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LOCAL AND SUPERLINEAR CONVERGENCE OF A CLASS OF VARIABLE METRIC--ETC(U)

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LOCAL AND SUPERLINEAR
CONVERGENCE OF A CLASS OF
VARIABLE METRIC METHODS

Klaus Ritter

Mathematics Research Center
University of Wisconsin-Madison
610 Walnut Street
Madison, Wisconsin 53706

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LOCAL AND SUPERLINEAR CONVERGENCE
OF A CLASS OF VARIABLE METRIC METHODS

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ABSTRACT

This paper considers a class of variable metric methods for unconstrained minimization problems. It is shown that with a step size of one each member of this class converges locally and superlinearly.

AMS (MOS) Subject Classification: 90C30

Key Words: Unconstrained minimization, variable metric method, local convergence, superlinear convergence.

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SIGNIFICANCE AND EXPLANATION

Many practical problems in operations research may be reduced to minimizing a function without constraints. Variable metric methods are successfully used in computing a sequence which converges to the minimum of a function. During each iteration a search direction and a step size are computed. In order to obtain fast convergence it is necessary that the chosen step size approximates the optimal step size, i.e., the step size which minimizes the function along the given search direction. This may require considerable computational effort. If an approximation to the minimum is known, however, it is often possible to increase the efficiency of an algorithm by showing that a step size of one is a sufficiently good approximation to the optimal step size. Such a situation arises, for instance, if as in the method of penalty functions a sequence of unconstrained problems is solved in order to obtain a solution to a more complicated optimization problem. In general the minimum of one penalty function is a good approximation to the minimum of the penalty function used next.

In this paper it is shown that variable metric methods converge rapidly with a step size of one if a good approximation of the solution is used as starting point.

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LOCAL AND SUPERLINEAR CONVERGENCE
OF A CLASS OF VARIABLE METRIC METHODS

Klaus Ritter

1. Introduction

If a variable metric method is used to compute a minimizer z of a function $F(x)$ it simultaneously generates a sequence of points $\{x_j\}$ and a sequence of matrices $\{H_j\}$. During each iteration a correction is added to H_j with the intent to construct an approximation to the inverse Hessian matrix of $F(x)$.

A large class of such methods has been introduced by Huang [8]. A restriction of the Huang class to update formulas which are of rank two, satisfy the quasi-Newton equation and maintain the symmetry of H_j leads to a class of methods proposed by Broyden [1] and Fletcher [6]. In [9] global and superlinear convergence has been established for each member of this subclass without the requirement of an optimal step size. However, in the context of a global convergence theory the Broyden-Fletcher-Goldfarb-Shanno-method [2], [6], [7], [10] appears to be the only one for which it can be shown that a step size of one is always acceptable after sufficiently many iterations.

Using a step size of one Broyden, Dennis and Moré [3] have shown that the Broyden-Fletcher-Goldfarb-Shanno-method and the Davidon-Fletcher-Powell-method [4], [5] converge superlinearly to a minimizer z of $F(x)$ provided that the initial point x_0 and the initial matrix H_0 are sufficiently close to z and the inverse Hessian matrix of $F(x)$ at z , respectively. It is the purpose of this paper to extend this result to all members of the Broyden class.

2. Preliminary results

Let $x \in E^n$ and let $F(x)$ be a real-valued function. If $F(x)$ is twice differentiable at a point x_j we denote the gradient and the Hessian matrix of $F(x)$ at x_j by $g_j = \nabla F(x_j)$ and $G_j = G(x_j)$, respectively. A prime is used for the transpose of a vector or a matrix. For any $x \in E^n$, $\|x\|$ denotes the Euclidean norm of x .

Throughout this paper we require the following assumption to be satisfied.

