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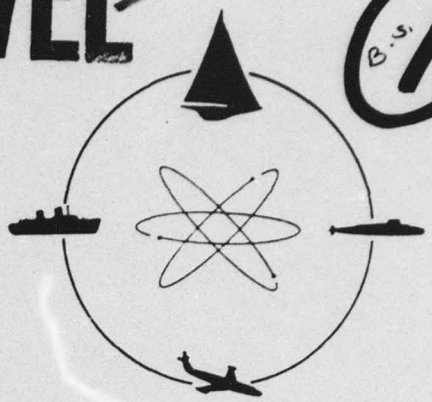

DAVIDSON LABORATORY

Report SIT-DL-78-9-2040

December 1978

ADDED MASS AND DAMPING OF THE HEAVING SURFACE EFFECT SHIP IN UNIFORM TRANSLATION

by

C.H. Kim and S. Tsakonas

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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The theoretical added mass and damping coefficients and the heave response show fair agreement with the corresponding experimental values.

Comparisons of the coupled aero-hydrodynamic and uncoupled analytical results with the experimental data prove that the uncoupled theory, dominant (Cont'd)

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KEYWORDS

Surface Effect Ship
Water Waves
Aero-hydrodynamic Forces
Bubble Pressure
Added Mass
Damping Factor
Heave Response

TABLE OF CONTENTS

NOMENCLATURE

A	escape area or area of nozzle throat
A_o	escape area at the equilibrium
A_c	cushion area
A_s	variation of the seal opening area
a	half length of the pressure patch or wave amplitude
b	half width of the pressure patch
C_a	non-dimensional added mass coefficient
C_d	non-dimensional damping coefficient
F_n	Froude number
g	gravitational acceleration
h	cushion height
i	$\sqrt{-1}$, or the suffix indicating the imaginary component
j	integer
K	nozzle flow coefficient or constant
k_j	wave number
L	nominal ship length
ℓ	integer
M''	added mass
m	integer
N	damping factor
n	integer
P	pressure
P_a	atmospheric pressure
P_b	bubble pressure
P_o	ambient pressure

Q	discharge
Q_o	discharge at equilibrium
r	distance or the suffix indicating the real component
s_m	distance
t	time
U	ship speed
V	the wave-induced volume
v	non-dimensional coefficient of the wave-induced volume
w	suffix indicating the water
x,y,z	coordinates
z	heave displacement
α	non-dimensional coefficient of the escape area
γ	ratio of specific heats
ϵ	a parameter
φ	the velocity potential
λ	scale ratio
Π	pressure
σ	circular frequency
τ	frequency, $\sigma U/g$
ρ, ρ_w	density of water
ρ_a	density of air
ζ	wave elevation

INTRODUCTION

The analysis is directed at the problem of finding the effect of the presence of the water wave upon the bubble (or cushion) pressure in the plenum of a surface-effect ship as it is forced in simple harmonic heaving motion while translating at constant speed over otherwise calm water.

The surface-effect ship undergoing the above-mentioned motion is a multi-parameter system influenced by the coupled aero- and hydrodynamic actions of water and air in the plenum which is under a continuous supply of air through a system of fans and leakage through the peripheral gap.

The analysis when linearized yields the aero-hydrodynamic forces due tc the forced heaving motion and thus provides the corresponding added mass and damping force coefficients in waves which is vital information for the evaluation of particular design characteristics of a ship at the desired speed and sea state.

The theory has been developed by referring to the works by Kaplan and Davis,[1] Doctors,[2] and Breslin and Hires.[3] The relevant experimental literature referenced here are the investigations due to Van Den Brug and Van Staveren,[4] Magnuson and Wolff,[5] Fein and Murray,[6] Moran, Fein and Ricci,[7] and Fridsma.[8,9]

Kaplan and Davis[1] assume that the free surface behaves as a rigid boundary at constant pressure whereas Doctors and Breslin, et al, assume that the deformation of water surface participates in the generation of the bubble pressure in conjunction with the actions of seals and fans, etc. The second cardinal assumption in the latter works[2,3] is that the deformation of the wave surface under the oscillatory rectangular pressure patch in uniform translation be used to display the way in which the motion of the water surface participates in the determination of the pressure variations in the plenum air.

Doctors deals with the three-dimensional rectangular pressure patch whereas Breslin and Hires idealize the problem to two-dimensional flow conditions.

*Superior numbers in text matter refer to similarly numbered references listed at the end of this report.

The analysis has been developed independently from that of Doctors,[2] although along the same lines, leading to expressions for the evaluation of the wave elevation, escape area and volume, quite different from those reported in Reference 2. In addition, a numerical approach has been developed valid for the entire range of frequency of practical interest, from very low to considerably high, in contrast to Doctors' numerical results which are restricted to the very low frequency regime.

In the dynamic analysis of the airflow phenomena, pressure and volume changes are assumed as by others[1-3] to occur sufficiently rapidly so that the adiabatic law governs the basic thermodynamic variations in the air cushion. The cushion pressure is assumed to be spatially uniform which is approximately valid according to Moran[10] and Tsakonas et al.[11] Since this work is intended to be applied to the high length-beam-ratio sidewalled surface effect ships,[5-7] it assumes also that the leakage of air takes place only at stern seal. The seal response dynamics are beyond the scope of the present investigation. It assumes a semi-rigid planing seal which has been the subject of the experimental study of the XR-5 SES Model.[5]

Following the calculation of the waves due to an oscillatory rectangular pressure patch in uniform translation, the bubble pressure is determined. The linearized equation of bubble pressure in plenum chamber and that of the heaving displacement are both derived by applying the perturbation method through the time derivative of the adiabatic law. The asymptotic behavior of the bubble pressures as the frequency $\to 0$ and ∞ is analyzed and discussed with and without water wave effects.

The motion-induced aero-hydrodynamic forces are calculated from the bubble pressure distribution on the cushion area as a function of Froude number and of reduced frequency. From its real and imaginary components, the added mass and damping coefficients are determined respectively. The asymptotic behavior of the added mass and damping as the frequency $\to 0$ and ∞ are evaluated and discussed both with and without water wave effects.

The scale effect, essential information for the extrapolation of the model measurements to the full-scale condition, is also the subject of the present analysis. The scale effect determined in the previous study[12] indicates a great dissimilitude in general. The scale effect ratio of the

pressure of the ship to the model as calculated here shows that the dissimilitude is small at low frequency but quite large at higher frequencies.

The present analysis does not include the sidewall hydrodynamic effects since experiments on the XR-5 SES[6,7] have shown slight effect of the sidewalls on added mass and damping.

A series of numerical computations has been done to determine the water waves, bubble pressure, heave added mass and damping coefficients, scale effect on bubble pressure, and the heave responses of the XR-5 SES model running in head waves.

The waves, escape area and volume which are induced by the oscillating pressure at uniform speed in otherwise calm water have been calculated for the model used by Doctors.[2] A computational procedure has been developed and established for any frequency and speed without limitation. The bubble pressure, added mass and damping coefficients have been computed for the XR-5 SES model, with and without incorporating the water elevation. A remarkable difference has been found between the two groups.

Experimental values[6] of the heave added mass and damping coefficients have been compared with the theoretical results. To study the validity of the theoratical added mass and damping coefficients, the heave response of the model has been calculated on the assumption that the pitch is restrained, using the theoretical added mass and damping factors with the experimentally[6] determined static stiffness and heave-exciting forces.

The analytical developments at limiting cases (i.e., high and low frequencies) have been derived and discussed in detail as well as the results of the comparison of predictions with measurements. The comparisons of the results of both the coupled aero-hydrodynamic analysis and the uncoupled analysis with the results of experiments have shown strongly that omission of the water waves in calculating the bubble pressure and the hydrodynamic forces is not a valid assumption.

This research was sponsored by the David Taylor Naval Ship Research and Development Center, General Hydromechanics Research Program, under Contract N00014-77-C-0061, Davidson Laboratory Project 4480/015.

THE VELOCITY POTENTIAL

The hydrodynamic boundary value problem for a pressure patch oscillating with constant amplitude and moving on the surface of a deep, incompressible, ideal fluid can be determined through the velocity potential method.

The right-handed Cartesian coordinate system is set up on the calm water surface with its vertical z-axis pointing upwards. See Figure 1a.

This problem is formulated as an unsteady boundary value problem[13,14] in the framework of the linearized conditions.

The pulsating pressure distribution of a rectangular shape on the calm water surface, which is a simple harmonic in time, is expressed as

$$p(x,y,0;t) = \begin{cases} \Pi(x,y)e^{i\sigma t} & |x| \leq a,\ |y| \leq b \\ 0 & |x| > a,\ |y| > b \end{cases} , \quad t > 0 \tag{1}$$

The deformation of water due to the above pressure distribution is assumed to be small and is represented by the velocity potential $\varphi(x,y,z;t)$ which is harmonic in space below the free surface; satisfying the d.e.,

$$\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0 , \qquad z < 0 , \ t > 0 \tag{2}$$

On the free surface the velocity potential φ and the surface elevation $\zeta = \zeta(x,y;t)$ satisfy the linearized kinematical and dynamical boundary conditions,

$$-\zeta_t + U\zeta_x + \varphi_z = 0 , \qquad z=0, \quad t>0 \tag{3}$$

and

$$\frac{p}{\rho} + g\zeta + \varphi_t - U\varphi_x = 0 , \qquad z=0, \quad t>0 \tag{4}$$

Eliminating ζ from Eqs.(3) and (4), we have the linearized free surface condition

$$\varphi_{tt} + U^2\varphi_{xx} - 2U\varphi_{xt} + g\varphi_z = \frac{1}{\rho}(Up_x - p_t), \quad z=0 , \ t > 0 \tag{5}$$

In addition to the above, φ and φ_t satisfy the condition of being

initially at rest on the surface

$$\varphi(x,y,0;0) = \varphi_t(x,y,0;0) = 0 \tag{6}$$

where $\varphi_t=0$ is valid under the assumption $p(x,y,0;0)=0$ [Eq.(1)][15], and on the bottom

$$\varphi_z = 0 \quad , \quad z = -\infty, \quad t > 0 \tag{7}$$

In the far field we suppose that φ and its first and second derivatives tend to zero for any given time; in fact, they tend to zero in such a way that Fourier transforms exist.

Now we seek the form of φ which satisfies the above conditions by employing Fourier transform technique. Define the double Fourier transforms

$$\bar{\varphi}(w,u,z;t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(x,y,z;t) e^{-i(wx+uy)} dxdy \tag{8}$$

and inversion

$$\varphi(x,y,z;t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{\varphi}(w,u,z;t) e^{i(wx+uy)} dwdu \tag{9}$$

Here w and u are the longitudinal and transverse wave numbers and are related to the circular wave number k and the wave angle θ by

$$\begin{aligned} w &= k\cos\theta \\ u &= k\sin\theta \end{aligned} \tag{10}$$

The Fourier transform (8) applied to Eq.(2) yields

$$\bar{\varphi}_{zz} - k^2\bar{\varphi} = 0 \tag{11}$$

The general solution of the above differential equation is

$$\bar{\varphi} = A(w,u;t)e^{|k|z} + B(w,u;t)e^{-|k|z} \tag{12}$$

where A and B are arbitrary constants. By making use of the Fourier transform of the bottom condition (7), we obtain $B=0$. Hence the above is reduced to

$$\bar{\varphi} = A(w,u;t)e^{|k|z} \tag{13}$$

To determine A, take the Fourier transform of the linearized free surface condition (5),

$$\bar{\varphi}_{tt} - U^2w^2\bar{\varphi} - i2Uw\bar{\varphi}_t + g\bar{\varphi}_z = i(Uw-\sigma)\frac{\bar{\Pi}}{\rho}e^{i\sigma t} \tag{14}$$

where

$$\bar{\Pi}(w,u) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Pi(x,y)e^{-i(wx+uy)}dxdy$$

$$= \frac{1}{2\pi}\int_{-b}^{b}d\eta\int_{-a}^{a}\Pi(\xi,\eta)e^{-i(w\xi+u\eta)}d\xi \tag{15}$$

Substituting $\bar{\varphi}$ from Eq.(13) in (14) yields

$$A_{tt}-U^2w^2A - i2UwA_t + g|k|A = i(Uw-\sigma)\frac{\bar{\Pi}}{\rho}e^{i\sigma t} \tag{16}$$

which by means of the Laplace transform and utilizing the initial condition

$$A(w,u;0) = A_t(w,u;0) = 0$$

yields the following solution in the transform plane

$$L(A) = \frac{(Uw-\sigma)\bar{\Pi}}{2\rho\sqrt{g|k|}}\left(\frac{1}{q-q_1} - \frac{1}{q-q_2}\right)L(e^{i\sigma t})$$

with

$$q_1 = i(Uw + \sqrt{g|k|})$$

$$q_2 = i(Uw - \sqrt{g|k|})$$

Inversion of the above is given by the convolution integral,

$$A = \sum_{\ell=\pm1}\frac{\ell\bar{\Pi}(Uw-\sigma)}{2\rho\sqrt{g|k|}}\int_0^t e^{i\sigma\tau}\,e^{i(\ell\sqrt{g|k|}+Uw)(t-\tau)}d\tau \tag{17}$$

Equation (17) after integration is substituted in (13) and we have

$$\bar{\varphi} = e^{i\sigma t}\sum_{\ell=\pm1}\frac{i\ell\bar{\Pi}(Uw-\sigma)}{2\rho\sqrt{g|k|}}e^{|k|z}\frac{\left[1-e^{i(Uw+\ell\sqrt{g|k|}-\sigma)t}\right]}{Uw+\ell\sqrt{g|k|}-\sigma} \tag{18}$$

The inverse transform of (18) is given by

$$\varphi = e^{i\sigma t}\frac{i}{8\pi^2\rho}\int_{-b}^{b} d\eta \int_{-a}^{a} d\xi \Pi(\xi,\eta) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{g|k|}} e^{|k|z+i(wx_o+uy_o)}(uw-\sigma)$$

$$\left(\frac{1-e^{i(Uw+\sqrt{g|k|}-\sigma)t}}{Uw+\sqrt{g|k|}-\sigma} - \frac{1-e^{i(Uw-\sqrt{g|k|}-\sigma)t}}{Uw-\sqrt{g|k|}-\sigma}\right)dwdu \qquad (19)$$

where $x_o = x - \xi$

$$y_o = y - \eta \qquad (20)$$

Now the variables (w,u) are transformed into (k,θ) by the relation (10). Replacing the elementary area $dwdu$ with

$$dwdu = kd\theta dk \qquad (10')$$

for the intervals $0 \le k \le \infty$, $-\pi \le \theta \le \pi$, we have

$$\varphi = e^{i\sigma t}\frac{i}{8\pi^2\rho}\int_{-b}^{b} d\eta \int_{-a}^{a} d\xi \Pi(\xi,\eta) \int_{-\pi}^{\pi} d\theta \int_{0}^{\infty} dk\sqrt{\frac{k}{g}}(Uk\cos\theta-\sigma)e^{kz+ik(x_o\cos\theta+y_o\sin\theta)}$$

$$\left(\frac{1-e^{i(Uk\cos\theta+\sqrt{gk}-\sigma)t}}{Uk\cos\theta+\sqrt{gk}-\sigma} - \frac{1-e^{i(Uk\cos\theta-\sqrt{gk}-\sigma)t}}{Uk\cos\theta-\sqrt{gk}-\sigma}\right) \qquad (21)$$

To fold up the integration interval $(-\pi,\pi)$ to $(0,\frac{\pi}{2})$, the integral $\int_{-\pi}^{\pi}$ is subdivided into four quadrants,

$$\int_{-\pi}^{\pi} = \int_{-\pi}^{-\pi/2} + \int_{-\pi/2}^{0} + \int_{0}^{\pi/2} + \int_{\pi/2}^{\pi}$$

and each integral is transformed into $\int_{0}^{\pi/2}$. For instance, $\int_{-\pi}^{-\pi/2} \to \int_{0}^{\pi/2}$ is achieved by setting $\theta = Y - \pi$, where Y is a new variable. It follows that $\cos\theta = -\cos Y$, $\sin\theta = -\sin Y$.

By replacing Y with θ and applying a similar procedure to $\int_{-\pi/2}^{0}$ and $\int_{\pi/2}^{\pi}$, we have the following folded form,

$$\varphi = e^{i\sigma t}\,\frac{i}{8\pi^2\rho}\int_{-b}^{b} d\eta \int_{-a}^{a} d\xi\, \Pi(\xi,\eta) \int_{0}^{\pi/2} d\theta \int_{0}^{\infty} dk \sqrt{\frac{k}{g}}\; e^{kz}$$

$$\left\{(Uk\cos\theta-\sigma)(e^{ikr_1} + e^{ikr_2})\left(\frac{1-e^{i\omega_1 t}}{\omega_1} - \frac{1-e^{i\omega_2 t}}{\omega_2}\right)\right.$$

$$\left.+(Uk\cos\theta+\sigma)(e^{-ikr_1}+e^{-ikr_2})\left(-\frac{1-e^{-i\omega_3 t}}{\omega_3} + \frac{1-e^{-i\omega_4 t}}{\omega_4}\right)\right\} \qquad (22)$$

where

$$r_1 = (x-\xi)\cos\theta + (y-\eta)\sin\theta$$

$$r_2 = (x-\xi)\cos\theta - (y-\eta)\sin\theta \qquad (23)$$

and

$$\left.\begin{matrix}\omega_1\\ \omega_2\end{matrix}\right\} = Uk\cos\theta \pm \sqrt{gk} - \sigma$$

$$\left.\begin{matrix}\omega_3\\ \omega_4\end{matrix}\right\} = Uk\cos\theta \pm \sqrt{gk} + \sigma \qquad (24)$$

STEADY STATE SOLUTION

In Eq.(22), φ is the solution of unsteady conditions which have been stated at the outset. The next step is to derive the steady state solution from Eq.(22) by eliminating the time-dependent terms. This is achieved in the following manner: 1) determine the wave numbers k_i from $\omega_i=0$, and 2) take the proper indenting paths in the neighborhood of k_i in the k-plane in order to have the exponentials vanish in time as $t \to \infty$. The integration paths determined as such are therefore not arbitrary but strictly bound to derivation of the steady solution and to the roots k_i.

Solving for the zeros of ω_i in Eq.(24), namely k_i, we have

$$\left.\begin{matrix} k_1 \\ k_2 \end{matrix}\right\} = \frac{\left(\mp 1 + \sqrt{1 + \frac{4U\sigma\cos\theta}{g}}\right)^2}{\frac{4U^2\cos^2\theta}{g}} \tag{25a}$$

$$\left.\begin{matrix} k_3 \\ k_4 \end{matrix}\right\} = \frac{\left(1 \mp \sqrt{1 - \frac{4U\sigma\cos\theta}{g}}\right)^2}{\frac{4U^2\cos^2\theta}{g}} \tag{26a}$$

or

$$\left.\begin{matrix} k_1 \\ k_2 \end{matrix}\right\} = \frac{1}{2} k_o \sec^2\theta \left[1+2\tau\cos\theta \mp \sqrt{1+4\tau\cos\theta}\right] \tag{25b}$$

$$\left.\begin{matrix} k_3 \\ k_4 \end{matrix}\right\} = \frac{1}{2} k_o \sec^2\theta \left[1-2\tau\cos\theta \mp \sqrt{1-4\tau\cos\theta}\right] \tag{26b}$$

with

$$k_o = \frac{g}{U^2} \quad , \quad \tau = \frac{U\sigma}{g} \tag{26c}$$

It is seen from (25a) that k_1 and k_2 are positive real and $k_1 \leqq k_2$ for interval $0 \leqq \theta \leqq \frac{\pi}{2}$. It is seen from (26a) that k_3 and k_4 are positive real and $k_3 \leqq k_4$ if

$$1 - \frac{4U\sigma\cos\theta}{g} \geqq 0$$

and k_3 and k_4 are complex with $\mathrm{Im}k_3<0$ and $\mathrm{Im}k_4>0$ (i.e., k_4= complex conjugate of k_3) if

$$1 - \frac{4U\sigma\cos\theta}{g} < 0$$

In other words k_3 and k_4 are positive real in

$$\theta_c \leq \theta \leq \frac{\pi}{2} \tag{27}$$

and complex in the interval

$$0 \leq \theta < \theta_c \tag{28}$$

where

$$\theta_c = \cos^{-1}\left(\frac{g}{4U\sigma}\right) \tag{29}$$

Now we consider the integrals with respect to k in the k-plane.

Since Taylor expansions of k and $\sqrt{gk}$ in the neighborhood of k_i, up to the first order, are

$$k \simeq k_i + (k-k_i) + 0(k^2)$$

$$\sqrt{gk} \simeq \sqrt{gk_i} + (k-k_i)\frac{\sqrt{g}}{2\sqrt{k_i}} \qquad (i=1,2,3,4) \tag{30}$$

we obtain the following expressions in the neighborhood of k_i:

$$i(Uk\cos\theta+\sqrt{gk}-\sigma) \simeq \begin{cases} i(Uk_1\cos\theta+\sqrt{gk_1}-\sigma)+i(k-k_1)\left(U\cos\theta + \dfrac{\sqrt{g}}{2\sqrt{k_1}}\right) \\ i(Uk_2\cos\theta+\sqrt{gk_2}-\sigma)+i(k-k_2)\left(U\cos\theta + \dfrac{\sqrt{g}}{2\sqrt{k_2}}\right) \end{cases}$$

$$i(Uk\cos\theta-\sqrt{gk}-\sigma) \simeq \begin{cases} i(Uk_1\cos\theta-\sqrt{gk_1}-\sigma)+i(k-k_1)\left(U\cos\theta - \dfrac{\sqrt{g}}{2\sqrt{k_1}}\right) \\ i(Uk_2\cos\theta-\sqrt{gk_2}-\sigma)+i(k-k_2)\left(U\cos\theta - \dfrac{\sqrt{g}}{2\sqrt{k_2}}\right) \end{cases} \tag{31}$$

$$i(Uk\cos\theta+\sqrt{gk}+\sigma) \simeq \begin{cases} i(Uk_3\cos\theta+\sqrt{gk_3}+\sigma)+i(k-k_3)\left(U\cos\theta+\dfrac{\sqrt{g}}{2\sqrt{k_3}}\right) \\ i(Uk_4\cos\theta+\sqrt{gk_4}+\sigma)+i(k-k_4)\left(U\cos\theta+\dfrac{\sqrt{g}}{2\sqrt{k_4}}\right) \end{cases}$$

$$i(Uk\cos\theta-\sqrt{gk}+\sigma) \simeq \begin{cases} i(Uk_3\cos\theta-\sqrt{gk_3}+\sigma)+i(k-k_3)\left(U\cos\theta-\dfrac{\sqrt{g}}{2\sqrt{k_3}}\right) \\ i(Uk_4\cos\theta-\sqrt{gk_4}+\sigma)+i(k-k_4)\left(U\cos\theta-\dfrac{\sqrt{g}}{2\sqrt{k_4}}\right) \end{cases} \quad (32)$$

From Eqs.(25a) and (26a), we have the positive real root k_i for the interval $0 \leq \theta \leq \frac{\pi}{2}$

$$\sqrt{k_1} = \frac{\sqrt{g}\left(-1+\sqrt{1+\frac{4U\sigma\cos\theta}{g}}\right)}{2U\cos\theta}$$

$$\sqrt{k_2} = \frac{\sqrt{g}\left(1+\sqrt{1+\frac{4U\sigma\cos\theta}{g}}\right)}{2U\cos\theta} \quad (33)$$

and for the interval $\theta_c \leq \theta \leq \frac{\pi}{2}$

$$\sqrt{k_3} = \frac{\sqrt{g}\left(1-\sqrt{1-\frac{4U\sigma\cos\theta}{g}}\right)}{2U\cos\theta}$$

$$\sqrt{k_4} = \frac{\sqrt{g}\left(1+\sqrt{1-\frac{4U\sigma\cos\theta}{g}}\right)}{2U\cos\theta} \quad (34)$$

Employing Eqs.(33) and (34), we have

$$i(Uk\cos\theta + \sqrt{gk} - \sigma) \sim \begin{cases} i(k-k_1)\,\dfrac{\sqrt{g}\sqrt{1+\dfrac{4U\sigma\cos\theta}{g}}}{2\sqrt{k_1}} \\ +i2\sqrt{gk_2} \end{cases}$$

$$i(Uk\cos\theta - \sqrt{gk} - \sigma) \sim \begin{cases} -i2\sqrt{gk_1} \\ i(k-k_2)\,\dfrac{\sqrt{g}\sqrt{1+\dfrac{4U\sigma\cos\theta}{g}}}{2\sqrt{k_2}} \end{cases} \qquad (35)$$

$$i(Uk\cos\theta + \sqrt{gk} + \sigma) \sim \begin{cases} i2\sqrt{gk_3} \\ i2\sqrt{gk_4} \end{cases}$$

$$i(Uk\cos\theta - \sqrt{gk} + \sigma) \sim \begin{cases} -i(k-k_3)\,\dfrac{\sqrt{g}\sqrt{1-\dfrac{4U\sigma\cos\theta}{g}}}{2\sqrt{k_3}} \\ +i(k-k_4)\,\dfrac{\sqrt{g}\sqrt{1-\dfrac{4U\sigma\cos\theta}{g}}}{2\sqrt{k_4}} \end{cases} \qquad (36)$$

Equation (35) shows that for the interval $0 \leqq \theta \leqq \frac{\pi}{2}$ the path of integration of the terms containing $e^{i\omega_1 t}$ and $e^{i\omega_2 t}$ in Eq.(22) in the complex k-plane should be deformed above the real axis in the neighborhood of both k_1 and k_2 in order to have the exponentials in t vanish as $t \to \infty$. Let this deformed path be denoted by L_1. (See Fig.1b.) Equation (36) shows that for the interval $\theta_c \leqq \theta < \frac{\pi}{2}$, the path of integration containing the factors $e^{-i\omega_3 t}$ and $e^{-i\omega_4 t}$ in Eq.(22) in the complex k-plane should be deformed above the real axis in the neighborhood of k_3 and below the real axis in the neighborhood of k_4 in order to have the exponentials in t vanish as $t \to \infty$. Let this path be denoted by L_2. Equation (24) shows that for the complex wave numbers k_3 and k_4 in the interval $0 \leqq \theta < \theta_c$, the integration

path requires no deformation in order to have the exponentials in t vanish as $t \to \infty$; for the path along the real axis the exponentials vanish as $t \to \infty$. Let this be denoted by L_3 . (See Figure 1b.)

From Eqs. (24),(25a) and (26a), we have the following relations,

$$-\frac{1}{2}\sqrt{k/g}\,U^2\cos^2\theta\left(\frac{1}{\omega_1}-\frac{1}{\omega_2}\right)=\frac{1}{k_1-k_2}\left(\frac{k_1}{k-k_1}-\frac{k_2}{k-k_2}\right)$$

$$-\frac{1}{2}\sqrt{k/g}\,U^2\cos^2\theta\left(\frac{1}{\omega_3}-\frac{1}{\omega_4}\right)=\frac{1}{k_3-k_4}\left(\frac{k_3}{k-k_3}-\frac{k_4}{k-k_4}\right) \qquad (37)$$

Then substituting the above relations in Eq.(22) and omitting the time factor $e^{i\sigma t}$ we have,

$$\varphi=\frac{i}{4\pi^2\rho U^2}\int_{-b}^{b}d\eta\int_{-a}^{a}d\xi\,\Pi(\xi,\eta)$$

$$\left\{-\int_0^{\pi/2}d\theta\sec^2\theta\int_{L_1}^{\infty}dk\,F_1(\theta,k)\right.$$

$$\left.+\int_{\theta_c}^{\pi/2}d\theta\sec^2\theta\int_{L_2}^{\infty}dk\,F_2(\theta,k)+\int_0^{\theta_c}d\theta\sec^2\theta\int_{L_3}^{\infty}dk\,F_2(\theta,k)\right\} \qquad (38)$$

where

$$\left.\begin{aligned}F_1(\theta,k)&=\left(Uk\cos\theta-\sigma\right)\left(e^{k(z+ir_1)}+e^{k(z+ir_2)}\right)\frac{1}{k_1-k_2}\left(\frac{k_1}{k-k_1}-\frac{k_2}{k-k_2}\right)\\F_2(\theta,k)&=\left(Uk\cos\theta+\sigma\right)\left(e^{k(z-ir_1)}+e^{k(z-ir_2)}\right)\frac{1}{k_3-k_4}\left(\frac{k_3}{k-k_3}-\frac{k_4}{k-k_4}\right)\end{aligned}\right\} \qquad (39)$$

The above φ is the ultimate form of the velocity potential for the steady state oscillation.

