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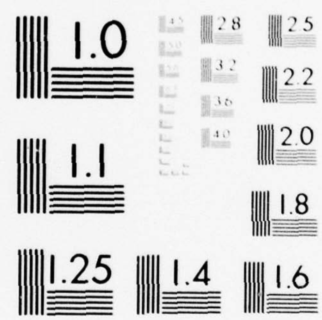

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STATISTICAL PROBLEMS IN THE CONTROL OF MULTI-ECHELON INVENTORY SYSTEMS

by

S. Zacks

Serial T-404
15 June 1979

The George Washington University
School of Engineering and Applied Science
Institute for Management Science and Engineering

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Problems of the robustness of adaptive control procedures against deviations of the empirical phenomena from the assumed models are of high practical and theoretical importance. The present paper studies the robustness of Bayes adaptive control of two-echelon inventory systems relative to erroneous assumptions concerning (i) the initial control parameters; (ii) the stationarity of the demand distribution and (iii) the nature of the demand distribution.

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Abstract
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Problems of the robustness of adaptive control procedures against deviations of the empirical phenomena from the assumed models are of high practical and theoretical importance. The present paper studies the robustness of Bayes adaptive control of two-echelon inventory systems relative to erroneous assumptions concerning (i) the initial control parameters; (ii) the stationarity of the demand distribution and (iii) the nature of the demand distribution.

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*Case Western Reserve University

1. Introduction

In a series of papers on two-echelon multi-station inventory systems [3,4,6,7] optimal ordering procedures were developed for systems modeled towards possible naval applications. In these models we assumed that ordering at the lower-echelon stations and at the upper-echelon depot is periodically done (at the beginning of every month) and that the lead times are fixed, one month from the depot to the lower echelon stations and two months from the wholesale supply centers to the depot. We further assumed that the monthly demand at each lower-echelon station is a random variable having a Poisson distribution, stationary in time. Moreover, the demand variables at different lower-echelon stations are mutually independent. The means of the corresponding demand distributions were assumed, however, to be unknown. Adaptive Bayes ordering policies were developed in [4,6] and proved in [7] to be optimal for the specific cost functions and objectives specified. These Bayes procedures were based on the assumption that the unknown means of the Poisson distributions, $\lambda_1,\lambda_2,\ldots,\lambda_k$, are priorly independent having prior gamma distributions, $\mathcal{G}(\frac{1}{\tau_i},\nu_i)$, with scale parameters τ_i and shape parameters ν_i, $i=1,\ldots,k$. The assumption about the prior independence of $\lambda_1,\ldots,\lambda_k$ was essential to the development. On the other hand, the assumption concerning the prior gamma distribution of the λ_i $(i=1,\ldots,k)$ is not essential but reasonable, since the gamma distribution is a conjugate prior to the Poisson distribution (see [1]). Moreover, the gamma priors yield negative-binomial predictive distributions (see [2]), which have been found empirically to fit well the demand distributions of many classes of items in the Polaris system (see [8]). The present study investigates certain robustness or sensitivity questions related to these Bayes control procedures. First, it is interesting to evaluate the sensitivty of the system to the choice of the initial prior parameters. Second, it is important to investigate the effects on the system of possible changes in the intensity of the demand distributions (the Poisson means) which may occur at unknown time points. Third, it is worthwhile to study the robustness of the control procedures to deviations of the demand distributions from the assumed Poisson distributions. Assuming that the demand distributions are indeed Poisson, or very close to Poisson, we may expect that the control procedures will be efficient if the choice of the prior parameters (τ_i, ν_i), $i = 1, \ldots, k$ is such that the prior mean $\nu_i \tau_i$ is close to the true value of λ_i and the prior variance $\nu \tau_i^2$, is small. The Bayes procedures are optimal in the sense of minimizing the prior risk under the assumed prior parameters. However, for the sensitivity analytis we compare the procedures by simulating certain realizations of the systems over a large number of periods, and comparing the moving averages of the actually accured costs. If the demand process is stationary the sequence of moving averages is ergodic, converging to the steady-state expected monthly cost. If the demand process is not stationary the sequence of moving averages will fluctuate and will reflect this lack of stability. However, we will demonstrate that even in such non-stationary cases, the Bayes adaptive control performs better than the optimal control which assumes a complete knowledge of the demand distribution. One can probably attain a similar degree of effectiveness by non-Bayes adaptive procedures, like the ones based on the maximum-likelihood estimation of the unknown parameters. We have not, however, compared Bayes and non-Bayes procedures. Finally, for the purpose of showing the possible effect on the moving-averages of the monthly costs of pronounced deviation of the demand distribution from the Poisson, we studied the behavior of the Bayes control procedure when the demand distribution is a compound Poisson, with mean at each month randomly determined according to a uniform distribution on some finite range. The general conclusion derived from the present investigation is that the Bayes adaptive procedures adjust quite fast to possible changes in the underlying demand distributions and generally provide effective controls. We provided in [4,6] also an approximate Bayes procedures. We show that these approximate procedures is effective only if the initial state of the system is not too unfavorable. The reader is refered to Section 6 for summary and conclusions based on certain simulations. In order to reduce somewhat the amount of computations needed for our investigation we have restricted attention to one-station two-echelon system, which will be specifically described in the next section.

2. The Two-Echelon System and its Statistical Control.

The system under consideration is comprised of a lower-echelon station, E, and an upper-echelon depot, D. Customers arrive at random at E and demand a random number of units of a specified item. Let X_i $(i=1,2,\dots)$ designate the total number of items demanded from E during the i-th month. It is assumed that X_i has a Poisson distribution with mean λ_i $(i=1,2,\dots)$, and $X_1, X_2, \dots$ are mutually independent, given $\lambda_1, \lambda_2, \dots$. Let Q_i denote the number of units in stock at E at the beginning of the i-th month. At the beginning of the i-th month E issues an order from D for Y_i items. $-Q_i \le Y_i \le S_i - Q_i$, where S_i denotes the total number of units at the system. A negative Y means that items are sent back to D. The lead-time for the flow of stock between E and D is $L_1 = 1$ month. At the beginning of the i-th month the depot, D, issues an order of Z_i units to be purchased from outside sources. It is assumed that the lead-time for this replenishment is $L_2 = 2$ months. The cost of procurement of new units is C*[$/unit]. An item demanded at E can be supplied to the customer at any time during the month. Unsatisfied demand at the end of a month is lost, with penalty for shortage of p[\$/unit]; (no backlogging). Items which are left at E undemanded accrue a carrying cost of C[\$/item]. The variables $(Q_i, S_i, X_i, Y_i, Z_{i-1}, Z_i)$, $i = 1,\dots,k$, designate the state of the system during the i-th month. The following recursive relations hold:

$$
\begin{aligned}
Q_{i+1} &= (Q_i + Y_i - X_i)^+ \\
S_{i+1} &= (S_i + Z_{i-1} - Y_i)^+ ,
\end{aligned}
\qquad i = 0,1,\dots \tag{2.1}
$$

where S_0 and Q_0 are the initial stock levels, $X_0 \equiv 0$, $Y_0 \equiv 0$, Z_{-1} and Z_0 are the prior pending procurement orders, and $a^+ = \max(0,a)$.

In [4, 6] we suggested to determine Y_i in a manner which minimizes the expected cost to the lower-echelon, due to surplus or shortages during the (i+1)st month. Furthermore, it was suggested that the ordering policy of the upper-echelon will be the minimal required for guaranteeing that the probability of shortage at the D will not exceed a specified risk level β, $0 < \beta < 1$. As shown in [4], if $\lambda_1 = \lambda_2 = \cdots = \lambda$, then the "desired" ordering level of E is $Y^0(\lambda, Q) = k^0(\lambda) - Q$, where

$$
k^0(\lambda) = \underline{\text{least non-negative integer}}\ k \ \underline{\text{satisfying}}:\ \mathrm{Pos}(k|2\lambda) - R \cdot \mathrm{Pos}(k|\lambda) \geq 0 , \tag{2.2}
$$

where $R = p/(c+p)$ and $\mathrm{Pos}(k|\mu)$ designates the c.d.f. of a Poisson distribution with mean μ.

It is easy to compute $k^0(\lambda)$. However, if λ is unknown and one applies the Bayes adaptive procedure, with prior gamma $\mathcal{G}(\frac{1}{\tau}, \nu)$ distribution then, the "desired" ordering level at the beginning of the i-th month is $K^0(T_{i-1}) - Q_i$, where $T_{i-1} = \sum_{i=0}^{i-1} X_i$, and

(2.3) $$K_i^0(T_{i-1}) = \underline{\text{least non-negative integer}}\ k$$
$$\underline{\text{satisfying}}:\ G(k|2\psi_{i+1},\nu + T_{i-1}) - RG(k|\psi_i,\nu + T_{i-1}) \geq 0 \quad ,$$

in which $G(k|\psi,\nu)$ designates the c.d.f. of a negative-binomial distribution with p.d.f.

(2.4) $$g(j|\psi,\nu) = \frac{\Gamma(\nu+j)}{\Gamma(j+1)\,\Gamma(\nu)}\,\psi^j(1-\psi)^\nu \quad , \quad j = 0,1,\ldots$$

$0 < \psi < 1$, and where $\psi_i = \tau/(1 + i\tau)$. The "desired" ordering level cannot be always attained since S_i may be smaller than $K^0(T_{i-1})$. We thus define

(2.5) $$Y_i^0 = \min(K^0(T_{i-1}),S_i) - Q_i \ , \quad i = 1,2,\ldots \ .$$

The upper-echelon ordering policy is defined as the least non-negative integer Z such that the predictive probability of $\{(S_i + Z_{i-1} - X_i)^+ - X_{i+1} + Z \geq K_{i+2}(T_{i-1} + X_i + X_{i+1})\}$, given T_{i-1}, is at least $\gamma = 1-\beta$. We have established in [4] that the Bayes adaptive ordering policy of the upper echelon satisfying the above requirement is

(2.6) $$Z_i^0(T_{i-1}) = \underline{\text{least non-negative integer}}\ Z\ \underline{\text{satisfying}}:$$
$$G(Z - K_i^0(T_{i-1})|\psi_i,\nu + T_{i-1}) + \sum_{j=0}^{S_i^*-1} g\,(j|\psi_i,\nu + T_{i-1})[G\,(Z- K_i^0(T_{i-1}) +$$
$$+ S_i^* - j|\psi_{i+1},\nu + T_{i+1} + j) - G\,(Z- K_i^0(T_{i-1})|\psi_{i+1},\nu + T_{i-1} + j)] \geq \gamma \ ,$$

where $S_i^* = S_i + Z_{i-1}$, $i = 1,2,\ldots$.