Assumption 1

There is a vector z such that $F(x)$ is twice continuously differentiable in some convex neighborhood of z , $\nabla F(z) = 0$, $G = G(z)$ is positive definite and the Lipschitz condition

$$(2.1) \quad \|G(x) - G(z)\| \leq L\|x - z\|,$$

where L is a positive constant, is satisfied for all x in some convex neighborhood of z .

Clearly the above assumption implies that there are constants $0 < \mu < \eta$ and a convex neighborhood $U(z)$ such that, for every $x \in U(z)$, the inequality (2.1) and the relation

$$\mu\|y\|^2 \leq y'G(x)y \leq \eta\|y\|^2 \quad \text{for all } y \in E^n$$

hold.

We consider the problem of determining a sequence

$$(2.2) \quad x_{j+1} = x_j - s_j, \quad j = 0, 1, 2, \dots$$

which converges to z .

If a variable metric method is used to compute the sequence (2.2), then an (n, n) -matrix H_j is associated with each x_j and the search direction s_j is determined by the relation

$$s_j = H_j g_j.$$

The matrix H_{j+1} , associated with x_{j+1} , is obtained by adding a rank two matrix to H_j in such a way that H_{j+1} satisfies the quasi-Newton equation

$$H_{j+1} d_j = p_j,$$

where

$$d_j = \frac{g_j - g_{j+1}}{\|s_j\|}, \quad p_j = \frac{s_j}{\|s_j\|}.$$

The various variable metric methods differ in the update procedure which is used to compute H_{j+1} from H_j . A large class of such methods has been studied by Broyden [1], Fletcher [6], and Huang [8]. In the following we will consider a subclass of these update procedures which has the property that if the initial matrix H_0 is symmetric all subsequent matrices H_j will be symmetric. It has been shown in [9] that the update formulas that correspond to this subclass can be written in the form

$$(2.3) \quad H_{j+1} = H_j + \frac{s_1 (d_j' p_j + d_j' H_j d_j) + s_2 d_j' H_j d_j}{d_j' p_j (s_1 d_j' p_j + s_2 d_j' H_j d_j)} p_j p_j' - s_1 \frac{p_j d_j' H_j + H_j d_j p_j'}{s_1 d_j' p_j + s_2 d_j' H_j d_j} - s_2 \frac{H_j d_j d_j' H_j}{s_1 d_j' p_j + s_2 d_j' H_j d_j},$$

where s_1 and s_2 are arbitrary parameters with $s_1^2 + s_2^2 > 0$.

Two well-known members of this class, namely the BFGS - Method (Broyden [2], Fletcher [6], Goldfarb [7], Shanno [10]) and the DFP - Method (Davidon [4], Fletcher, Powell [5]) can be obtained by choosing $s_1 = 1, s_2 = 0$ and $s_1 = 0, s_2 = 1$, respectively. The choice $s_1 = -s_2$ results in the rank one update formula

$$H_{j+1} = H_j + \frac{(p_j - H_j d_j)(p_j' - d_j' H_j)}{(p_j' - d_j' H_j) d_j}.$$

However this method is known to be numerically unstable. It will be excluded in the following.

If H_j is positive definite it has been shown in [9] that H_j can be written in the form

$$(2.4) \quad H_j = \frac{p_j p_j'}{p_j' q_j p_j} + \frac{q_j q_j'}{w_j' q_j} + \sum_{i=3}^n \frac{p_{ij} p_{ij}'}{d_{ij}' p_{ij}},$$

where

$$1) \quad p_j = \frac{H_j q_j}{\|H_j q_j\|}, \quad q_j = \frac{1}{\|H_j q_j\|},$$