SURFACE ELEVATION

The surface elevation at a point on the calm water surface, z=0, is evaluated by assuming uniform pressure distribution $\Pi = \Pi(\xi,\eta)$ of a rectangular shape, omitting the time factor $e^{i\sigma t}$,

$$\zeta = -\frac{\Pi}{\rho g} + \frac{U}{g}\varphi_x - \frac{i\sigma}{g}\varphi \quad , \quad z=0 \tag{40}$$

where the first term is the static elevation while the remainder represents the elevation due to the dynamic effect.

Let the velocity potential φ in Eq.(38) be written in the following form, omitting the factor $e^{i\sigma t}\frac{\Pi}{4\pi^2\rho U^2}$,

$$\begin{aligned}\varphi = &-\int_0^{\pi/2} d\theta \sec^2\theta \left(\frac{k_1}{k_1-k_2}\varphi_{1,1}(\theta) - \frac{k_2}{k_1-k_2}\varphi_{1,2}(\theta)\right)\\ &+\int_{\theta_c}^{\pi/2} d\theta \sec^2\theta \left(\frac{k_3}{k_3-k_4}\varphi_{2,1}(\theta) - \frac{k_4}{k_3-k_4}\varphi_{2,2}(\theta)\right)\\ &+\int_0^{\theta_c} d\theta \sec^2\theta \left(\frac{k_3}{k_3-k_4}\varphi_{3,1}(\theta) - \frac{k_4}{k_3-k_4}\varphi_{3,2}(\theta)\right)\end{aligned} \tag{41}$$

with

$$\varphi_{1,1}(\theta) = \int_{-b}^{b} d\eta \int_{-a}^{a} d\xi \int_{L_1}^{\infty} dk\,(Uk\cos\theta-\sigma)\,\frac{e^{k(z+ir_1)}+e^{k(z+ir_2)}}{k-k_1}$$

$$\varphi_{1,2}(\theta) = \int_{-b}^{b} d\eta \int_{-a}^{a} d\xi \int_{L_1}^{\infty} dk\,(Uk\cos\theta-\sigma)\,\frac{e^{k(z+ir_1)}+e^{k(z+ir_2)}}{k-k_2}$$

$$\varphi_{2,1}(\theta) = \int_{-b}^{b} d\eta \int_{-a}^{a} d\xi \int_{L_2}^{\infty} dk\,(Uk\cos\theta+\sigma)\,\frac{e^{k(z-ir_1)}+e^{k(z-ir_2)}}{k-k_3}$$

$$\varphi_{2,2}(\theta) = \int_{-b}^{b} d\eta \int_{-a}^{a} d\xi \int_{L_2}^{\infty} dk\,(Uk\cos\theta+\sigma)\,\frac{e^{k(z-ir_1)}+e^{k(z-ir_2)}}{k-k_4} \tag{42}$$

[Cont'd]

$$\varphi_{3,1}(\theta) = \int_{-b}^{b} d\eta \int_{-a}^{a} d\xi \int_{0}^{\infty}_{L_3} dk\,(U k\cos\theta+\sigma)\,\frac{e^{k(z-ir_1)} + e^{k(z-ir_2)}}{k-k_3}$$

$$\varphi_{3,2}(\theta) = \int_{-b}^{b} d\eta \int_{-a}^{a} d\xi \int_{0}^{\infty}_{L_3} dk\,(U k\cos\theta+\sigma)\,\frac{e^{k(z-ir_1)} + e^{k(z-ir_2)}}{k-k_4} \qquad (42)$$

where

$$r_1 = (x-\xi)\cos\theta + (y-\eta)\sin\theta$$

$$r_2 = (x-\xi)\cos\theta - (y-\eta)\sin\theta$$

The first subscript in φ_{ij} denotes the path of integration as identified in Figure 2 and the second denotes the singular point along the given line from left to right. To evaluate the integrals systematically, rearrange the *above in the following form:*

$$I_{1,1}(x,y,z;\theta,k_1) = \int_{-b}^{b} d\eta \int_{-a}^{a} d\xi \int_{0}^{\infty}_{\subset} dk\,(U k\cos\theta-\sigma)\,\frac{e^{k(z+ir_1)}}{k-k_1}$$

$$I'_{1,1}(x,y,z;\theta,k_1) = \int_{-b}^{b} d\eta \int_{-a}^{a} d\xi \int_{0}^{\infty}_{\subset} dk\,(U k\cos\theta-\sigma)\,\frac{e^{k(z+ir_2)}}{k-k_1}$$

$$J_{1,1}(x,y,z;\theta,k_1) = \frac{\partial}{\partial x} I_{1,1}$$

$$J'_{1,1}(x,y,z;\theta,k_1) = \frac{\partial}{\partial x} I'_{1,1} \qquad (43a)$$

$$I_{1,2}(x,y,z;\theta,k_2) = \int_{-b}^{b} d\eta \int_{-a}^{a} d\xi \int_{0}^{\infty}_{\subset} dk\,(U k\cos\theta-\sigma)\,\frac{e^{k(z+ir_1)}}{k-k_2}$$

$$I'_{1,2}(x,y,z;\theta,k_2) = \int_{-b}^{b} d\eta \int_{-a}^{a} d\xi \int_{0}^{\infty}_{\subset} dk\,(U k\cos\theta-\sigma)\,\frac{e^{k(z+ir_2)}}{k-k_2}$$

$$J_{1,2}(x,y,z;\theta,k_2) = \frac{\partial}{\partial x} I_{1,2}$$

$$J'_{1,2}(x,y,z;\theta,k_2) = \frac{\partial}{\partial x} I'_{1,2} \qquad (43b)$$

$$I_{2,1}(x,y,z;\theta,k_3) = \int_{-b}^{b} d\eta \int_{-a}^{a} d\xi \int_{0}^{\infty} dk\,(Uk\cos\theta+\sigma)\,\frac{e^{k(z-ir_1)}}{k-k_3}$$

$$I'_{2,1}(x,y,z;\theta,k_3) = \int_{-b}^{b} d\eta \int_{-a}^{a} d\xi \int_{0}^{\infty} dk\,(Uk\cos\theta+\sigma)\,\frac{e^{k(z-ir_2)}}{k-k_3}$$

$$J_{2,1}(x,y,z;\theta,k_3) = \frac{\partial}{\partial x} I_{2,1}$$

$$J'_{2,1}(x,y,z;\theta,k_3) = \frac{\partial}{\partial x} I'_{2,1} \qquad (43c)$$

$$I_{2,2}(x,y,z;\theta,k_4) = \int_{-b}^{b} d\eta \int_{-a}^{a} d\xi \int_{0}^{\infty} dk\,(Uk\cos\theta+\sigma)\,\frac{e^{k(z-ir_1)}}{k-k_4}$$

$$I'_{2,2}(x,y,z;\theta,k_4) = \int_{-b}^{b} d\eta \int_{-a}^{a} d\xi \int_{0}^{\infty} dk\,(Uk\cos\theta+\sigma)\,\frac{e^{k(z-ir_2)}}{k-k_4}$$

$$J_{2,2}(x,y,z;\theta,k_4) = \frac{\partial}{\partial x} I_{2,2}$$

$$J'_{2,2}(x,y,z;\theta,k_4) = \frac{\partial}{\partial x} I'_{2,2} \qquad (43d)$$

$$I_{3,1}(x,y,z;\theta,k_3) = \int_{-b}^{b} d\eta \int_{-a}^{a} d\xi \int_{0}^{\infty} dk\,(Uk\cos\theta+\sigma)\,\frac{e^{k(z-ir_1)}}{k-k_3}$$

$$I'_{3,1}(x,y,z;\theta,k_3) = \int_{-b}^{b} d\eta \int_{-a}^{a} d\xi \int_{0}^{\infty} dk\,(Uk\cos\theta+\sigma)\,\frac{e^{k(z-ir_2)}}{k-k_3}$$

$$J_{3,1}(x,y,z;\theta,k_3) = \frac{\partial}{\partial x} I_{3,1}$$

$$J'_{3,1}(x,y,z;\theta,k_3) = \frac{\partial}{\partial x} I'_{3,1} \qquad (43e)$$

$$I_{3,2}(x,y,z;\theta,k_4) = \int_{-b}^{b} d\eta \int_{-a}^{a} d\xi \int_{0}^{\infty} dk\,(Uk\cos\theta+\sigma)\,\frac{e^{k(z-ir_1)}}{k-k_4}$$

$$I'_{3,2}(x,y,z;\theta,k_4) = \int_{-b}^{b} d\eta \int_{-a}^{a} d\xi \int_{0}^{\infty} dk\,(Uk\cos\theta+\sigma)\,\frac{e^{k(z-ir_2)}}{k-k_4}$$

$$J_{3,2}(x,y,z;\theta,k_4) = \frac{\partial}{\partial x} I_{3,2}$$

$$J'_{3,2}(x,y,z;\theta,k_4) = \frac{\partial}{\partial x} I'_{3,2} \qquad (43f)$$

The integrations in Equations (43a) through (43f) are given in Appendices A-C. Substituting Eqs.(43a)-(43f) in Eq.(42), and omitting the same factor as that in (41) we have φ in the following form,

$$\varphi = -\int_{0}^{\pi/2} d\theta\sec^2\theta \left[\frac{k_1}{k_1-k_2}(I_{1,1}+I'_{1,1}) - \frac{k_2}{k_1-k_2}(I_{1,2}+I'_{1,2})\right]$$

$$+\int_{\theta_c}^{\pi/2} d\theta\sec^2\theta \left[\frac{k_3}{k_3-k_4}(I_{2,1}+I'_{2,1}) - \frac{k_4}{k_3-k_4}(I_{2,2}+I'_{2,2})\right]$$

$$+\int_{0}^{\theta_c} d\theta\sec^2\theta \left[\frac{k_3}{k_3-k_4}(I_{3,1}+I'_{3,1}) - \frac{k_4}{k_3-k_4}(I_{3,2}+I'_{3,2})\right] \qquad (44)$$

In a similar manner we have φ_x in the following form:

$$\varphi_x = -\int_{0}^{\pi/2} d\theta\sec^2\theta \left[\frac{k_1}{k_1-k_2}(J_{1,1}+J'_{1,1}) - \frac{k_2}{k_1-k_2}(J_{1,2}+J'_{1,2})\right]$$

$$+\int_{\theta_c}^{\pi/2} d\theta\sec^2\theta \left[\frac{k_3}{k_3-k_4}(J_{2,1}+J'_{2,1}) - \frac{k_4}{k_3-k_4}(J_{2,2}+J'_{2,2})\right]$$

$$+\int_{0}^{\theta_c} d\theta\sec^2\theta \left[\frac{k_3}{k_3-k_4}(J_{3,1}+J'_{3,1}) - \frac{k_4}{k_3-k_4}(J_{3,2}+J'_{3,2})\right] \qquad (45)$$

Substituting $(I_{i,j}+I'_{i,j})$ and $(J_{i,j}+J'_{i,j})$ (which are given in Appendices A, B, C) in the above equations (44) and (45), making use of Equation (40) and

omitting the static elevation and time factor, the elevation ζ is determined in non-dimensional form,

$$\frac{pg\zeta}{\Pi} = -\frac{1}{4\pi^2 U^2}\left\{\int_0^{\pi/2} d\theta\, \frac{\sec^3\theta}{\sin\theta}\left\{\frac{\sigma^2}{k_1 k_2}(I_1 - I_1') + \sum_{\ell=1}^{2}(-1)^{1+\ell}\frac{(Uk_\ell\cos\theta-\sigma)^2}{k_\ell(k_1-k_2)}[I_2(k_\ell)-I_2'(k_\ell)]\right\}\right.$$

$$+\int_{\theta_c}^{\pi/2} d\theta\, \frac{\sec^3\theta}{\sin\theta}\left\{\frac{\sigma^2}{k_3 k_4}(I_1^*-I_1'^*) + \sum_{\ell=3}^{4}(-1)^{1+\ell}\frac{(Uk_\ell\cos\theta+\sigma)^2}{k_\ell(k_3-k_4)}[I_4(k_\ell) - I_4'(k_\ell)]\right\}$$

$$\left.+\int_0^{\theta_c} d\theta\, \frac{\sec^3\theta}{\sin\theta}\left\{\frac{\sigma^2}{k_3 k_4}(I_1^*-I_1'^*) + \sum_{\ell=3}^{4}(-1)^{1+\ell}\frac{(Uk_\ell\cos\theta+\sigma)^2}{k_\ell(k_3-k_4)}[I_5(k_\ell) - I_5'(k_\ell)]\right\}\right\} \tag{46}$$

where $*$ indicates the complex conjugate, and

$$I_1 - I_1' = \sum_{m=1}^{8}(-1)^m\left(\log|s_m| + \begin{matrix}0\\ i\pi\end{matrix}\right), \quad s_m \gtrless 0 \tag{47}$$

$$I_2(k_\ell) - I_2'(k_\ell) = \sum_{m=1}^{8}(-1)^{m+1} e^{ik_\ell s_m}\left\{E_1[ik_\ell s_m] - \begin{matrix}0\\ i2\pi\end{matrix}\right\}, \quad s_m \gtrless 0$$

$$\ell = 1,2 \tag{48}$$

$$I_4(k_3) - I_4'(k_3) = \sum_{m=1}^{8}(-1)^{m+1} e^{-ik_3 s_m}\left\{E_1[-ik_3 s_m] - \begin{matrix}i2\pi\\ 0\end{matrix}\right\}, \quad s_m \gtrless 0 \tag{49}$$

$$I_4(k_4)-I_4'(k_4) = \sum_{m=1}^{8} (-1)^{m+1} e^{-ik_4 s_m}\left\{E_1[-ik_4 s_m]+\begin{matrix}0\\ i2\pi\end{matrix}\right\},\ s_m \gtrless 0 \qquad (50)$$

$$I_5(k_3)-I_5'(k_3)=\begin{cases} \sum_{m=1}^{8} (-1)^{m+1} e^{-ik_3 s_m}\left\{E_1[-ik_3 s_m]-\begin{matrix}i2\pi\\ 0\end{matrix}\right\}, s_m \gtrless 0,\ \mathrm{Re}k_3 > 0 \\ \sum_{m=1}^{8} (-1)^{m+1} e^{-ik_3 s_m}\left\{E_1[-ik_3 s_m]-\begin{matrix}0\\ 0\end{matrix}\right\},\ s_m \gtrless 0,\ \mathrm{Re}k_3 < 0 \end{cases} \qquad (51)$$

$$I_5(k_4)-I_5'(k_4)=\begin{cases} \sum_{m=1}^{8} (-1)^{m+1} e^{-ik_4 s_m}\left\{E_1[-ik_4 s_m]+\begin{matrix}0\\ i2\pi\end{matrix}\right\}, s_m \gtrless 0,\ \mathrm{Re}k_4 > 0 \\ \sum_{m=1}^{8} (-1)^{m+1} e^{-ik_4 s_m}\left\{E_1[-ik_4 s_m]+\begin{matrix}0\\ 0\end{matrix}\right\},\ s_m \gtrless 0,\ \mathrm{Re}k_4 < 0 \end{cases} \qquad (52)$$

s_m as well as the derivations of the above formulas are given in *Appendices A, B, C and D.*

The functions (I_1-I_1'), (I_2-I_2), etc. in (47) to (52) contain sums of logarithmic functions. It might seem that each function has singularities at points (a,b), (-a,b), (a,-b) and (-a,-b) where a pair of s_m vanish (see Eq.A-5). But one can verify that the function has no singularities when all the terms are taken together or as one might also put it, any singularities in the individual terms are cancelled due to the alternating series. Thus, the solution is regular everywhere.

As a special case, consider the wave elevation due to a uniformly moving pressure distribution without oscillation. Then according to Eqs. (25b), (26b) and (26c) we have,

$$\tau = 0\ ,\ k_1=k_3=0$$
$$k_2=k_4=k_o\sec^2\theta \qquad (53)$$

Hence, Eq.(46) is reduced to

$$\frac{\rho g \zeta}{\Pi} = -\frac{1}{4\pi^2}\int_0^{\pi/2}\frac{\sec\theta}{\sin\theta}\left\{I_2(k_2)-I_2'(k_2)+I_4(k_4)-I_4'(k_4)\right\} \tag{53a}$$

and further since $\{I_2(k_2)-I_2'(k_2)\}$ and $\{I_4(k_4)-I_4'(k_4)\}$ are complex conjugates of each other, the above is expressed in the form

$$\frac{\rho g \zeta}{\Pi} = -\frac{1}{2\pi^2}\int_0^{\pi/2} d\theta\,\frac{\sec\theta}{\sin\theta}\,\mathrm{Re}\left\{I_2(k_o\sec^2\theta) - I_2'(k_o\sec^2\theta)\right\} . \tag{53b}$$

The resultant elevation is the sum of the dynamic and static terms as shown in Eq.(40). The static pressure Π is discontinuous at $|x|=a$. The value of the discontinuity is defined as

$$\Pi(\pm a,y) = \tfrac{1}{2}[\Pi(\pm a + 0,y) + \Pi(\pm a - 0,y)] . \tag{54}$$

Hence, the total non-dimensional elevation is obtained by adding the following to the dynamic term (46) or (53b),

$$\frac{\rho g \zeta}{\Pi}_{\text{static}} = \begin{cases} -\frac{1}{2} & |x|=a \\ -1 & |x|<a \end{cases} \tag{54a}$$

ESCAPE AREA AND VOLUME

The escape area A at the stern section and the volume V under the plenum area are obtained by integrating the wave elevation ζ of Eq.(46).

$$A = \int_{-b}^{b} \zeta dy$$

$$V = \int_{-a}^{a}\int_{-b}^{b} \zeta dy dx \tag{55}$$

Referring to Eq.(46), it is seen that the above integrations have to use the following component integrals:

For the Escape Area –

$$\int_{-b}^{b}(I_1 - I_1')dy = \frac{1}{i\sin\theta}\left(I_3 + I_3'\right)_{-b}^{b}$$

$$\int_{-b}^{b}\left(I_2(k_\ell) - I_2'(k_\ell)\right)dy = \frac{1}{ik_\ell\sin\theta}\left\{-(I_1 + I_1') + I_2(k_\ell) + I_2'(k_\ell)\right\}_{-b}^{b}$$

$$\int_{-b}^{b}\left(I_4(k_\ell) - I_4'(k_\ell)\right)dy = \frac{1}{-ik_\ell\sin\theta}\left\{-(I_1^* + I_1'^*) + I_4(k_\ell) + I_4'(k_\ell)\right\}_{-b}^{b} \tag{56}$$

For the Volume –

$$\int_{-a}^{a}\int_{-b}^{b}(I_1 - I_1')dydx = -\frac{1}{\cos\theta\sin\theta}\left(I_6 + I_6'\right)_{-b,-a}^{b,a}$$

$$\int_{-a}^{a}\int_{-b}^{b}\left(I_2(k_\ell) - I_2'(k_\ell)\right)dydx = \frac{1}{k_\ell^2\sin\theta\cos\theta}\left\{k_\ell(I_3 + I_3') + (I_1 + I_1') - \left(I_2(k_\ell) + I_2'(k_\ell)\right)\right\}_{-b,-a}^{b,a}$$

$$\int_{-a}^{a}\int_{-b}^{b}\left(I_4(k_\ell) - I_4'(k_\ell)\right)dydx = \frac{1}{k_\ell^2\sin\theta\cos\theta}\left\{k_\ell(I_3^* + I_3'^*) + (I_1^* + I_1'^*) - \left(I_4(k_\ell) + I_4'(k_\ell)\right)\right\}_{-b,-a}^{b,a} \tag{57}$$

where

$$I_6 = \int_0^\infty dk\,\frac{e^{kz}}{k^3}\left(e^{iks_1} - e^{iks_2} + e^{iks_3} - e^{iks_4}\right)$$

$$I_6' = \int_0^\infty dk\,\frac{e^{kz}}{k^3}\left(-e^{iks_5} + e^{iks_6} - e^{iks_7} + e^{iks_8}\right) \tag{58}$$

These are evaluated in Eq.(A-33) in Appendix A.

Using Eqs.(46), (56) and (57), and omitting the time factor $e^{i\sigma t}$, the non-dimensional escape area and volume are written in the following forms:

$$\alpha = \frac{\rho g A}{\Pi L} = \frac{i}{4\pi^2 U^2 L} \Bigg\{ \int_0^{\pi/2} d\theta \, \frac{\sec^3\theta}{\sin^2\theta} \left\{ \frac{\sigma^2}{k_1 k_2} \left(I_3 + I_3'\right) + \sum_{\ell=1}^{2} (-1)^{1+\ell} \frac{(Uk_\ell \cos\theta - \sigma)^2}{k_\ell^2 (k_1 - k_2)} \left[-\left(I_1 + I_1'\right) + I_2(k_\ell) + I_2'(k_\ell) \right] \right\}$$

$$- \int_{\theta_c}^{\pi/2} d\theta \, \frac{\sec^3\theta}{\sin^2\theta} \left\{ \frac{\sigma^2}{k_3 k_4} \left(I_3^* + I_3'^*\right) + \sum_{\ell=3}^{4} (-1)^{1+\ell} \frac{(Uk_\ell \cos\theta + \sigma)^2}{k_\ell^2 (k_3 - k_4)} \left[-\left(I_1^* + I_1'^*\right) + I_4(k_\ell) + I_4'(k_\ell) \right] \right\}$$

$$- \int_{0}^{\theta_c} d\theta \, \frac{\sec^3\theta}{\sin^2\theta} \left\{ \frac{\sigma^2}{k_3 k_4} \left(I_3^* + I_3'^*\right) + \sum_{\ell=3}^{4} (-1)^{1+\ell} \frac{(Uk_\ell \cos\theta + \sigma)^2}{k_\ell^2 (k_3 - k_4)} \left[-\left(I_1^* + I_1'^*\right) + I_5(k_\ell) + I_5'(k_\ell) \right] \right\} \Bigg\} \tag{59}$$

$$v = \frac{\rho g V}{\Pi L^2} = \frac{1}{4\pi^2 U^2 L^2} \Bigg\{ \int_0^{\pi/2} d\theta \, \frac{\sec^4\theta}{\sin^2\theta} \left\{ \frac{\sigma^2}{k_1 k_2} \left(I_6 + I_6'\right) - \sum_{\ell=1}^{2} (-1)^{1+\ell} \frac{(Uk_\ell \cos\theta - \sigma)^2}{k_\ell^3 (k_1 - k_2)} \left[k_\ell \left(I_3 + I_3'\right) + \left(I_1 + I_1'\right) - \right. \right.$$

$$\left. \left. - \left(I_2(k_\ell) + I_2'(k_\ell) \right) \right] \right\}$$

$$+ \int_{\theta_c}^{\pi/2} d\theta \, \frac{\sec^4\theta}{\sin^2\theta} \left\{ \frac{\sigma^2}{k_3 k_4} \left(I_6^* + I_6'^*\right) - \sum_{\ell=3}^{4} (-1)^{1+\ell} \frac{(Uk_\ell \cos\theta + \sigma)^2}{k_\ell^3 (k_3 - k_4)} \left[k_\ell \left(I_3^* + I_3'^*\right) + \left(I_1^* + I_1'^*\right) - \right. \right.$$

$$\left. \left. - \left(I_4(k_\ell) + I_4'(k_\ell) \right) \right] \right\}$$

$$+ \int_{0}^{\theta_c} d\theta \, \frac{\sec^4\theta}{\sin^2\theta} \left\{ \frac{\sigma^2}{k_3 k_4} \left(I_6^* + I_6'^*\right) - \sum_{\ell=3}^{4} (-1)^{1+\ell} \frac{(Uk_\ell \cos\theta + \sigma)^2}{k_\ell^3 (k_3 - k_4)} \left[k_\ell \left(I_3^* + I_3'^*\right) + \left(I_1^* + I_1'^*\right) - \right. \right.$$

$$\left. \left. - \left(I_5(k_\ell) + I_5'(k_\ell) \right) \right] \right\} \Bigg\} \tag{60}$$

In the above Eqs.(59) and (60), the symbols for the integral limits have been omitted.

Since pairs of s_m (Eqs.A-5) vanish at the corner points such as $s_1=s_6=0$ at x=a, y=b, etc., the logarithmic functions and the exponential integrals in I_1, I_1', $I_2(k_\ell)$, etc., become evidently singular at the points. Examination shows, however, that the sums $\left\{-(I_1+I_1')+I_2(k_\ell)+I_2'(k_\ell)\right\}$, $\left\{-(I_1^*+I_1'^*)+I_4(k_\ell)+I_4'(k_\ell)\right\}$, etc., are all bounded at these corner points and each I_3, I_3', I_6, I_6' is bounded as well at these corner points. The regularity of the above expressions in Eqs.(59) and (60) are the expected results which derive from the integration of the regular function ζ in Eq.(46).

For the special case $\sigma \to 0$,* the escape area at the stern and the volume under the plenum area are obtained from Eqs.(59) and (60) by using the conditions (53):

$$\alpha = \frac{\rho g A}{\Pi L}_{\sigma\to 0} = -\frac{1}{2\pi^2 L k_o}\int_0^{\pi/2} d\theta\,\frac{\cos\theta}{\sin^2\theta}\,\mathrm{Im}\left\{-(I_1+I_1')+\left[I_2(k_o\sec^2\theta)+I_2'(k_o\sec^2\theta)\right]\right\}_{-b}^{b} \tag{61a}$$

and

$$v = \frac{\rho g V}{\Pi L^2}_{\sigma\to 0} = -\frac{1}{2\pi^2 L^2 k_o^2}\int_0^{\pi/2} d\theta\,\frac{\cos\theta}{\sin^2\theta}\,\mathrm{Re}\left\{k_o\sec^2\theta(I_3+I_3')+(I_1+I_1')-\left[I_2(k_o\sec^2\theta)+I_2'(k_o\sec^2\theta)\right]\right\}_{-b,-a}^{b,a} \tag{61b}$$

The resultant escape area and the volume are obtained by adding the static terms to the dynamic terms, (59) to (61b).

Using Eq.(54), the static escape area at the stern and the static volume are given in non-dimensional form by

$$\alpha_{st} = \frac{\rho g A}{\Pi L}_{static} = -\frac{b}{L}$$

and

$$v_{st} = \frac{\rho g V}{\Pi L^2}_{static} = -\frac{4ab}{L^2} \tag{61c}$$

where a and b are half-length and beam of the rectangular pressure distribution and L is a nominal length of the SES or ACV.

*See foonote, p.24.

The asymptotic expression of α and v for the frequency $\sigma \to \infty$* are obtained from Eq.(59) and Eq.(60), respectively. Since $\theta \to \pi/2$ for $\sigma \to \infty$ in Eq.(29), the second integrals in Eqs.(59) and (60) vanish for $\sigma \to \infty$. Also, the terms contained in the summation symbols Σ should vanish for $\sigma \to \infty$ due to the expressions for the wave numbers as shown in Eq.(25). The terms with $*$ in Eq.(59) and Eq.(60) are the complex conjugates of those which are independent of the frequency σ and the speed U.

In view of the above remarks, we have the asymptotic expression of the resultant α and v for $\sigma \to \infty$

$$\alpha_r = -\frac{1}{2\pi^2 L} \int_0^{\pi/2} d\theta \, \frac{\sec\theta}{\sin^2\theta} \, \mathrm{Im}(I_3 + I_3') - \frac{b}{L}$$

$$\alpha_i = 0 \tag{62a}$$

$$v_r = \frac{1}{2\pi^2 L^2} \int_0^{\pi/2} d\theta \, \frac{\sec^2\theta}{\sin^2\theta} \, \mathrm{Re}(I_6 + I_6') - \frac{4ab}{L^2}$$

$$v_i = 0 \quad . \tag{62b}$$

where the suffixes r and i indicate the real and imaginary components, respectively. In the above integrations, the symbols for the integral limits have been omitted.

From the expressions of α and v, it is seen that they are in phase with the pressure Π at the frequencies $\sigma \to 0$ and $\sigma \to \infty$.

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*It should be noted that in the limiting cases when $\sigma \to 0$ or $\sigma \to \infty$, then σ and τ can be interchanged.

THE RELATIONSHIP BETWEEN BUBBLE PRESSURE AND HEAVE

The air pressure in the plenum of the SES (surface effect ship) is perturbed by the small vertical oscillatory motion heave $ze^{i\sigma t}$. The first order perturbed pressure $p_b e^{i\sigma t}$ which is dynamic and known as bubble pressure will be obtained by relating it to the first order heave $ze^{i\sigma t}$ in the following analysis.

Since the volume and pressure changes are assumed to occur rapidly, the state of air pressure and volume in the plenum is assumed to satisfy the adiabatic law,

$$pv^{\gamma} = \text{const.} \tag{63}$$

where γ is the ratio of specific heats which is 1.4 for air. The differentiation of the above with respect to time t gives

$$\frac{\partial p}{\partial t} + \gamma \frac{p}{v} \frac{\partial v}{\partial t} = 0 \quad . \tag{64}$$

Expanding p and v in the foregoing up to the first order, we have

$$\begin{aligned} p &= p^{(0)} + \epsilon\, p^{(1)} \\ v &= v^{(0)} + \epsilon\, v^{(1)} \end{aligned} \tag{65}$$

where $p^{(0)}$ and $v^{(0)}$ are the pressure and volume in the equilibrium stage $p^{(1)}$ and $v^{(1)}$ are the first order perturbed pressure and volume, respectively, and ϵ is a parameter.

