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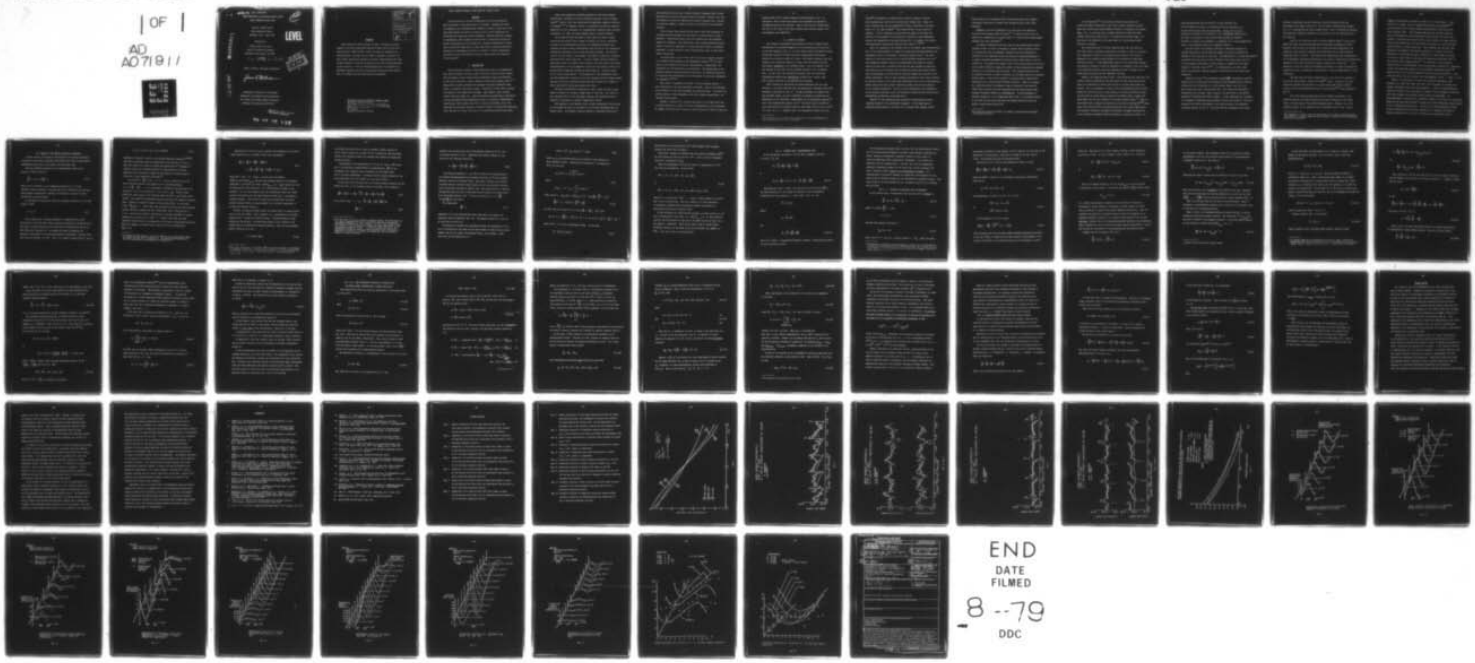

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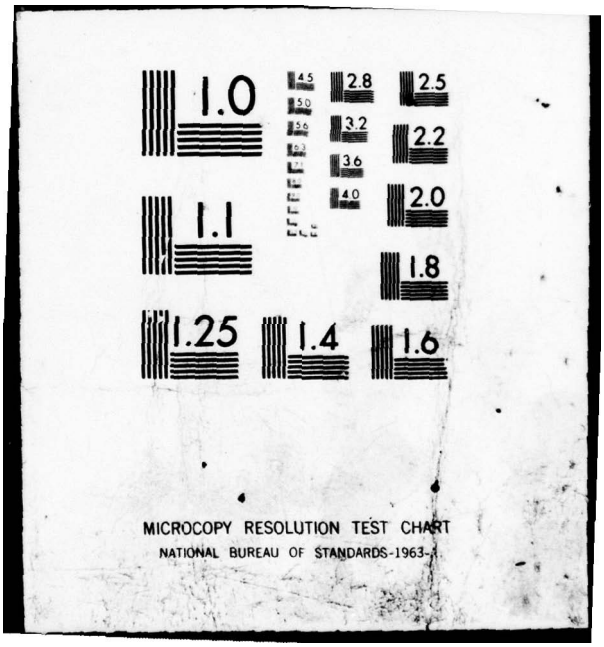
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THREE-DIMENSIONAL FLOW THROUGH HIGHLY-LOADED AXIAL COMPRESSOR BLADE ROWS