- ii) $w_j \in \text{span}\{q_j, q_{j+1}\}$ such that $w_j' p_j = 0$, $w_j' q_j > 0$ and $q_j = H_j w_j$ has norm one,
 iii) the vectors $d_{3j}, \dots, d_{nj}$ are orthogonal to p_j and q_j and are such that

$$d_{ij}' H_j d_{kj} = 0, \quad i, k = 3, \dots, n, \quad i \neq k.$$

and

$$p_{ij} = H_j d_{ij}, \quad i = 3, \dots, n, \quad \text{has norm one.}$$

Then every H_{j+1} determined by (2.3) has the form (see [9])

$$(2.5) \quad H_{j+1} = \frac{p_j p_j'}{d_j' p_j} + \omega_j \frac{u_j u_j'}{w_j' u_j} + \sum_{i=3}^n \frac{p_{ij} p_{ij}'}{d_{ij}' p_{ij}},$$

where the vector u_j is uniquely determined by the conditions

$$(2.6) \quad u_j \in \text{span}\{q_j, p_j\}, \quad \|u_j\| = 1, \quad d_j' u_j = 0, \quad w_j' u_j > 0.$$

The parameter ω_j depends on the particular numbers β_1 and β_2 used in (2.3). More precisely,

$$(2.7) \quad \omega_j = \gamma_j \|q_j - \frac{d_j' q_j}{d_j' p_j} p_j\|$$

with

$$(2.8) \quad \gamma_j = \frac{\beta_1 d_j' p_j + \beta_2 \frac{(d_j' p_j)^2}{d_j' q_j p_j}}{\beta_1 d_j' p_j + \beta_2 d_j' H_j d_j}.$$

This shows that if H_j is positive definite and

$$d_j' p_j = \frac{q_j' p_j - q_{j+1}' p_j}{\|s_j\|} > 0, \quad \text{i.e.,} \quad q_{j+1}' p_j < q_j' p_j$$

then H_{j+1} is positive definite if and only if $\gamma_j > 0$.

3. Superlinear convergence

Throughout this section we will assume that Assumption 1 is satisfied and that the sequence $\{x_j\}$ is generated by the following algorithm.

Algorithm

Step 0: Choose numbers β_1, β_2 with $\beta_1 + \beta_2 \neq 0$, a vector x_0 , and a symmetric positive definite matrix H_0 . Compute $q_0 = VF(x_0)$. If $q_0 = 0$, stop; otherwise set $j = 0$ and go to Step 1.

Step 1: Set

$$s_j = H_j q_j \quad \text{and} \quad x_{j+1} = x_j - s_j.$$

Compute $q_{j+1} = VF(x_{j+1})$. If $q_{j+1} = 0$, stop; otherwise go to Step 2.

Step 2: Compute H_{j+1} by (2.3). Replace j with $j+1$ and go to Step 1.

We will show that for every choice of β_1 and β_2 with $\beta_1 + \beta_2 \neq 0$ there are numbers

$$\delta = \delta(\beta_1, \beta_2) \quad \text{and} \quad \delta^* = \delta^*(\beta_1, \beta_2)$$

such that

$$\|H_0^{-1}\| \leq \delta \quad \text{and} \quad \|x_0 - z\| \leq \delta^*$$

imply that every H_j is a well-defined positive definite matrix and that the algorithm either terminates after a finite number of iterations or generates a sequence $\{x_j\}$ which converges superlinearly to z .

The convergence proof is based on an estimate for the trace of the matrix

$$(3.1) \quad G^{1/2} H_j G^{1/2} + G^{-1/2} B_j G^{-1/2}$$

where $G^{1/2}$ is the square root of the positive definite matrix G , $G^{-1/2} = (G^{1/2})^{-1}$ and $B_j = H_j^{-1}$. It follows immediately from (2.4) and (2.5) that

$$(3.2) \quad B_j = \frac{p_j q_j q_j'}{q_j' p_j} + \frac{w_j w_j'}{w_j' q_j} + \sum_{i=3}^n \frac{d_{ij} d_{ij}'}{d_{ij}' p_{ij}}$$

and

$$(3.3) \quad B_{j+1} = \frac{d_j d_j'}{d_j' p_j} + \frac{1}{\omega_j} \frac{w_j w_j'}{w_j' u_j} + \sum_{i=3}^n \frac{d_{ij} d_{ij}'}{d_{ij}' p_{ij}}.$$