Substituting Eq.(65) into (64) and collecting the first terms only, we then have

$$\frac{\partial p^{(1)}}{\partial t} = -\gamma \frac{p^{(0)}}{v^{(0)}} \frac{\partial v^{(1)}}{\partial t} \tag{66}$$

where

$$\begin{aligned} p^{(1)} &= p_b e^{i\sigma t} \\ v^{(1)} &= v_b e^{i\sigma t} = \text{the perturbed first order volume} \\ p^{(0)} &= p_o + p_a = \text{total pressure} \\ v^{(0)} &= v_o \end{aligned} \tag{67}$$

and

p_o = air pressure in equilibrium or ambient pressure

p_a = atmospheric pressure

v_o = the volume in equilibrium

Thus, Eq.(66) is written with omission of the time factor in the following form

$$i\sigma p_b = -\gamma \frac{p_o+p_a}{v_o} i\sigma v_b \tag{68}$$

where $i\sigma v_b$ is the variation of the volume flux which is positive in the direction outward normal to the plenum chamber as indicated in Figure 1c and is the sum of the flux normal across air-water interface, plenum ceiling, fans and seal apertures. The above relation and the normal volume flux have been used in References 3, 11, 12 which confirm that the use of (68) is identical to that of the method of momentum change in Reference 1. The normal volume flux[11] has been expressed by the surface integral of the normal velocity which is derived by the Green's function technique.

1. Over the Air-Water Interface

Equations (60), (61b) and (61c) give the volume due to the water wave elevation per unit pressure, distributed uniformly over the air-water interface. Hence, the contribution to the volume flux due to the bubble pressure is given by

$$i\sigma v_b = -i\sigma v \frac{L^2}{\rho_w g} p_b \tag{69}$$

where v is the non-dimensional volume due to wave elevation per unit uniform pressure over the rectangular patch and ρ_w is the water density. The minus sign indicates the sense of the flow direction.

2. Across the Plenum Ceiling

The variation of the volume flux normal outward to the plenum chamber across the ceiling, due to its heaving displacement z, is given by

$$i\sigma v_b = +i\sigma A_c z \tag{70}$$

where A_c is the cushion area or the area of ceiling. The + sign indicates the sense of the flow direction.

3. Across the Fan Aperture

The variation of the volume flux which flows into the plenum across the fan aperture is

$$i\sigma v_b = - n \frac{\partial Q_f}{\partial p} p_b \tag{71}$$

where $\frac{\partial Q_f}{\partial p}$ is the discharge by fan per unit pressure per pair of fans. The minus sign in Eq.(71) indicates the sense of the flow direction and n is the number of pair of fans.

4. Across the Stern-Seal Aperture

The variation of the volume flux flowing outward from the plenum across the seal aperture in the plenum wall is estimated by applying the perturbation procedure to the nozzle discharge equation.

$$Q = k A \sqrt{\frac{2(p_1 - p_2)}{\rho_a}} \tag{72}$$

where k is the flow coefficient which varies between 0.6 and 0.8.* Hydrodynamic studies have shown that for an ideal fluid passing through a narrow slit or a circular opening $k \sim 0.611$ which has been experimentally verified.

A is the area of the nozzle throat or seal opening

p_1 is the pressure in the upstream or in the plenum

p_2 is the pressure at the nozzle or seal and identical to the atmospheric pressure p_a

ρ_a is the air density

Q is the discharge .

To evaluate the increase of volume flux from its equilibrium condition, we expand the terms Q, A and $p_1 - p_2$ up to the first order perturbed terms assuming that the terms are functions of ϵ . Namely, we have the following:

$$\begin{aligned} Q &= Q_o + \epsilon Q^{(1)} \\ A_s &= A_o + \epsilon A^{(1)} \\ p_1 - p_2 &= p_o + \epsilon p^{(1)} \end{aligned} \tag{73}$$

where Q_o, A_o and p_o are the zeroth order term, whereas $Q^{(1)}$, $A^{(1)}$ and $p^{(1)}$

*As the "nozzle" is formed by the water surface and the seal, the coefficient k will depend on Froude number and on Reynolds number for small opening areas, A , as measured at zero speed.[3]

are the first order terms. The first order terms are redesignated by

$$A^{(1)} = A_s e^{i\sigma t} \quad ; \quad Q^{(1)} = Q_s e^{i\sigma t} \quad ; \quad p^{(1)} = p_b e^{i\sigma t} \quad . \tag{74}$$

Substituting (73) into (72) and making use of the following expansion

$$\sqrt{\frac{2(p_o + \varepsilon p^{(1)})}{\rho_a}} = \sqrt{\frac{2p_o}{\rho_a}}\left(1 + \frac{\varepsilon\, p^{(1)}}{2p_o}\right) \quad , \tag{75}$$

we have the expression for Q, from which the first order equation is obtained in the following form

$$Q_s = k\sqrt{\frac{2p_o}{\rho_a}}\left(\frac{A_o}{2p_o}\, p_b + A_s\right) \tag{76}$$

The variation A_s of the seal opening area A in (76) is due to heave z and bubble pressure p_b. Namely,

$$A_s = \frac{\partial A_s}{\partial z}\, z + \left(\frac{\partial A_s}{\partial p} - \alpha\, \frac{L}{\rho_w g}\right) p_b \tag{77}$$

where

$\frac{\partial A_s}{\partial z}$ = the rate of change of seal opening due to seal elasticity per unit heave

$\frac{\partial A_s}{\partial p}$ = the rate of change of seal opening due to seal elasticity per unit pressure

$-\alpha\, \frac{L}{\rho_w g}$ = the rate of change of opening due to the wave elevation per unit pressure, in way of the stern seal. α is the non-dimensional escape area which has been given in Eqs.(59),(61a) (61c), and (62a).

Substituting (77) into (76), we have the following volume flux which flows outward in the normal direction to the plenum wall across the seal aperture:

$$i\sigma v_b = k\sqrt{\frac{2p_o}{\rho_a}}\left[\left(\frac{A_o}{2p_o} + \frac{\partial A_s}{\partial p} - \alpha\, \frac{L}{\rho_w g}\right) p_b + \frac{\partial A_s}{\partial z}\, z\right] \quad . \tag{78}$$

Thus the itemization of the variation of volume fluxes has been completed.

The sum of the first order volume fluxes, Eqs.(69), (70), (71) and (78), is now substituted into Eq.(68). Solving for the unknown bubble pressure p_b, we have the non-dimensional expression in the following form

$$\frac{\frac{p_b}{p_o}}{\frac{z}{h}} = \frac{-\gamma\left(1+\frac{p_a}{p_o}\right)\left(1-\frac{i}{\sigma}\frac{k}{A_c}\frac{\partial A_s}{\partial z}\sqrt{\frac{2p_o}{\rho_a}}\right)}{1-\gamma\left(1+\frac{p_a}{p_o}\right)\frac{p_o}{v_o}\left[\frac{vL^a}{\rho_w g}+\frac{i}{\sigma}n\left|\frac{\partial Q_f}{\partial p}\right|+\frac{i}{\sigma}k\sqrt{\frac{2p_o}{\rho_a}}\left(\frac{A_o}{2p_o}+\frac{\partial A_s}{\partial p}-\alpha\frac{L}{\rho_w g}\right)\right]} \tag{79}$$

where the relation $\frac{\partial Q_f}{\partial p} = -\left|\frac{\partial Q_f}{\partial p}\right|$ and $v_o = A_c h$ have been used, while h indicates the bubble (or cushion) height.

The dynamic property of the stern seal $\frac{\partial A_s}{\partial z}$ and $\frac{\partial A_s}{\partial p}$ is unknown. At the present stage of investigation, we assume the seal is rigid. This assumption will evidently simplify the aero-hydrodynamic analysis. In addition to this, the analysis will be practically applied to the model test which uses a semi-rigid seal, e.g., in the case of the XR-5 SES model. Reference 3 reports the investigation of the seal stiffness on the aero-hydrodynamic bubble pressure of a SES model.

The above assumption reduces the expression of the bubble pressure to the following form,

$$\frac{\frac{p_b}{p_o}}{\frac{z}{h}} = \frac{-\gamma\left(1+\frac{p_a}{p_o}\right)}{1-\gamma\left(1+\frac{p_a}{p_o}\right)\frac{p_o}{v_o}\left[\frac{vL^a}{\rho_w g}+\frac{i}{\sigma}\left\{n\left|\frac{\partial Q_f}{\partial p}\right|+k\sqrt{\frac{2p_o}{\rho_a}}\left(\frac{A_o}{2p_o}-\alpha\frac{L}{\rho_w g}\right)\right\}\right]} \tag{80}$$

If the water wave induced volume v and area α are set equal to zero, the above relation will represent the bubble pressure due to heave of an air cushion vehicle above the ground.

Although Eq.(80) is derived by means of perturbation analysis keeping only the first-order term, two limiting values of the bubble pressure under extreme conditions can be obtained:

a) When translation velocity U and frequency σ approach to zero with openings closed aft and no fans in operation (closed box), then

$$\lim_{\substack{U\to 0\\ \sigma\to 0}} \frac{\frac{p_b}{p_o}}{\frac{z}{h}} \cong \rho_w g \,. \tag{81b}$$

b) When the heaving vehicle is in translatory motion, $U \neq 0$, with all the openings in effect, then when frequency $\sigma \to 0$*

$$\lim_{\sigma \to 0} \frac{\frac{p_b}{p_o}}{\frac{z}{h}} \to 0 \qquad \text{with } 90^\circ \text{ phase lag.} \tag{81b}$$

where the following information has been used, namely, for $\sigma \to 0$, according to (59) to (61):

1) α_r and v_r are negative real constant values

2) α_i and v_i are linearly proportional to σ

The above relation is also equally valid for the uncoupled aerodynamic analysis, namely, for $v=\alpha=0$.

It is apparent that the asymptotic expression of the bubble pressure Eq.(80) for $\sigma \to \infty$ is dependent on the behavior of the wave-induced volume v for $\sigma \to \infty$ on the air-water interface. Since the asymptotic expression of v, Eq. (62b), is v_r = a negative constant and $v_i = 0$, we have with water wave effects

$$\lim_{\sigma \to \infty} \frac{p_b}{p_o} \Big/ \frac{z}{h} = \frac{-\gamma\left(1 + \frac{p_a}{p_o}\right)}{1 + \gamma\left(1 + \frac{p_a}{p_o}\right) \frac{p_o}{v_o} \frac{L^2}{\rho_w g} |v_r|_{\sigma \to \infty}} , \tag{82a}$$

whereas with no water wave effects,

$$\lim_{\sigma \to \infty} \frac{p_b}{p_o} \Big/ \frac{z}{h} = -\gamma\left(1 + \frac{p_a}{p_o}\right) . \tag{82b}$$

Both limiting values of pressure signatures show a 180-degree phase lag.

*It should be noted that in the limiting cases when $\sigma \to 0$ or $\sigma \to \infty$, then σ and τ can be interchanged.

ADDED MASS AND DAMPING COEFFICIENTS

The real and imaginary components of the bubble pressure Eq.(80) are given by

$$\left\{\frac{p_b/p_o}{z/h}\right\}_r = \frac{-\gamma(1+p_a/p_o)\left[1-\gamma(1+p_a/p_o)\,\frac{p_o Q_r}{v_o}\right]}{\left[1-\gamma(1+p_a/p_o)\,\frac{p_o Q_r}{v_o}\right]^2+\left[\gamma(1+p_a/p_o)\,\frac{p_o Q_i}{v_o}\right]^2}$$

$$\left\{\frac{p_b/p_o}{z/h}\right\}_i = \frac{-\left[\gamma(1+p_a/p_o)\right]^2\,\frac{p_o Q_i}{v_o}}{\left[1-\gamma(1+p_a/p_o)\,\frac{p_o Q_r}{v_o}\right]^2+\left[\gamma(1+p_a/p_o)\,\frac{p_o Q_i}{v_o}\right]^2} \qquad (83)$$

where

$$Q_r = v_r\,\frac{L^2}{\rho_w g}+\frac{k}{\sigma}\,\alpha_i\,\frac{L}{\rho_w g}\sqrt{\frac{2p_o}{\rho_a}}$$

$$Q_i = v_i\,\frac{L^2}{\rho_w g}+\frac{1}{\sigma}\left\{n\left|\frac{\partial Q_f}{\partial p}\right|+k\sqrt{\frac{2p_o}{\rho_a}}\left(\frac{A_o}{2p_o}-\alpha_r\,\frac{L}{\rho_w g}\right)\right\} \qquad (84)$$

The integration of the bubble pressure (83) over the cushion area yields the aero-hydrodynamic force due to heaving motion of a surface effect ship in translation on the calm water surface. The real and imaginary components of the force are designated as aero-hydrodynamic inertial and damping forces from which the added mass M'' and damping factor N are defined in the following form,

$$M'' = \frac{p_{b_r} A_c}{\sigma^2 z}$$

$$N = -\frac{p_{b_i} A_c}{\sigma z} \qquad (85)$$

where the minus sign comes from the $\mathrm{Re}(p_b A_c e^{i\sigma t})$.

The non-dimensional added mass and damping coefficients of Eqs.(85) can be defined in the following form,[7]

$$C_a = \frac{M''}{\frac{1}{2}\rho_w L^3} \quad ; \quad C_d = \frac{N}{\frac{1}{2}\rho_w U L^2} \tag{86}$$

The added mass M'' and damping factor N for $\sigma \to 0$ are derived by knowing the behavior of the α_r, α_i, v_r and v_i at $\sigma \to 0$;

$$\frac{\alpha_i}{\sigma} = \frac{d\alpha_i}{d\sigma} > 0$$

$$\frac{v_i}{\sigma} = \frac{dv_i}{d\sigma} > 0$$

$$v_r = -|v_r| \quad ; \quad \alpha_r = -|\alpha_r| \quad . \tag{87}$$

The results are given by

$$\underset{\sigma \to 0}{M''} = \frac{E A_c^2}{D^2}$$

$$\underset{\sigma \to 0}{N} = \frac{A_c^2}{D} \tag{88}$$

where

$$D = n\left|\frac{\partial Q_f}{\partial p}\right| + k\sqrt{\frac{2p_o}{\rho_a}}\left(\frac{A_o}{2p_o} + \underset{\sigma \to 0}{|\alpha_r|}\ \frac{L}{\rho_w g}\right)$$

$$E = -\frac{v_o}{\gamma(p_a+p_o)} - \underset{\sigma \to 0}{|v_r|}\ \frac{L^2}{\rho_w g} + \underset{\sigma \to 0}{\frac{d\alpha_i}{d\sigma}}\ k\sqrt{\frac{2p_o}{\rho_a}}\ \frac{L}{\rho_w g} \tag{89}$$

where D is always positive, whereas E may take positive or negative value. Thus the damping factor N is always positive and the added mass may be positive or negative (i.e., the motion of the vehicle and the added water mass are in opposite directions).[4]

Definite signs are obtained when the water wave effects are omitted. Namely,

$$D = n\left|\frac{\partial Q_f}{\partial p}\right| + k\sqrt{\frac{2p_o}{\rho_a}}$$

$$E = -\frac{v_o}{\gamma(p_a+p_o)} \tag{90}$$

From Eqs.(90) we have always the negative added mass and positive damping factor for the SES without water wave effects or ground effect machine at $\sigma \rightarrow 0$.

It is apparent from Eqs.(88), (89) and (90) that the added mass M'' and damping factor N at the vanishingly low frequency are proportional to the cushion area squared and dependent on the contributions by the fan, seal leakage, wave-induced area and volume and the cushion volume and ambient pressure which are chosen in the preliminary design stage.

SCALE EFFECT ON THE BUBBLE PRESSURE

Reference 12 presents the scale effect on the real component of the bubble pressure in the two-dimensional SES against the scale ratio λ = ship length/model length. It shows that the scale effect is larger for the low frequency than for the high frequency.

A similar study can be carried out for the present three-dimensional SES model.

Since $p_o Q_r / v_o$ and $p_o Q_i / v_o$ in (83) are dimensionless, the scale effect on the magnitude of the pressure (or the components) can be estimated by scaling the ambient pressure p_o by λp_o. For the real component the ratio of the coefficient for ship and model is

$$\left(\frac{p_b/p_o}{z/h}\right)_{r_{ship}} \Big/ \left(\frac{p_b/p_o}{z/h}\right)_{r_{model}}$$

$$= \frac{\gamma(1 + p_a/\lambda p_o)\left[1-\gamma(1 + p_a/\lambda p_o) p_o Q_r/v_o\right]}{\left[1-\gamma(1 + p_a/\lambda p_o) p_o Q_r/v_o\right]^2 + \left[\gamma(1 + p_a/\lambda p_o) p_o Q_i/v_o\right]^2} \Big/$$

$$\frac{\gamma(1 + p_a/p_o)\left[1-\gamma(1 + p_a/p_o) p_o Q_r/v_o\right]}{\left[1-\gamma(1 + p_a/p_o) p_o Q_r/v_o\right]^2 + \left[\gamma(1 + p_a/p_o) p_o Q_i/v_o\right]^2} \qquad (91)$$

Similar expressions can be obtained for the imaginary component as well as for the resultant magnitude.

NUMERICAL CALCULATIONS AND DISCUSSION

Water Waves - Elevation, Escape Area and Volume

The model used in Reference 2 has been chosen to compare the analytical values of wave elevation, escape area at stern and volume on the air-water interface in the developments of that reference and the present study.

A series of calculations have been carried out for the elevation of the free surface along the wall $y=b$ at Froude number $F_n=0.5$ for different values of frequencies $\tau=0$, and $\tau=0.2375$ and $\tau=0.2750$ which bracket the critical frequency $\tau=0.25$. The results are shown in Figures 2a and 2b and compare very favorably with those of Doctors who defines $F_n = v/\sqrt{4ag}$. Figure 2c is is a comparison of the values of wave elevation of the corner point $x/a=-1$ and $y/b=1$ in terms of frequency τ, evaluated by the present analysis and by that of Reference 2. In the low frequency regime the agreement is excellent, whereas outside that the discrepancy is quite substantial especially in the imaginary part of the elevation values.

Similar calculations were carried out for the escape area, α, at the stern and the volume, v, on the air-water interface for the entire frequency range of practical interest. A comparison of the results of both analytical solutions is shown in Figures 3a and 3b for the escape area coefficient, α, and volume coefficient, v, respectively, for a series of frequencies τ. The agreement is rather satisfactory up to $\tau=1.5$ except for the real value of the v coefficient. It is interesting to notice that the value of the escape area oscillates with τ, whereas the volume coefficient v behaves relatively monotonically. Sudden changes of the values of α and v are exhibited at the critical $\tau=0.25$ as should be expected. Furthermore, it is interesting to note that as $\tau \to 0$, the imaginary part of both quantities tends to zero whereas the real parts tend to finite negative values. These limiting values can be readily obtained from Equations (61a), (61b), and (62a), (62b). As the frequency approaches higher values, the imaginary parts of α and v diminish showing a trend to zero asymptote, whereas the real components tend to a negative non-zero asymptote. The non-zero values are less than the static values of $\alpha_r=-0.25$ and $v_r=-0.25$. It is apparent, therefore, that the dynamic values of α and v in Eqs.(62a) and (62b) are positive and their magnitudes are less than their corresponding static values (Eq.61c).

Similar calculations have been carried out for XR-5 SES model at $F_n = 0.72$, for which extensive experimental work is available.[5-7] Figure 4 is a sketch of the model and Table I presents its principal particulars. The results for α and v are exhibited in Figures 5a and 5b versus the frequency τ. The trends of the results of both models are, in general, similar except that the oscillatory behavior of α has been reduced considerably in the second model.

Bubble Pressures

The values of bubble pressures in the plenum calculated from the expressions given by Eqs.(80), (81) and (82), for the XR-5 SES model at $F_n=0.72$ are graphically exhibited in Figures 6a and 6b for a series of reduced frequencies τ.

Figure 6a shows the values of bubble pressure and phase shift angle versus τ in the presence or absence of water free surface effects. In the limit as $\tau \to 0$, the expression given by Eq.(81) shows the pressure amplitude reduces to zero with phase lag of 90 degrees. The pressure amplitude generally increases as τ increases tending to the asymptotic value given by Eqs.(82a) and (82b) for both cases, with and without the free surface wave effects.

Figure 6b gives the entire picture of bubble pressure versus τ ranging from zero to 600. It is seen that the amplitude reaches its asymptote, $\gamma(1+ p_a/p_o)$ at $\tau = 200$, whereas the phase angle may reach its asymptote at $\tau = 800$. It is to be noted, however, that the values of the bubble pressure in the uncoupled condition (i.e., without water wave effects) are, in general, much larger than those obtained under the coupled condition. In fact, the asymptotic value of the former is about thirty times larger than the latter.

Added Mass and Damping Coefficients

The heave added mass and damping coefficients C_a, C_d of the XR-5 SES model have been calculated and are shown in Figures 7a and 7b. The experimental values[6] for the model are also compared with the theoretical values in Figure 7a. Both results show the same trend and the experimental damping coefficients agree better with the present theory when wave effects are taken into account. The asymptotic values of the coefficients obtained from Eqa.(85)

and (88) as $\tau \to 0$ are the same for both coupled and uncoupled conditions. The imaginary part of the bubble pressure from which the dampling factor N is determined is linearly proportional to the frequency τ (or σ) at low frequency for both coupled and uncoupled conditions (see Eq.85). The experiments[4] also find this component linearly proportional to the frequency for an ACV model over the ground. The added mass coefficient is, in general, negative for the specific model except in a narrow region as indicated in Figure 7a.

The non-dimensional coefficients C_a and C_d diminish as τ increases. As τ approaches ∞, the coefficients tend to vanish for both coupled and uncoupled conditions (see Figure 7b).

Scale Effect on the Bubble Pressure

The scale effect on the bubble pressure amplitude has been calculated for the XR-5 SES model at $F_n = 0.72$, and the results are shown in Figure 8. It is seen that the scale effect is increasing with increasing frequency at constant scale ratio λ and increases also with λ at constant frequency. The scale effect increases with both increasing frequency and scale ratio. For instance, the ship pressure at $\tau=4.5$ is underestimated by 14% when the model scale ratio $\lambda=10$, which corresponds to about a 133 long-ton ship.*

The previous work[12] on the scale effect is concerned with the real component of the pressure only. It shows that the scale effect is larger at low frequency, a trend which is opposite to the present result. However, it is to be noted that the analysis of Reference 12 deals with the two-dimensional theory and hence it is difficult to compare with the present three-dimensional analysis.

Heave Response

Since the theoretical heave added mass and damping coefficients are available, the heave response with pitch restrained has been calculated by utilizing test results[6] for the restoring and exciting forces of the XR-5 SES model.

Figure 9a shows the mean line drawn among the experimental values of the non-dimensional heave exciting force[6] versus the non-dimensional encounter frequency $\sigma\sqrt{L/g}$. The extrapolation has been made by assuming that the heave-exciting is identical to the restoring force at zero frequency. The

*These conclusions reached after the set of calculations performed for the XR-5 SES for given fan characteristics and discharge should be used with caution and cannot be generalized before extensive calculations are performed.

mean line has been cut-off at the frequency beyond which the data are unreliable.

The experimental added mass and damping coefficients have been obtained from the three test points indicated in Figures 18 and 19 of Reference 6. Using these values, three heave responses have been calculated. Figure 9b compares the semi-analytical and the experiment-based calculations. The semi-analytical response is higher than the experiment-based calculation results due to the fact that the theoretical damping coefficients are lower than the experimental values.

The theoretical calculations show a similar trend to that of the heave with pitch free (x symbols) which is obtained from Reference 7. The natural pitching frequency of the model is 5 rad/sec. It is seen in Figure 9b that the theoretical heave at the natural pitching frequency is about 50% of the heave with pitch free. This kind of characteristic heave response in coupled and uncoupled conditions for an SES model has been found by Fridsma[8,9] at Davidson Laboratory.

Numerical Procedures

The computational work has been carried out at the Stevens Institute Computer Center on the PDP-10 high-speed digital computer. Program structure with Flow-Chart of the main subroutine VLLI, program listing and descriptions of input and output are given in Appendix E.

The exponential integral of complex argument, which is the basic element in all the analytical expressions, is discussed in detail in Appendix D with the view to utilizing high-speed digital computers.

Furthermore for convenience in the numerical integration of Eqs.(46) and (59) to (62), a transformation is introduced to simplify the structure of the singular behavior at θ_c (critical point, see Eq.29), and then the "Lagrange interpolation method" has been employed for the evaluation of the finite contribution of the high-order singularity.

SUMMARY AND CONCLUSIONS

A mathematical model has been developed for the evaluation of the added mass and damping coefficients of a heaving surface effect ship in uniform translation on a calm water surface.

The analysis furnishes the variation of bubble pressure in the plenum and its scale effect in terms of reduced frequency τ in the entire range of practical interest, and also provides information on the wave elevation of the free surface, the escape area at the stern and the volume on the air-water interface.

A program has been written in FORTRAN IV language and applied to the PDP-10 high-speed digital computer.

The following basic assumptions were made:

1. The deformation of the water surface participates in the generation of the bubble pressure in conjunction with the action of seals and fans.

2. The deformation of the free surface under the oscillatory rectangular pressure patch in uniform translation is used to display the way in which the motion of the water surface participates in the evaluation of the pressure variations in the plenum air.

3. In the dynamic analysis of the air flow phenomena, pressure and volume changes are assumed to occur sufficiently rapdily so that the adiabatic law governs the basic thermodynamic variations in the air cushion.

4. The bubble (or cushion) pressure is assumed to be spatially homogeneous.

5. The leakage of the air takes place only at fan and stern apertures.

6. The perturbed (or bubble) pressure and air volume have the same order of magnitudes as the heave displacement.

Expressions for the wave elevation, escape area and volume have been derived and their limiting values at $\tau \to 0$ and $\tau \to \infty$ have been studied.

The linear equations of bubble pressure and heave motion have been developed, applying perturbation procedure to the adiabatic law, and their asymptotic values at $\tau \to 0$ and $\tau \to \infty$ have been established.

A series of numerical calculations for the SES model have been carried out for the wave elevation, escape area, volume, bubble pressure, added mass, damping factor, the scale effect on the bubble pressure and the heave response with pitch restrained.

All the above results have been compared with available experimental data and their behavior as functions of the frequency discussed.

The following conclusions can be drawn from this investigation:

1. The assumption of a rigid-free surface is not valid. Considerable differences have been observed with and without the free surface waves.

2. The bubble pressure in the plenum has zero amplitude with 90° phase lag as $\tau \to 0$, whereas at $\tau \to \infty$ it reaches a constant value with 180° phase lag.

3. The added mass and damping factors as $\tau \to 0$ reach asymptotically finite values whereas as $\tau \to \infty$ they vanish.

4. The scale effect on the bubble pressure increases with increasing frequency and scale ratio. (More extensive calculations are needed before generalizing this.)

5. The present method can be applied to predict the heave response with restrained pitch motion utilizing experimental values of the restoring and exciting forces.

6. The added mass and damping factor compare favorably with the experimental data. Unfortunately a very limited number of experiments are at present available and before more experiments are conducted, no general conclusion as to the accuracy of the theoretical prediction can be drawn.

It is expected that the present method can be extended to calculate 1) the heave exciting force of a captive SES model uniformly running in head regular waves and 2) the heave restoring force acting on an SES model in uniform translation.