Grant No. AFOSR 75-2784

Final Scientific Report

November, 1974 - June, 1978

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FOREWORD

This document has been prepared in order to provide an overall summary of the work accomplished between November 1974 and June 1978 under Grant No. AFOSR 75-2784. While the report is intended to be self-contained, an attempt has been made to maintain its brevity not only through appropriate reference to earlier Interim Reports but also through reference to material arising from this work already published in the literature. Prof. Sir William Hawthorne of the University of Cambridge, England, has been co-investigator during the Grant period; Prof. J.E. McCune has been the principal investigator.

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FINAL SCIENTIFIC REPORT, AFOSR GRANT NO. AFOSR 75-2784

ABSTRACT

A brief account of the work accomplished over the three-and-one-half year extent of the subject Grant is provided. Although the report is arranged in part chronologically, emphasis is given to that part of the work which appears not only to be new but also to have significant implications for future gas turbine technology. Wherever possible, examples of experimental and/or numerical verification of theoretical results are noted; shortcomings are also pointed out. Physical explanations of phenomena of particular importance, of which some seem to have gone previously unnoticed in turbomachinery practice are offered. Finally, suggestions are made for future work aimed at practical applications and extensions of this research.

I. INTRODUCTION

Early analytical studies of three-dimensional flows in turbomachinery were limited primarily to what we shall call here "strictly linear" theory, with the blades of each cascade being represented by small perturbations of the inlet conditions - usually modeled as purely axial annular flows.[1-5] These initial studies represent the analogue (for axial compressor blade rows), of classical thin airfoil theory. They were, in that sense, necessarily limited to very light wheel loadings: small swirls, small pressure ratios across any stage, etc. Such restrictions precluded direct application of these preliminary three-dimensional studies to practical turbomachinery. Nevertheless, their results could not be entirely overlooked, since they indicated the potential existence of physical effects of qualitative importance which had been up to that time omitted in turbomachine analysis and design.

These three-dimensional phenomena appeared to have their maximum significance, according to the then available analyses, in the transonic regime[1,2] and for blade rows operating with significant spanwise variation in loading.[3,4,1] It remained, however, to establish the relative practical importance of such conclusions for turbomachinery operating under realistic conditions, i.e. under the high loadings typical of useful machines.

By the time the work described in this report was ready to begin, the notion had evolved[5] that it might be possible to use as a reference, in place of the undisturbed inlet flow, the azimuthally-average flow, treating the latter exactly (nonlinearly) with the help of the axisymmetric through-flow equations, such as those described by Marble. The θ-variation of the flow field, it was argued, could then be treated as a perturbation about this "exact" mean flow. If such a scheme could be worked out along these lines, the overall large turning of the flow would be described two-dimensionally by the axisymmetric theory and thus could be chosen to be unrestricted in magnitude through any given blade row - pressure ratios of 2.0 or more being typical. On perturbation of the axisymmetric model, the three-dimensional structure of the flow through practical (highly-loaded) cascades could then be described with sufficient accuracy for a number of direct applications within gas turbine technology.

The basic justification of this idea is that a blade row with a large number of blades, B, can have a large effect on the flow, as a result of the concerted effect of all the blades, whereas each individual blade is required to contribute, by itself, comparatively little.

This approach to the analytic study of axial turbomachine fluid mechanics has formed the basis of virtually all of the work performed under the subject Grant. The somewhat intricate methods of describing effectively

the perturbations about such a highly nonlinear "reference flow" as that mentioned above have been devised during this period. Further, they have been gradually improved by applying the proposed procedure over a variety of practical examples, a number of which are described in the following sections.

By and large, these initial efforts seem to have been successful in providing increased insight into the nature of complex flows in turbomachinery. Functionally useful analytical methods have been developed within this framework. More importantly, their various applications have served to reveal the importance of certain physical effects not formerly accessible to ready evaluation and/or interpretation, and, in some cases, simply unrecognized.

