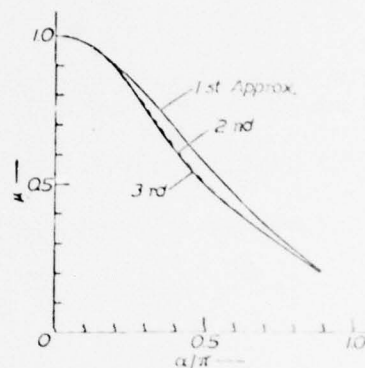


Fig. (3) - μ vs. α/π for a mode I crack.

B. Mode II

Using the property $p_{\mu+1}^m(0)/p_{\mu}^m(0) \rightarrow (-1)$ for a large m , it is seen that asymptotically:

$$R_m \rightarrow (-1/m), \quad S_m \rightarrow (-1/m^2), \quad T_m \rightarrow (-1/m^2), \quad U_m \rightarrow -(1-\nu)/m$$

The second equations of the coupled dual series (11) and (12) can be written as

$$\sum_1 \frac{1}{m} F_m \sin m\phi = \sum_1 \left[\left(\frac{1}{m} + \frac{U_m}{1-\nu}\right) F_m + \frac{T_m}{1-\nu} E_m \right] \sin m\phi \quad \alpha < \phi < \pi \quad (46)$$

$$-R_0 E_0 + \sum_1 \frac{1}{m} E_m \cos m\phi = \sum_1 \left[\left(\frac{1}{m} + R_m\right) E_m + S_m F_m \right] \cos m\phi \quad \alpha < \phi < \pi \quad (47)$$

After the change of variables similar to (42) with $\sin m\phi = (-1)^{m+1} \sin m\omega$, $E_m^* = (-1)^m E_m$ and $F_m^* = (-1)^{m+1} F_m$. The dual series take the form

$$\sum_1 F_m^* \sin m\omega = 0, \quad x_0 < \omega < \pi \quad (48)$$

$$\sum_1 \frac{1}{m} F_m^* \sin m\omega = \sum_1 c_{1m} \sin m\omega, \quad 0 < \omega < x_0$$

$$\sum E_m^* \cos m\omega = 0, \quad x_0 < \omega < \pi \quad (49)$$

$$-R_0 E_0^* + \sum_1 \frac{1}{m} E_m^* \cos m\omega = \sum_1 c_{2m} E_m^* \cos m\omega, \quad 0 < \omega < x_0$$

where

$$c_{1m} = (1-\nu)^{-1} (U_m F_m^* - T_m E_m^*) + F_m^*/m, \quad (50)$$

$$c_{2m} = m^{-1} + R_m - S_m F_m^*/E_m^*$$

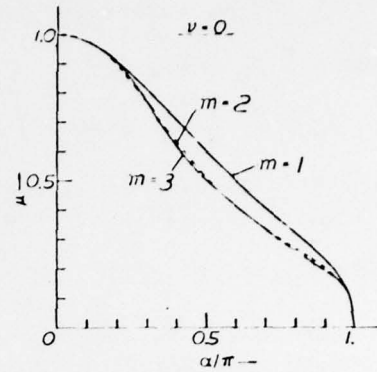


Fig. (4) - μ vs. α/π for a mode II or a mode III crack.

For a large m , both c_{1m} and $c_{2m} \rightarrow 1/m^2$.

As a first approximation, we take only one term, $m=1$, in the infinite series on the right hand sides of (48) and (49). From (22) we obtain

$$F_1^* = (1-\beta)(3+\beta)[F_1^* - (1-\nu)^{-1}(U_1 F_1^* + T_1 E_1^*)]/4 \quad (51)$$

where E_1^* can be determined from (29), (31) and (36).

$$E_1^* = \frac{2}{\pi} \int_0^{x_0} h(u) \cos u du = \frac{2}{\pi} \int_0^{\pi} h(u) \frac{du}{d\xi} (t + s \cos \xi) d\xi = 2tc_0 + sc_1 \quad (52)$$

For the cosine series (49), we obtain two equations from (45)

$$(-R_0 - \ln s - 2c_{21}t^2)c_0 - stc_{21}c_1 = 0$$

$$-2stc_{21}c_0 + (1-s^2c_{21})c_1 = 0$$

These two equations and (51) constitute a system of linear equations in c_0 , c_1 , F_1^* . For a non-trivial solution, we must have

$$\begin{vmatrix} R_0 + \ln s + 2t^2(1+R_1) & st(1+R_1) & -ts_1 \\ 2st(1+R_1) & -1+s^2(1+R_1) & -ss_1 \\ -tT_1/2 & -sT_1/4 & -(1-\nu)/(1-\beta)/(3+\beta) + (1-\nu-U_1)/4 \end{vmatrix} = 0 \quad (53)$$

This is the equation for the determination of μ .