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TABLE 1

PRINCIPAL CHARACTERISTICS OF XR-5 SES MODEL

Length overall, ft	LOA = 15.58
Length of bubble, ft	L = 13.83
Beam of bubble, ft	b = 2.12
Height of bubble, ft	h = 0.72
Test displacement, lb	Δ = 298
Nozzle factor	k = 0.65
Fan flow rate per pair of fans, $ft^5/lb\text{-}sec$	$\lvert\partial Q_f/\partial p\rvert$ = 0.03
No. of pairs of fans	n = 2
Leak area at equilibrium	$A_\ell^{(o)}$ = 0.333
Froude number	F_n = 0.72
Encounter frequency, rad/sec	ω_e = 3 ~ 22
Natural pitching frequency, rad/sec	ω_x = 5
Heave restoring force, lb/ft	B = 1350

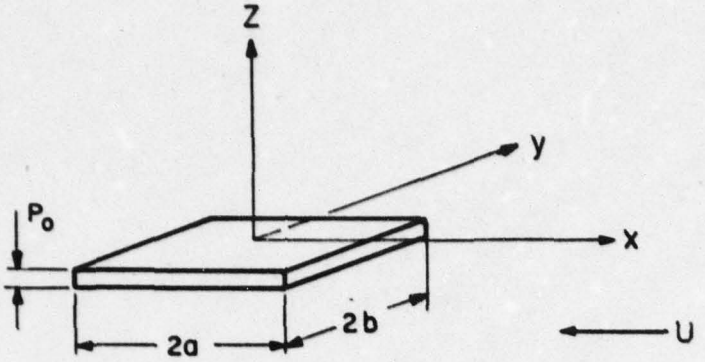

FIG. 1a. RECTANGULAR PRESSURE DISTRIBUTION

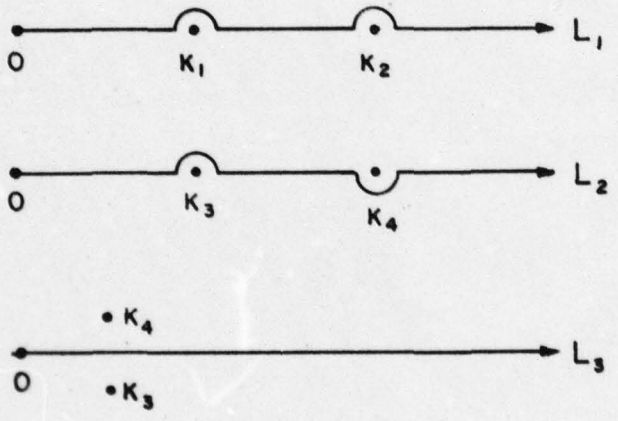

FIG. 1b. INTEGRATION PATHS

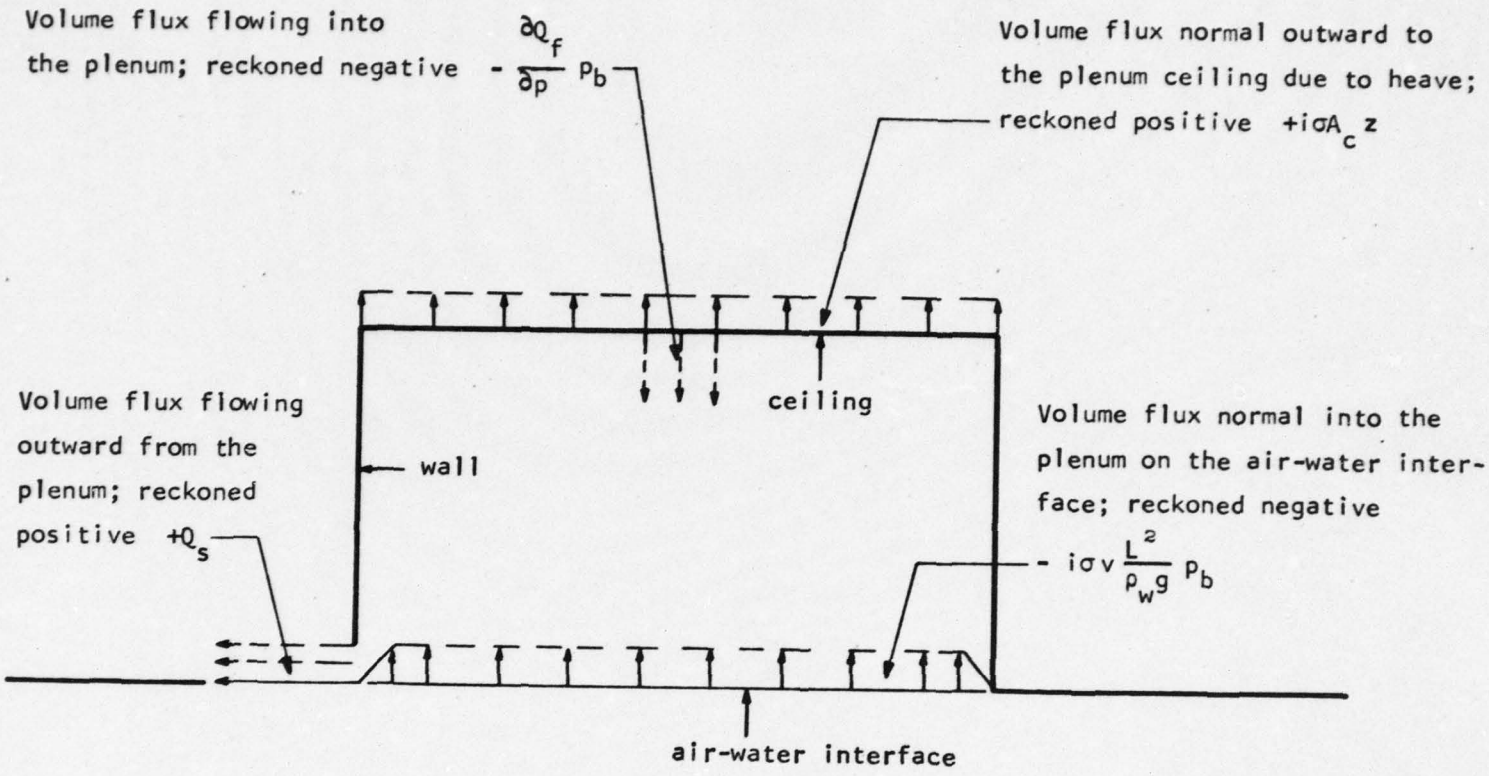

FIGURE 1c. SKETCH OF THE SENSES OF THE FLOW DIRECTION OF THE VOLUME FLUX NORMAL TO THE PLENUM WALL, CEILING AND AIR-WATER INTERFACE

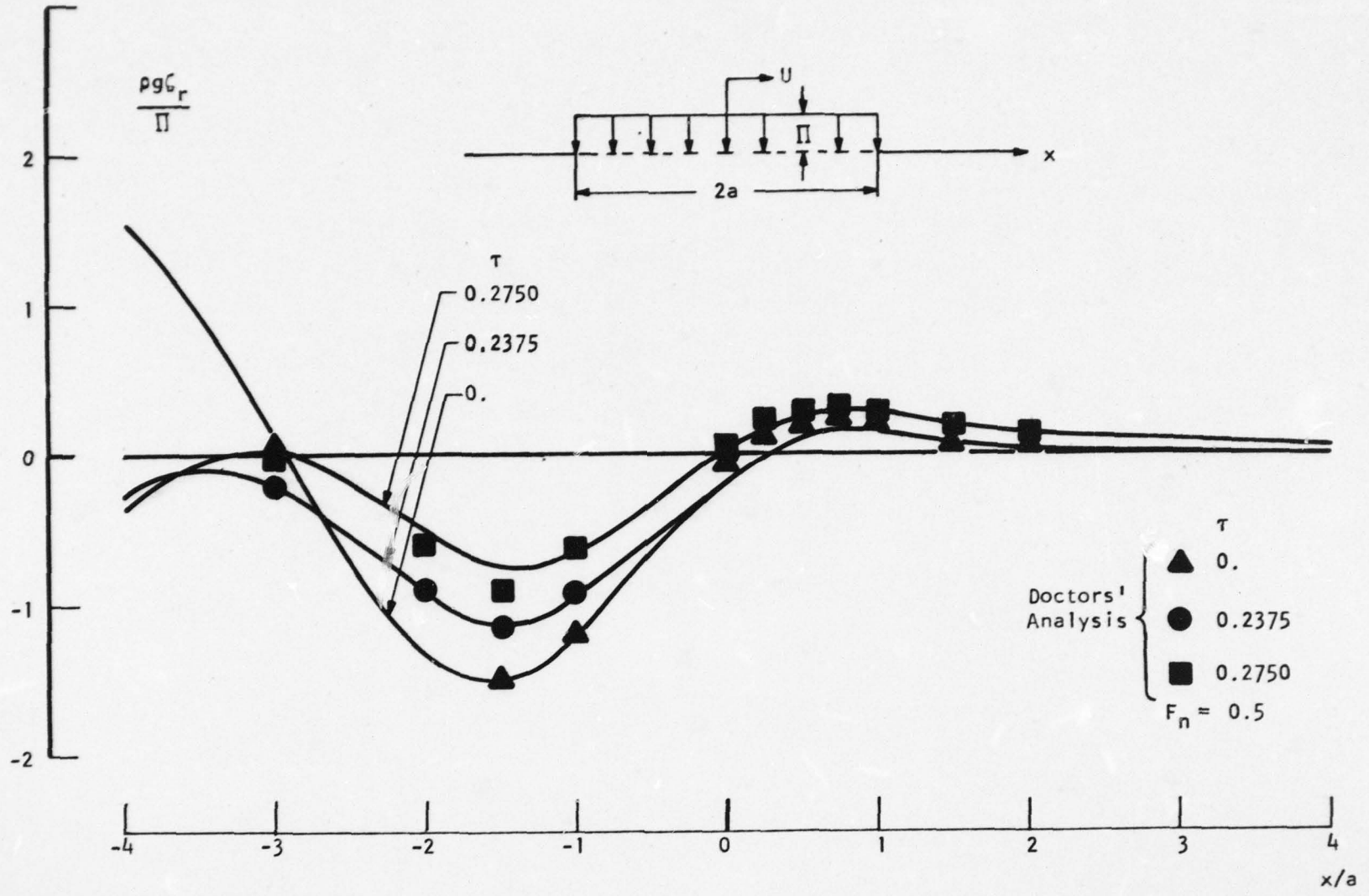

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FIGURE 2a. ELEVATION ALONG THE LONGITUDINAL SECTION y/b=1

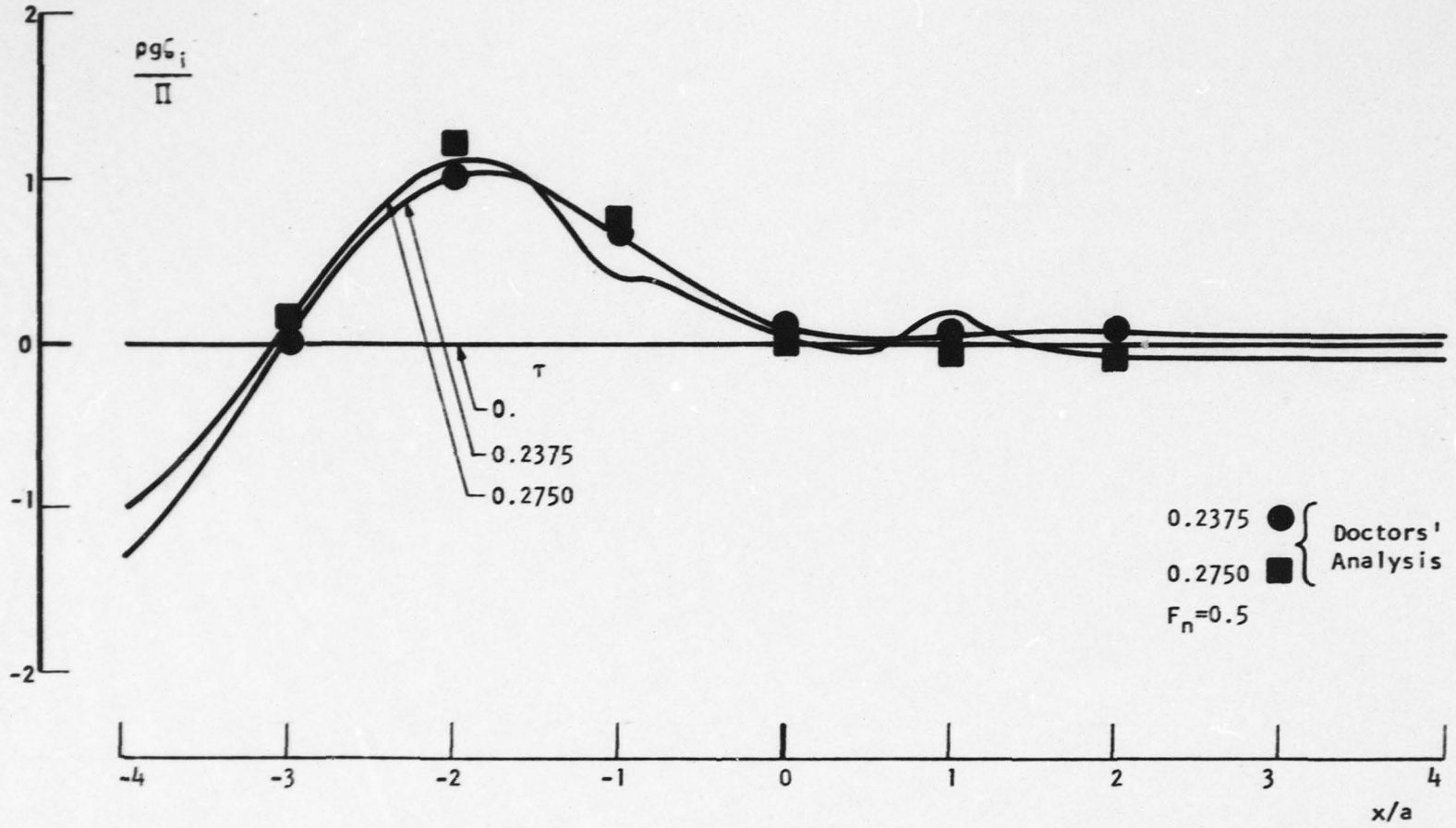

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FIGURE 2b. ELEVATION ALONG THE LONGITUDINAL SECTION y/b=1

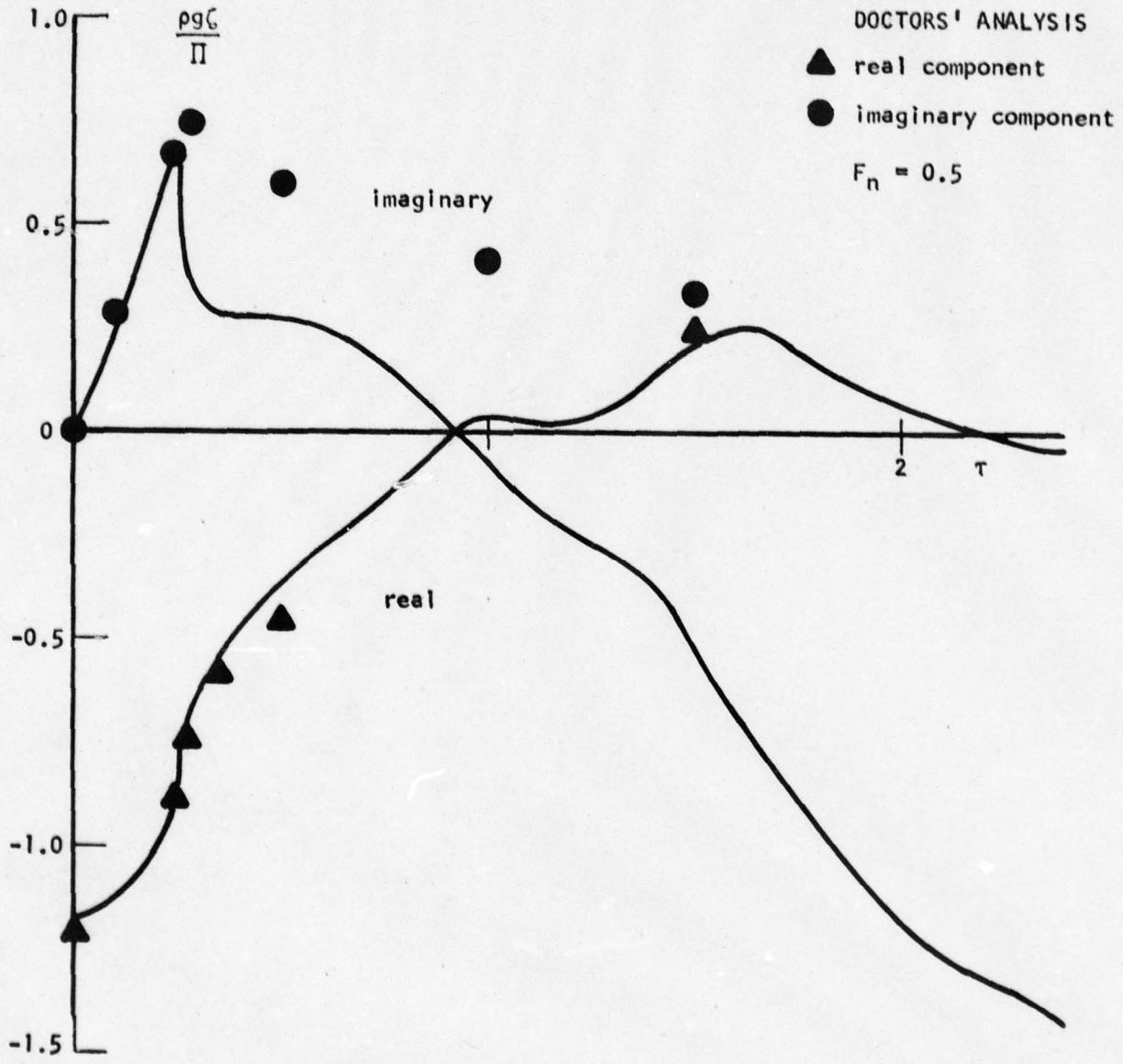

FIGURE 2c. ELEVATION VS. τ AT x/a = -1 AND y/b = 1

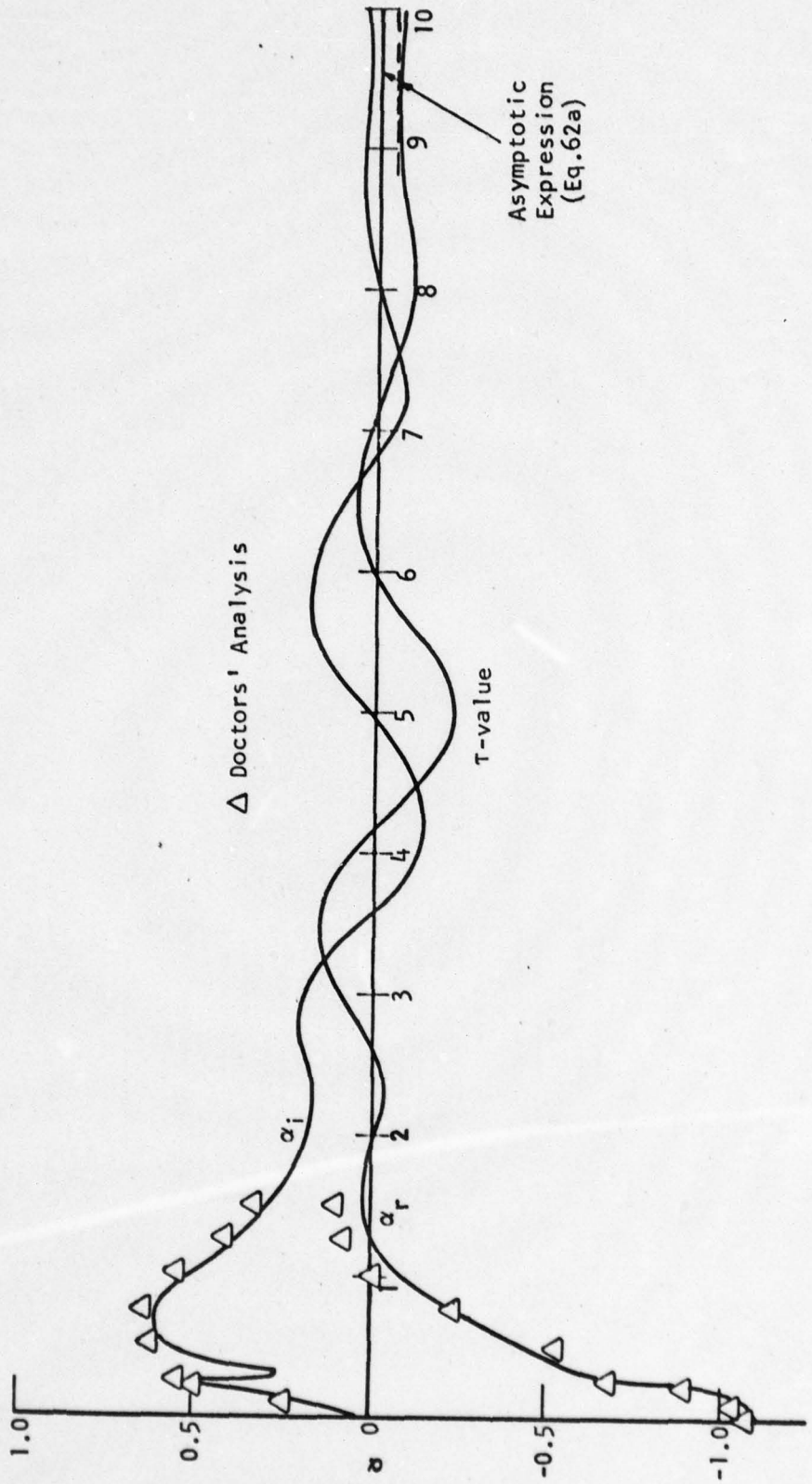

FIGURE 3a. ESCAPE AREA COEFFICIENT α AT STERN FOR THE MODEL IN REF.[2] VERSUS τ FOR $F_n = 0.5$

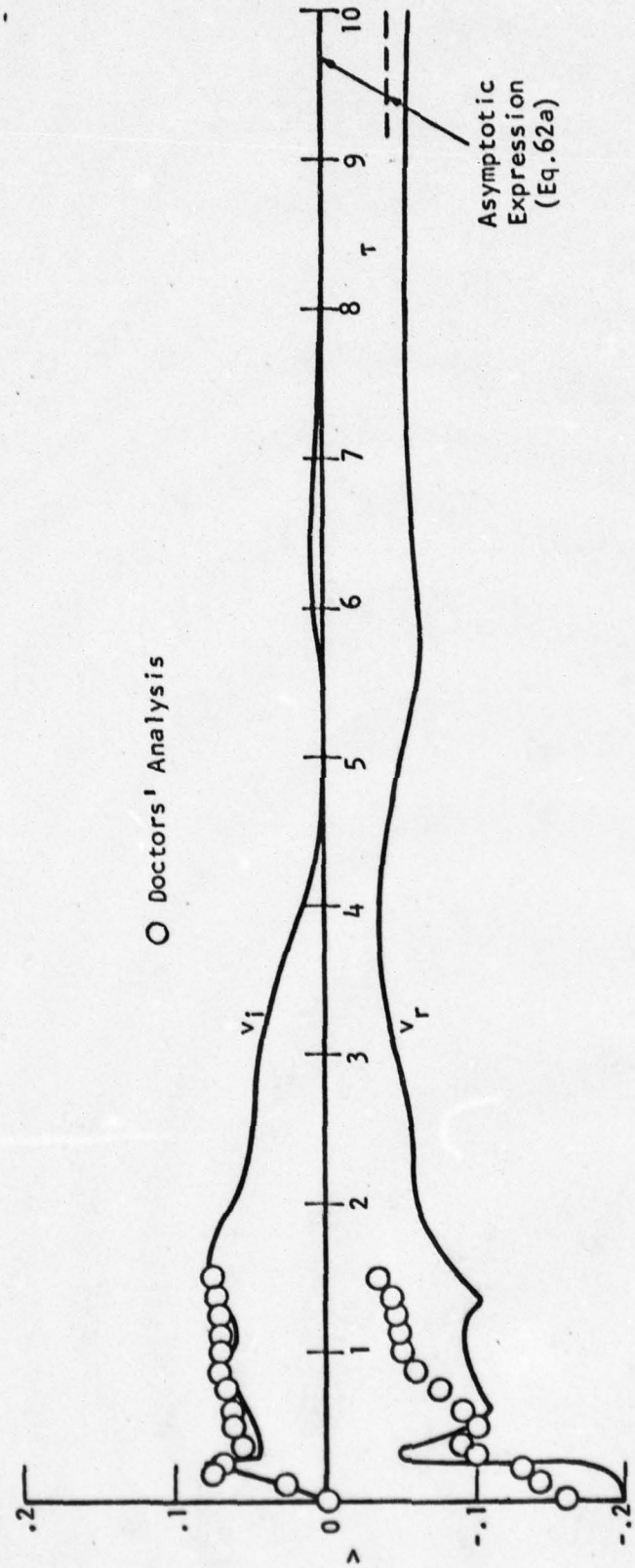

FIGURE 3b. VOLUME COEFFICIENT v VERSUS τ
FOR F_n = 0.5 FOR THE MODEL IN REF. [2]

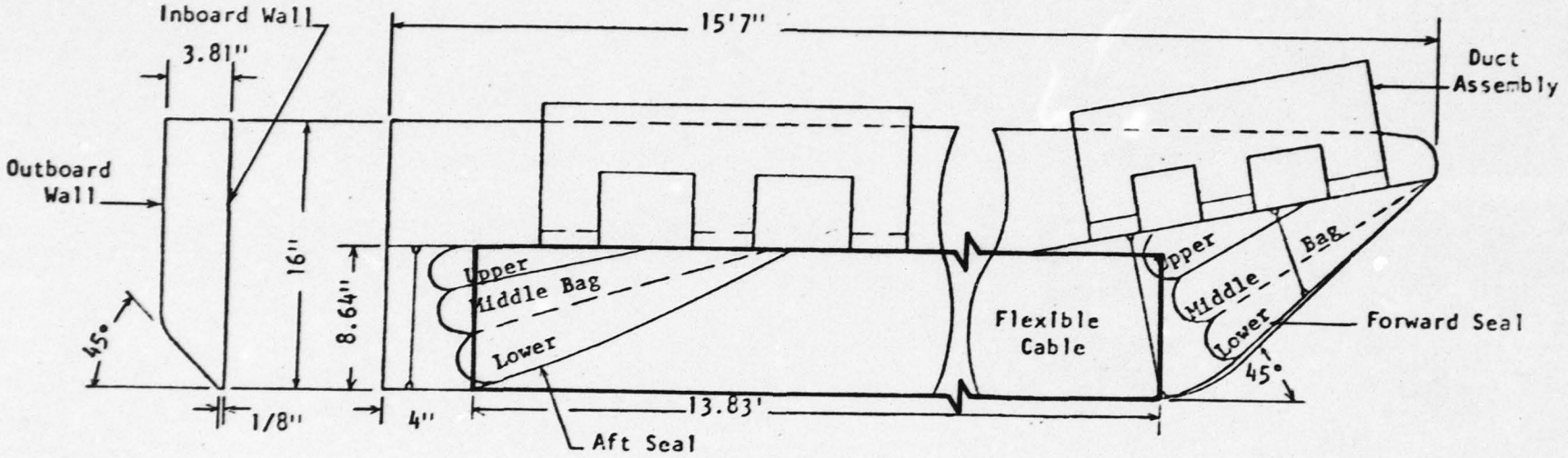

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FIGURE 4. SKETCH OF SES MODEL INCLUDING BOW AND STERN SEAL CONFIGURATIONS AND MIDSTATION SIDEWALL SECTION