In Table 1 we present a simulated two-echelon system with parameters $\lambda = 10$, $\nu = 9.75$, $\tau = 1.1$, $c = .5$[\$], $p = 10.$[\$], $c^* = 5.$[\$] and $\gamma = .85$. We denote by IQ, IS and IX the values of Q_i, S_i and X_i. KOP designates $K_i^0(T_{i-1})$, IOR designates Y_i^0 defined in (2.3) and IZN is equal to $Z_i^0(T_{i-1})$ defined in (2.6). The cost is designated by CST and is equal to $C^* \cdot \text{IZN} + C \cdot \max(0,\text{IQ} - \text{IX}) - p \cdot \min(0,\text{IQ} - \text{IX})$. Figure 1 shows a (computer) graph of the moving-averages of the simulated monthly cost, CST. It is interesting to notice that in the present example, the initial stock level Q_1 and S_1 are small compared to the "desired" stock level, which is $k^0(10) = 28$. Although λ is unknown there is a relatively quick convergence of $K_i^0(T_{i-1})$ to $k^0(10)$. There is also an immediate correction of the stock level S_1 by a large order Z_1. As seen in Figure 1, the moving-averages of CST converge rapidly to the limit of their expectations. As shown in the appendix, if the system's stock level S_i stabilizes on a level higher than $k^0(\lambda)$ the limiting value of the average monthly cost is approximately

$$(2.7) \qquad \bar{c} = \lambda c^* + (c+p)[\bar{Q} \operatorname{Pos}(\bar{Q}|\lambda) - \lambda \operatorname{Pos}(\bar{Q}-1|\lambda)] - p(\bar{Q}-\lambda) \quad ,$$

where $\bar{Q} = k^0(\lambda) - \lambda$. For the parametersof Table 1, $\bar{Q} = 18$ and $\bar{c} = 54.93$.

As descussed in [4], there is a high correlation in the simulated data, between the order level Z_i and the demand level at the previous month, X_{i-1}. Indeed, in Table 1 the correlation between Z_i and X_{i-1} for all $i \geq 3$ is .954. It **was** therefore recommended in [4] to determine Z_1 and Z_2 exactly (according to (2.6)) and to set $Z_i = X_{i-1}$ for every $i \geq 3$. This policy will be called the "approximate upper-echelon ordering policy." In Table 2 and Figure 2 we present the simulated system for the same demand values, X_i, as in Table 1 but with the approximate upper-echelon policy. We see that the results are quite similar and the moving-averages of CST converge to the same limit. The convergence is however somewhat slower. The graph in Figure 2 is above that of Figure 1 for $i \leq 40$.

3. The Effects of the Prior Parameters.

In the present section we study the effect of the prior parameters on the control procedure. In order to test the possible effect of an apparently wrong choice of the prior parameter we have performed a simulation run similar to that of Tables 1 and 2, but with the prior parameter $\nu = 1.0$ rather than $\nu = 9.75$. In Table 3 we present the results with an exact upper-echelon policy, while in Table 4 we show the influence of the approximate upper echelon policy. As seen in Table 3 and Figure 3, the inventory system adjusts itself very fast to the difference between the assumed prior expectation of λ (which is 1.1) and the actual value of λ. The moving averages of CST converge to the same limit value as those of Tables 1 and 2. This is not the case, however, when we apply the approximate upper-echelon policy, as seen in Table 4 and Figure 4. One has to apply the exact upper-echelon policy for about $n = 10$ months before switching to the approximate policy, namely, using X_{i-1} for Z_i. Exact ordering for $n = 2$ months only is insufficient due to the effect of the low initial stocks and relatively high demand. As a result, the Q_i values in Table 4 are often close to zero. The general indication is that as long as the stock levels Q_i and S_i are too small one should apply the exact Bayes ordering policy of the upper-echelon. Once the system stabilizes with S_i values greater than $K_i(T_{i-1})$ values, one can switch to the approximate policy.

4. The Effects of Non-Stationarity.

Non-stationarity can manifest itself in different forms. We investigate here the effects of sudden unexpected changes in the intensity λ, of the Poisson process of demand. More specifically, we present in Table 5 a simulation of an inventory system with $\lambda_i = 10$ for $1 \leq i \leq 19$; $\lambda_i = 15$ for $20 \leq i \leq 39$ and $\lambda_i = 20$ for $40 \leq i$. The statistician is, however, unaware of these abrupt changes in λ and continues to use the Bayes adaptive policy, with an approximate upper-echelon ordering. As expected, since the Bayes procedure is adaptive, it reacts gradually to the fact that the observed demand tends to be larger after the change in λ than before. As a consequence, we see in Table 5 that the $K_i(T_{i-1})$ values increase steadily after $i = 20$. They do not reach, however, the optimal values $k^0(15) = 41$ and $k^0(20) = 50$. In Figure 5 we see the graph of the moving-averages of the monthly cost, CST. We cannot compare this graph with that of Figure 2 when $i \geq 20$. This is due to the fact that the expected monthly costs associated with the optimal policy, based on a complete knowledge of the epochs and magnitude of change in λ, jumps at every epoch of change to a new steady state limit. We can compare, however, the results presented in Table 5 and Figure 5 to those obtained when the statistican knows the initial value of λ and is not aware of possible changes in λ. In other words, the same optimal policy, based on the exact initial λ, is employed all the time. In Table 6 and Figure 6 we present the simulation results of such a case. We see in Figure 6 that the moving-averages of the CST are below those of Figure 5 for $1 \leq i \leq 20$ (as expected), but for $i > 20$ the adaptive Bayes procedure performs better. This indicates that when there is a possibility of non-stationarity a non-adaptive optimal policy based on the assumption of stationarity is likely to perform worse than an adaptive procedure, which corrects itself according to the observations on the actual demand. We remark here that, as seen in Table 5, the Bayes adaptive policy applied reacts rather slowly to unanticipated abrupt changes in λ. Indeed this policy is not optimal in such situations, since it has been designed for stationary cases in which λ is unknown but fixed. The derivation of a Bayes adaptive policy for cases of abrupt changes in λ, at unknown epochs, is an important subject for future research.

5. The Effect of Erroneous Demand Models.

One of the important components of the parameteric control model is the form of the demand distribution. In the previous sections we studied various characteristics of control procedures derived for the model of Poisson distribution of demand. It is interesting to investigate how robust are these adaptive control procedures against deviations from the Poisson model. An exhaustive study of the sensitivity of the Bayes adaptive procedures requires more development in terms of various alternatives to the Poisson distribution than what we present here. We focus attention in the present study only on the following alternative. We consider a compound Poisson distribution in which the mean, λ, is uniformly distributed over the interval [10, 20]. This provides a discrete distribution with a p.d.f.

$$(5.1) \qquad f(j) = \frac{1}{10 \cdot j!} \int_{10}^{20} e^{-\lambda} \lambda^{j} \, d\lambda = \frac{1}{10}[\mathrm{Pos}(j|10) - \mathrm{Pos}(j|20)], \; j = 0,1,\ldots .$$

Notice that $\sum_{j=0}^{\infty} (1-\mathrm{Pos}(j|\lambda))$ is the expected value, λ, of the Poisson random variable. Hence, $\sum_{j=0}^{\infty} f(j) = 1$. Moreover, the expected value and variance of a random varialbe, J, having the above p.d.f. is $E\{\Lambda\} = 15$ and

$$(5.2) \qquad \mathrm{Var}\{J\} = E\{\mathrm{Var}\{J|\Lambda\}\} + \mathrm{Var}\{E\{J|\Lambda\}\} = E\{\Lambda\} + \mathrm{Var}\{\Lambda\} = 15 + \frac{100}{12} = 23.33\ldots$$

Accordingly, we may try to compare the behavior of the Bayes adaptive control procedure, for a Poisson distribution under a demand distribution specified by (5.1) to that of a Poisson with mean $\lambda = 15$. In Table 7 and Figure 7 we see the performance of the Bayes adaptive control under the compound Poisson demand distribution and the exact upper-echelon ordering. We see that $K_i(T_{i-1})$ approaches indeed $k^0(15) = 41$. The upper-echelon policy of exact Bayes adaptive control, according to (2.6), guarantees that the system stock level, S_i, is generally above the "desirable" level $K_i(T_{i-1})$. The moving-averages of the monthly cost, CST, converge consequently to a limiting expected cost, which is approximately equal to 100.5. Notice that if the demand distribution were Poisson with mean $\lambda = 15$ then the limiting expected cost, according to (2.7) would be $\bar{c} = 81.71$. Thus, there is 25% increase in the limiting expected cost due to the fact that the actual demand distribution is not Poisson. It is of interest to test how would the approximate upper-echelon ordering policy perform in the present case, compared to that of the exact upper-echelon policy. In Table 8 and Figure 8 we present a simulation with the approximate policy, parallel to that of Table 7. We see that in the case of an approximate upper-echelon policy the system stock levels, S_i, are generally below the "desirable" one, and shortages in the lower-echelon are prevalent. The limiting expected monthly cost is increased, as seen in Figure 8 to 135. This is about 35% increase over that of the exact upper-echelon policy. The indication is that the approximate upper-echelon ordering policy isnot robust against considerable deviations of the actual demand distribution from the assumed one. It is safer to use the exact policy.