Observe that by (2.6)

$$u_j = \frac{q_j + \alpha_j p_j}{\|q_j + \alpha_j p_j\|}, \quad \alpha_j = \frac{-d_j' q_j}{d_j' p_j}.$$

Therefore, we have $\|q_j + \alpha_j p_j\| w_j' u_j = w_j' q_j$ and by (2.7)

$$(3.4) \quad \omega_j \frac{u_j u_j'}{w_j' u_j} = \gamma_j \frac{(q_j + \alpha_j p_j)' (q_j + \alpha_j p_j)}{w_j' q_j}$$

$$(3.5) \quad \frac{1}{\omega_j} \frac{w_j w_j'}{w_j' u_j} = \frac{1}{\gamma_j} \frac{w_j w_j'}{w_j' q_j}.$$

Let ψ_j denote the trace of the matrix (3.1). Since the trace of a matrix is equal to the sum of its diagonal elements it follows from (2.4), (2.5) and (3.2) through (3.5) that we have the following relation between ψ_{j+1} and ψ_j .

$$(3.6) \quad \begin{aligned} \psi_{j+1} &= \psi_j - \frac{p_j' G p_j + \rho_j^2 q_j' G^{-1} q_j}{\rho_j g_j' p_j} + \frac{p_j' G p_j + d_j' G^{-1} d_j}{d_j' p_j} \\ &\quad - \frac{q_j' G q_j + w_j' G^{-1} w_j}{w_j' q_j} + \frac{1}{\gamma_j} \frac{\gamma_j^2 (q_j + \alpha_j p_j)' G (q_j + \alpha_j p_j) + w_j' G^{-1} w_j}{w_j' q_j} \\ &= \psi_j + \left(\frac{p_j' G p_j + d_j' G^{-1} d_j}{d_j' p_j} - 2 \right) \\ &\quad + (\gamma_j^{-1} - 1) \frac{q_j' G q_j}{w_j' q_j} + \left(\frac{1}{\gamma_j} - 1 \right) \frac{w_j' G^{-1} w_j}{w_j' q_j} - \left(\frac{p_j' G p_j + \rho_j^2 q_j' G^{-1} q_j}{\rho_j g_j' p_j} - 2 \right) \\ &\quad + \gamma_j \left(\frac{(q_j + \alpha_j p_j)' G (q_j + \alpha_j p_j)}{w_j' q_j} - \frac{q_j' G q_j}{w_j' q_j} \right). \end{aligned}$$

In the following five lemmas we will establish some properties of ψ_j and the terms on the right hand side of (3.6) which will enable us to prove the key result that the sequence $\{\psi_j\}$ is bounded.

Lemma 1

Let H_j be positive definite. Then

$$i) \quad \psi_j \geq 2n$$

$$ii) \quad \|H_j\| \leq \psi_j \|G^{-1}\| \quad \text{and} \quad \|H_j^{-1}\| \leq \psi_j \|G\|.$$

Proof:

Let $x, y \in E^n$ be such that $y'x \neq 0$. Set $v = y - Gx$. Then (see [9]),

$$(3.7) \quad \frac{x'Gx + y'G^{-1}y}{y'x} = 2 + \frac{v'G^{-1}v}{y'x}.$$

The first statement of the lemma follows immediately from this equality. By definition ψ_j is equal to the sum of the eigenvalues of the two matrices $G^{1/2}H_jG^{1/2}$ and $G^{-1/2}H_j^{-1}G^{-1/2}$. Since both matrices are positive definite we have

$$\|G^{1/2}H_jG^{1/2}\| \leq \psi_j \quad \text{and} \quad \|G^{-1/2}H_j^{-1}G^{-1/2}\| \leq \psi_j.$$

This completes the proof of the lemma.