As stated above, the new theory has lent itself in several instances to an enhanced understanding of new experimental results.[7,8,9] In other cases it has stimulated the search, experimentally or numerically, for verification of the predictions it has provided. The significance of the phenomena gradually being uncovered in this way is of potential importance to the design and understanding of the performance of practical turbomachinery components.[10,11,12]

This report is organized as follows: In the following section, a brief chronology of the work performed is provided. Some of the highlights of the results, as they have developed, are indicated there. In Section III certain aspects of these highlights, and their possible implications, are outlined in more detail.

Finally, in Section IV, a selected description of the many lines open to future research as a result of the work accomplished so far is provided. An example is offered which illustrates some features of a class of important

problems which can be treated along the lines developed so far. In particular, areas in which the results can be extended and applied in turbomachine practice are outlined. Some of this work has already begun. (In fact, at the time of this final writing certain aspects appear to be encouragingly near completion.)

II. REVIEW OF PROGRESS

The concept of describing three-dimensional flows in highly-loaded turbomachines by the method suggested in the Introduction was tried first in the work leading to Ref. 13. In that initial study an isolated,* moving, rectilinear cascade of lifting lines with incompressible flow moving through it was used as a model of a rotor. The overall turning of the flow caused by the cascade was taken to be large, with the restriction that the spanwise variation of the loading be small. The "axisymmetric" flow was replaced in that model by the pitchwise average of the rectilinear flow. In that sense, the reference flow became that associated with corresponding highly-loaded rectilinear "actuator-disc" models, with weak spanwise variation of the specified mean turning.

Useful methods for describing the three-dimensional flow as a perturbation of the above nonlinear (but two-dimensional) "reference flow" were developed as described in Ref. 13. In rapid succession the results were then extended (again, for small variations of loading, but high work configurations) to an isentropic, compressible case in rectilinear geometry,[14] and to the incompressible annular case.[15] The latter study provided the first opportunity for including important centrifugal effects due to swirl[16] (cf. Refs. 8, 9, 11). Somewhat later, the compressible annular case was

*The assumption of an isolated row allowed use of the approximation that the flow was steady in rotor coordinates.

treated[10] and applied to a highly-loaded transonic cascade of lifting lines representing a ducted fan operating near constant work. These results confirmed that the three-dimensional effects identified in the early works, i.e. for light loading,[1-5] were indeed present in more practical examples. In fact, deviations of predicted flow angles at the blade row, relative to those expected in the absence of significant three-dimensional effects, were magnified by increased loading (Fig. 1), roughly in accordance with expectations arising from strictly linear theory.[5,7,10,12]

After the extension of the original work to the annular case (incompressible, Ref. 15) it became possible to compare our predictions with data obtained from experimental study of flow behind a 23-bladed transonic rotor.[7] The tests referred to there were carried out in the MIT Blowdown Compressor Facility.[17] (The comparison between a theory of incompressible flow behind an annular cascade and data from a transonic blade row were made possible by adjusting the annular area in the former case to obtain the proper mean axial-velocity ratio corresponding to the mean density and area changes that occurred in the actual experiment.) Since we were primarily interested in flow effects induced by vorticity present in the blade wakes downstream of the rotor, compressibility effects other than those grossly affecting the mean flow were not expected to be of primary importance. (As will be seen subsequently, this assumption is not entirely justified, since the interaction between pressure and vorticity disturbances is important in swirling flow.)

Results[7] of this comparison were sufficiently encouraging to warrant continued research in the established framework. At the same time, not unexpected shortcomings of the theory (as it existed then) emerged clearly.

These results led to increased efforts toward obtaining a more complete theoretical description of actual flows through practical axial turbomachinery.

Meanwhile, the work accomplished up to that point was reported in the first Interim Report under the Grant.[18] At about the same time, our results were summarized in an invited paper at a Project Squid (-NASA-AFOSR) workshop, and later published.[19]

It should be emphasized once more that the three-dimensional analyses reported in Refs. 18 and 19 correspond to airfoil and/or propeller theory in an ideal gas, with the important refinement that the perturbations introduced by the individual blades are pictured as being superimposed on a complex, swirling (axisymmetric)"reference flow" rather than on a nearly uniform one. This meant, for example, that in the initial works the vorticity in the "wakes" behind the blades was limited to that associated with spanwise variation of circulation along the blades. Such loading variations lead to trailing vorticity, or shed circulation, in the wakes, much as in classical airfoil theory. This type of vorticity is aligned with the (mean) downstream flow and is often called "Beltrami vorticity". When it is present to a sufficient extent it produces a type of structure in the velocity field* quite distinct from that associated with the lossy-viscous wakes normally considered dominant in turbomachinery. It was accordingly of decisive importance, in the comparisons made in Ref. 7, to discover whether or not such velocity field structures were indeed observed in the flow behind a nonuniformly-loaded rotor.