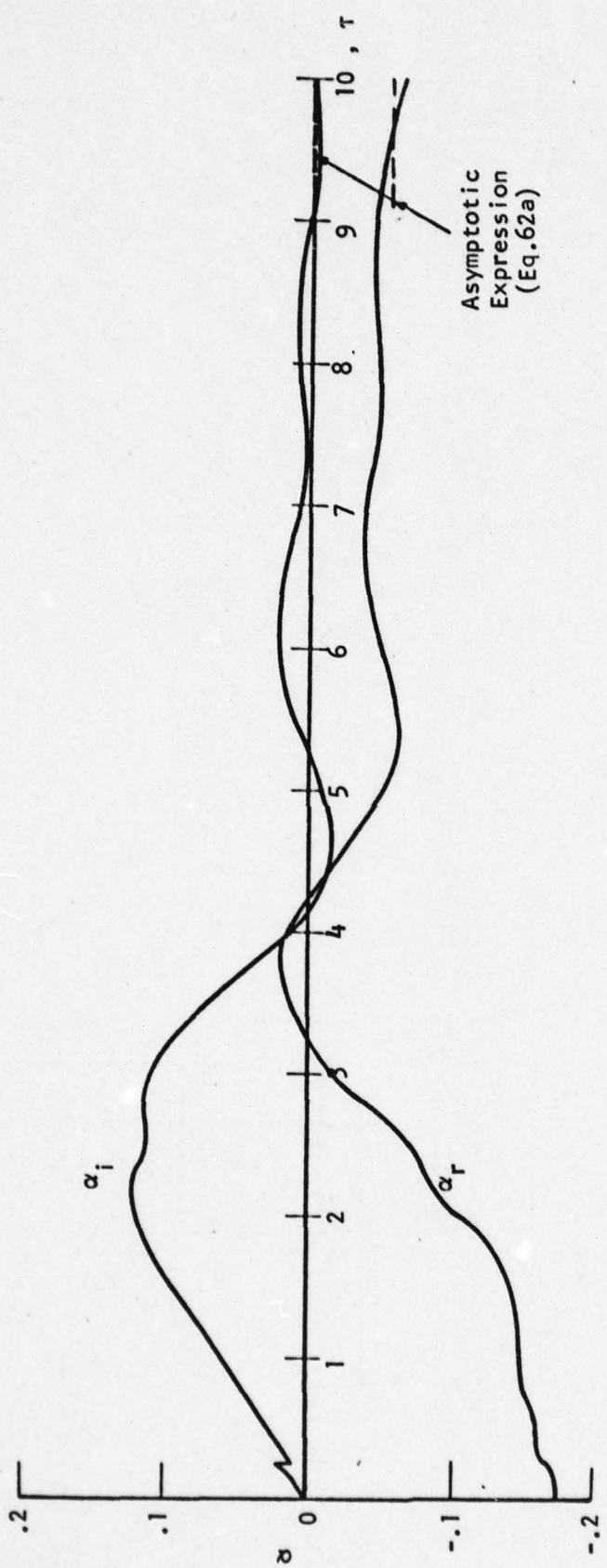

FIGURE 5a. ESCAPE AREA COEFFICIENT α AT STERN VERSUS τ
AT $F_n = 0.72$ for XR-5 SES MODEL

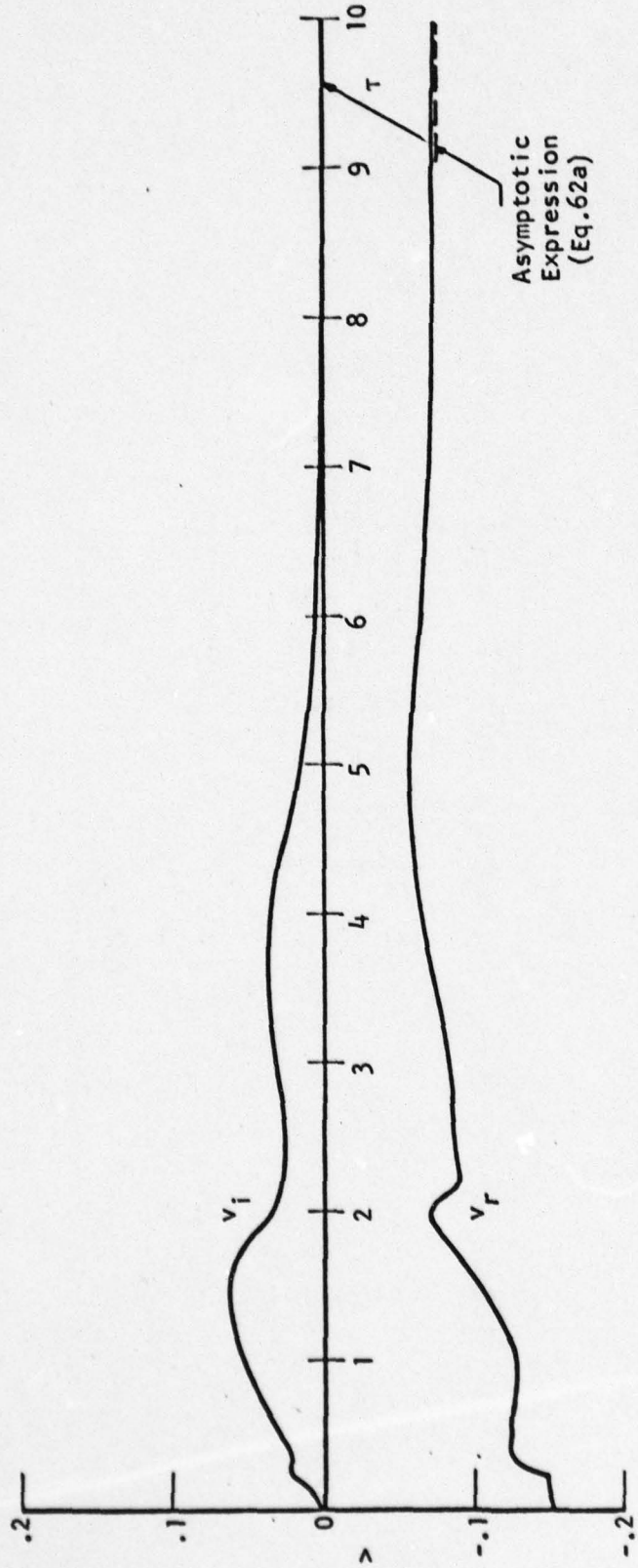

FIGURE 5b. VOLUME COEFFICIENT ν VERSUS τ
AT F_n = 0.72 FOR XR-5 SES MODEL

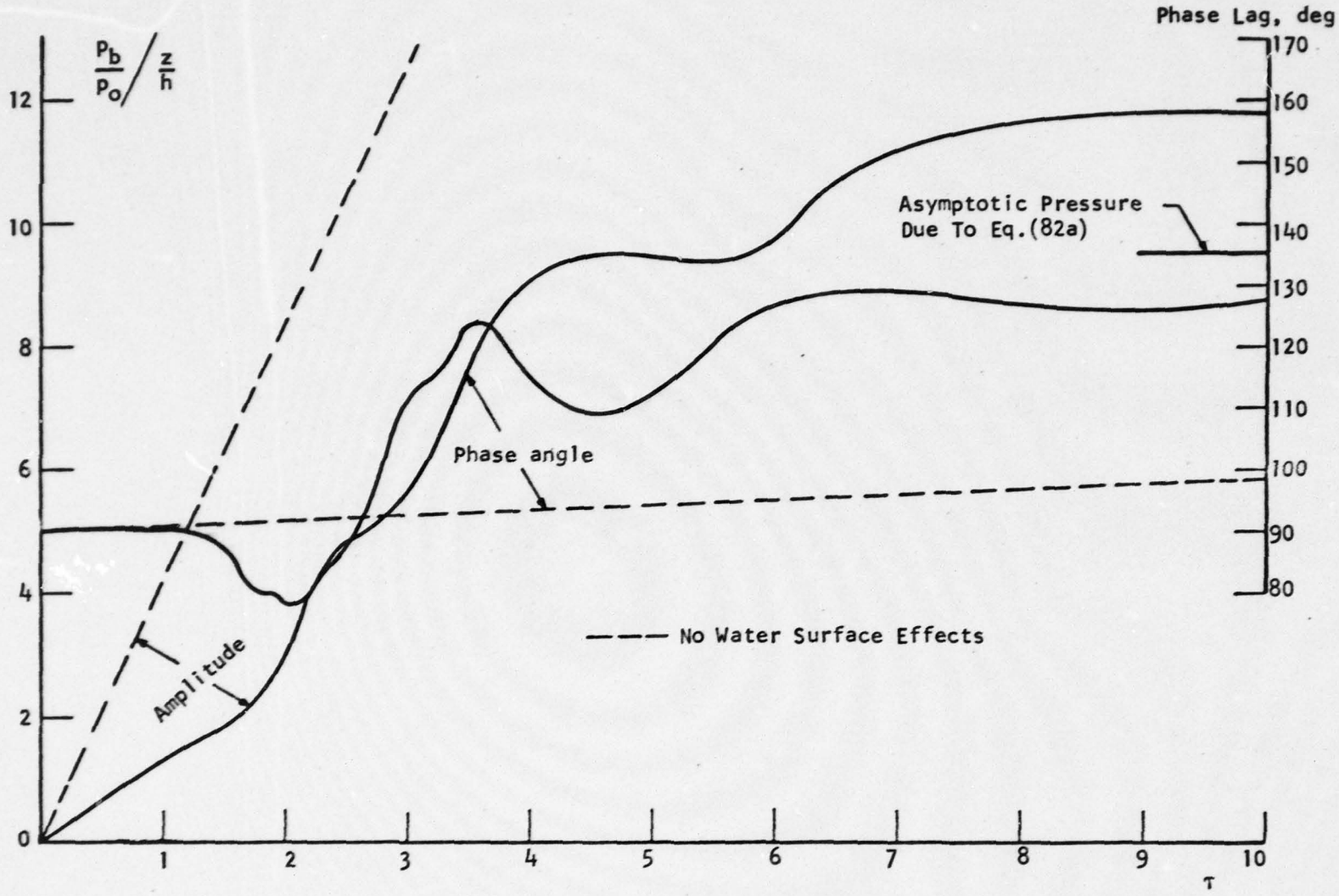

FIGURE 6a. BUBBLE PRESSURE COEFFICIENTS VERSUS τ
AT $F_n = 0.72$ FOR XR-5 SES MODEL

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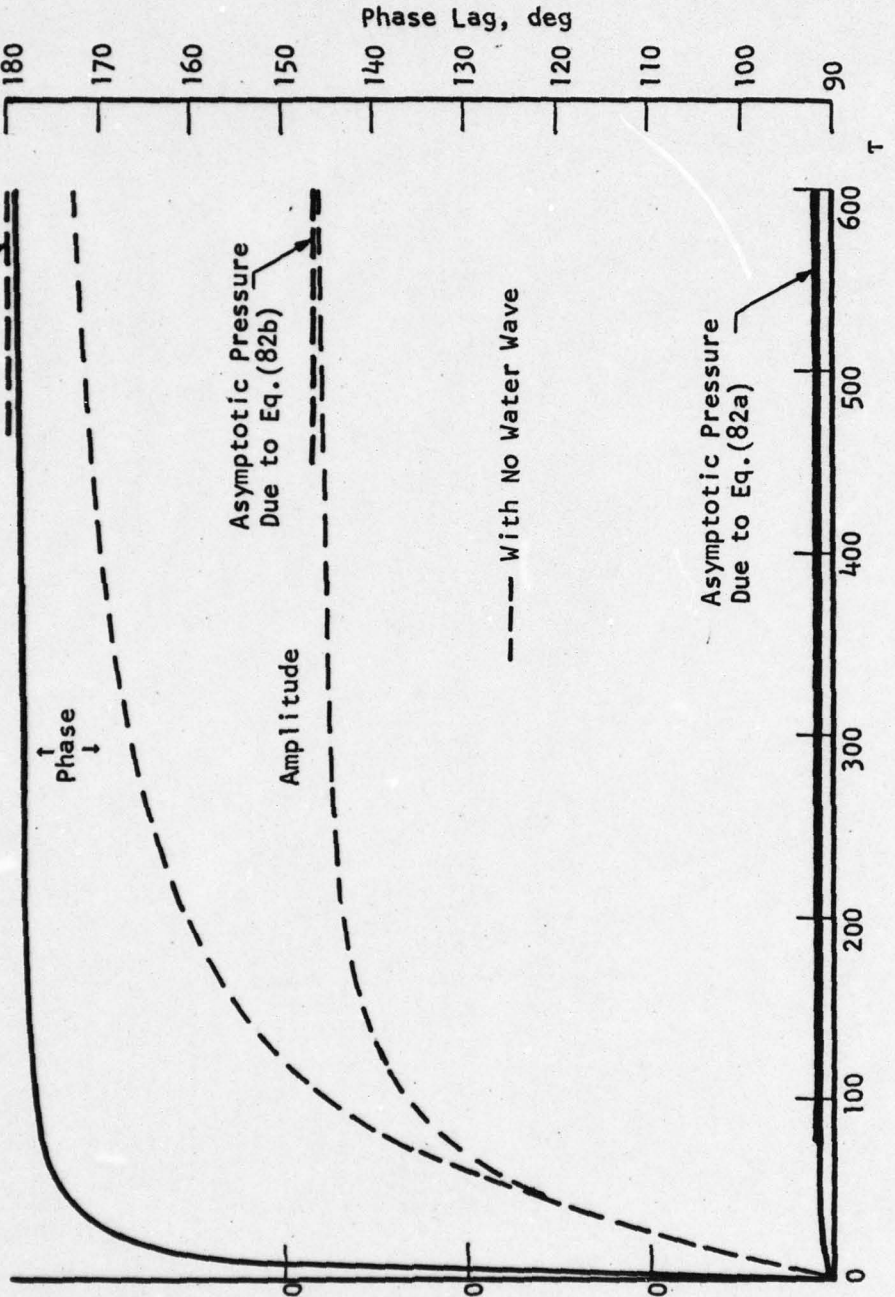

FIGURE 6b. BUBBLE PRESSURE IN THE HIGH FREQUENCY REGION

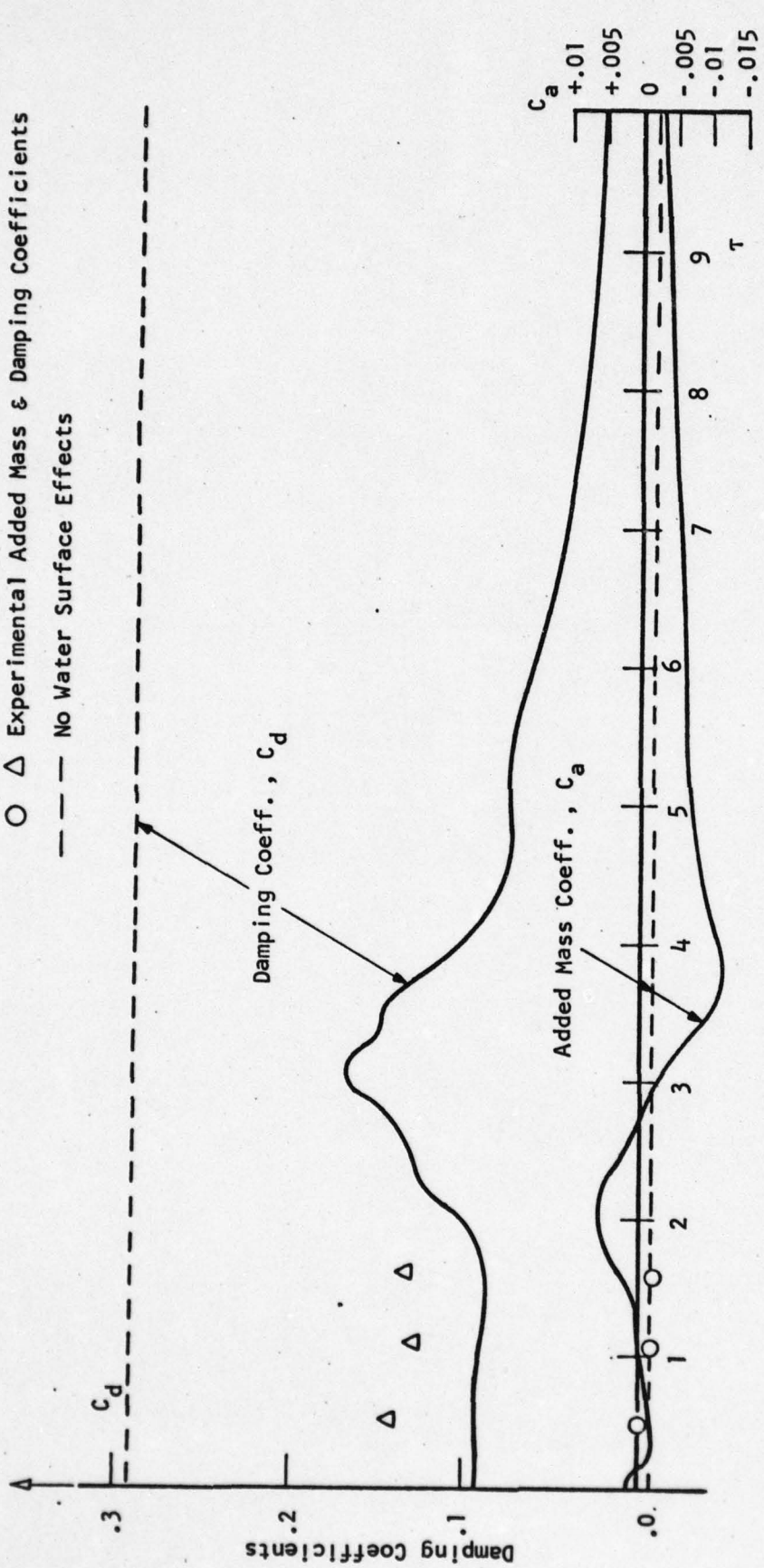

FIGURE 7a. ADDED MASS AND DAMPING COEFFICIENTS VERSUS τ
AT F_n = 0.72 FOR XR-5 SES MODEL

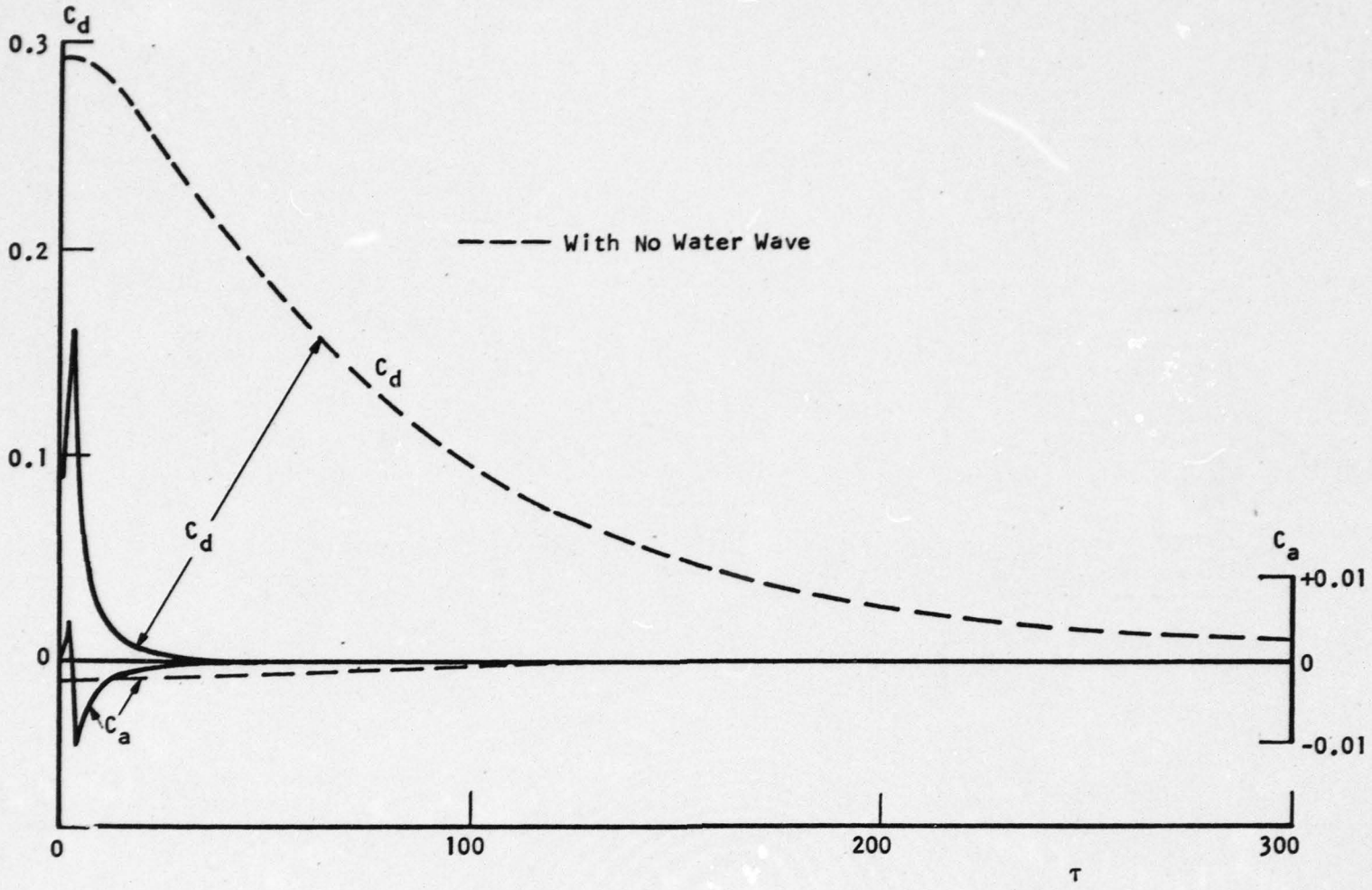

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FIGURE 7b. ADDED MASS AND DAMPING COEFFICIENTS (C_a,C_d) IN THE HIGH FREQUENCY REGION

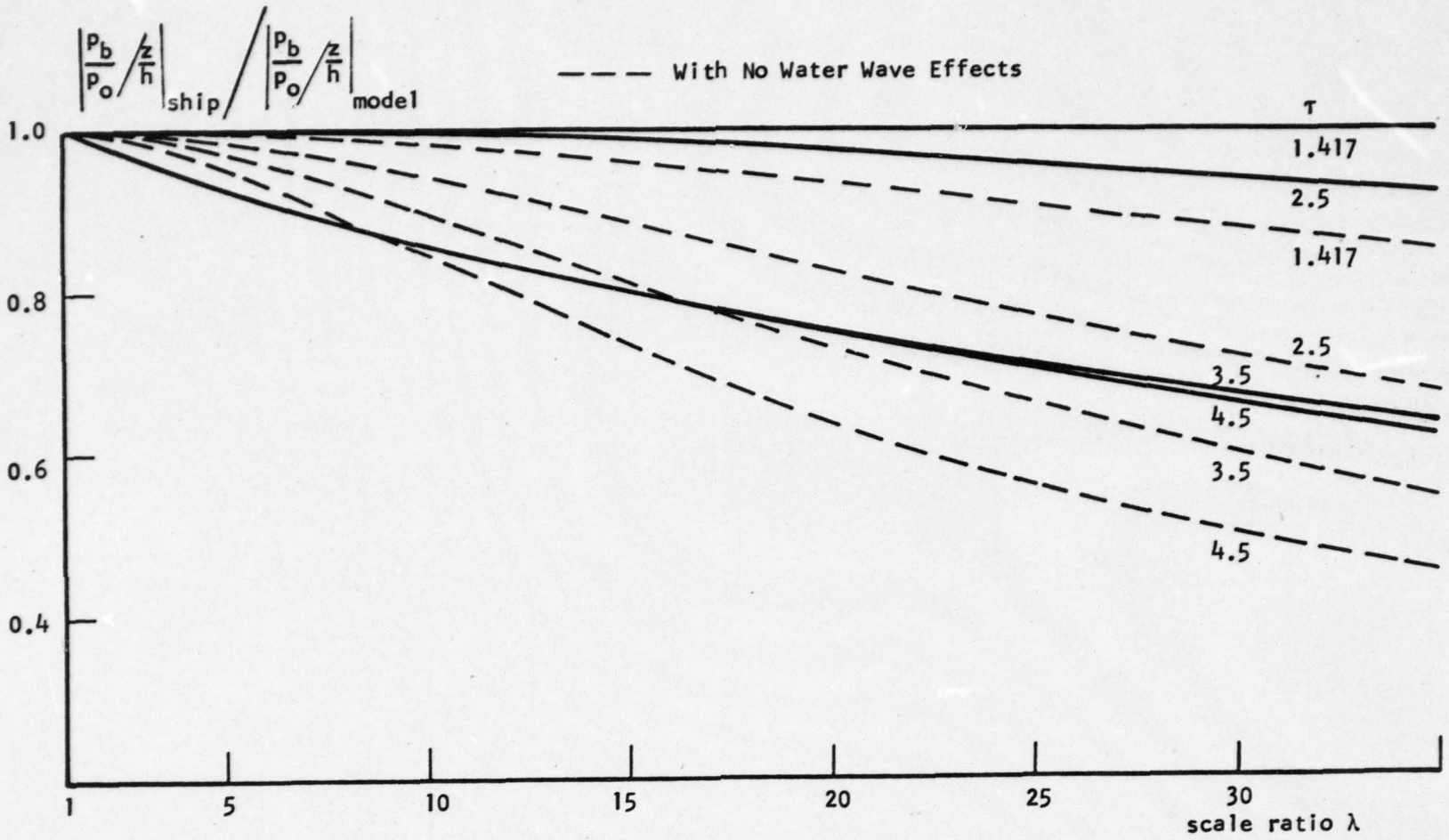

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FIGURE 8. SCALE EFFECT ON THE BUBBLE PRESSURE VERSUS SCALE RATIO λ AT F_n = 0.72 FOR XR-5 SES MODEL

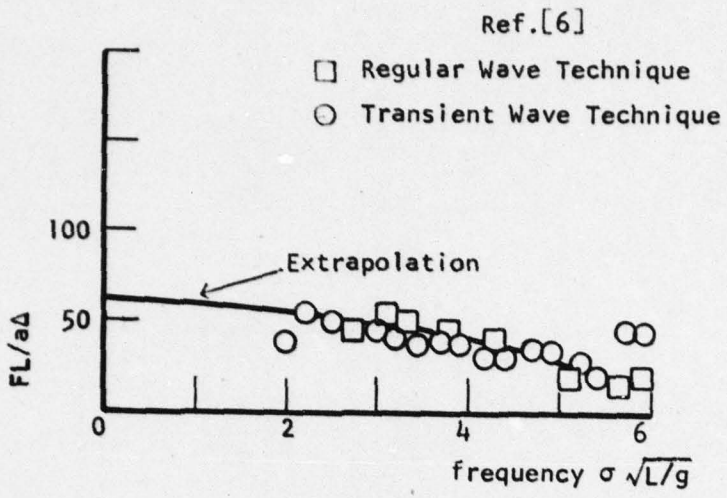


FIGURE 9a. HEAVE-EXCITING FORCE OF XR-5 SES MODEL AT F_n=0.72

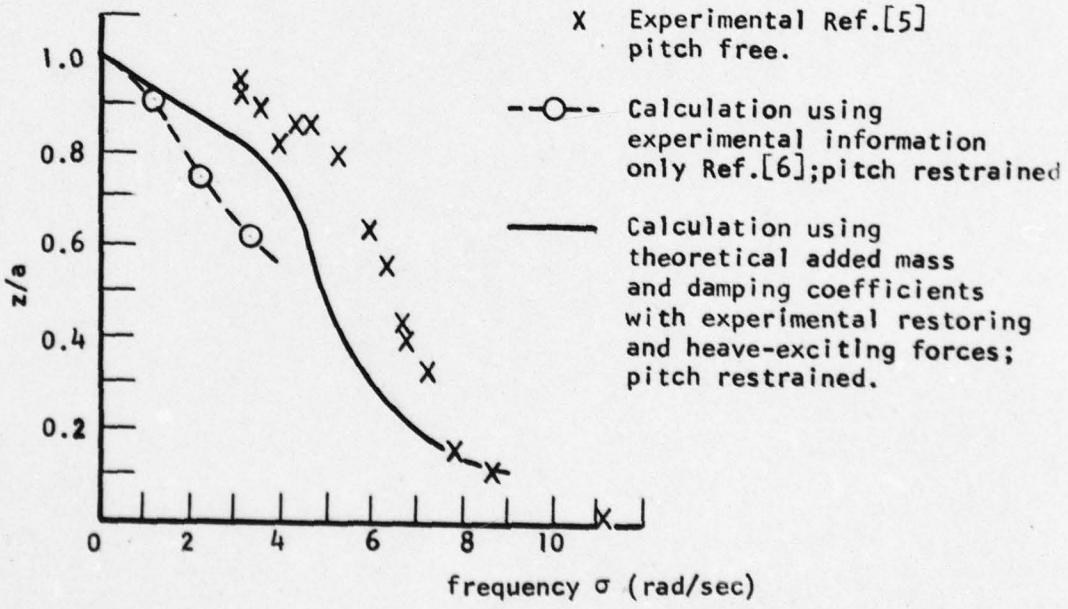


FIGURE 9b. HEAVE TO WAVE AMPLITUDE RATIO OF XR-5 SES MODEL AT F_n = 0.72

APPENDIX A

CALCULATION OF $I_{1,j}$, $I'_{1,j}$, $J_{1,j}$, $J'_{1,j}$ $(j=1,2)$

Integration of $I_{1,j}$ and $I'_{1,j}$ in Eqs. (43a) and (43b) with respect to ξ and η yields

$$I_{1,j} = -\frac{U}{\sin\theta}\int_0^\infty dk\,\frac{e^{kz}\left(e^{iks_1}-e^{iks_2}+e^{iks_3}-e^{iks_4}\right)}{k(k-k_j)} + \frac{\sigma}{\sin\theta\cos\theta}\int_0^\infty dk\,\frac{e^{kz}(\quad''\quad)}{k^2(k-k_j)} \qquad (A\text{-}1)$$

$$I'_{1,j} = \frac{U}{\sin\theta}\int_0^\infty dk\,\frac{e^{kz}\left(-e^{iks_5}+e^{iks_6}-e^{iks_7}+e^{iks_8}\right)}{k(k-k_j)} - \frac{\sigma}{\sin\theta\cos\theta}\int_0^\infty dk\,\frac{e^{kz}(\quad''\quad)}{k^2(k-k_j)} \qquad (A\text{-}2)$$