6. Summary and Conclusions.

In the present study we have shown that

1. The exact Bayes adaptive ordering procedures adjust very quickly to the actual demand, even if the initial prior parameters are wrongly chosen or if the initial stock values are inappropriate. The moving-averages of the actual monthly costs rapidly converge, in the stationary case, to the limiting expected monthly cost.

2. When the system stock level, S_i, is above the "desired" stock level $K_i(T_{i-1})$, the approximate upper-echelon ordering, $Z_i = X_{i-1}$, provides an effective simplification without much additional cost. On the other hand, if the system stock level, S_i, is too low the

upper-echelon ordering should be according to the exact Bayes predictive policy.

3. In cases of abrupt changes in the demand distributions at unknown time points, it is better to apply adaptive procedures than optimal procedures based on the assumption of stationarity of demand.

4. The exact Bayes ordering policy protects better than the approximate one against possible discrepancy between the model and the actual demand distribution.

Several questions discussed previously require further research. In particular, the development of optimal adaptive policies for non-stationary demand. We have applied here the Bayes adaptive procedures designed for stationary Poisson demand with unknown mean, λ. If λ changes at unknown time points these policies are not optimal any more. How much more complicated the optimal policies would be? These are open problems for further research.

7. Tables and Graphs.

Table 1: Simulated Two-Echelon System Under Poisson Demand, $\lambda = 10$, Bayes Control $\nu = 9.75$, $\tau = 1.1$, $\gamma = .85$; Exact Upper-Echelon Ordering.

I	IQ	IS	IX	KOP	IOR	IZN	CST
1	5	10	4	37	5	52	260.50
2	6	6	6	24	0	0	0.00
3	0	52	14	22	22	0	140.00
4	8	38	14	26	13	11	115.00
5	12	24	10	28	12	13	91.00
6	14	25	10	28	11	10	52.00
7	15	33	11	28	13	10	52.00
8	17	32	15	28	11	11	56.00
9	13	27	13	30	14	18	90.00
10	14	25	6	30	11	14	74.00
11	19	37	10	29	10	4	24.50
12	19	41	12	29	10	10	53.50
13	17	33	7	29	12	12	65.00
14	22	36	10	28	6	5	31.00
15	18	38	13	28	10	10	52.50
16	15	30	9	29	14	15	78.00
17	20	31	5	29	9	9	52.50
18	24	41	7	28	4	3	23.50
19	21	43	10	27	6	5	30.50
20	17	36	10	27	10	10	53.50
21	17	31	13	27	10	11	57.00
22	14	28	10	28	14	14	72.00
23	18	29	14	28	10	10	52.00
24	14	29	13	28	14	14	70.50
25	15	26	13	28	11	13	66.00
26	13	27	8	29	14	15	77.50
27	19	32	14	28	9	6	32.50
28	14	33	10	29	15	16	82.00
29	19	29	6	29	10	10	56.50
30	23	39	8	28	5	4	27.50
31	20	41	11	28	8	8	44.50
32	17	34	8	28	11	11	59.50
33	20	34	10	28	8	8	45.00
34	18	35	10	28	10	10	54.00
35	18	33	10	28	10	10	54.00
36	18	33	10	28	10	10	54.00
37	18	33	7	28	10	10	55.50
38	21	36	10	28	7	7	40.50
39	18	36	7	28	10	10	55.50
40	21	36	13	28	7	7	39.00
41	15	33	11	28	13	13	67.00
42	17	29	10	28	11	11	58.50
43	18	32	16	28	10	10	51.00
44	12	27	9	28	15	16	81.50
45	18	28	3	28	10	9	52.50

(Table 1, cont.)

I	IQ	IS	IX	KOP	IOR	IZN	CST
46	25	41	7	28	3	3	24.00
47	21	43	8	27	6	6	36.50
48	19	38	8	27	3	8	45.50
49	19	36	19	27	3	7	35.00
50	8	25	6	28	17	21	106.00
51	19	26	3	28	7	6	33.00
52	23	44	10	27	4	1	11.50
53	17	40	10	27	10	10	53.50
54	17	31	6	27	10	10	55.50
55	21	35	10	27	6	6	35.50
56	17	35	12	27	10	10	52.50
57	15	29	13	27	12	12	61.00
58	14	26	11	27	12	13	66.50

FIGURE 1: Moving Averages of Monthly Cost; Bayes Control, $\nu = 9.75$, $\tau = 1.1$, $p = 10[\$]$, $c = 5[\$]$, $c^* = 5[\$]$; Exact Upper Echelon Ordering.

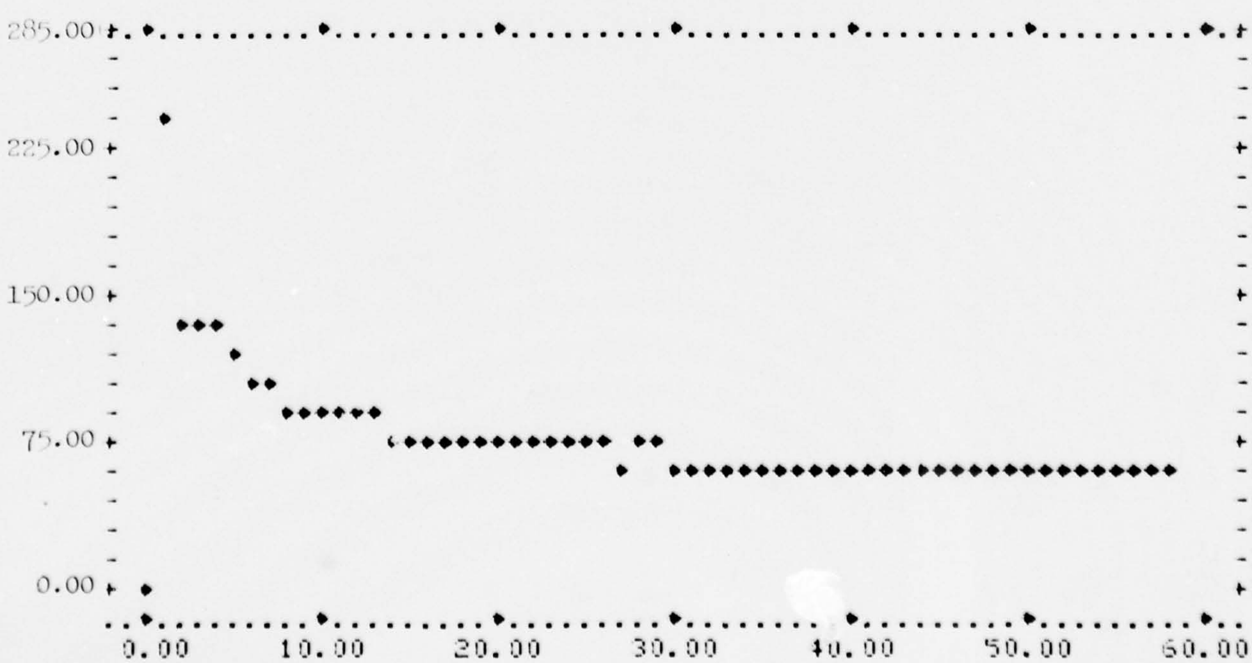


TABLE 2: Simulated Two-Echelon System Under Poisson Demand, $\lambda = 1.1$, Bayes Control $\nu = 9.75$, $\tau = 1.1$, $\gamma = .85$; Approximate Upper Echelon Ordering.

I	IQ	IS	IX	KOP	IOR	IZN	CST
1	5	10	4	37	5	52	260.50
2	6	6	6	24	0	31	155.00
3	0	52	14	22	22	6	170.00
4	3	69	14	26	13	14	130.00
5	12	61	10	28	16	14	71.00
6	13	65	10	28	10	10	54.00
7	13	69	11	28	10	10	53.50
8	17	68	15	28	11	11	56.00
9	13	63	13	30	17	15	75.00
10	17	61	6	30	13	13	70.50
11	24	70	10	29	5	6	37.00
12	19	73	12	29	10	10	53.50
13	17	67	7	29	12	12	65.00
14	22	70	10	28	6	7	41.00
15	18	72	13	28	10	10	52.50
16	15	66	9	29	14	13	68.00
17	20	67	5	29	9	9	52.50
18	24	75	7	28	4	5	33.50
19	21	77	10	27	6	7	40.50
20	17	72	10	27	10	10	53.50
21	17	69	13	27	10	10	52.00
22	14	66	10	28	14	13	67.00
23	13	66	14	28	10	10	52.00
24	14	65	13	28	14	14	70.50
25	15	62	13	28	13	13	66.00
26	15	63	8	29	14	13	68.50
27	21	68	14	28	7	8	43.50
28	14	67	10	29	15	14	72.00
29	19	65	6	29	10	10	56.50
30	23	73	8	28	5	6	37.50
31	20	75	11	28	8	8	44.50
32	17	70	8	28	11	11	59.50
33	20	70	10	28	8	8	45.00
34	18	71	10	28	10	10	54.00
35	13	69	10	28	10	10	54.00
36	13	69	10	28	10	10	54.00
37	18	69	7	28	10	10	55.50
38	21	72	10	28	7	7	40.50
39	18	72	7	28	10	10	55.50
40	21	72	13	28	7	7	39.00
41	15	69	11	28	13	13	67.00
42	17	65	10	28	11	11	58.50
43	13	68	16	28	10	10	51.00
44	12	63	9	28	16	16	81.50
45	19	64	3	28	9	9	53.00

(Table 2, cont.)