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*The expected structure included, for example, sawtooth-like variations in the radial velocity.

In the experiment[20,7] these unique structures were found to be clearly discernible downstream of the rotor at radial stations for which there was a sufficiently large spanwise variation of loading on the blading (Figs. 1-4). At the same time, at radial stations for which the blade loading variation was small (Fig. 1), the wake flow structure (Figs. 5-7) was found to be dominated by that more typical of viscous wakes behind airfoils or cascades.

The conclusions drawn from this comparison were that the theory at that stage could be considered (partially) successful, in that it accounted for a significant portion of the observed wake structure, and that the experiment thus provided strong evidence for the presence of downstream trailing vorticity. A three-dimensional theory of the type being developed was therefore necessary in order to contribute to estimates of the flow angles at the individual blades, since such estimates now had to include the induced angle associated with the "Beltrami" vorticity.

However, the comparison made in Ref. 7 also indicated that there were substantial regions downstream of the rotor where the wake flow structure was found to be dominated by effects more typical of viscous wakes behind the blading. This is describable in terms of the corresponding vorticity (which, unlike the Beltami Vorticity, arises because of viscous interaction between the fluid and the solid surfaces formed by the blades) which is swept into these wakes from the flow affected by the blade boundary layers.[11] Study of this effect were, in large part, motivated by the work of Kerrebrock[16,20] which shows that vorticity-and/or entropy-perturbations must couple significantly with pressure (and hence flow-angle) disturbances in highly swirling flow. In the resulting aerodynamic modelling designed to include this effect, the

swrirling downstream flow was pictured as being threaded with vortex filaments (oriented in the spanwise direction immediately aft of each blade) lying within the blade wakes. Because the vorticity field can be related to the variation of the thermodynamic properties of the fluid, the presence of such viscous wakes necessarily implies a variation of entropy from streamline to streamline, or, in the incompressible limit, a corresponding variation of stagnation pressure.

For this reason, the available blade-loss coefficients measured for cascades operating at typical incidence angles were used to establish the strength of wake vorticity of this type. The combination of a strong centrifugally accelerated field with the vorticity arising from the blade wakes turns out to lead (perhaps surprisingly, at first glance) to significant flow angle changes at the blade, (Fig. 8)[11,12]. In turbomachinery, in other words, there is an important "induced incidence" associated with the viscous wakes themselves!

Meanwhile, Greitzer[8] had developed an approximate, beautifully intuitive, analysis which includes many of the important three-dimensional flow features arising from the interaction of a total pressure defect (e.g. due to an inlet distortion) with a swirling mean flow. His results were verified by experimental data obtained using an annular rig constructed at the Whittle Laboratory,[8] Cambridge University, England. In that experiment the existense of a systematic downstream static pressure variation was observed. This work motivated even further development of our theory, based on the analytical techniques applied in Ref. 11. That work directly complements and extends

Greitzer's prediction, and also allows more detailed comparison with the experimental data (Figs. 9,10,11,12). For the lowest circumferential harmonic (i.e. n= 1) representing the total pressure defect, there is substantial agreement with Greitzer's analysis, especially in the region not too far downstream from the blade-row.

Even so, for free-vortex swirl, the more complete theory[11] clearly also shows that while the resulting downstream disturbances (including the static pressure) grow initially in strength for a certain distance downstream of the blade-row, they then decay inversely with (some power of) the axial distance downstream (Figs. 13,14,15). This analysis also demonstrates that the amplitude of the downstream disturbances decreases as the number of significant circumferential harmonics of the inlet distortion increases (compare Fig. 16 with 13).* It is essential to note that these fluid mechanical phenomena are beyond the predictive capability of the two dimensional cascade theory. They are simply absent in that description.