Differentiation of $I_{1,j}$ (A-1) and $I'_{1,j}$ (A-2) with respect to x yields

$$J_{1,j} = \frac{U\cos\theta}{i\sin\theta}\int_0^\infty dk\,\frac{e^{kz}\left(e^{iks_1}-e^{iks_2}+e^{iks_3}-e^{iks_4}\right)}{k-k_j} - \frac{\sigma}{i\sin\theta}\int_0^\infty dk\,\frac{e^{kz}(\quad''\quad)}{k(k-k_j)} \qquad (A\text{-}3)$$

$$J'_{1,j} = -\frac{U\cos\theta}{i\sin\theta}\int_0^\infty dk\,\frac{e^{kz}\left(-e^{iks_5}+e^{iks_6}-e^{iks_7}+e^{iks_8}\right)}{k-k_j} + \frac{\sigma}{i\sin\theta}\int_0^\infty dk\,\frac{e^{kz}(\quad''\quad)}{k(k-k_j)} \qquad (A\text{-}4)$$

where $j = 1,2$ and

$$s_1 = (x\cos\theta + y\sin\theta) - (a\cos\theta + b\sin\theta)$$

$$s_2 = (x\cos\theta + y\sin\theta) - (a\cos\theta - b\sin\theta)$$

$$s_3 = (x\cos\theta + y\sin\theta) + (a\cos\theta + b\sin\theta)$$

$$s_4 = (x\cos\theta + y\sin\theta) + (a\cos\theta - b\sin\theta)$$

$$s_5 = (x\cos\theta - y\sin\theta) - (a\cos\theta + b\sin\theta)$$

$$s_6 = (x\cos\theta - y\sin\theta) - (a\cos\theta - b\sin\theta)$$

$$s_7 = (x\cos\theta - y\sin\theta) + (a\cos\theta + b\sin\theta)$$

$$s_8 = (x\cos\theta - y\sin\theta) + (a\cos\theta - b\sin\theta) \qquad \text{(A-5)}$$

Since

$$\frac{1}{k(k-k_j)} = -\frac{1}{k_j}\frac{1}{k} + \frac{1}{k_j}\frac{1}{k-k_j}$$

$$\frac{1}{k^2(k-k_j)} = -\frac{1}{k_j}\frac{1}{k^2} - \frac{1}{k_j^2}\frac{1}{k} + \frac{1}{k_j^2}\frac{1}{k-k_j} \qquad \text{(A-6)}$$

Equations (A-1) through (A-4) are expressed in the following forms,

$$I_{1,j} = \left(\frac{U}{k_j\sin\theta} - \frac{\sigma}{k_j^2\sin\theta\cos\theta}\right)(I_1 - I_2) - \frac{\sigma}{k_j\sin\theta\cos\theta} I_3$$

$$I'_{1,j} = \left(-\frac{U}{k_j\sin\theta} + \frac{\sigma}{k_j^2\sin\theta\cos\theta}\right)(I'_1 - I'_2) + \frac{\sigma}{k_j\sin\theta\cos\theta} I'_3$$

$$J_{1,j} = \frac{\sigma}{ik_j\sin\theta} I_1 + \left(\frac{U\cos\theta}{i\sin\theta} - \frac{\sigma}{ik_j\sin\theta}\right) I_2$$

$$J'_{1,j} = -\frac{\sigma}{ik_j\sin\theta} I'_1 + \left(-\frac{U\cos\theta}{i\sin\theta} + \frac{\sigma}{ik_j\sin\theta}\right) I'_2 \qquad \text{(A-7)}$$

where $I_1 = \int_0^\infty dk \, \frac{e^{kz}\left(e^{iks_1} - e^{iks_2} + e^{iks_3} - e^{iks_4}\right)}{k}$

$$I_2 = \int_0^\infty dk \, \frac{e^{kz}\left(\quad\text{"}\quad\right)}{k-k_j} \qquad \text{(A-8)}$$

[Cont'd]

$$I_3 = \int_0^\infty dk \, \frac{e^{kz}\left(e^{iks_1} - e^{iks_2} + e^{iks_3} - e^{iks_4}\right)}{k^3} \qquad \text{(A-8)}$$

$$I_1' = \int_0^\infty dk \, \frac{e^{kz}\left(-e^{iks_5} + e^{iks_6} - e^{iks_7} + e^{iks_8}\right)}{k}$$

$$I_2' = \int_0^\infty dk \, \frac{e^{kz}\left(\quad " \quad \right)}{k - k_j}$$

$$I_3' = \int_0^\infty dk \, \frac{e^{kz}\left(\quad " \quad \right)}{k^2} \qquad \text{(A-9)}$$

Integration of I_1, I_3, I_1', I_3'

To transform $kz + iks_1$, $kz + iks_2$, etc., we write

$$\zeta = x\cos\theta + y\sin\theta - iz$$

$$\zeta' = x\cos\theta - y\sin\theta - iz$$

$$\zeta_1 = a\cos\theta + b\sin\theta$$

$$\zeta_2 = a\cos\theta - b\sin\theta \qquad \text{(A-10)}$$

and we have

$$z + is_1 = i(\zeta - \zeta_1), \quad z + is_5 = i(\zeta' - \zeta_1)$$

$$z + is_2 = i(\zeta - \zeta_2), \quad z + is_6 = i(\zeta' - \zeta_2)$$

$$z + is_3 = i(\zeta + \zeta_1), \quad z + is_7 = i(\zeta' + \zeta_1)$$

$$z + is_4 = i(\zeta + \zeta_2), \quad z + is_8 = i(\zeta' + \zeta_2) \qquad \text{(A-11)}$$

Substituting the above in I_1, I_3, I_1', I_3', we have

$$I_1(\zeta) = \int_0^\infty \frac{dk}{k}\left[e^{ik(\zeta-\zeta_1)} - e^{ik(\zeta-\zeta_2)} + e^{ik(\zeta+\zeta_1)} - e^{ik(\zeta+\zeta_2)}\right] \qquad \text{(A-12)}$$

$$I_3(\zeta) = \int_0^\infty \frac{dk}{k^2}\left[\quad " \quad \right] \qquad \text{(A-13)}$$

$$I_1'(\zeta') = \int_0^\infty \frac{dk}{k}\left[-e^{ik(\zeta'-\zeta_1)} + e^{ik(\zeta'-\zeta_2)} - e^{ik(\zeta'+\zeta_1)} + e^{ik(\zeta'+\zeta_2)}\right] \quad \text{(A-14)}$$

$$I_3'(\zeta') = \int_0^\infty \frac{dk}{k^2}\left[\qquad '' \qquad \right] \quad \text{(A-15)}$$

Consider the first integral $I_1(\zeta)$. $I_1(\zeta)$ in Eq.(A-12) is analytic in the lower half plane and $I_1(\zeta) \to 0$ for $\zeta \to i\infty$. Hence, differentiating with respect to ζ, we have

$$\frac{dI_1}{d\zeta} = i\int_0^\infty dk\left[e^{ik(\zeta-\zeta_1)} - e^{ik(\zeta-\zeta_2)} + e^{ik(\zeta+\zeta_1)} - e^{ik(\zeta+\zeta_2)}\right]. \quad \text{(A-16)}$$

Integration with respect to k is given by

$$\frac{dI_1}{d\zeta} = -\left(\frac{1}{\zeta-\zeta_1} - \frac{1}{\zeta-\zeta_2} + \frac{1}{\zeta+\zeta_1} - \frac{1}{\zeta+\zeta_2}\right). \quad \text{(A-17)}$$

Integrating with respect to ζ, we have

$$I_1 = -\left(\log\frac{\zeta-\zeta_1}{\zeta-\zeta_2} + \log\frac{\zeta+\zeta_1}{\zeta+\zeta_2}\right) + K \quad \text{(A-18)}$$

where K = integration constant.

Since both I_1 in (A-12) and the logarithmic terms in (A-18) vanish for $\zeta \to i\infty$, we have $K=0$.

Hence,

$$I_1 = -\log\left(\frac{\zeta-\zeta_1}{\zeta-\zeta_2}\ \frac{\zeta+\zeta_1}{\zeta+\zeta_2}\right) \quad \text{(A-19a)}$$

and

$$I_1\Big|_{z=0} = -\left(\log|s_1| + {0 \atop i\pi}\right) - \left(\log|s_3| + {0 \atop i\pi}\right) + \left(\log|s_2| + {0 \atop i\pi}\right) + \left(\log|s_4| + {0 \atop i\pi}\right),\ s_m \gtrless 0 \quad \text{(A-19b)}$$

A similar procedure is applied to I_3, I_1' and I_3':

$$I_3 = -i\left(-\zeta I_1 + \zeta_1 \log\frac{\zeta+\zeta_1}{\zeta-\zeta_1} + \zeta_2 \log\frac{\zeta-\zeta_2}{\zeta+\zeta_2}\right) \quad \text{(A-20a)}$$

$$I_3\Big|_{z=0} = -i\Big[-(x\cos\theta + y\sin\theta)\, I_1\Big|_{z=0} + \zeta_1\{(\log|s_3|+{}^{0}_{i\pi}) - (\log|s_1|+{}^{0}_{i\pi})\} + \zeta_2\{(\log|s_2|+{}^{0}_{i\pi}) - (\log|s_4|+{}^{0}_{i\pi})\}\Big], \quad s_m \gtrless 0 \qquad \text{(A-20b)}$$

$$I_1' = -\left(\log\frac{\zeta'-\zeta_2}{\zeta'-\zeta_1} + \log\frac{\zeta'+\zeta_2}{\zeta'+\zeta_1}\right) \qquad \text{(A-21a)}$$

$$I_1'\Big|_{z=0} = (\log|s_5|+{}^{0}_{i\pi}) + (\log|s_7|+{}^{0}_{i\pi}) - (\log|s_6|+{}^{0}_{i\pi}) - (\log|s_8|+{}^{0}_{i\pi}), \; s_m \gtrless 0 \qquad \text{(A-21b)}$$

$$I_3' = -i\left(-\zeta' I_1' + \zeta_1 \log\frac{\zeta'-\zeta_1}{\zeta'+\zeta_1} + \zeta_2 \log\frac{\zeta'+\zeta_2}{\zeta'-\zeta_2}\right) \qquad \text{(A-22a)}$$

$$I_3'\Big|_{z=0} = -i\Big[-(x\cos\theta - y\sin\theta)\, I_1'\Big|_{z=0} + \zeta_1\{(\log|s_5|+{}^{0}_{i\pi}) - (\log|s_7|+{}^{0}_{i\pi})\} + \zeta_2\{(\log|s_8|+{}^{0}_{i\pi}) - (\log|s_6|+{}^{0}_{i\pi})\}\Big], \quad s_m \gtrless 0 \qquad \text{(A-22b)}$$

Integration of I_2 and I_2'

I_2 (A-8) and I_2' (A-9) are identical with each other in form; therefore it is necessary to consider the integral

$$G = \int_{0\curvearrowright}^{\infty} \frac{e^{ik(s_1-iz)}}{k-k_j}\, dk \qquad \text{(A-23)}$$

Transforming the above by translation

$$w = k - k_j \qquad \text{(A-24)}$$

and rotation

$$u = -iw(s_1 - iz) \qquad \text{(A-25)}$$

we have

$$G = \int_{ik_j s_1 + k_j z}^{-i\infty(s_1 - iz)} \frac{e^{-u}}{u}\, du \quad . \tag{A-26}$$

The above is expressed by the exponential integral,[17]

$$G = e^{ik_j s_1 + k_j z} \left\{ E_1\left[ik_j s_1 + k_j z\right] - \frac{0}{i2\pi} \right\} \text{ for } s_1 \gtrless 0 \tag{A-27}$$

Hence for z=0,

$$I_2 = \sum_{m=1}^{4} (-1)^{m+1}\, e^{ik_j s_m} \left\{ E_1\left[ik_j s_m\right] - \frac{0}{i2\pi} \right\} \text{ for } s_m \gtrless 0 \quad . \tag{A-28}$$

$$I_2' = -\sum_{m=5}^{8} (-1)^{m+1}\, e^{ik_j s_m} \left\{ E_1\left[ik_j s_m\right] - \frac{0}{i2\pi} \right\} \text{ for } s_m \gtrless 0 \quad . \tag{A-29}$$

where $j = 1,2$.

CALCULATION OF I_6 AND I_6'

We calculate the following integrals which are used for evaluation of the volume in Eq.(59):

$$I_6 = \int_0^\infty \frac{dk}{k^3}\, e^{kz}\left(e^{iks_1} - e^{iks_2} + e^{iks_3} - e^{iks_4}\right)$$

$$I_6' = \int_0^\infty \frac{dk}{k^3}\, e^{kz}\left(-e^{iks_5} + e^{iks_6} - e^{iks_7} + e^{iks_8}\right) \tag{A-30}$$

The same procedure as that used previously in the Integration of I_1, I_1', I_3 and I_3' is used.

The I_6 and I_6' are similarly expressed in the following form:

$$I_6(\zeta) = \int_0^\infty \frac{dk}{k^3}\left[e^{ik(\zeta-\zeta_1)} - e^{ik(\zeta-\zeta_2)} + e^{ik(\zeta+\zeta_1)} - e^{ik(\zeta+\zeta_2)}\right]$$

$$I_6'(\zeta') = \int_0^\infty \frac{dk}{k^3}\left[-e^{ik(\zeta'-\zeta_1)} + e^{ik(\zeta'-\zeta_2)} - e^{ik(\zeta'+\zeta_1)} + e^{ik(\zeta'+\zeta_2)}\right] \tag{A-31}$$

For $\begin{matrix}\zeta \\ \zeta'\end{matrix} \to i\infty$, $\begin{matrix}I_6 \\ I_6'\end{matrix} \to 0$, and analytic in ζ and ζ'.

Differentiating $I_6(\zeta)$ and $I_6'(\zeta')$ three times with respect to ζ and ζ', and integrating them with respect to k, we have

$$\frac{d^3 I_6}{d\zeta^3} = \frac{1}{\zeta-\zeta_1} - \frac{1}{\zeta-\zeta_2} + \frac{1}{\zeta+\zeta_1} - \frac{1}{\zeta+\zeta_2}$$

$$\frac{d^3 I_6'}{d\zeta'^3} = -\frac{1}{\zeta'-\zeta_1} + \frac{1}{\zeta'-\zeta_2} - \frac{1}{\zeta'+\zeta_1} + \frac{1}{\zeta'+\zeta_2} \qquad \text{(A-32)}$$

Integration of the above derivatives with respect to ζ and ζ' yields each time a pair of integration constants. The first two pairs of constants vanish, whereas the last pair results are $\frac{1}{2}(\zeta_1^2-\zeta_2^2)$ and $-\frac{1}{2}(\zeta_1^2-\zeta_2^2)$. Hence, the pure constant terms in $I_6(\zeta)$ and $I_6'(\zeta')$ disappear and the I_6 and I_6' are finally expressed in the following form:

$$I_6\Big|_{z=0} = -\frac{1}{2}(x\cos\theta+y\sin\theta)^2 \, I_1\Big|_{z=0} - (x\cos\theta+y\sin\theta)\zeta_1\left[\left(\log|s_1|+{}^0_{i\pi}\right)-\left(\log|s_3|+{}^0_{i\pi}\right)\right]$$

$$+(x\cos\theta+y\sin\theta)\zeta_2\left[\left(\log|s_2|+{}^0_{i\pi}\right)-\left(\log|s_4|+{}^0_{i\pi}\right)\right]$$

$$+\frac{1}{2}\left[\zeta_1^2\left\{\left(\log|s_1|+{}^0_{i\pi}\right)+\left(\log|s_3|+{}^0_{i\pi}\right)\right\} -\right.$$

$$\left.-\zeta_2^2\left\{\left(\log|s_2|+{}^0_{i\pi}\right)+\left(\log|s_4|+{}^0_{i\pi}\right)\right\}\right] \quad \text{for } s_m \gtrless 0$$

$$I_6'\Big|_{z=0} = -\left\{\frac{1}{2}(x\cos\theta-y\sin\theta)^2 \, I_1'\Big|_{z=0} - (x\cos\theta-y\sin\theta)\zeta_1\left[\left(\log|s_5|+{}^0_{i\pi}\right)-\left(\log|s_7|+{}^0_{i\pi}\right)\right]\right.$$

$$+(x\cos\theta-y\sin\theta)\zeta_2\left[\left(\log|s_6|+{}^0_{i\pi}\right)-\left(\log|s_8|+{}^0_{i\pi}\right)\right]$$

$$+\frac{1}{2}\left[\zeta_1^2\left\{\left(\log|s_5|+{}^0_{i\pi}\right)+\left(\log|s_7|+{}^0_{i\pi}\right)\right\} -\right.$$

$$\left.\left.-\zeta_2^2\left\{\left(\log|s_6|+{}^0_{i\pi}\right)+\left(\log|s_8|+{}^0_{i\pi}\right)\right\}\right]\right\} \quad \text{for } s_m \gtrless 0 \; . \qquad \text{(A-33)}$$

APPENDIX B

CALCULATION OF $I_{2,\ell}$, $I'_{2,\ell}$, $J_{2,\ell}$, $J'_{2,\ell}$ (ℓ=1,2 ; j=3,4)

Integrating $I_{2,\ell}$ and $I'_{2,\ell}$ with respect to ξ and η in (43c) and (43d), and making use of (A-5) and (A-6), we have

$$I_{2,\ell} = \left(\frac{U}{k_j \sin\theta} + \frac{\sigma}{k_j^2 \sin\theta\cos\theta}\right)(I_1^* - I_4) + \frac{\sigma}{k_j \sin\theta\cos\theta} I_3^* \tag{B-1}$$

$$I'_{2,\ell} = -\left(\frac{U}{k_j \sin\theta} + \frac{\sigma}{k_j^2 \sin\theta\cos\theta}\right)(I_1'^* - I'_4) - \frac{\sigma}{k_j \sin\theta\cos\theta} I'^*_3 \tag{B-2}$$

$$J_{2,\ell} = \frac{\sigma}{ik_j \sin\theta} I_1^* - \left(\frac{U\cos\theta}{i\sin\theta} + \frac{\sigma}{ik_j \sin\theta}\right) I_4 \tag{B-3}$$

$$J'_{2,\ell} = -\frac{\sigma}{ik_j \sin\theta} I_1'^* + \left(\frac{U\cos\theta}{i\sin\theta} + \frac{\sigma}{ik_j \sin\theta}\right) I'_4 \tag{B-4}$$

where the * indicates the complex conjugate, and,

$$I_4 = \begin{cases} \int_0^\infty dk \, \dfrac{e^{kz}(\quad\quad'')}{k-k_3} \\ \int_0^\infty dk \, \dfrac{e^{kz}(\quad\quad'')}{k-k_4} \end{cases} \tag{B-5}$$

$$I'_4 = \begin{cases} \int_0^\infty dk \, \dfrac{e^{kz}(-e^{-iks_5} + e^{-iks_6} - e^{-iks_7} + e^{-iks_8})}{k-k_3} \\ \int_0^\infty dk \, \dfrac{e^{kz}(\quad\quad'')}{k-k_4} \end{cases} \tag{B-6}$$

Integration of I_4 and I_4'

It is apparent from (B-5) and (B-6) that an evaluation of two integrals is necessary:

$$G_1 = \int_{0\curvearrowright}^{\infty} \frac{dk}{k-k_3}\, e^{-ik(s_1+iz)}$$

$$G_2 = \int_{0\curvearrowright}^{\infty} \frac{dk}{k-k_4}\, e^{-ik(s_1+iz)} \qquad \text{(B-7)}$$

Transforming G_1 and G_2 by translation,

$$w = k - k_j \qquad \text{(B-8)}$$

and by rotation

$$u = iw(s_1+iz) \qquad \text{(B-9)}$$

we have,

$$\left.\begin{matrix} G_1 \\ G_2 \end{matrix}\right\} = e^{-ik_j s_1 + k_j z} \int_{-ik_j s_1 + k_j z}^{i\infty(s_1+iz)} \frac{e^{-u}}{u}\, du \qquad \text{(B-10)}$$

Thus, by applying the formula for exponential integral,[17] we have

$$G_1 = e^{-ik_3 s_1 + k_3 z} \left\{E_1[-ik_3 s_1 + k_3 z] - \begin{matrix} i2\pi \\ 0 \end{matrix}\right\},\ s_1 \gtrless 0$$

$$G_2 = e^{-ik_4 s_1 + k_4 z} \left\{E_1[-ik_4 s_1 + k_4 z] + \begin{matrix} 0 \\ i2\pi \end{matrix}\right\},\ s_1 \gtrless 0 \qquad \text{(B-11)}$$

Due to the above expressions (B-11), the I_4 and I_4' in (B-5) and (B-6) for $z=0$, are

$$I_4 = \begin{cases} \sum_{m=1}^{4} (-1)^{m+1}\, e^{-ik_3 s_m} \left\{E_1[-ik_3 s_m] - \begin{matrix} i2\pi \\ 0 \end{matrix}\right\},\ s_m \gtrless 0 \\ \sum_{m=1}^{4} (-1)^{m+1}\, e^{-ik_4 s_m} \left\{E_1[-ik_4 s_m] + \begin{matrix} 0 \\ i2\pi \end{matrix}\right\},\ s_m \gtrless 0 \end{cases} \qquad \text{(B-12)}$$

$$I_4' = \begin{cases} -\sum_{m=5}^{8} (-1)^{m+1}\, e^{-ik_3 s_m} \left\{E_1[-ik_3 s_m] - \begin{matrix} i2\pi \\ 0 \end{matrix}\right\},\ s_m \gtrless 0 \\ -\sum_{m=5}^{8} (-1)^{m+1}\, e^{-ik_4 s_m} \left\{E_1[-ik_4 s_m] + \begin{matrix} 0 \\ i2\pi \end{matrix}\right\},\ s_m \gtrless 0 \end{cases} \qquad \text{(B-13)}$$

APPENDIX C

CALCULATION OF INTEGRALS FOR COMPLEX k_3, k_4

We evaluate the integrals $I_5(k_\ell)$ and $I_5'(k_\ell)$ of the complex arguments in Eqs.(46 , (51) and (52).

Consider

$$G_3 = \int_0^\infty dk \; \frac{e^{-ik(s_1+iz)}}{k-k_j} \tag{C-1}$$

where k_j = complex, k_3 = complex conjugate of k_4 and $\mathrm{Im}k_3 < 0$. And $\mathrm{Re}k_j \gtrless 0$.

Translating and rotating by

$$w = k-k_j \quad \text{and} \quad u = iw(s_1+iz) \tag{C-2}$$

we have

$$G_3 = \int_{-ik_j s_1 + k_j z}^{i\infty(s_1+iz)} \frac{e^{-u}}{u} \, du \tag{C-3}$$

For k_3, with z=0, we have the following relations:

If $s_1 > 0$,

$$\mathrm{Re}(-ik_3 s_1) < 0$$

$$\mathrm{Im}(-ik_3 s_1) \lesseqgtr 0 \quad \text{for Re } k_3 \gtrless 0$$

If $s_1 < 0$,

$$\mathrm{Re}(-ik_3 s_1) > 0$$

$$\mathrm{Im}(-ik_3 s_1) \gtrless 0 \quad \text{for Re } k_3 \gtrless 0$$

For k_4, with z=0, we have the following relations:

If $s_1 > 0$,

$$\mathrm{Re}(-ik_4 s_1) > 0$$

$$\mathrm{Im}(-ik_4 s_1) \lesseqgtr 0 \quad \text{for } \mathrm{Re}\,k_4 \gtreqless 0$$

If $s_1 < 0$,

$$\mathrm{Re}(-ik_4 s_1) < 0$$

$$\mathrm{Im}(-ik_4 s_1) \gtreqless 0 \quad \text{for } \mathrm{Re}\,k_4 \gtreqless 0$$

In addition to the above for both k_3 and k_4 with z=0,

$$\mathrm{Re}(is_1\infty) = 0$$
$$\mathrm{Im}(is_1\infty) \gtreqless 0 \quad \text{for} \quad s_1 \gtreqless 0$$

Thus, applying formulas of exponential integrals,[17] we have the following results for z=0,

For $\mathrm{Re}\,k_3 \gtreqless 0$

$$G_3 = \begin{cases} e^{-ik_3 s_1}\{E_1[-ik_3 s_1] - \begin{matrix} i2\pi \\ 0 \end{matrix}\}, & s_1 \gtreqless 0 \\ e^{-ik_3 s_1}\{E_1[-ik_3 s_1] - \begin{matrix} 0 \\ 0 \end{matrix}\}, & s_1 \gtreqless 0 \end{cases} \qquad \text{(C-4)}$$

and for

$\mathrm{Re}\,k_4 \gtreqless 0$

$$G_3 = \begin{cases} e^{-ik_4 s_1}\{E_1[-ik_4 s_1] + \begin{matrix} 0 \\ i2\pi \end{matrix}\}, & s_1 \gtreqless 0 \\ e^{-ik_4 s_1}\{E_1[-ik_4 s_1] + \begin{matrix} 0 \\ 0 \end{matrix}\}, & s_1 \gtreqless 0 \end{cases} \qquad \text{(C-5)}$$

Thus for z=0, the I_5 and I_5' are given:

$\mathrm{Re}\,k_3 \gtreqless 0$

$$I_5 = \begin{cases} \sum_{m=1}^{4} (-1)^{m+1} e^{-ik_3 s_m}\{E_1[-ik_3 s_m] - \begin{matrix} i2\pi \\ 0 \end{matrix}\} & s_m \gtreqless 0 \end{cases} \qquad \text{(Cont'd)}$$

$$I_5 = \sum_{m=1}^{4} (-1)^{m+1} e^{-ik_3 s_m} \left\{ E_1[-ik_3 s_m] - \begin{matrix} 0 \\ 0 \end{matrix} \right\}, \quad s_m \gtrless 0$$

for $\underline{\mathrm{Re}k_4 \gtrless 0}$

$$I_5 = \begin{cases} \sum_{m=1}^{4} (-1)^{m+1} e^{-ik_4 s_m} \left\{ E_1[-ik_4 s_m] + \begin{matrix} 0 \\ i2\pi \end{matrix} \right\}, & s_m \gtrless 0 \\ \sum_{m=1}^{4} (-1)^{m+1} e^{-ik_4 s_m} \left\{ E_1[-ik_4 s_m] + \begin{matrix} 0 \\ 0 \end{matrix} \right\}, & s_m \gtrless 0 \end{cases} \qquad \text{(C-6)}$$

for $\underline{\mathrm{Re}k_3 \gtrless 0}$

$$I_5' = \begin{cases} -\sum_{m=5}^{8} (-1)^{m+1} e^{-ik_3 s_m} \left\{ E_1[-ik_3 s_m] - \begin{matrix} i2\pi \\ 0 \end{matrix} \right\}, & s_m \gtrless 0 \\ -\sum_{m=5}^{8} (-1)^{m+1} e^{-ik_3 s_m} \left\{ E_1[-ik_3 s_m] - \begin{matrix} 0 \\ 0 \end{matrix} \right\}, & s_m \gtrless 0 \end{cases}$$

for $\underline{\mathrm{Re}k_4 \gtrless 0}$

$$I_5' = \begin{cases} -\sum_{m=5}^{8} (-1)^{m+1} e^{-ik_4 s_m} \left\{ E_1[-ik_4 s_m] + \begin{matrix} 0 \\ i2\pi \end{matrix} \right\}, & s_m \lessgtr 0 \\ -\sum_{m=5}^{8} (-1)^{m+1} e^{-ik_4 s_m} \left\{ E_1[-ik_4 s_m] + \begin{matrix} 0 \\ 0 \end{matrix} \right\}, & s_m \gtrless 0 \end{cases} \qquad \text{(C-7)}$$

APPENDIX D

EVALUATION OF EXPONENTIAL INTEGRAL OF COMPLEX ARGUMENT

To calculate the exponential integral of a complex argument $E_1[-ik_\ell s_m]$ as given in Eqs.(51) and (52), we put

$$z = -ik_\ell s_m \tag{D-1}$$

where k_ℓ represents a complex wave number k_3 or k_4 and the s_m has been given in Eq.(A-5), Appendix A.

The exponential integral is given by[17]

$$E_1[z] = -\gamma - \log z - \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n \cdot n!} \qquad (|\arg z| < \pi) \tag{D-2}$$

Let the above z be given in polar coordinates

$$z = r\, e^{i\alpha} \tag{D-3}$$

$$\text{with } r = |-ik_\ell s_m| \ , \ \alpha = \tan^{-1} \frac{\mathrm{Im}(-ik_\ell s_m)}{\mathrm{Re}(-ik_\ell s_m)} \tag{D-4}$$

Substituting z Eq.(D-3) into Eq.(D-2), we have

$$E_1[-ik_\ell s_m] = -C(r,\alpha) - iS(r,\alpha) \tag{D-5}$$

where

$$C(r,\alpha) = \gamma + \log r + \sum_{n=1}^{\infty} \frac{(-1)^n r^n \cos n\alpha}{n \cdot n!}$$

$$S(r,\alpha) = \alpha + \sum_{n=1}^{\infty} \frac{(-1)^n r^n \sin n\alpha}{n \cdot n!} \tag{D-6}$$

where $\gamma = 0.5772$.