Lemma 2

Let H_j be positive definite. For every $0 < \lambda < 1$ there are constants $\tau_1 > 2n$ and $\tau_1^* > 0$ such that, for every j ,

$$\psi_j \leq \tau_1 \quad \text{and} \quad \|x_j - z\| \leq \tau_1^*$$

imply

$$(3.8) \quad \|x_{j+1} - z\| \leq \lambda \|x_j - z\|.$$

Proof:

Because

$$\frac{p_j'Gp_j + p_j^2q_j'G^{-1}q_j}{p_j'q_jp_j} - 2 \leq \psi_j - 2n$$

it follows from (3.7) and Lemma 1 that

$$(3.9) \quad \|p_jq_j - Gp_j\|^2 = O(\psi_j - 2n) \quad \text{and} \quad \|s_j\| = O(\|q_j\|).$$

Therefore, x_j and x_{j+1} are in $U(z)$ for τ_1 and τ_1^* sufficiently small and we obtain from Taylor's theorem the relation

$$(3.10) \quad q_{j+1} = q_j - Gs_j - E_j s_j$$

where

$$E_j = \int_0^1 G(x_j - ts_j) dt - G$$

and

$$(3.11) \quad \begin{aligned} \|E_j\| &\leq \max_{0 \leq t \leq 1} \|G(x_j - ts_j) - G\| \\ &\leq \max_{0 \leq t \leq 1} \|L(x_j - t(x_j - x_{j+1}) - z)\| \\ &\leq L \max(\|x_j - z\|, \|x_{j+1} - z\|). \end{aligned}$$

Therefore,

$$(3.12) \quad \begin{aligned} \frac{\|q_{j+1}\|}{\|q_j\|} &\leq \left\| \frac{q_j}{\|q_j\|} - G \frac{s_j}{\|q_j\|} \right\| + \|E_j\| \frac{\|s_j\|}{\|q_j\|} \\ &= \frac{\|s_j\|}{\|q_j\|} \|\rho_j q_j - G p_j\| + \|E_j\| \frac{\|s_j\|}{\|q_j\|}. \end{aligned}$$

Since Taylor's theorem and Assumption 1 imply

$$(3.13) \quad \|q_j\| = o(\|x_j - z\|) \quad \text{and} \quad \|x_j - z\| = o(\|q_j\|),$$

it follows from (3.9), (3.11) and (3.12) that $\|x_{j+1} - z\| \leq \lambda \|x_j - z\|$ for τ_1 and τ_1^* sufficiently small.

Lemma 3

Let H_j be positive definite. Then there are constants $2n < \tau_2 \leq \tau_1$ and $0 < \tau_2^* \leq \tau_1^*$ such that, for every j ,

$$\psi_j \leq \tau_2 \quad \text{and} \quad \|x_j - z\| \leq \tau_2^*$$

imply

i) H_{j+1} is well-defined and positive definite.

$$\text{ii) } |1 - \gamma_j| = O\left(\frac{p_j' G p_j - \rho_j^2 q_j' G^{-1} q_j}{\rho_j q_j' p_j} - 2\right) + O(\|x_j - z\|) .$$

Proof:

By (2.5) and (2.7), H_{j+1} is well-defined and positive definite if $d_j' p_j > 0$ and $\gamma_j > 0$. Using (3.10) we obtain

$$d_j' p_j = p_j' G p_j + p_j' E_j p_j .$$

Since by (3.8) and (3.11)

$$(3.14) \quad \|E_j\| = O(\|x_j - z\|)$$

this shows that $d_j' p_j > 0$ for τ_2 and τ_2^* sufficiently small.