I	IQ	IS	IX	KOP	ICR	IZN	CST
46	25	77	7	28	3	3	24.00
47	21	79	8	27	6	7	41.50
48	19	74	8	27	3	8	45.50
49	19	73	19	27	8	8	40.00
50	8	62	6	28	20	19	96.00
51	22	64	3	28	6	6	39.50
52	25	80	10	27	2	3	22.50
53	17	76	10	27	10	10	53.50
54	17	69	6	27	10	10	55.50
55	20	73	10	27	6	6	35.50
56	17	73	12	27	10	10	52.50
57	15	67	13	27	12	12	61.00
58	14	64	11	27	13	13	66.50

FIGURE 2: Moving Averages of Monthly Cost; Bayes Control, $\nu = 9.75$, $\tau = 1.1$, $p = 10[\$]$, $c = .5[\$]$, $c' = 5[\$]$; Approximate Upper Echelon Ordering.

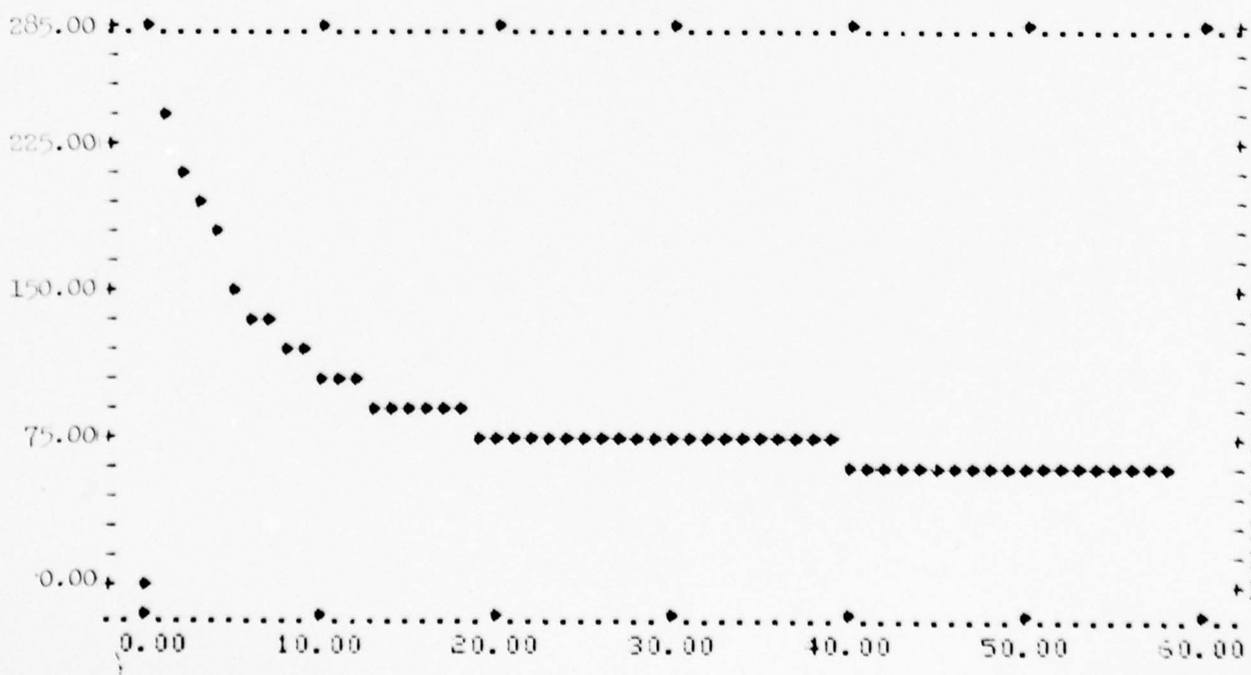


TABLE 3: Simulated Two-Echelon Systems Under Poisson Demand, $\lambda = 10$, Bayes Control $\nu = 1.0$, $\tau = 1.1$, $\gamma = .85$; Exact Upper Echelon Ordering.

I	IQ	IS	IX	KOP	IOR	IZN	CST
1	5	10	4	7	2	2	10.50
2	3	6	6	11	3	12	90.00
3	0	2	14	14	2	11	195.00
4	0	0	14	20	0	26	270.00
5	0	0	10	24	0	19	195.00
6	0	16	10	24	16	10	150.00
7	6	25	11	25	19	12	110.00
8	14	24	15	26	10	13	75.00
9	9	21	13	27	12	17	125.00
10	8	21	6	28	13	15	76.00
11	15	32	10	27	12	4	22.50
12	17	37	12	27	10	10	52.50
13	15	29	7	27	12	12	64.00
14	20	32	10	27	7	7	40.00
15	17	34	13	27	10	10	52.00
16	14	28	9	27	13	13	67.50
17	18	29	5	27	9	9	51.50
18	22	37	7	27	5	5	32.50
19	20	39	10	26	6	5	30.00
20	16	34	10	26	10	11	58.00
21	16	29	13	26	10	10	51.50
22	13	27	10	27	14	14	71.50
23	17	27	14	27	10	10	51.50
24	13	27	13	27	14	14	70.00
25	14	24	13	28	10	15	75.50
26	11	25	8	28	14	13	66.50
27	17	32	14	28	11	8	41.50
28	14	31	10	28	14	14	72.00
29	18	29	6	28	10	10	56.00
30	22	37	8	28	6	6	37.00
31	20	39	11	27	7	7	39.50
32	16	34	8	28	12	12	64.00
33	20	33	10	27	7	7	40.00
34	17	35	10	27	10	10	53.50
35	17	32	10	27	10	10	53.50
36	17	32	10	27	10	10	53.50
37	17	32	7	27	10	10	55.00
38	20	35	10	27	7	6	35.00
39	17	35	7	27	10	10	55.00
40	20	34	13	27	7	7	38.50
41	14	31	11	27	13	13	66.50
42	16	27	10	27	11	11	58.00
43	17	30	16	27	10	10	50.50
44	11	25	9	28	14	18	91.00
45	16	26	8	28	10	9	51.50

(Table 3, cont.)

I	IQ	IS	IX	KOP	IOR	IZN	CST
46	23	41	7	27	4	1	13.00
47	20	43	8	27	7	7	41.00
48	19	36	8	27	8	8	45.50
49	19	35	19	27	3	3	40.00
50	8	24	6	27	16	19	96.00
51	18	26	3	27	8	6	37.50
52	23	42	10	27	4	3	21.50
53	17	33	10	27	10	10	53.50
54	17	31	6	27	10	10	55.50
55	21	35	10	27	6	6	35.50
56	17	35	12	27	10	10	52.50
57	15	29	13	27	12	12	61.00
58	14	26	11	27	12	13	66.50

Figure 3: Moving Average Of Monthly Cost, Bayes Control, $\nu = 1$, $\tau = 1.1$, $\gamma = .85$, $p = 10$[\$], $c = .5$[\$], $c^* = 5$[\$]; Exact Upper Echelon Ordering.

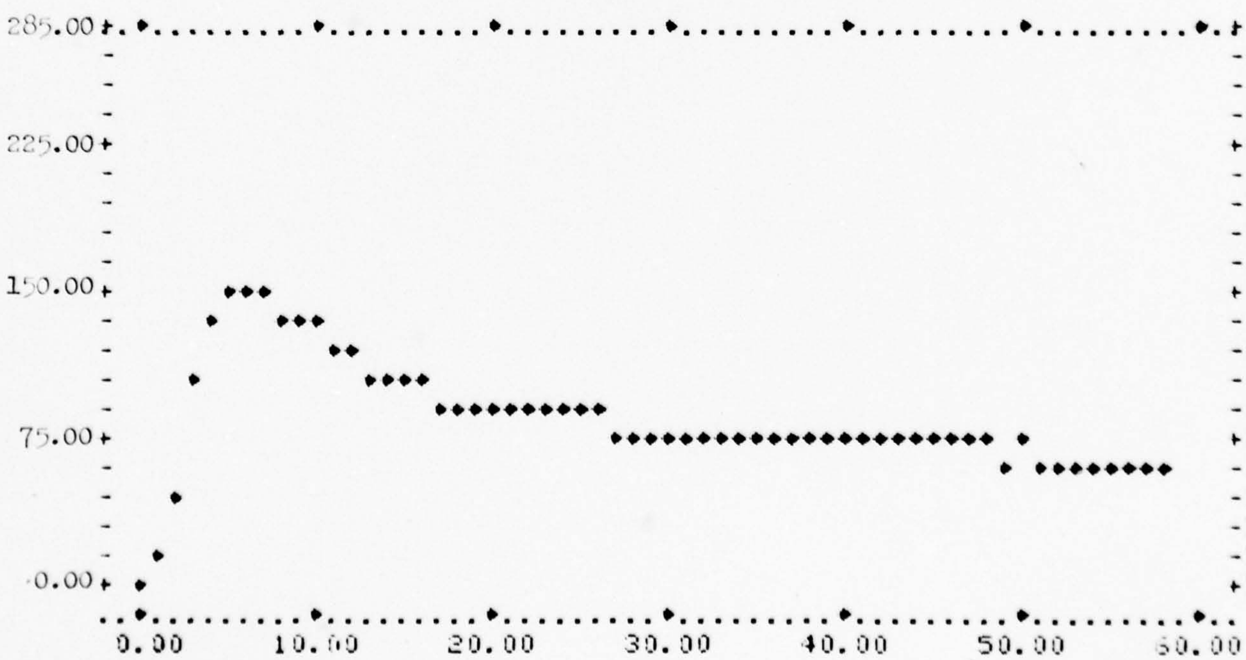


TABLE 4: Simulated Two-Echelon System under Poisson Demand, $\lambda = 10$, Bayes Control $\nu = 1.0$, $\tau = 1.1$, $\gamma = .85$; Approximate Upper Echelon Ordering.