Introducing a new variable β ,

$$\beta = \alpha + \pi \tag{D-7}$$

we have

$$z^n = (-1)^n [r^n \cos n\beta + i\, r^n \sin n\beta] \tag{D-8}$$

Hence, Eq.(D-5) is given by

$$E_1[-ik_\ell s_m] = -C(r,\beta) - iS(r,\beta) \tag{D-9}$$

where

$$C(r,\beta) = \gamma + \log r + \sum_{n=1}^{\infty} \frac{r^n \cos n\beta}{n \cdot n!}$$

$$S(r,\beta) = (\beta - \pi) + \sum_{n=1}^{\infty} \frac{r^n \sin n\beta}{n \cdot n!} \tag{D-10}$$

The exponential integral of a complex argument $-ik_\ell s_m$ ($k_\ell = k_{\ell_r} + k_{\ell_i}$) in Eq.(D-1) approaches the exponential integral of a pure imaginary argument $-ik_{\ell_r} s_m$ when $k_{\ell_i} \to 0$ ($k_{\ell i} \gtrless 0$):

$$\lim_{k_{\ell_i} \to 0} E_1[-ik_\ell s_m] = E_1[-ik_{\ell_r} s_m] = -Ci(k_{\ell_r} s_m) - i[Si(k_{\ell_r} s_m) - \frac{\pi}{2}] \quad \text{for } k_{\ell_r} > 0, s_m > 0 \tag{D-11}$$

where Ci and Si represent the cosine and sine integral, respectively.

Evaluation of the exponential integral of pure imaginary argument is carried out by making use of the standard subroutine "SICI"[6] which has been given in the Scientific Subroutine Package. The program computes the sine and cosine integrals

$$Si(x) \quad \text{and} \quad Ci(x) \quad , \quad x > 0$$

and thus by making use of the above, we compute the exponential integral

$$E_1(ix) = -Ci(x) + i[Si(x) - \frac{\pi}{2}] \quad , \; x > 0$$

and $E_1(-ix)$ with $x > 0$ by the complex conjugate of $E_1(ix)$.

Evaluation of Eqs.(51) and (52)

Since

$$e^{-ik_{\ell}s_m} = e^{k_{\ell_i}s_m}\, e^{-ik_{\ell_r}s_m} \tag{D-12}$$

we have

$$e^{-ik_{\ell}s_m} E_1[-ik_{\ell}s_m] = -e^{k_{\ell_i}s_m}\{[\cos(k_{\ell_r}s_m)C(r,\beta)+\sin(k_{\ell_r}s_m)S(r,\beta)] + i[-\sin(k_{\ell_r}s_m)C(r,\beta)+\cos(k_{\ell_r}s_m)S(r,\beta)]\} \tag{D-13}$$

$$-i2\pi e^{-ik_3 s_m} = -2\pi e^{k_{3_i}s_m}[\sin(k_{3_r}s_m)+i\cos(k_{3_r}s_m)] \quad \text{for } k_{3_r} > 0, \quad s_m > 0 \tag{D-14}$$

and

$$i2\pi e^{-ik_4 s_m} = 2\pi e^{k_{4_i}s_m}[\sin(k_{4_r}s_m)+i\cos(k_{4_r}s_m)] \quad \text{for } k_{4_r} > 0, \quad s_m < 0 \tag{D-15}$$

Substituting Eqs.(D-13) to (D-15) in (51) and (52), one obtains the expressions for $I_5(k_{\ell}) - I_5'(k_{\ell})$.

$$e^{-ik_3 s_m}\{E_1[-ik_3 s_m] - {}^{i2\pi}_{0}\}\ ,\ s_m \gtrless 0\ ,\ \text{Re}\ k_3 > 0$$

$$= -e^{k_{3_i}s_m}\{\cos(k_{3_r}s_m)C(r,\beta) + \sin(k_{3_r}s_m)[S(r,\beta)+ {}^{2\pi}_{0}]\}$$

$$-ie^{k_{3_i}s_m}\{-\sin(k_{3_r}s_m)C(r,\beta)+ \cos(k_{3_r}s_m)[S(r,\beta)+{}^{2\pi}_{0}]\} \tag{D-16}$$

$$e^{-ik_4 s_m}\{E_i[-ik_4 s_m] + i{}^{0}_{2\pi}\},\ s_m \gtrless 0\ ,\ \text{Re}\ k_4 > 0$$

$$= -e^{k_{4_i}s_m}\{\cos(k_{4_r}s_m)C(r,\beta) + \sin(k_{4_r}s_m)[S(r,\beta)- {}^{0}_{2\pi}]\}$$

$$-ie^{k_{4_i}s_m}\{-\sin(k_{4_r}s_m)C(r,\beta)+ \cos(k_{4_r}s_m)[S(r,\beta)- {}^{0}_{2\pi}]\} \tag{D-17}$$

Since Eq.(D-13) is formally identical to the complex conjugate of $(\varphi_{or} + i\psi_{or})$ in the Appendix of Reference 19, and since the formula gives expressions for both small and large r for a rapid numerical calculation, the expression of the $\varphi_{or}+i\psi_{or}$ has been used.

For small r, Eq.(D-13) remains unchanged.

For large r, Eq.(D-13) takes the following form

$$e^{-ik_{\ell}s_m} E_1[-ik_{\ell}s_m] = -\sum_{n=1}^{N} (n-1)! \frac{\cos n\beta}{r^n}$$

$$+i\sum_{n=1}^{N} (n-1)! \frac{\sin n\beta}{r^n} \qquad \text{(D-18)}$$

where N = *truncation number.*

Integration of Eqs. 46, 59 and 60

Integration with respect to θ in (46) (59) and (60) has been carried out numerically by employing the trapezoidal rule. For the singular behavior which appears in the neighborhood of $\theta = \frac{\pi}{2}$, a change of variable, adopted in Reference 2, has been employed for convenience.

$$\theta' = \frac{\sin\theta}{\cos^2\theta}$$

which gives

$\theta' \to 0$ for $\theta \to 0$

$\theta' \to \infty$ for $\theta \to \frac{\pi}{2}$

indicating the singular behavior has been shifted to $\theta' \to \infty$ at which the integrand tends to zero.

For the integration with respect to θ', the upper limit has been truncated at 77 as being sufficient.

In the neighborhood of the critical point θ_c, however, when the τ values nearly satisfy the relation (29), singular behavior occurs also when $k_3 - k_4 \to 0$ as it is seen from Eq. (26b):

$$\frac{1}{k_3 - k_4} = - \frac{1}{k_o \sec^2\theta \sqrt{1 - 4\tau\cos\theta}}$$

$$(1 - 4\tau \cos\theta) \to 0 \quad \text{for} \quad \theta \to \theta_c$$

In these cases the Lagrange interpolation method[18] has been employed. The integration has been assumed to have the form

$$I = \int_{\theta_c - 2\delta}^{\theta_c + 2\delta} \frac{M(\theta)}{(\theta - \theta_c)^3} \, d\theta$$

where $\delta = 0.00033$ has been finally adopted and $M(\theta) = (\theta - \theta_c)^3$. [Integrand of Eq. (46) or (59) or (60)].

APPENDIX E

FLOW DIAGRAM AND PROGRAM SYSTEM

With the following input,

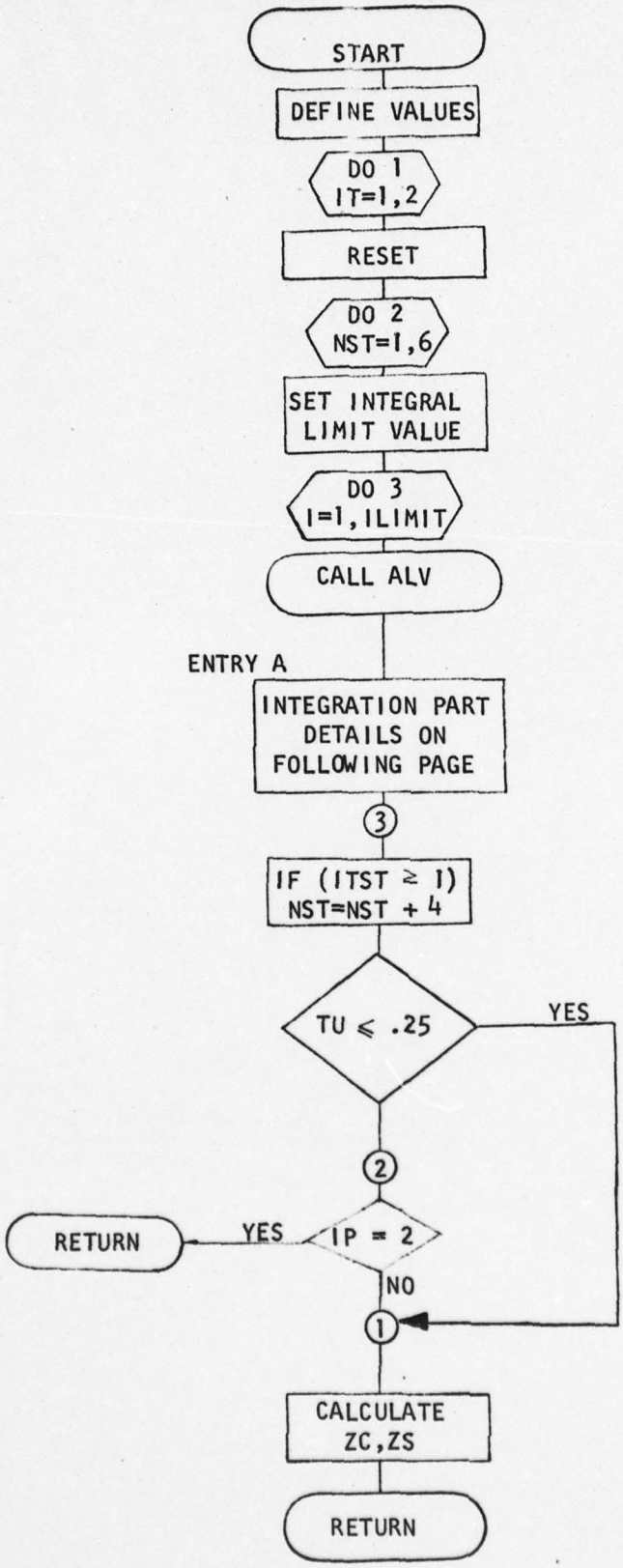
IN PROGRAM	IN TEXT	
TU	τ	= reduced frequency
FAN	$\left\|\frac{\partial Q_f}{\partial p}\right\|$	= fan discharge rate for a pair of fans
SK	k	= nozzle flow factor
ROW	ρ_w	= water density
SELA	A_o	= seal leakage at the equilibrium
BH	h	= cushion height
DS	Δ	= displacement
XNF	n	= number of pair of fans
SL	L	= nominal ship length
SB	B	= nominal ship beam
A	a	= half cushion length
B	b	= half cushion width
FN	F_n	= Froude number

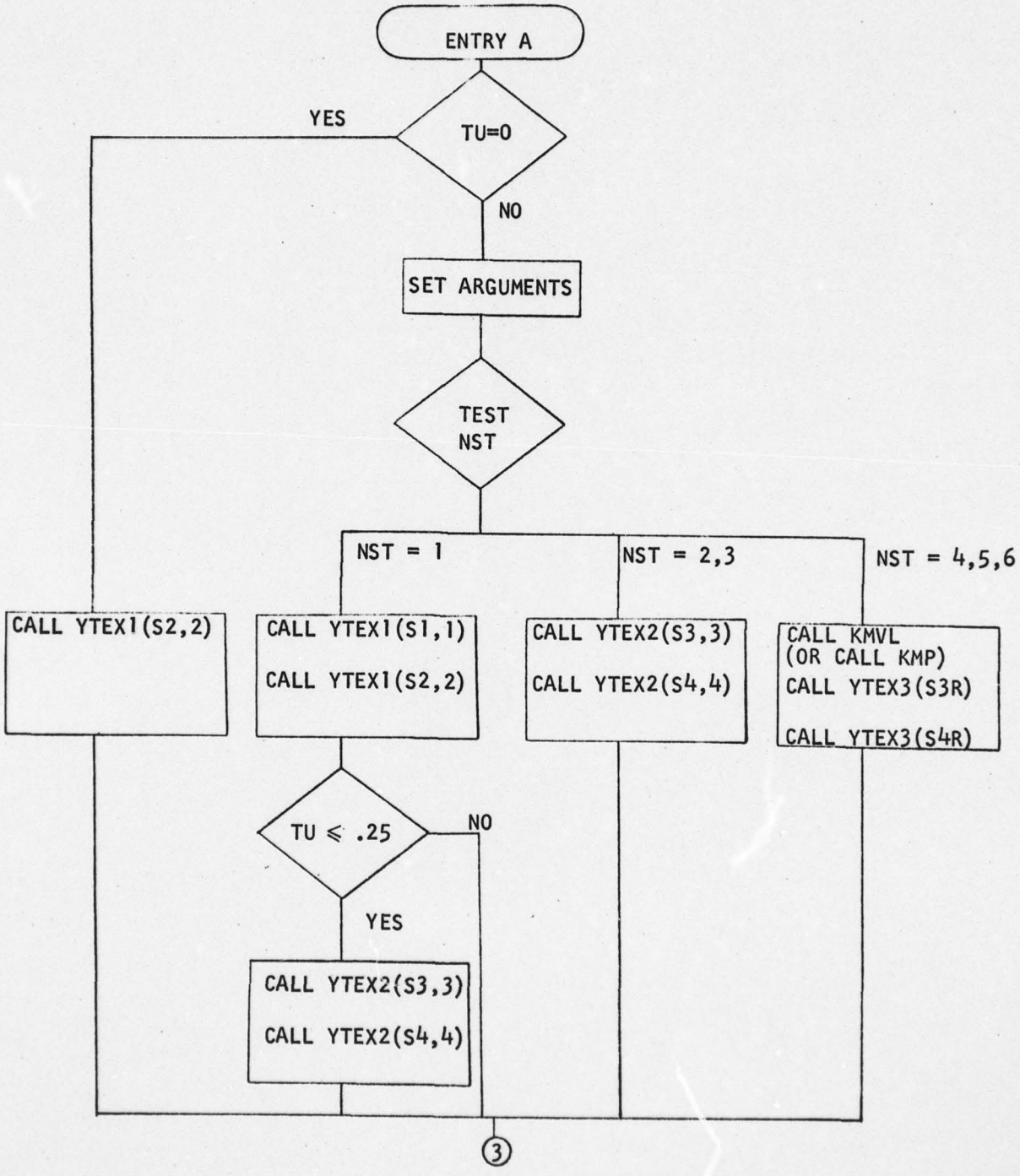
one obtains the following results:

RV	v_r	= real part of the volume coefficient
IV	v_i	= imaginary part of the volume coefficient
RA	α_r	= real part of the escape area coefficient
IA	α_i	= imaginary part of the escape area coefficient
AMP	$\left\|\frac{P_b}{P_o}\Big/\frac{z}{h}\right\|$	= amplitude of the bubble pressure coefficient
PHA	phase lag	= phase lag of the bubble pressure
ADM	C_a	= added mass coefficient
DAM	C_d	= damping coefficient

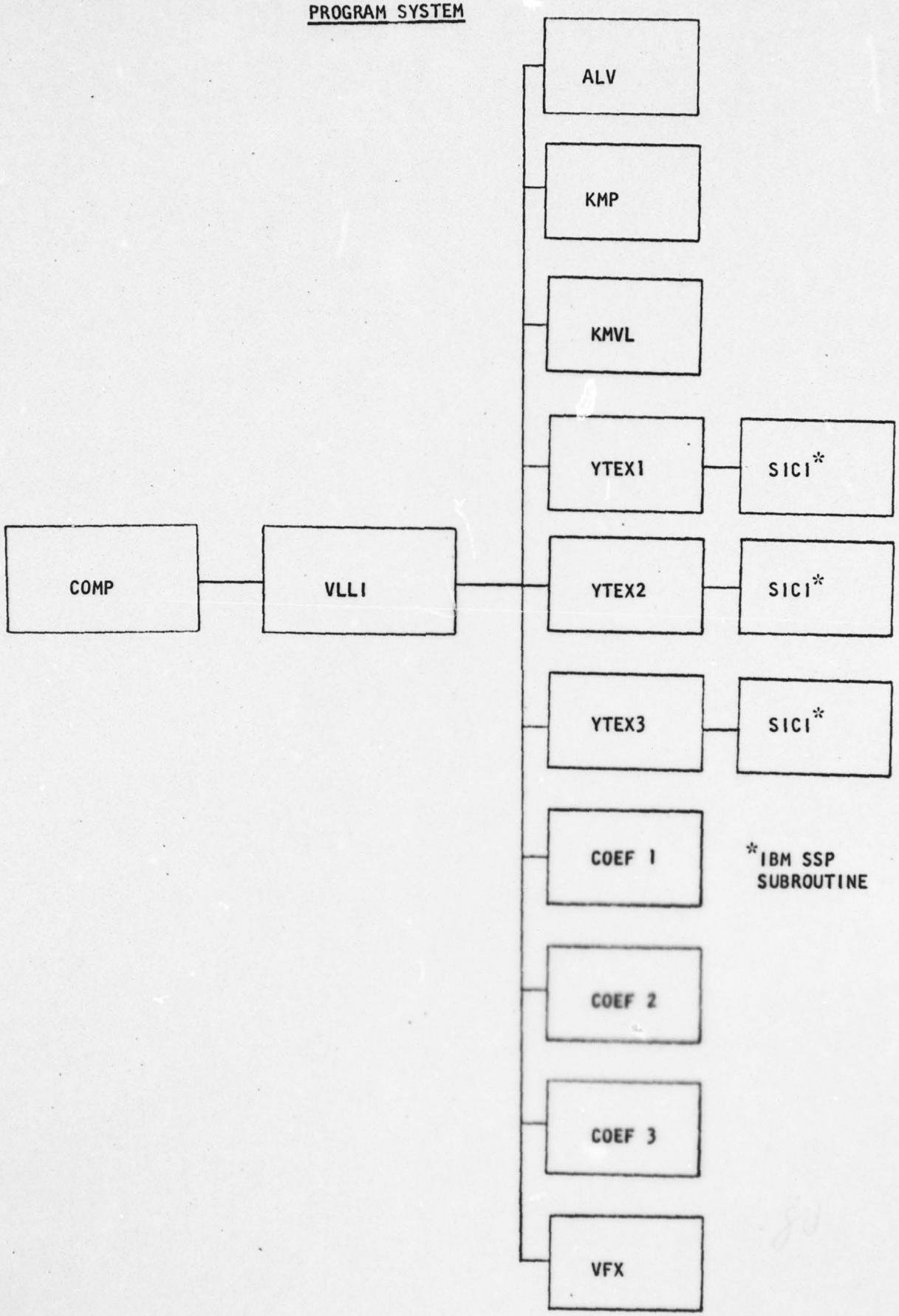
Appendix E includes three sections: Flow Diagram and Program System for Subroutine VLLI, INPUT-OUTPUT SAMPLE and PROGRAM LISTING.

FLOW DIAGRAM (SUBROUTINE VLLI)




ENTRY A
TU=0
YES
NO
SET ARGUMENTS
TEST NST
NST = 1
NST = 2,3
NST = 4,5,6
CALL YTEX1(S2,2)
CALL YTEX1(S1,1)
CALL YTEX1(S2,2)
CALL YTEX2(S3,3)
CALL YTEX2(S4,4)
CALL KMVL
(OR CALL KMP)
CALL YTEX3(S3R)
CALL YTEX3(S4R)
TU ≤ .25
NO
YES
CALL YTEX2(S3,3)
CALL YTEX2(S4,4)
3

PROGRAM SYSTEM



APPENDIX E
(Cont'd)

INPUT OUTPUT SAMPLE

```
 RUN ACVP

 >>  DEFINITIONS
     RV = REAL PART VOLUME
     IV = IMAGE PART VOLUME
     RA = REAL PART ESCAPE AREA
     IA = IMAGE PART ESCAPE AREA
     AMP= BUBLE PRESSURE AMPLITUDE
     PHA= BUBLE PRESSURE PHASE
     ADM= ADDED MASS COEFFICIENT
     DAM= DAMPING COEFICIENT

   >> INPUT FAN,SK,ROW,SELA,BH,DS,XNF
0.03 0.65 1.94 0.333 0.72 298. 2.

   >> INPUT SL,SB,A,B,FN
13.83 2.12 6.915 1.06 0.72

   >> INPUT TU, IF ANY. OTHERWISE KEY IN -1.
10.

 TU =     10.00  RV =      -0.073  IV =      0.001
                 RA =      -0.064  IA =      0.003
                 AMP=       8.439  PHA=   -158.138
                 ADM= -.2813E-02   DAM=  .2177E-01

   >> INPUT TU, IF ANY. OTHERWISE KEY IN -1.
-1.

END OF EXECUTION
```

APPENDIX E
(Cont'd)

PROGRAM LISTING

```
00100    CCCC     MAIN PROGRAM FOR COMPUTION OF VOLUM FLUX AND
00200    CCCC     ESCAPE AREA OF ACV HEAVING IN UNIFORM STREAM
00300    CCCC     IN ADDITION CALCULATE BUBLE PRESSURE, ADDED MASS
00400    CCCC     AND DAMPING COEFFICIENTS
00500    CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
00600             PROGRAM COMP
00700             COMMON /V1/SL,SB,A,B,FN,TU,EPSI,GAM,BTA
00800             COMMON /V2/BET,ZT1,ZT2,ZT3,ZT4
00900             COMMON /V3/ ALF1,ALF2,ALF3,ALF4,NST12,NST13,ILM
01000             COMMON /V4/NTS1,NTS2,NTS3,KT1,KT2,KT3,KT12,KT13
01100             COMMON /C1/ EPS(3),GAB(3)
01200             DATA EPS/.001,.0001,.0001/,GAB/.03325,.011,.011/
01300             DATA BTA,ZT1,ZT2,ZT3,ZT4/.05,.04,.06,.1,2./
01400             DATA ALF1,ALF2,NTS1,NTS2,NTS3,KT1/.033,.05,20,20,30,10/
01500             DATA GAMA,PA,ROA,G/1.4,2048.,0.00237,32.2/
01600             ALF3=ZT3
01700             ALF4=ZT4
01800             KT2=NTS2
01900             KT3=NTS3
02000    CCCC     SL = NOMINAL SHIP LENGTH
02100    CCCC     SB = NOMINAL SHIP BEAM
02200    CCCC     A  = HALF CUSHION LENGTH
02300    CCCC     B  = HALF CUSION WIDTH
02400    CCCC     FN = FROUD NUMBER
02500    CCCC     FAN = FAN DISCHARGE RATE FOR PAIR OF FANS
02600    CCCC     XNF = NUMBER OF PAIR OF FANS
02700    CCCC     SELA = SEAL LEAKAGE AT THE EQUILIBRIUM
02800    CCCC     BH = BUBLE HEIGHT
02900    CCCC     ROW = WATER DENSITY
03000    CCCC     DS = DISPLACEMENT
03100    CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
03200             TYPE 600
03300    600      FORMAT('  >>  DEFINITIONS'//'      RV = REAL PART VOLUME'
/
03400          1           '      IV = IMAGE PART VOLUME'/
03500          2           '      RA = REAL PART ESCAPE AREA'/
03600          3           '      IA = IMAGE PART ESCAPE AREA'/
03700          4           '      AMP= BUBLE PRESSURE AMPLITUDE'/
03800          5           '      PHA= BUBLE PRESSURE PHASE'/
03900          6           '      ADM= ADDED MASS COEFFICIENT'/
04000          7           '      DAM= DAMPING COEFICIENT'/)
04100             TYPE 300
04200    300      FORMAT('    >> INPUT FAN,SK,ROW,SELA,BH,DS,XNF')
04300             ACCEPT 100,FAN,SK,ROW,SELA,BH,DS,XNF
04400             TYPE 400
04500    400      FORMAT('    >> INPUT SL,SB,A,B,FN')
04600             ACCEPT 100,SL,SB,A,B,FN
04700             BL=2.*A
04800             BB=2.*B
04900    10       CONTINUE
05000             TYPE 500
05100    500      FORMAT('    >> INPUT TU, IF ANY. OTHERWISE KEY IN -1.')
05200             ACCEPT 100,TU
05300             IF(TU.LT.0.) CALL EXIT
05400    100      FORMAT(7F)
```

```
05500           VR=0.
05600           VI=0.
05700           CALL VLLI(VR,VI,1)
05800           AI=0.
05900           AR=0.
06000           CALL VLLI(AI,AR,2)
06100           IF(TU.EQ.0.) GOTO 20
06200           SP=FN*SQRT(BL*G)
06300           SIG=IU*G/SP
06400           SIG2=SIG*SIG
06500           FC=SK/SIG
06600           BL2=BL*BL
06700           CA=BL*BB
06800           VOL=CA*BH
06900           PO=DS/CA
07000           POVO=PO/VOL
07100           FCTOR=PO/BH*CA
07200           COF=.5*ROW*BL2*BL
07300           COP=.5*ROW*SP*BL2
07400           C1=GAMA*(1.+PA/PO)
07500           C2=SQRT(2.*PO/ROA)
07600           C3=SQRT(2.*PO*ROA)
07700           C4=BL2/ROW/G
07800           C5=BL/ROW/G
07900           QR=VR*C4+FC*AI*C5*C2
08000           QI=VI*C4-FC*AR*C5*C2+FC*SELA/C3+XNF*FAN/SIG
08100           DLQR=POVO*QR
08200           DLQI=POVO*QI
08300           PB=(1.-C1*DLQR)**2+(C1*DLQI)**2
08400           PBR=-C1*(1.-C1*DLQR)/PB
08500           PBI=-C1*C1*DLQI/PB
08600           ADMC=PBR*FCTOR/SIG2/COF
08700           DAMC=-PBI*FCTOR/SIG/COF
08800           AMP=SQRT(PBR*PBR+PBI*PBI)
08900           PHAS=ATAN2(PBI,PBR)*57.29577
09000           TYPE 200,TU,VR,VI,AR,AI,AMP,PHAS,ADMC,DAMC
09100   200     FORMAT(2X,'TU = 'F8.2,'  RV = 'F10.3,'  IV ='F10.3/
09200          1          15X,'  RA = 'F10.3,'  IA ='F10.3/
09300          2       15X,'  AMP= 'F10.3,'  PHA= 'F10.3/
09400          3          15X,'  ADM= 'E10.4,'  DAM= 'E10.4/)
09500           GOTO 10
09600   20      AMP=0.
09700           PHAS=-90.
09800           TYPE 200,TU,VR,VI,AR,AI,AMP,PHAS
09900           GOTO 10
10000           END
```

```
00100    CCCC     SUBPROGRAM FOR COMPUTATION OF
00200    CCCC     VOLUME FLUX, IF IP = 1
00300    CCCC     ESCAPE AREA, IF IP = 2
00400             SUBROUTINE VLLI(ZC,ZS,IP)
00500             DIMENSION THR(3),THL(3),FRR(3),FIR(3),FRL(3),FIL(3)
00600             COMMON/MYTEX/ SM(8)
00700             DIMENSION ZCC(2),ZSS(2),FTR(3),FTI(3)
00800             COMMON/LOG/C1,C2,R1,R2
00900             COMMON /V1/SL,SB,A,B,FN,TU,EPSI,GAM,BTA
01000             COMMON /V2/BET,ZT1,ZT2,ZT3,ZT4
01100             COMMON /V3/ALF1,ALF2,ALF3,ALF4,NTS12,NTS13,ILM
01200             COMMON /V4/NTS1,NTS2,NTS3,KT1,KT2,KT3,KT12,KT13
01300             COMMON /CF1/ TV,S1,S2,S3,S4,SGC2,SGC0,NST,FDT,CN,SN,CSN
01400             COMMON /CF11/ SK05,TTM,TM
01500             COMMON /CF2/ EP(4),EI(4),EPD(4),EID(4),EPR(4,8),EPI(4,8)
01600             COMMON /C1/ EPSS(3),GAB(3)
01700             DATA X11,X44,X15/11.,144.,315./
01800             ZCC(1)=0.
01900             ZCC(2)=0.
02000             ZSS(2)=0.
02100             ZSS(1)=0.
02200             P=3.14159
02300             TP=2.*P
02400             P2=P*P
02500             SL2=SL*SL
02600             SBSL=.5*SB/SL
02700             ABL2=4.*A*B/SL2
02800             G=32.2
02900             SLMN=77.
03000             BLF=0.000333
03100             EPSI=EPSS(IP)
03200             GAM=GAB(IP)
03300             BET=GAM
03400             LP=3
03500             LP1=LP+1
03600             X=-A
03700             Y=B-EPSI
03800             GAM3=3.*GAM
03900             KT12=KT1+KT2
04000             KT13=KT12+KT3
04100             RMA=SLMN-BET-ZT1*(KT1-1)-ZT2*KT2-ZT3*KT3
04200             MOO=RMA/ZT4
04300             RMB=MOO*ZT4
04400             DELTA=RMA-RMB
04500             KT14=KT13+MOO
04600             KT15=KT14+1
04700             NTS12=NTS1+NTS2
04800             NTS13=NTS12+NTS3
04900             IF(TU.LT..25) GO TO 127
05000             GC=1./(4.*TU)
05100             TCR=ACOS(GC)
05200             TVCR=SIN(TCR)/COS(TCR)**2
05300             EPS=LP*BLF
05400             RTA=SLMN-TVCR-EPS-ALF1*(NTS1-1)-ALF2*NTS2-ALF3*NTS3
05500             LO=RTA/ALF4
```

```
05600            RTB=LO*ALF4
05700            DELTB=RTA-RTB
05800            NTS14=NTS13+LO
05900            NTS15=NTS14+1
06000            RA=TVCR-EPS-GAM3
06100            LQM=RA/BTA
06200            RB=LQM*BTA
06300            DEL=RA-RB
06400            LQM1=LQM+1
06500            SMA=TVCR-BET-ZT1*(KT1-1)-ZT2*KT2-ZT3*KT3
06600            LOO=SMA/ZT4
06700            SMB=LOO*ZT4
06800            DELTC=SMA-SMB-.005
06900            LT14=KT13+LOO
07000            LT15=LT14+1
07100            ITST=2
07200            KT115=KT15
07300            IF(TVCR.GT.77.) ITST=1
07400            IF(TVCR.LE.66.) ITST=0
07500            IF(ITST.EQ.2) KT115=LT15
07600    127     CONTINUE
07700            SFC=1./(180.*BLF**2)
07800            SFCC=1./(180.*GAM**2)
07900            SP=FN*SQRT(G*SL)
08000            SIG=TU*G/SP
08100            SIG2=SIG**2
08200            SP2=SP**2
08300            FN2=FN**2
08400            SGC2=G/SP2
08500            FCT=4.*P2*SP2*SL2
08600            IF(IP.EQ.2) FCT=FCT/SL
08700            FCTR=FN**4/(2.*P2)
08800            IF(IP.EQ.2) FCTR=2.*P2*SL
08900    C       'IT' STANDS FOR CONTR REPET OF INTGR FOR X=-X:GIVES RSUL
T
09000            DO 1 IT=1,2
09100            CT3=0.
09200            CT4=0.
09300            SII=0.
09400            SIR=0.
09500            ZC1=0.
09600            ZS1=0.
09700            DO 2 NST=1,6
09800            IF(NST.EQ.1.OR.NST.EQ.6) GOTO 2216
09900            IF(ITST.GE.1) GOTO 2
10000    2216    CONTINUE
10100            IF(NST.EQ.1) TV=BET
10200            IF(NST.EQ.2) TV=BLF+TVCR
10300            IF(NST.EQ.3) TV=TVCR+EPS+BET
10400            IF(NST.EQ.4) TV=TVCR-EPS
10500            IF(NST.EQ.5) TV=GAM
10600            IF(NST.EQ.6) TV=TV+DEL-GAM
10700            IF(NST.EQ.6.AND.ITST.GE.1) TV=GAM
10800            IF(NST.EQ.2) THR(1)=TV
10900            IF(NST.EQ.4) THL(1)=TV
11000            IF(NST.EQ.1) ILM=KT15
11100
```

```
11200           IF(NST.EQ.2.OR.NST.EQ.4) ILM=3
11300           IF(NST.EQ.3) ILM=NTS15
11400           IF(NST.EQ.5) ILM=3
11500           IF(NST.EQ.6.AND.ITST.EQ.0) ILM=LQM1
11600           IF(NST.EQ.6.AND.ITST.GE.1) ILM=KT115
11700           DO 3 I=1,ILM
11800           IN=I+1
11900           CALL COEF1(TU)
12000           IF(IP.EQ.2) CSN=CSN/SGC2/CN
12100           IF(IP.EQ.2) SGCO=SGCO*CN
12200           FDTSG=FDT*SGCO
12300   122     CONTINUE
12400           S21=S2-S1
12500           S43=S4-S3
12600           R1=X*CN+Y*SN
12700           R2=X*CN-Y*SN
12800           C1=A*CN+B*SN
12900           C2=A*CN-B*SN
13000           SM(1)=R1-C1
13100           SM(2)=R1-C2
13200           SM(3)=R1+C1
13300           SM(4)=R1+C2
13400           SM(5)=R2-C1
13500           SM(6)=R2-C2
13600           SM(7)=R2+C1
13700           SM(8)=R2+C2
13800           CALL ALV(IP,Q1,Q2,Q3,Q4,Q5,Q6)
13900           IF(TU.EQ.0.) GO TO 100
14000           S1S=S1**2
14100           S2S=S2**2
14200           S3S=S3**2
14300           S4S=S4**2
14400           S1Q=S1S*S1
14500           S2Q=S2S*S2
14600           S3Q=S3S*S3
14700           S4Q=S4S*S4
14800           F3=SIG2/(S1*S2)
14900           FC33=F3
15000           F4=(SP*S1*CN-SIG)**2/(-S1Q*S21)
15100           F5=(SP*S2*CN-SIG)**2/(-S2Q*S21)
15200           F1=-SGCO*(SIG-SP*S1*CN)**2/(S1S*S21)
15300           F2=SGCO*(SIG-SP*S2*CN)**2/(S2S*S21)
15400           IF(TU.LE..25) GOTO 1231
15500           IF(NST.GE.4.OR.NST.EQ.1.OR.S43.LT..1E-7) GOTO 123
15600   1231    FC3=SIG2/(S3*S4)
15700           FC4=(SP*S3*CN+SIG)**2/(-S3Q*S43)
15800           FC5=(SP*S4*CN+SIG)**2/(-S4Q*S43)
15900           IF(IP.EQ.2)  F3=SGCO*(SIG+SP*S3*CN)**2/(S3S*S43)
16000           IF(IP.EQ.2) F4=-SGCO*(SIG+SP*S4*CN)**2/(S4S*S43)
16100           IF(IP.EQ.2) FC4=-SIG2*SGCO/(S3*S4)
16200   123     CONTINUE
16300           IF(IP.EQ.2) FC3=SIG2*SGCO/(S1*S2)
16400           IF(NST.EQ.2.OR.NST.EQ.3) GOTO 19
16500           IF(NST.GE.4) GOTO 33
16600           CALL YTEX1(S1,1,EPR,EPI)
16700           CALL YTEX1(S2,2,EPR,EPI)
16800           IF(TU.LE..25) GOTO 66
```