By (2.4)

$$d_j' H_j d_j = \frac{(d_j' p_j)^2}{\rho_j q_j' p_j} + \frac{(d_j' q_j)^2}{w_j' q_j} .$$

Therefore it follows from (2.8) that

$$(3.15) \quad 1 - \gamma_j = \frac{(d_j' q_j)^2}{w_j' q_j d_j' p_j} \left[\beta_1 + \beta_2 \frac{d_j' p_j}{\rho_j q_j' p_j} + \beta_2 \frac{(d_j' q_j)^2}{w_j' q_j d_j' p_j} \right]^{-1} .$$

By definition,

$$(3.16) \quad |d_j' q_j| = \left| \frac{q_j' q_j - q_{j+1}' q_j}{\|s_j\|} \right| = \frac{|q_{j+1}' q_j|}{\|s_j\|} \leq \frac{\|q_j\|}{\|s_j\|} \frac{\|q_{j+1}\|}{\|q_j\|} ,$$

$$(3.17) \quad \left| \frac{d_j' p_j}{\rho_j q_j' p_j} - 1 \right| = \left| \frac{q_j' p_j - q_{j+1}' p_j}{q_j' p_j} - 1 \right| \leq \frac{\|q_j\|}{q_j' p_j} \frac{\|q_{j+1}\|}{\|q_j\|} .$$

It follows from part ii) of Lemma 1 that the sequences $\{1/w_j' q_j\}$, $\{1/d_j' p_j\}$, $\{\|q_j\|/\|s_j\|\}$ and $\{\|q_j\|/q_j' p_j\}$ are bounded. Because $\beta_1 + \beta_2 \neq 0$ we deduce, therefore, from (3.9), (3.12), and (3.14) through (3.17) that

$$(3.18) \quad |1 - \gamma_j| = 0 \left(\frac{\|g_{j+1}\|^2}{\|g_j\|^2} \right) < 1$$

for τ_2 and τ_2^* sufficiently small. Furthermore, using (3.18) and (3.12) we obtain

$$|1 - \gamma_j| = 0(\|\rho_j g_j - G p_j\|^2 + \|E_j\|^2)$$

which by (3.7) and (3.14) implies the second part of the lemma.

Lemma 4

Let H_j be positive definite and $0 < t < 1$. There are constants $2n < \tau_3 \leq \tau_2$, $0 < \tau_3^* \leq \tau_2^*$ and $\delta_1 > 0$ such that, for every j ,

$$\psi_j \leq \tau_3 \quad \text{and} \quad \|x_j - z\| \leq \tau_3^*$$

imply

$$\begin{aligned} \text{i)} \quad & \frac{p_j' G p_j + d_j' G^{-1} d_j}{d_j' p_j} - 2 \leq \delta_1 \|x_j - z\| \\ \text{ii)} \quad & \gamma_j \frac{(q_j + \alpha_j p_j)' G (q_j + \alpha_j p_j) - q_j' G q_j}{w_j' q_j} \leq \delta_1 \|x_j - z\| \\ \text{iii)} \quad & (\gamma_j - 1) \frac{q_j' G q_j}{w_j' q_j} + \left(\frac{1}{\gamma_j} - 1 \right) \frac{w_j' G^{-1} w_j}{w_j' q_j} \\ & - t \left(\frac{p_j' G p_j + \rho_j^2 q_j' G^{-1} q_j}{\rho_j q_j' p_j} - 2 \right) \leq \delta_1 \|x_j - z\|. \end{aligned}$$

Proof:

It follows from (3.10) that

$$d_j - G p_j = E_j p_j$$

which by (3.7) and (3.14) implies the first part of the lemma. In order to prove the second part we observe that

$$\begin{aligned}
(q_j + a_j p_j)' G (q_j + a_j p_j) - q_j' G q_j &= 2 a_j p_j' G q_j + a_j^2 p_j' G p_j \\
&= -2 \frac{d_j' q_j}{d_j' p_j} (d_j' - p_j' E_j) q_j + \left(\frac{d_j' q_j}{d_j' p_j} \right)^2 p_j' (d_j - E_j p_j) \\
&= - \frac{(d_j' q_j)^2}{d_j' p_j} + 2 \frac{d_j' q_j}{d_j' p_j} p_j' E_j q_j - \left(\frac{d_j' q_j}{d_j' p_j} \right)^2 p_j' E_j p_j \\
&= o(\|E_j\|) .
\end{aligned}$$