I	IQ	IS	IX	KOP	IOR	IZN	CST
1	5	10	4	7	2	2	10.50
2	3	6	6	11	3	12	90.00
3	0	2	14	14	2	6	170.00
4	0	0	14	20	0	14	210.00
5	0	0	10	24	0	14	170.00
6	0	4	10	24	4	10	150.00
7	0	8	11	25	8	10	160.00
8	0	7	15	26	7	11	205.00
9	0	2	13	27	2	15	205.00
10	0	0	6	28	0	13	125.00
11	0	9	10	27	9	6	130.00
12	0	12	12	27	12	10	170.00
13	0	6	7	27	6	12	130.00
14	0	9	10	27	9	7	135.00
15	0	11	13	27	11	10	180.00
16	0	5	9	27	5	13	155.00
17	0	6	5	27	6	9	95.00
18	1	14	7	27	13	5	85.00
19	7	16	10	26	9	7	65.00
20	6	11	10	26	5	10	90.00
21	1	8	13	26	7	10	170.00
22	0	5	10	27	5	13	165.00
23	0	5	14	27	5	10	190.00
24	0	4	13	27	4	14	200.00
25	0	1	13	28	1	13	195.00
26	0	2	8	28	2	13	145.00
27	0	7	14	28	7	8	180.00
28	0	6	10	28	6	14	170.00
29	0	4	6	28	4	10	110.00
30	0	12	8	28	12	6	110.00
31	4	14	11	27	10	8	110.00
32	2	9	8	28	6	11	105.00
33	1	9	10	27	8	8	130.00
34	0	10	10	27	10	10	150.00
35	0	8	10	27	8	10	150.00
36	0	8	10	27	8	10	150.00
37	0	8	7	27	8	10	120.00
38	1	11	10	27	10	7	125.00
39	1	11	7	27	10	10	110.00
40	4	11	13	27	7	7	125.00
41	0	8	11	27	8	13	175.00
42	0	4	10	27	4	11	155.00
43	0	7	16	27	7	10	210.00
44	0	2	9	28	2	16	170.00
45	0	3	3	28	3	9	75.00

(Table 4, cont.)

I	IQ	IS	IX	KOP	IOR	IZN	CST
46	0	16	7	27	16	3	35.00
47	9	18	8	27	9	7	35.50
48	10	13	8	27	3	8	41.00
49	5	12	19	27	7	8	180.00
50	0	1	6	27	1	19	155.00
51	0	3	3	27	3	6	60.00
52	0	19	10	27	19	3	115.00
53	9	15	10	27	6	10	60.00
54	5	8	6	27	3	10	60.00
55	2	12	10	27	10	6	110.00
56	2	12	12	27	10	10	150.00
57	0	6	13	27	6	12	190.00
58	0	3	11	27	3	13	175.00

FIGURE 4: Moving Average of Monthly Cost; Bayes Control, $\nu = 1$, $\tau = 1.1$, $p = 10[\$]$, $c = .5[\$]$, $c^* = 5[\$]$; Exact Upper Echelon Ordering.

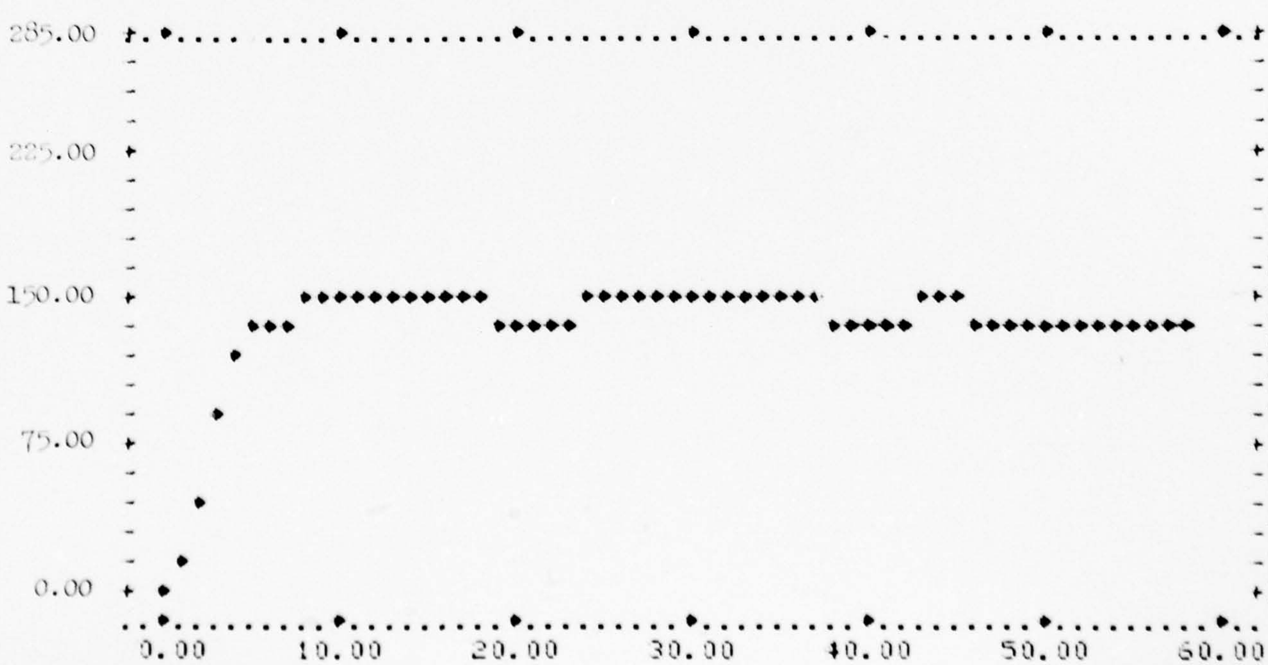


TABLE 5: Simulated Two-Echelon System Under Poisson Demand With Increasing Intensity; $\lambda_i = 10$ $(i \le 19)$, $\lambda_i = 15$ $(20 \le i \le 39)$, $\lambda_i = 20$ $(i \ge 40)$; *Bayes Control* $\nu = 9.75$, $\tau = 1.1$, $\gamma = .85$; Approximate Upper Echelon Ordering.

I	IQ	IS	IX	KOP	IOR	IZN	CST
1	5	10	4	37	5	52	260.50
2	6	6	6	24	0	0	0.00
3	0	52	14	22	22	6	170.00
4	8	38	14	26	18	14	130.00
5	12	30	10	28	16	14	71.00
6	18	34	10	28	10	10	54.00
7	18	38	11	28	10	10	53.50
8	17	37	15	28	11	11	56.00
9	13	32	13	30	17	15	75.00
10	17	30	6	30	13	13	70.50
11	24	39	10	29	5	6	37.00
12	19	42	12	29	10	10	53.50
13	17	36	7	29	12	12	65.00
14	22	39	10	28	6	7	41.00
15	18	41	13	28	10	10	52.50
16	15	35	9	29	14	13	68.00
17	20	36	5	29	9	9	52.50
18	24	44	7	28	4	5	33.50
19	21	46	10	27	6	7	40.50
20	17	41	18	27	10	10	60.00
21	9	30	15	28	19	18	150.00
22	13	25	19	29	12	15	135.00
23	6	24	22	30	18	19	255.00
24	2	17	14	31	15	22	230.00
25	3	22	17	31	19	14	210.00
26	5	27	12	32	22	17	155.00
27	15	29	12	32	14	12	61.50
28	17	34	15	32	15	12	61.00
29	17	31	14	32	14	15	76.50
30	17	29	17	32	12	14	70.00
31	12	27	16	32	15	17	125.00
32	11	25	12	33	14	16	90.00
33	13	30	18	33	17	12	110.00
34	12	28	16	33	16	18	130.00
35	12	24	23	33	12	16	190.00
36	1	19	12	34	12	23	225.00
37	7	23	14	34	16	12	130.00
38	9	32	11	34	22	14	90.00
39	21	33	22	34	12	11	65.00
40	11	25	10	34	14	22	110.50
41	15	26	20	34	11	10	100.00
42	6	28	17	35	22	20	210.00
43	11	21	24	35	10	17	215.00
44	0	17	25	35	17	24	370.00
45	0	9	23	36	9	25	355.00

(Table 5, cont.)

I	IQ	IS	IX	KOP	IOR	IZN	CST
46	0	10	25	37	10	23	365.00
47	0	10	19	37	10	25	315.00
48	0	14	17	37	14	19	265.00
49	0	22	17	37	22	17	255.00
50	5	24	17	38	19	17	205.00
51	7	24	19	38	17	17	205.00
52	5	22	19	38	17	19	235.00
53	3	20	23	38	17	19	295.00
54	0	16	31	38	16	23	425.00
55	0	4	14	39	4	31	295.00
56	0	13	22	39	13	14	290.00
57	0	22	23	39	22	22	340.00
58	0	13	13	40	13	23	295.00

FIGURE 5: Moving Averages of Monthly Cost; Bayes Control, $\nu = 9.75$, $\tau = 1.1$, $\gamma = .85$; $p = 10[\$]$, $c = .5[\$]$, $c^* = 5[\$]$; Poisson Demand With Increasing Intensity; Approximate Upper Echelon Ordering.