AD-A070 886 STEVENS INST OF TECH HOBOKEN N J DAVIDSON LAB F/G 13/10
ADDED MASS AND DAMPING OF THE HEAVING SURFACE EFFECT SHIP IN UN--ETC(U)
DEC 78 C H KIM, S TSAKONAS N00014-77-C-0061
UNCLASSIFIED SIT-DL-78-9-2040 NL

2 OF 2
AD
A070886

END
DATE
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8-79
DDC

```
16900            DELAC=DELTA
17000            CALL COEF2(1,2)
17100            IF(IP.EQ.2) GOTO 133
17200            CALL VFX(F3,F4,F5,Q1,Q2,Q3,Q4,Q5,Q6,1,2,YC1,YS1)
17300            GOTO 305
17400    133     CONTINUE
17500            YC1=F1*(EP(1)+EPD(1))+F2*(EP(2)+EPD(2))
17600            YS1=F1*(EI(1)+EID(1))+F2*(EI(2)+EID(2))
17700            YC1=YC1-(F1+F2)*Q1+FC3*Q3
17800            YS1=YS1-(F1+F2)*Q2+FC3*Q4
17900            YC1=YC1*FDT
18000            YS1=YS1*FDT
18100            GOTO 305
18200    19      CALL YTEX2(S3,3,EPR,EPI)
18300            CALL YTEX2(S4,4,EPR,EPI)
18400            CALL COEF2(3,4)
18500            IF(IP.EQ.2) GOTO 191
18600            CALL VFX(FC3,FC4,FC5,Q1,Q2,Q3,Q4,Q5,Q6,3,4,YC1,YS1)
18700            GOTO 192
18800    191     CONTINUE
18900            YC1=F3*(EP(3)+EPD(3))+F4*(EP(4)+EPD(4))
19000            YS1=F3*(EI(3)+EID(3))+F4*(EI(4)+EID(4))
19100            YC7=-(F3+F4)*Q1+FC4*Q3
19200            YS7=(F3+F4)*Q2-FC4*Q4
19300            YC1=(YC1+YC7)*FDT
19400            YS1=(YS1+YS7)*FDT
19500    192     CONTINUE
19600            IF(NST.EQ.3) GOTO 506
19700            FRR(I)=YC1*(THR(I)-TVCR)**3
19800            FIR(I)=YS1*(THR(I)-TVCR)**3
19900            ALF=BLF
20000            GOTO 306
20100    506     CONTINUE
20200            DELAC=DELTB
20300            GOTO 305
20400    33      TM=-TM
20500            CTU=TM**0.5
20600            S3R=SK05*TTM
20700            S4I=SK05*CTU
20800            S4R=S3R
20900            S3I=-S4I
21000            IF(IP.EQ.2) GOTO 335
21100            CALL KMVL(S4R,S4I,SIG,SP,CN,FD1,FD2,FD3,FD4R,FD4I)
21200            GOTO 336
21300    335     CALL KMP(S4R,S4I,SGCO,SIG,SP,CN,FK1,FK2,AC,BD)
21400    336     CONTINUE
21500            CALL YTEX3(S3R,S3I,3,EPR,EPI)
21600            CALL YTEX3(S4R,S4I,4,EPR,EPI)
21700            CALL COEF2(3,4)
21800            IF(IP.EQ.2) GOTO 337
21900            YC3=FD4R*(EP(3)+EPD(3)+EP(4)+EPD(4))-FD4I*(EI(3)
22000           1 +EID(3)-EI(4)-EID(4))
22100            YS3=FD4R*(EI(3)+EID(3)+EI(4)+EID(4))+FD4I*(EP(3)
22200           1 +EPD(3)-EP(4)-EPD(4))
22300            YC8=(FD1*Q1+FD2*Q3+FD3*Q5)
22400            YS8=(-FD1*Q2-FD2*Q4-FD3*Q6)
22500            YC1=(YC3+YC8)*FDTSG
```

```
22600            YS1=(YS3+YS8)*FDTSG
22700            GOTO 338
22800    337     YC3=AC*(EP(3)+EPD(3))-BD*(EI(3)+EID(3))+
22900         1       AC*(EP(4)+EPD(4))+BD*(EI(4)+EID(4))
23000            YS3=BD*(EP(3)+EPD(3))+AC*(EI(3)+EID(3))
23100         1             -BD*(EP(4)+EPD(4))+AC*(EI(4)+EID(4))
23200            YC8=-FK1*Q1+FK2*Q3
23300            YS8=FK1*Q2-FK2*Q4
23400            YC1=(YC3+YC8)*FDT
23500            YS1=(YS3+YS8)*FDT
23600    338     CONTINUE
23700            IF(NST.GE.5) GOTO 508
23800            FRL(I)=YC1*(THL(I)-TVCR)**3
23900            FIL(I)=YS1*(THL(I)-TVCR)**3
24000            ALF=BLF
24100            GOTO 306
24200    508     CONTINUE
24300            IF(ITST.GE.1) GO TO 125
24400            IF(NST.EQ.6) GOTO 136
24500            THR(I)=SIN(TV)**3
24600            FTR(I)=YC1*THR(I)
24700            FTI(I)=YS1*THR(I)
24800            ALF=GAM
24900            GOTO 306
25000    136     CONTINUE
25100            DELAC=BTA
25200            GOTO 305
25300    125     CONTINUE
25400            IF(ITST.EQ.1) DELAC=DELTA
25500            IF(ITST.EQ.2) DELAC=DELTC
25600            GOTO 305
25700    C       COMPUTATION OF V-FLX FOR TU<.25 IN BELOW
25800    66      CONTINUE
25900            CALL YTEX2(S3,3,EPR,EPI)
26000            CALL YTEX2(S4,4,EPR,EPI)
26100            CALL COEF2(1,4)
26200            DELAC=DELTA
26300            IF(IP.EQ.2) GOTO 665
26400            YC1=F4*(EP(1)+EPD(1))-F5*(EP(2)+EPD(2))
26500            YS1=F4*(EI(1)+EID(1))-F5*(EI(2)+EID(2))
26600            YC2=+FC4*(EP(3)+EPD(3))-FC5*(EP(4)+EPD(4))
26700            YS2=+FC4*(EI(3)+EID(3))-FC5*(EI(4)+EID(4))
26800            YC1=YC1+(-F4+F5)*Q1+(-F4*S1+F5*S2)*Q3+F3*Q5
26900            YS1=YS1+(-F4+F5)*Q2+(-F4*S1+F5*S2)*Q4+F3*Q6
27000            YC2=YC2+(-FC4+FC5)*Q1+(-FC4*S3+FC5*S4)*Q3+FC3*Q5
27100            YS2=YS2-(-FC4+FC5)*Q2-(-FC4*S3+FC5*S4)*Q4-FC3*Q6
27200            YC1=(YC1+YC2)*FDTSG
27300            YS1=(YS1+YS2)*FDTSG
27400            GOTO 305
27500    665     CONTINUE
27600            YC1=F1*(EP(1)+EPD(1))+F2*(EP(2)+EPD(2))
27700            YS1=F1*(EI(1)+EID(1))+F2*(EI(2)+EID(2))
27800            YC2=F3*(EP(3)+EPD(3))+F4*(EP(4)+EPD(4))
27900            YS2=F3*(EI(3)+EID(3))+F4*(EI(4)+EID(4))
28000            YC1=YC1-(F1+F2)*Q1+FC3*Q3
28100            YS1=YS1-(F1+F2)*Q2+FC3*Q4
28200            YC2=YC2-(F3+F4)*Q1+FC4*Q3
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28300               YS2=YS2+(F3+F4)*Q2-FC4*Q4
28400               YC1=(YC1+YC2)*FDT
28500               YS1=(YS1+YS2)*FDT
28600               GOTO 305
28700     100       CONTINUE
28800     C         CAL OF VLL FOR TAU=0.
28900               CALL YTEX1(S2,2,EPR,EPI)
29000               CALL COEF2(2,2)
29100               YC1=CSN*FCTR*FCT*FDT*(EP(2)+EPD(2)-S2*Q3-Q1)
29200               IF(IP.EQ.2) YS1=CSN*FDT*(EI(2)+EID(2)-Q2)
29300               IF(IP.EQ.2) YC1=0.
29400               DELAC=DELTA
29500     305       CONTINUE
29600               IF(I.EQ.1.OR.I.EQ.ILM) YC1=0.5*YC1
29700               IF(I.EQ.1.OR.I.EQ.ILM) YS1=0.5*YS1
29800               IF(I.EQ.1) ALF=BET
29900               IF(I.EQ.1.AND.ITST.EQ.0.AND.NST.EQ.6) ALF=DEL
30000               ZC1=ZC1+YC1*ALF
30100               ZS1=ZS1+YS1*ALF
30200               NST1=NST
30300               ALF=ALFC(IN,NST1,DELAC,ITST)
30400     306       TV=TV+ALF
30500               IF(NST.EQ.2) THR(I+1)=TV
30600               IF(NST.EQ.4) THL(I+1)=TV
30700     3         CONTINUE
30800     991       FORMAT(2X,2E10.4)
30900               IF(TU.LE.0.25) GOTO 11
31000               IF(NST.EQ.5)CT3=SFCC*(-X11*FTR(3)+X44*FTR(2)-X15*FTR(1))
31100               IF(NST.EQ.5)CT4=SFCC*(-X11*FTI(3)+X44*FTI(2)-X15*FTI(1))
31200               IF(NST.NE.6) GOTO 2
31300               SIR1=X11*(FRL(1)-FRR(3))
31400               SIR2=-X44*(FRL(2)-FRR(2))
31500               SIR3=X15*(FRL(3)-FRR(1))
31600               SIR=SFC*(SIR1+SIR2+SIR3)
31700               SII1=X11*(FIL(1)-FIR(3))
31800               SII2=-X44*(FIL(2)-FIR(2))
31900               SII3=X15*(FIL(3)-FIR(1))
32000               SII=SFC*(SII1+SII2+SII3)
32100     2         CONTINUE
32200     11        ZC=(ZC1+SIR+CT3)/FCT
32300               ZS=(ZS1+SII+CT4)/FCT
32400               IF(TU.EQ.0..AND.IP.EQ.2) ZS=ZS1/FCTR
32500               X=-X
32600               ZCC(IT)=ZC
32700               ZSS(IT)=ZS
32800               IF(IP.EQ.2) GOTO 99
32900     1         CONTINUE
33000     99        CONTINUE
33100               ZC=(ZCC(2)-ZCC(1))*2.-ABL2
33200               ZS=(ZSS(2)-ZSS(1))*2.
33300               IF(IP.EQ.2) ZS=ZS-SBSL
33400               IF(IP.EQ.2) ZC=-ZC-ABL2
33500               RETURN
33600               END
```

```
00100            SUBROUTINE KMP(S4R,S4I,SGCO,SIG,SP,CN,FK1,FK2,
00200          1 AC,BD)
00300    3       FORMAT(6E12.4)
00400            XX=2.*SP*SIG*CN
00500            SIG2=SIG**2
00600            S4R2=S4R**2
00700            S4I2=S4I**2
00800            XY2=S4R2+S4I2
00900            XP=2.*S4R
01000            XY4=XY2**2
01100            FK2=-SGCO*SIG2/XY2
01200            AA=(SP*S4R*CN+SIG)**2-(SP*S4I*CN)**2
01300            BB=2.*(SP*S4R*CN+SIG)*SP*S4I*CN
01400            DD=2.*S4I*(S4R2-S4I2)
01500            CC=4.*S4R*S4I2
01600            CCDD=CC**2+DD**2
01700            AC=(AA*CC-BB*DD)/CCDD*SGCO
01800            BD=-(BB*CC+AA*DD)/CCDD*SGCO
01900            FK1=2.*AC
02000            RETURN
02100            END
02200            SUBROUTINE ALV(IP,Q1,Q2,Q3,Q4,Q5,Q6)
02300            COMMON/MYTEX/ SM(8)
02400            DIMENSION ASM(8),AL(8),G1(8)
02500            COMMON/LOG/C1,C2,R1,R2
02600    C       Q1=RE(I1+I1D),Q2=IM(I1+I1D),Q3=RE(I3+I3D),Q4=IM(I3+I3D)
02700    C       Q5=RE(I9+I9D),Q6=IM(I9+I9D)
02800            PI=3.14159
02900            DO 1  M=1,8
03000            ASM(M)=ABS(SM(M))
03100            IF(ASM(M))  3,3,4
03200    4       AL(M)=ALOG(ASM(M))
03300            GO TO 1
03400    3       CONTINUE
03500    1       CONTINUE
03600            AP1=-AL(1)+AL(2)-AL(3)+AL(4)
03700            AP2=AL(5)-AL(6)+AL(7)-AL(8)
03800            AP=AP1+AP2
03900            AP3=AL(3)-AL(1)
04000            AP4=AL(2)-AL(4)
04100            AP5=AL(5)-AL(7)
04200            AP6=AL(8)-AL(6)
04300            DO 31 K=1,8
04400            IF(SM(K).LT.0.) G1(K)=PI
04500            IF(SM(K).GE.0.) G1(K)=0.
04600    31      CONTINUE
04700            AQ1=G1(3)-G1(1)
04800            AQ2=G1(2)-G1(4)
04900            AQ3=G1(5)-G1(7)
05000            AQ4=G1(8)-G1(6)
05100            SII1=-G1(1)+G1(2)-G1(3)+G1(4)
05200            SII1D=G1(5)-G1(6)+G1(7) -G1(8)
05300            Q2=SII1+SII1D
05400            SII3=R1*AP1-C1*AP3-C2*AP4
05500            SII3D=R2*AP2-C1*AP5-C2*AP6
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05600            Q4=SII3+SII3D
05700            Q1=AP
05800            SRI3=-R1*SII1+C1*AQ1+C2*AQ2
05900            SRI3D=-R2*SII1D+C1*AQ3+C2*AQ4
06000            Q3=SRI3+SRI3D
06100            IF(IP.EQ.2) RETURN
06200            SR9=-.5*R1**2*AP1-R1*C1*(AL(1)-AL(3))+R1*C2*(AL(2)-AL(4)
06300           1 )+0.5*C1**2*(AL(1)+AL(3))-0.5*C2**2*(AL(2)+AL(4))
06400            SI9=-.5*R1**2*SII1-R1*C1*(G1(1)-G1(3))
06500           1 +R1*C2*(G1(2)-G1(4))+0.5*C1**2*(G1(1)+G1(3))-.5*C2**2*
06600           2 (G1(2)+G1(4))
06700            SR9D=-0.5*R2**2*AP2+R2*C1*(AL(5)-AL(7))-R2*C2*(AL
06800           1 (6)-AL(8))-0.5*C1**2*(AL(5)+AL(7))+0.5*C2**2*(AL(6)+AL(8
                 ))
06900            SI9D=-.5*R2**2*SII1D+R2*C1*(G1(5)-G1(7))-R2*C2*(G1(6)
07000           1 -G1(8))-0.5*C1**2*(G1(5)+G1(7))+.5*C2**2*(G1(6)+G1(8))
07100            Q5=SR9+SR9D
07200            Q6=SI9+SI9D
07300            RETURN
07400            END
07500            SUBROUTINE KMVL(S4R,S4I,SIG,SP,CN,A1,A2,A3,A4,A5)
07600       C    COMPUTES COEF OF KOMPLEX WAVE NUMBERS
07700       C    A1=FD1,A2=FD2,A3=FD3,A4=FD4R,A5=FD4I
07800            SIG2=SIG**2
07900            SP2=SP**2
08000            CN2=CN**2
08100            X=S4R
08200            Y=S4I
08300            X2=X**2
08400            X3=X2*X
08500            Y2=Y**2
08600            Y3=Y2*Y
08700            Y4=Y3*Y
08800            XY2=X2+Y2
08900            XY2P=XY2**2
09000            XY3P=XY2**3
09100            XZ=XY2*2.*X
09200            XY=3.*X2-Y2
09300            A=SP*SIG*CN
09400            B=SP2*CN2
09500            U=-6.*X2*Y2+2.*Y4
09600            V=2.*Y*X3-6.*X*Y3
09700            W=U**2+V**2
09800            G=(SP*X*CN+SG)**2-(SP*Y*CN)**2
09900            H=2.*(SP*X*CN+SIG)*SP*Y*CN
10000            A3=SIG2/XY2
10100            A4=(G*U+H*V)/W
10200            A5=(G*V-H*U)/W
10300            A2=-2.*(A4*X+A5*Y)
10400            A1=-2.*A4
10500            RETURN
10600            END
10700            SUBROUTINE YTEX1(SK,K,EPR,EPI)
10800            COMMON/MYTEX/ SM(8)
10900            DIMENSION EPR(4,8),EPI(4,8)
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```
11000    12      FORMAT(/2X,' C,S,CI,SI'/)
11100    13      FORMAT(10E12.4)
11200    14      FORMAT(/2X,'SKM'/)
11300            P=3.14159
11400            TP=P*2.
11500            DO 6 M=1,8
11600            SKM=ABS(SK*SM(M))
11700            C=COS(SKM)
11800            S=SIN(SKM)
11900            CALL SICI(SI,CI,SKM)
12000            IF(SM(M)) 7,8,9
12100    7       EPR(K,M)=-C*CI-S*SI - TP*S
12200            EPI(K,M)=S*CI-C*SI - TP*C
12300            GO TO 6
12400    8       CONTINUE
12500            GO TO 6
12600    9       EPR(K,M)=-C*CI-S*SI
12700            EPI(K,M)=-S*CI+C*SI
12800    6       CONTINUE
12900    11      FORMAT(/3X,'SINGULAR'/)
13000            RETURN
13100            END
13200            SUBROUTINE YTEX2(SK,K,EPR,EPI)
13300            COMMON/MYTEX/ SM(8)
13400            DIMENSION EPR(4,8),EPI(4,8)
13500    12      FORMAT(/2X,'C,S,CI,SI'/)
13600    13      FORMAT(10E12.4)
13700    14      FORMAT(/2X,'SKM'/)
13800            P=3.14159
13900            TP=2.*P
14000            DO 6 M=1,8
14100            SKM=ABS(SK*SM(M))
14200            C=COS(SKM)
14300            S=SIN(SKM)
14400            CALL SICI(SI,CI,SKM)
14500            IF(SM(M)) 7,8 , 9
14600    7       EPR(K,M)=-C*CI-S*SI
14700            EPI(K,M)=-S*CI+C*SI
14800            IF(K.EQ.4) GO TO 19
14900            GO TO 6
15000    8       CONTINUE
15100            GO TO 6
15200    19      EPR(K,M)=EPR(K,M)-S*TP
15300            EPI(K,M)=EPI(K,M)+C*TP
15400            GO TO 6
15500    9       EPR(K,M)=-C*CI-S*SI
15600            EPI(K,M)=S*CI-C*SI
15700            IF(K.EQ.3) GO TO 18
15800            GO TO 6
15900    18      EPR(K,M)=EPR(K,M)-TP*S
16000            EPI(K,M)=EPI(K,M)-TP*C
16100    6       CONTINUE
16200    11      FORMAT(/2X,'SINGL'/)
16300            RETURN
16400            END
```

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16500            SUBROUTINE YTEX3(SR,SI,K,EPR,EPI)
16600            DIMENSION EPR(4,8),EPI(4,8)
16700            COMMON/MYTEX/ SM(8)
16800    2       FORMAT(1X,10G10.4)
16900            P=3.14159
17000            TP=P*2.
17100            IT=0
17200            IF(K.EQ.4.AND.SR.GT.0.) IT=3
17300            IF(K.EQ.3.AND.SR.GT.0.) IT=1
17400            DO 60 M=1,8
17500            IF(SM(M).EQ.0.) GO TO 60
17600            X=-SR*SM(M)
17700            XY=-X
17800            Y=SI*SM(M)
17900            X2=X**2
18000            Y2=Y**2
18100            AX=ABS(X)
18200            BA=(X2+Y2)**0.5
18300            C=COS(AX)
18400            S=SIN(AX)
18500            IF(Y.GE.80.) Y=80.
18600            IF(Y.LE.-80.) Y=-80.
18700            E=EXP(Y)
18800            BB=BA
18900            AT=ATAN2(X,Y)
19000            BC=AT+P
19100            IF(SR.EQ.0.) BC=P
19200            BE=0.
19300            BD=0.
19400            IF(BA-6.) 28,30,30
19500    28      MM=1
19600    25      BD=BD+BB*COS(FLOAT(MM)*BC)
19700            BE=BE+BB*SIN(FLOAT(MM)*BC)
19800            BB=BB*BA*FLOAT(MM)/FLOAT((MM+1)**2)
19900            MM=MM+1
20000            IF(BB.GT.0.1E-5*BA) GO TO 25
20100            BD=(0.577+ALOG(BA)+BD)
20200            BE=(BE+AT)
20300            GO TO 26
20400    30      DO 31 MM=1,5
20500            BD=BD+COS(FLOAT(MM)*BC)/BB
20600            BE=BE+SIN(FLOAT(MM)*BC)/BB
20700            BB=BA*BB/FLOAT(MM)
20800    31      CONTINUE
20900    26      CONTINUE
21000            IF(SR.EQ.0.) GO TO 69
21100            GAS=TP*E*S
21200            GAC=TP*E*C
21300            IF(BA-6.) 70,77,77
21400            IF(BA-6.) 70, 77, 77
21500    70      GA=E*(C*(-BD)+SIGN(S,XY)*(-BE))
21600            GB=E*(-SIGN(S,XY)*(-BD)+C*(-BE))
21700            IF(K.EQ.3.AND.SR.GT.0..AND.SM(M).GT.0.) GA=GA-GAS
21800            IF(K.EQ.3.AND.SR.GT.0..AND.SM(M).GT.0.) GB=GB-GAC
21900            IF(K.EQ.4.AND.SR.GT.0..AND.SM(M).LT.0.) GA=GA-GAS
```

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22000           IF(K.EQ.4.AND.SR.GT.0..AND.SM(M).LT.0.) GB=GB+GAC
22100           GO TO 80
22200    77     IF(IT.EQ.1.AND.SM(M).GT.0.) GO TO 92
22300           IF(IT.EQ.3.AND.SM(M).LT.0.) GO TO 93
22400           GO TO 94
22500    92     GA=-BD-E*TP*S
22600           GB=BE-E*TP*C
22700           GO TO 80
22800    93     GA=-BD-E*TP*S
22900           GB=BE+E*TP*C
23000           GO TO 80
23100    94     GA=-BD
23200           GB=BE
23300           GO TO 80
23400    69     CONTINUE
23500           GA=-E*BD
23600           GB=0.
23700    80     EPR(K,M)=GA
23800           EPI(K,M)=GB
23900    60     CONTINUE
24000    99     FORMAT(/2X,'SINFULAR'/)
24100           RETURN
24200           END
```

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