Because of (3.18) and (3.14) this equality implies the second part of the lemma for τ_3 and τ_3^* sufficiently small. Finally we have

$$\begin{aligned}
(3.19) \quad (\gamma_j - 1) \frac{q_j' G q_j}{w_j' q_j} + \frac{1 - \gamma_j}{\gamma_j} \frac{w_j' G^{-1} w_j}{w_j' q_j} &= \\
(\gamma_j - 1) \left[\frac{\gamma_j - 1}{\gamma_j} \frac{w_j' G^{-1} w_j}{w_j' q_j} + \frac{q_j' G q_j - w_j' G^{-1} w_j}{w_j' q_j} \right] .
\end{aligned}$$

Since

$$|q_j' G q_j - w_j' G^{-1} w_j| = o(\|w_j - G q_j\|) ,$$

the last part of the lemma follows from (3.19), (3.7) and Lemmas 1 and 3.

Lemma 5

There are constants $\tau > 2n$ and $\tau^* > 0$ such that

$$\psi_0 \leq \tau \quad \text{and} \quad \|x_0 - z\| \leq \tau^*$$

imply

- i) H_j is well-defined and positive definite for $j = 0, 1, 2, \dots$
- ii) $\psi_j \leq \tau_3$ for $j = 0, 1, 2, \dots$
- iii) $\sum_{j=0}^{\infty} \|x_j - z\| < \infty$.

Proof.

Choose $\lambda = 0.5$ in Lemma 2 and let τ and τ^* be such that

$$(3.20) \quad \tau + 6\delta_1 \tau^* \leq \tau_3 \quad \text{and} \quad \tau^* \leq \tau_3^* .$$

We will show by induction that then for every j the following statements hold.

$$(3.21) \quad H_{j+1} \text{ is well-defined and positive definite for } i = 0, \dots, j$$

$$(3.22) \quad \|x_{i+1} - z\| \leq 0.5 \|x_i - z\|, \quad i = 0, \dots, j$$

$$(3.23) \quad \psi_{j+1} \leq \psi_0 + 3\delta_1 \sum_{i=0}^j \|x_i - z\| \leq \tau_3 .$$

Let $j = 0$. Since (3.20) implies $\tau \leq \tau_3 \leq \tau_2 \leq \tau_1$ and $\tau^* \leq \tau_3^* \leq \tau_2^* \leq \tau_1^*$, it follows from Lemma 3 and the definition of H_0 that H_1 is positive definite. Furthermore, Lemma 2 gives the inequality (3.22) and Lemma 4 in conjunction with (3.6) implies the relation (3.23). Now assume that (3.21) through (3.23) are satisfied for some $j - 1 > 0$. Since (3.22) and (3.23) give the inequalities.

$$\|x_j - z\| < \|x_0 - z\| \leq \tau_2^* \quad \text{and} \quad \psi_j \leq \tau_2$$

it follows from Lemma 3 that H_{j+1} is positive definite. Moreover, Lemma 2 implies

$$\|x_{j+1} - z\| \leq 0.5 \|x_j - z\| .$$

Using Lemma 4 and (3.6) we obtain

$$\begin{aligned} \psi_{j+1} &\leq \psi_j + 3\delta_1 \|x_j - z\| \\ &\leq \psi_0 + 3\delta_1 \sum_{i=0}^{j-1} \|x_i - z\| + 3\delta_1 \|x_j - z\| \\ &\leq \psi_0 + 3\delta_1 \sum_{i=0}^j (0.5)^i \|x_0 - z\| \\ &\leq \psi_0 + 6\delta_1 \|x_0 - z\| \leq \tau_3 . \end{aligned}$$

Since (3.23) implies that the sum

$$\sum_{j=0}^{\infty} \|x_j - z\|$$

is finite the proof of the lemma is complete.