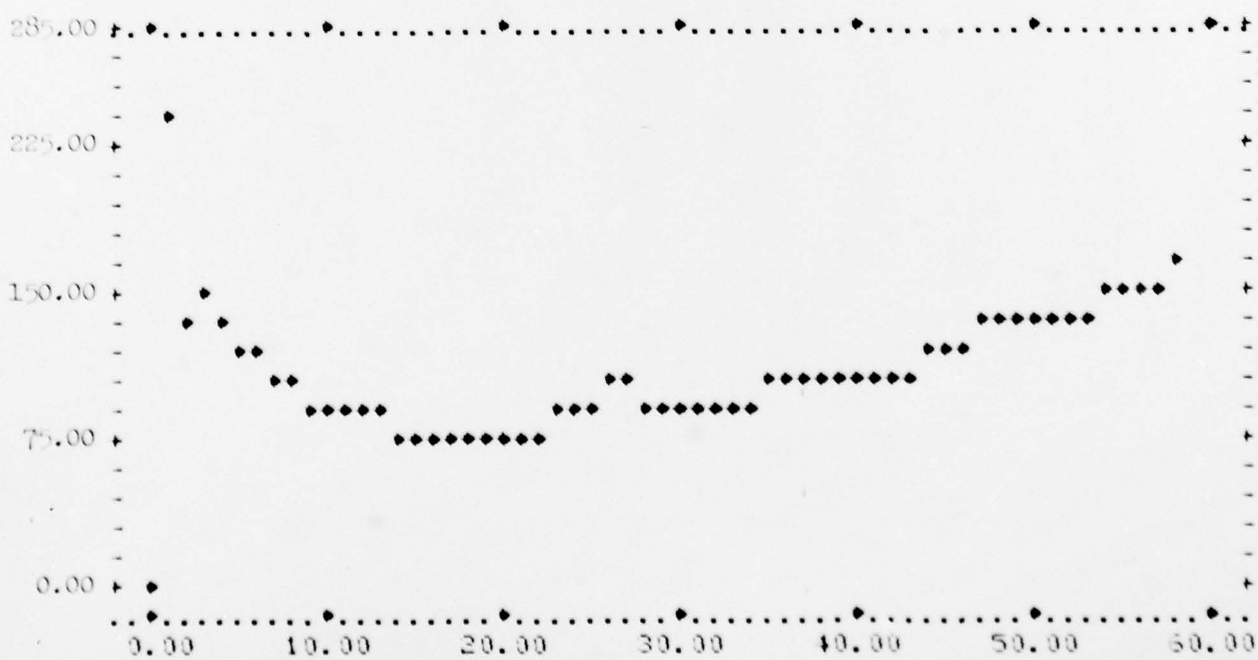


TABLE 6: Simulated Two-Echelon System Under Poisson Demand With Increasing Intensity; $\lambda_i = 10$ $(i \leq 19)$, $\lambda_i = 15$ $(20 \leq i \leq 39)$, $\lambda_i = 20$ $(i \geq 40)$; Lower Echelon Control Optimal For Poisson with $\lambda = 10$; Approximate Upper Echelon Control.

I	IQ	IS	IX	KOP	IOR	IZN	CST
1	5	10	4	28	5	43	215.50
2	6	6	6	28	0	0	0.00
3	0	43	14	28	28	6	170.00
4	14	29	14	28	14	14	70.00
5	14	21	10	28	7	14	72.00
6	11	25	10	28	14	10	50.50
7	15	29	11	28	13	10	52.00
8	17	28	15	28	11	11	56.00
9	13	23	13	28	10	15	75.00
10	10	21	6	28	11	13	67.00
11	15	30	10	28	13	6	32.50
12	18	33	12	28	10	10	53.00
13	16	27	7	28	11	12	64.50
14	20	30	10	28	8	7	40.00
15	18	32	13	28	10	10	52.50
16	15	26	9	28	11	13	68.00
17	17	27	5	28	10	9	51.00
18	22	35	7	28	6	5	32.50
19	21	37	10	28	7	7	40.50
20	18	32	18	28	10	10	50.00
21	10	21	15	28	11	18	140.00
22	6	16	19	28	10	15	205.00
23	0	15	22	28	15	19	315.00
24	0	8	14	28	8	22	250.00
25	0	13	17	28	13	14	240.00
26	0	18	12	28	18	17	205.00
27	6	20	12	28	14	12	120.00
28	6	25	15	28	17	12	130.00
29	10	22	14	28	12	15	115.00
30	8	20	17	28	12	14	160.00
31	3	18	16	28	15	17	215.00
32	2	16	12	28	14	16	180.00
33	4	21	18	28	17	12	200.00
34	3	19	16	28	16	18	220.00
35	3	15	23	28	12	16	280.00
36	0	10	12	28	10	23	235.00
37	0	14	14	28	14	12	200.00
38	0	23	11	28	23	14	180.00
39	12	24	22	28	12	11	155.00
40	2	16	10	28	14	22	190.00
41	6	17	20	28	11	10	190.00
42	0	19	17	28	19	20	270.00
43	2	12	24	28	10	17	305.00
44	0	8	25	28	8	24	370.00
45	0	0	23	28	0	25	355.00

(Table 6, cont.)

I	IQ	IS	IX	KOP	ICR	IZN	CST
46	0	1	25	28	1	23	365.00
47	0	1	19	28	1	25	315.00
48	0	5	17	28	5	19	265.00
49	0	13	17	28	13	17	255.00
50	0	15	17	28	15	17	255.00
51	0	15	19	28	15	17	275.00
52	0	13	19	28	13	19	285.00
53	0	11	23	28	11	19	325.00
54	0	7	31	28	7	23	425.00
55	0	0	14	28	0	31	295.00
56	0	9	22	28	9	14	290.00
57	0	18	23	28	18	22	340.00
58	0	9	18	28	9	23	295.00

FIGURE 6: Moving Averages of Monthly Cost; Fixed Control Under Poisson Demand With Increasing Intensity.

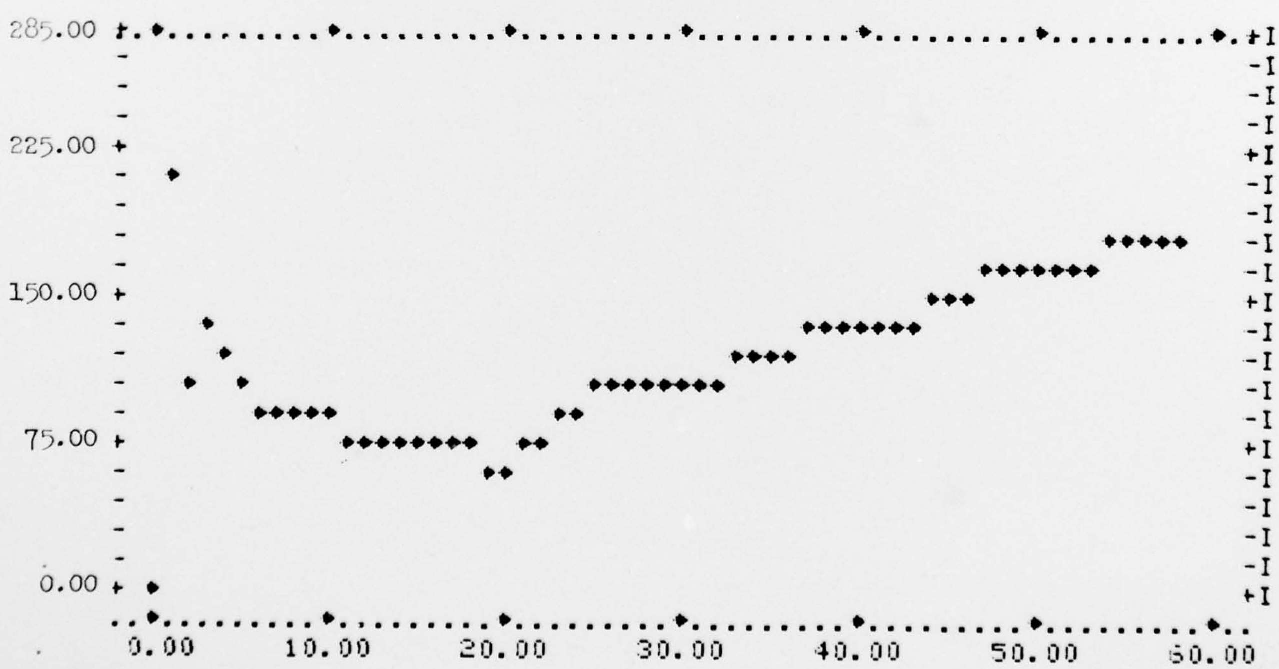


TABLE 7: Simulated Two-Echelon Systems Under Compound Poisson Demand With Random Uniform Intensity $(10 \leq \lambda \leq 20)$; Bayes Control $\nu = 9.75$, $\tau = 1.1$, $\gamma = .85$; Exact Upper Echelon Ordering.

I	IQ	IS	IX	KOP	IOR	JZN	CST
1	5	10	6	7	2	2	20.00
2	1	4	16	15	3	20	250.00
3	0	0	18	24	0	25	305.00
4	0	2	13	30	2	30	280.00
5	0	14	12	31	14	15	195.00
6	2	32	15	31	29	12	190.00
7	16	32	13	33	16	13	91.50
8	19	31	18	33	12	13	65.50
9	13	31	16	34	18	21	135.00
10	15	28	13	35	13	18	91.00
11	15	36	17	35	20	13	85.00
12	18	37	6	36	18	18	96.00
13	30	44	10	34	4	3	25.00
14	24	52	21	34	10	10	51.50
15	13	34	18	35	21	23	165.00
16	16	26	21	36	10	19	145.00
17	5	28	22	37	23	23	285.00
18	6	25	12	38	19	24	180.00
19	13	36	14	38	23	12	70.00
20	22	46	11	38	16	14	75.50
21	27	47	12	37	10	10	57.50
22	25	49	17	37	12	12	64.00
23	20	42	19	37	17	17	85.50
24	18	35	15	38	17	20	101.50
25	20	37	14	38	17	15	78.00
26	23	43	11	38	15	14	76.00
27	27	47	21	38	11	11	58.00
28	17	40	25	38	21	22	190.00
29	13	26	10	39	13	26	131.50
30	16	33	13	39	22	10	51.50
31	25	51	13	38	13	12	66.00
32	25	48	20	38	13	13	67.50
33	18	40	7	39	21	21	110.50
34	32	46	17	38	6	6	37.50
35	21	50	11	38	17	17	90.00
36	27	45	23	38	11	11	57.00
37	15	39	17	39	24	24	140.00
38	22	33	15	39	11	17	88.50
39	18	42	27	39	21	15	165.00
40	12	32	16	39	20	28	180.00
41	16	31	12	39	15	16	82.00
42	19	47	13	39	20	12	63.00
43	26	50	7	39	13	13	74.50
44	32	55	10	39	7	6	41.00
45	29	58	13	39	10	10	55.50

(Table 7, cont.)