We can proceed in a similar manner to higher order approximations. In Figure 4, we have shown results for $\nu = 0$ for the first three approximations. The difference between the second and third approximations is very small. Hence we stopped at the third. Since the results did not differ from the second to the third approximation, we only plotted the third approximation in Figure 5 for different Poisson's ratios.

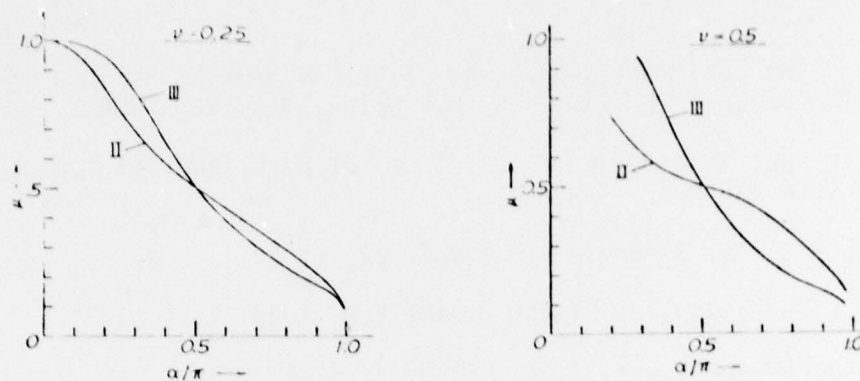


Fig. (5) - The third approximation of μ vs. α/π for a mode II or a mode III crack.

C. Mode III

In comparing equations (16), (17) with (11) and (12), we see that the coupled dual series for mode III are obtained from those for mode II by interchanging E_m and F_m , R_m and U_m , S_m and T_m . The same approximate method for mode II can be applied to mode III. For $\nu = 0$, mode II and mode III have the same results which are plotted in Figure 4. For $\nu = 0.25$ and $\nu \rightarrow 0.5$, only results from the third approximation are plotted in Figure 5.

CONCLUSIONS

The equivalence between the potential problem and the crack problem under mode I or the frictionless punch problem has been known for a long time. The wide applicability of potential problems has been a prime motivation in obtaining the singularities at the apex by various methods. In this paper, however, we have presented a method of solution for mode II and III conditions. The method gives results to any desired accuracy and, to the best of our knowledge, is not yet available in the open literature.

For modes other than opening mode, the results show that the stress singularities are dominated by the angle of the apex as well as the elastic constant of the material. The results further indicate that when the apex angle is greater than 180° , the stress singularity is stronger than one half enhancing the tendency of crack front to straighten out. Similarly, when the apex angle is less than 180° , the stress singularity is less severe than one half and, again, this will tend to retard the growth at the apex until the crack front straightens out.

It should be noted that the definition of modes II and III is quite arbitrary for the configuration under consideration, since the crack fronts away from the apex are under mixed mode conditions of the conventional type.

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 1940	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER (9) Technical
4. TITLE (and Subtitle) APEX SINGULARITIES FOR CORNER CRACKS UNDER OPENING, SLIDING, AND TEARING MODES		5. TYPE OF REPORT & PERIOD COVERED Summary Report, no specific reporting period
7. AUTHOR(s) B./Noble, M. A. Hussain S. L./Pu		6. PERFORMING ORG. REPORT NUMBER
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53706		8. CONTRACT OR GRANT NUMBER(s) DAAG29-75-C-0024
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P. O. Box 12211 Research Triangle Park, North Carolina 27709		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 3 - Applications of Mathematics
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) (12) 16p		12. REPORT DATE March 1979
		13. NUMBER OF PAGES 12
		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited. (14) MRC-TSR-1940		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Stress intensity factor, Corner singularity, Three-dimensional crack, Dual series		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) Power singularities at the apex of a flat, wedge-shaped, angular-sector crack in a three-dimensional solid are studied. Using Boussinesq stress functions of elasticity theory, the problem is reduced to an eigenvalue problem of a dual series. The stress singularity is found to be stronger or weaker than one half, depending upon whether the apex angle is greater or less than 180°. This tends to accelerate or retard the crack growth at the apex until the crack front straightens out.		