Using the above results we can now prove that the sequence $\{x_j\}$ generated by the algorithm converges locally and superlinearly to z .

Theorem

Let Assumption 1 be satisfied. For every choice of β_1 and β_2 with $\beta_1 + \beta_2 \neq 0$, there are numbers

$$\delta(\beta_1, \beta_2) \quad \text{and} \quad \delta^*(\beta_1, \beta_2)$$

such that the inequalities

$$\|H_0^{-G^{-1}}\| \leq \delta(\beta_1, \beta_2) \quad \text{and} \quad \|x_0 - z\| \leq \delta^*(\beta_1, \beta_2)$$

imply that the sequences $\{x_j\}$ and $\{H_j\}$ in the algorithm are well-defined and have the following properties.

- i) H_j is positive definite for all j .
- ii) Either $x_j = z$ for some j or

$$\frac{\|x_{j+1} - z\|}{\|x_j - z\|} \rightarrow 0 \quad \text{as } j \rightarrow \infty$$

$$\sum_{j=0}^{\infty} \left(\frac{\|x_{j+1} - z\|}{\|x_j - z\|} \right)^2 \quad \text{is finite}$$

$$\{H_j\} \quad \text{and} \quad \{H_j^{-1}\} \quad \text{are bounded.}$$

Proof:

It follows immediately from Assumption 1 that there is some neighborhood $U_0(z) \subset U(z)$ such that $x \in U_0(z)$ and $x \neq z$ imply $\nabla F(x) \neq 0$. Choose $\delta^*(\beta_1, \beta_2) \leq \tau^*$ such that $\|x - z\| \leq \delta^*(\beta_1, \beta_2)$ implies $x \in U_0(z)$. Furthermore choose $\delta(\beta_1, \beta_2)$ such that

$$\|H_0^{-G^{-1}}\| \leq \delta(\beta_1, \beta_2) \quad \text{implies} \quad \psi_0 \leq \tau.$$

By (3.7) a $\delta(\beta_1, \beta_2)$ with this property exists for every $\tau > 2n$.

With $\delta(\beta_1, \beta_2)$ and $\delta^*(\beta_1, \beta_2)$ chosen in this way we deduce from Lemma 2 and Lemma 5 that H_j is well-defined and positive definite and $x_j \in U_0(z)$ for every j . Therefore, it follows

from Step 1 of the algorithm that $\{x_j\}$ is well-defined. Furthermore $g_j = 0$ if and only if $x_j = z$.

Let $\{x_j\}$ be an infinite sequence. By Lemma 1 and Lemma 5, the sequences $\{H_j\}$ and $\{H_j^{-1}\}$ are bounded.

With $0 < t < 1$ we deduce from (3.6) and Lemma 4 that, for every j ,

$$(3.24) \quad \psi_{j+1} \leq \psi_j + 3\delta_1 \|x_j - z\| + (t-1) \left(\frac{p_j' G p_j + \rho_j^2 q_j' G q_j}{\rho_j q_j' p_j} - 2 \right).$$

Since by Lemma 1, $\psi_j \geq 2n$ it follows from (3.24) and part iii) of Lemma 5 that

$$\sum_{j=0}^{\infty} \left(\frac{p_j' G p_j + \rho_j^2 q_j' G^{-1} q_j}{\rho_j q_j' p_j} - 2 \right) < \infty,$$

which by (3.7) implies that

$$(3.25) \quad \sum_{j=0}^{\infty} \|\rho_j q_j - G p_j\|^2 < \infty.$$

Using (3.12), (3.25), (3.14) and part iii) of Lemma 5 we obtain

$$\sum_{j=0}^{\infty} \left(\frac{\|q_{j+1}\|}{\|q_j\|} \right)^2 < \infty.$$

In view of (3.13) this inequality completes the proof of the theorem.

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