46	21	46	11	39	13	13	95.00
47	28	45	10	39	11	11	64.00
48	29	53	13	38	9	9	53.00
49	25	51	8	38	13	13	73.50
50	30	52	20	38	8	7	40.00
51	18	45	25	38	20	21	175.00
52	13	27	10	39	14	26	131.50
53	17	38	11	38	21	9	48.00
54	27	53	13	38	11	11	62.00
55	25	49	17	38	13	13	69.00
56	21	43	17	38	17	17	87.00
57	21	39	8	38	17	17	91.50
58	30	48	21	38	8	8	44.50

FIGURE 7: Moving Averages of Monthly Cost; Compound Poisson Demand; Bayes Contro, $\nu = 9.75$, $\tau = 1.1$, $\gamma = .85$, $p = 10[\$]$, $c = .5[\$]$, $c^* = 5[\$]$; Exact Upper Echelon Ordering.

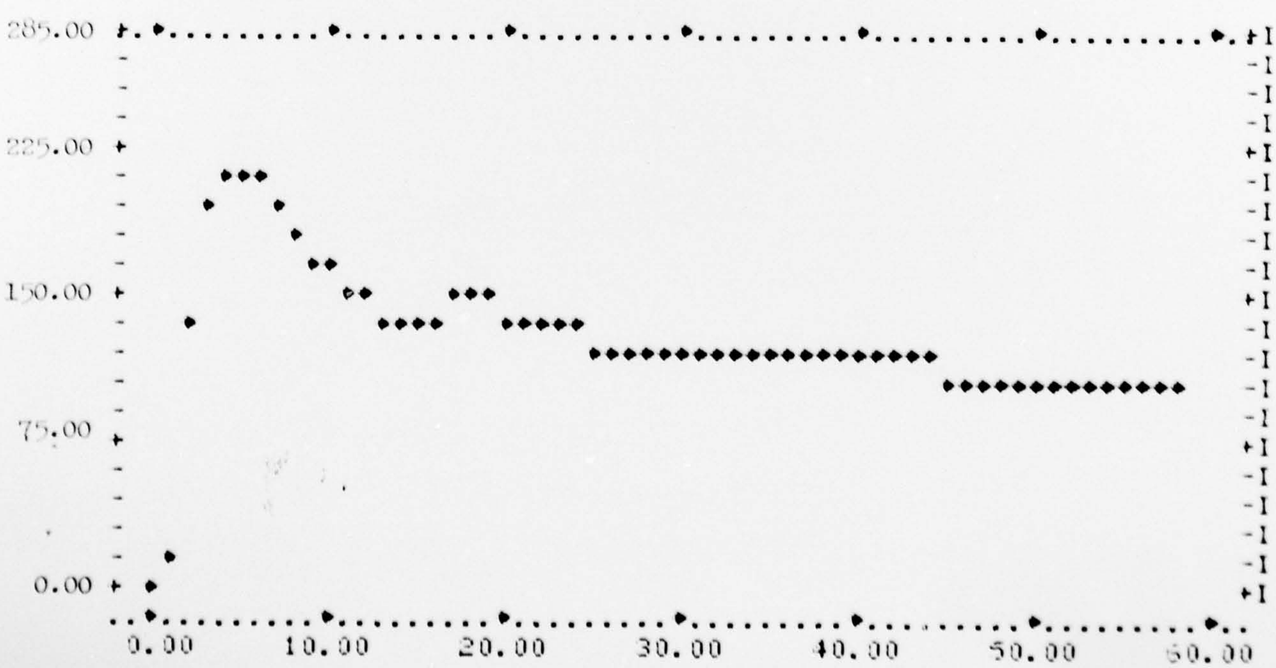


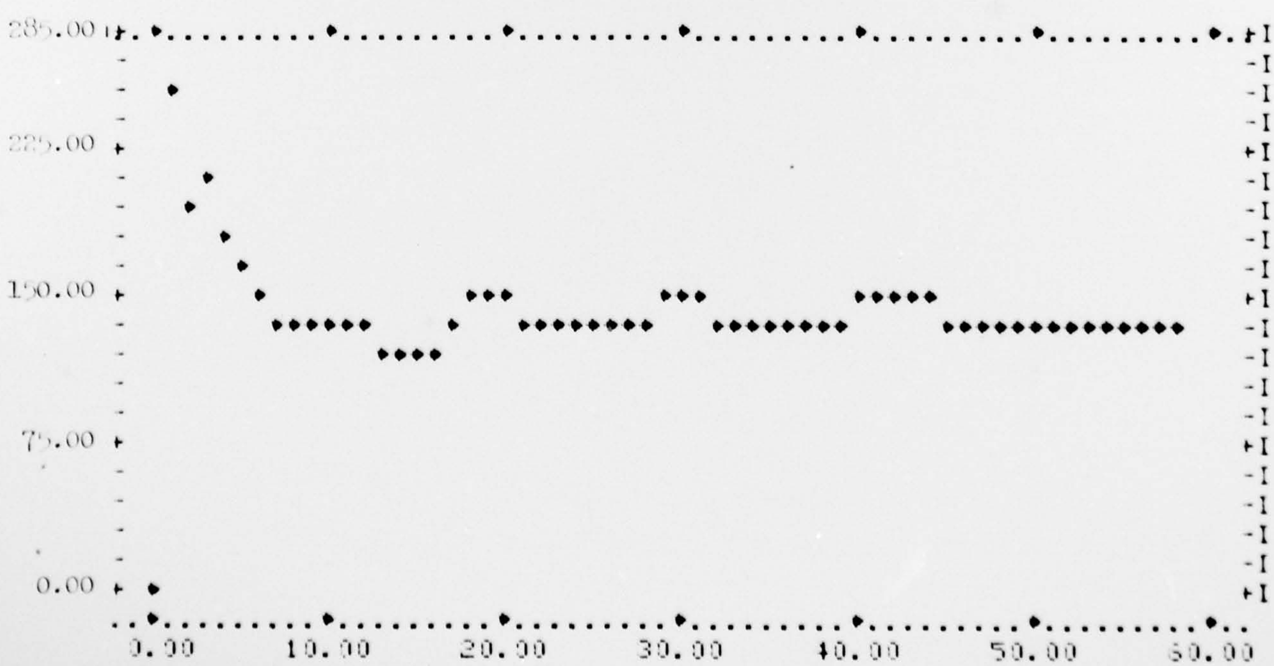
TABLE 8: Simulated Two-Echelon Systems Under Compound Poisson Demand With Random Uniform Intensity $(10 \leq \lambda \leq 20)$; Bayes Control $\nu = 9.75$, $\tau = 1.1$, $\gamma = .85$; Approximate Upper Echelon Ordering.

I	IQ	IS	IX	KOP	IOR	IZN	CST
1	5	10	6	37	5	52	270.00
2	4	4	16	27	0	0	120.00
3	0	40	13	32	32	16	260.00
4	14	22	13	36	8	18	90.50
5	9	25	12	36	16	13	95.00
6	13	31	15	35	18	12	80.00
7	16	29	13	36	13	15	76.50
8	16	28	18	36	12	13	85.00
9	10	25	16	37	15	18	150.00
10	9	22	13	37	13	16	120.00
11	9	27	17	37	18	13	145.00
12	10	26	6	38	16	17	87.00
13	20	33	10	36	13	6	35.00
14	23	40	21	35	12	10	51.00
15	14	25	18	37	11	21	145.00
16	7	17	21	37	10	18	230.00
17	0	17	22	38	17	21	325.00
18	0	13	12	39	13	22	230.00
19	1	22	14	39	21	12	190.00
20	8	30	11	39	22	14	100.00
21	19	31	12	38	12	11	58.50
22	19	33	17	38	14	12	61.00
23	16	27	19	38	11	17	115.00
24	8	20	15	39	12	19	165.00
25	5	22	14	39	17	15	165.00
26	8	27	11	39	19	14	100.00
27	16	31	21	38	15	11	105.00
28	10	24	25	39	14	21	255.00
29	0	10	10	40	10	25	225.00
30	0	21	13	39	21	10	180.00
31	3	33	13	39	25	13	115.00
32	20	30	20	39	10	13	65.00
33	10	23	7	39	13	20	101.50
34	16	29	17	39	13	7	45.00
35	12	32	11	39	20	17	35.50
36	21	28	23	39	7	11	75.00
37	5	22	17	39	17	23	235.00
38	5	16	15	39	11	17	135.00
39	1	24	27	39	23	15	335.00
40	0	14	16	40	14	27	295.00
41	0	13	12	40	13	16	200.00
42	1	28	13	40	27	12	180.00
43	15	31	7	40	16	13	59.00
44	24	36	10	39	12	7	42.00
45	26	39	18	39	13	10	54.00

(Table 8, cont.)

I	IQ	IS	IX	KOP	IOR	IZN	CST
46	21	28	11	39	7	13	35.00
47	17	27	10	39	10	11	58.50
48	17	35	13	39	13	10	52.00
49	22	33	8	39	11	13	72.00
50	25	35	20	38	10	8	42.50
51	15	28	25	39	13	20	200.00
52	8	11	10	39	8	25	195.00
53	1	21	11	39	20	10	150.00
54	10	35	13	39	25	11	35.00
55	22	32	17	39	10	13	67.50
56	15	26	17	39	11	17	105.00
57	9	22	8	39	13	17	85.50
58	14	31	21	38	17	8	110.00

FIGURE 8: Moving Averages of Monthly Cost; Compound Poisson Demand; Bayes Control, $\nu = 9.75$, $\tau = 1.1$, $\gamma = .85$, $p = 10[\$]$, $c = .5[\$]$, $c^* = 5[\$]$; Approximate Upper Echelon Ordering.



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9. Appendix.

9.1 Derivation of the Limiting Average Cost.

The expected monthly cost associated with Y_i^0 and Z_i^0 is

$$(9.1)\quad E\{CST\} = c^* E\{Z_i^0\} + E\{c[(Q_i + Y_i^0 - X_i)^+ - X_{i+1}]^+ - p[(Q_i + Y_i^0 - X_i)^+ - X_{i+1}]^-\} .$$

If $S_i \geq K_i(T_{i-1})$ then $Q_i + Y_i^0 = K_i(T_{i-1})$. Furthermore, we have seen that after the system adjusts itself, $Z_i^0 \approx X_{i-1}$ and $K_i(T_{i-1}) \approx k^0(\lambda)$. Accordingly, the limiting expected monthly cost is approximately

$$(9.2)\quad M(\lambda) = c^*\lambda + E\{c[(k^0(\lambda) - X_i)^+ - X_{i+1}]^+ - p[(k^0(\lambda) - X_i)^+ - X_{i+1}]^-\} .$$

If we replace $k^0(\lambda) - X_i$ in (9.2) by its expectation, $\bar{Q} = k^0(\lambda) - \lambda$, we obtain formula (2.6) namely

$$(9.3)\quad \bar{c} = \lambda(c^* + p) + (c+p)[\bar{Q}\, \mathrm{Pos}(\bar{Q}|\lambda) - \lambda\, \mathrm{Pos}[(\bar{Q}-1|\lambda)] - p\bar{Q} .$$

To evaluate this approximation we develop the formula of $M(\lambda)$. From (9.2) we obtain

$$(9.4)\quad M(\lambda) = c^*\lambda + c \sum_{j=0}^{k^0} p(j|\lambda) \sum_{i=0}^{k^0-j} p(i|\lambda)(k^0 - j - i) + p \sum_{j=0}^{k^0} p(j|\lambda) \sum_{i=k^0-j+1}^{\infty} p(i|\lambda)(i-k^0+j) + \lambda p \sum_{j=k^0+1}^{\infty} p(j|\lambda) ,$$

where $k^0 \equiv k^0(\lambda)$. Notice that

$$(9.5)\quad \sum_{i=1}^{c} p(i|\lambda)(c-i) = c\, \mathrm{Pos}(c|\lambda) - \lambda\, \mathrm{Pos}(c-1|\lambda) = (c-\lambda)\, \mathrm{Pos}(c-1|\lambda) + cp(c|\lambda).$$

and

$$(9.6)\quad \sum_{i=c+1}^{\infty} (i-c)\, p(i|\lambda) = (\lambda-c) - (\lambda-c)\, \mathrm{Pos}(c-1|\lambda) + cp(c|\lambda) .$$

Hence, (9.4) obtains the form

(9.7)
$$M(\lambda) = (c^* + p)\lambda + (c+p) \sum_{j=0}^{k^0-1} p(j|\lambda) \, [(k^0-j-\lambda) \cdot$$
$$\mathrm{Pos}(k^0-j-1|\lambda) + (k^0-j) \, p(k^0-j|\lambda)] -$$
$$p[(k^0-\lambda) \, \mathrm{Pos}(k^0-1|\lambda) + k^0 p(k^0|\lambda)] \quad .$$

Finally, we notice that

(9.8)
$$\sum_{j=0}^{k^0-1} p(j|\lambda) \, \mathrm{Pos}(k^0-j-1|\lambda) = P\{X_1 + X_2 \le k^0 - 1\} = \mathrm{Pos}(k^0 - 1|2\lambda) \quad ,$$

where X_1 and X_2 are independent identically distributed Poisson r.v.'s with mean λ. Similarly,

(9.9)
$$\sum_{j=0}^{k^0-1} jp(j|\lambda) \, \mathrm{Pos}(k^0-j-1|\lambda) = \lambda \sum_{j=0}^{k^0-2} p(j|\lambda) \, \mathrm{Pos}(k^0-2-j|\lambda) = \lambda \mathrm{Pos}(k^0-2|2\lambda).$$

Accordingly, from (9.8) and (9.9) we prove that

(9.10)
$$M(\lambda) = \lambda(c^*+p) + (c+p)[(k^0-2\lambda) \, \mathrm{Pos}(k^0-2|2\lambda) + (k^0-\lambda) \, p(k^0-1|2\lambda)] +$$
$$+ (c+p)[k^0 p(k^0|2\lambda) - k^0 p(0|\lambda) \, p(k^0|\lambda) - \lambda p(k^0-1|2\lambda) +$$
$$+ \lambda p(0|\lambda) \, p(k^0-1|\lambda)] - p(k^0-\lambda) \, \mathrm{Pos}(k^0-1|\lambda) - pk^0 \, p(k^0|\lambda) \quad .$$

This function is considerably more complicated than $\bar{c}$ given by (2.6) or (9.3). In the following table we give a few numerical comparisons between $\bar{c}$ and $M(\lambda)$. We see that for λ not exceeding 10

Table 9.1: Numerical Comparison of $\bar{c}$ and $M(\lambda)$ for $\lambda = 5(5)30$ and $c^* = 5$, $c = .5$ and $p = 10$.

λ	$\bar{c}$	$M(\lambda)$
5.	28.57448	28.08917
10.	54.92688	54.14090
15.	81.70587	79.75215
20	107.50767	105.69582
25	133.29885	131.40261
30	159.00808	156.94519

the approximation to $M(\lambda)$ given by $\bar{c}$ is very good. For values of λ greater than 10 the approximation is not as good, but the relative error for $\lambda \geq 20$ is less than 2%. Since (9.3) is considerably simpler than (9.10) it seems justifiable in many circumstances.

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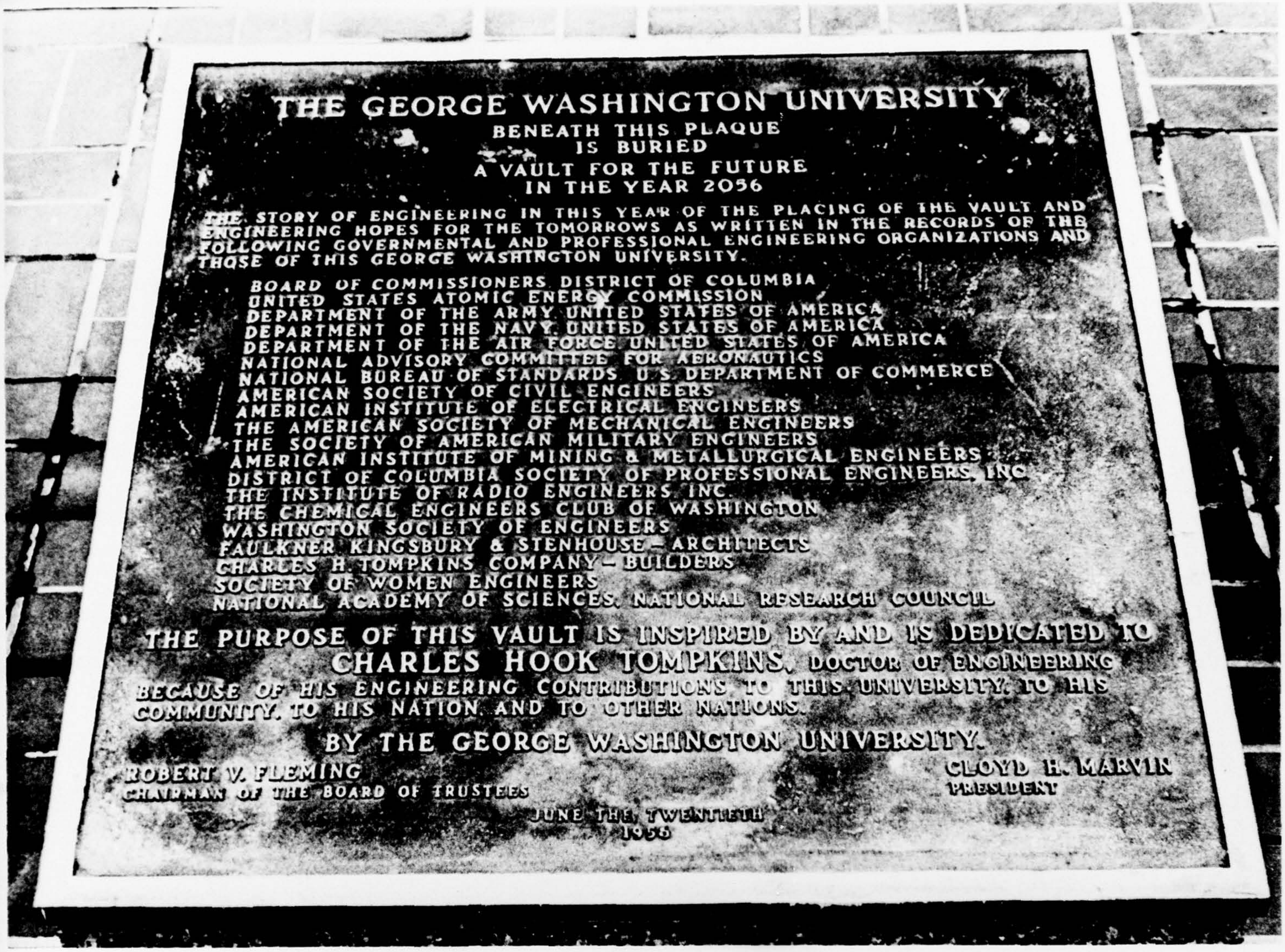

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