The work under the Grant accomplished up to this point was reported in the second Interim Report submitted to AFOSR.[26] Further, results on the inlet distortion problem were published in the ASME transactions,[9] while initial results on the three-dimensional effects of viscous wakes[11,26] were published in Ref. 12.

Up to the stage described above we had confined our analysis to an isolated blade-row (or single actuator-disc) model involving not only steady flows but also nearly "free-vortex" downstream swirl. However, one of the important results of the comparisons with experimental data in Ref. 7 was to

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*This suggests, of course, that the significance of the effects discussed above may diminish in relative importance as the number of blades in the corresponding blade row is increased.

suggest the possibility of removal of some of these restrictions. It was noticed, for example, that the original Beltrami-flow calculations[13-15] were successful even in regions departing significantly from free-vortex swirl (constant work). This strongly suggested that the basic procedure used for describing the three-dimensional blade-to-blade flow (due to the presence of discrete wake structure of any type) relative to a mean swirling flow of high loading was in no way limited in principle to rotors with small radial variation in blade circulation. This confirmed that the basic idea underlying the suggested "perturbative technique" is related to the fact, mentioned in the Introduction, that a blade-row of a large number of blades, B, can have a large effect on the flow even though each individual blade is only lightly loaded comparatively.[19,21] On this basis, a more ambitious analytic study of strong Beltrami flow was undertaken successfully.[21] Comparison of this newly-developed analysis with the available experimental data[17,20] obtained from M.I.T. blow-down compressor facility was then carried out. That study confirmed, even more strongly, the trend apparent in Ref. 7. (Fig. 17 & 18).

Finally, the opportunity to extend the analytical techniques we had developed up to that point under the Grant to the treatement of unsteady flow through turbomachine blade rows became apparent. A study of three-dimensional effects in unsteady flows, resulting, for example, from the passage of distorted inlet flow through a (rectilinear) moving cascade was undertaken. The three-dimensional "bending and/or torsional" motions of the blades (in rectilinear geometry) have now been analyzed[23] in terms of the three-dimensional characteristics of the corresponding unsteady flow. Initial work on three-dimensional flutter analysis indicates that the flutter boundaries of a compressor array may be changed significantly as a result of unsteady three-dimensional effects.

III EXAMPLE OF THE GENERAL ANALYTICAL TECHNIQUES

In this section, we present an illustration of our general techniques as applied to the study of unsteady three-dimensional flows through typical turbomachine blade rows of the annular and rectilinear type.

The Euler momentum equations for an incompressible fluid can be written in "Lamb's form" as:

$$\frac{\partial \underset{\sim}{V}}{\partial t} - \underset{\sim}{V} \times \underset{\sim}{\Omega} = -\nabla\left(\frac{p_o}{\rho}\right) + \underset{\sim}{F} \tag{3.1}$$

where $\underset{\sim}{V}$ is the velocity, p_o the stagnation pressure, $\underset{\sim}{\Omega} = \nabla \times \underset{\sim}{V}$ the (absolute) vector vorticity, t the time and $+\underset{\sim}{F}$ is a body force per unit mass (local imposed acceleration) reacting on the fluid as a result of forces concentrated at blade surfaces.

In the absence of sources or sinks the continuity condition is, for this example, simply

$$\nabla \cdot \underset{\sim}{V} = 0 \tag{3.2}$$

Let us assume that a velocity potential, ϕ, represents the irrotational part of the flow field, while a vector $\underset{\sim}{A}$ describes the solenoidal part. The effect of bound vorticity on the fluid will be introduced by the use of the "body force" denoted by $\underset{\sim}{F}$. As pointed out above $\underset{\sim}{F}$ represents the reaction on the fluid of the forces experienced by the blades. Clearly, then, away from the blading, $\underset{\sim}{F}$ is zero. Thus, it is useful to write, with $\underset{\sim}{V} = \nabla\phi + \underset{\sim}{A}$,

$$\underset{\sim}{\Omega} = \nabla \times \underset{\sim}{A} = (\nabla\sigma \times \nabla\tau) + (\nabla\alpha \times \nabla G)\hat{H}'(\alpha) \qquad (3.3)$$

employing a "multiple" version of the Clebsch-Hawthorne* approach.[18,22,26] The idea of using more than one combination of the type $\nabla\sigma \times \nabla\tau$, etc.,..., was first found to be useful in steady flow applications.[26] Perhaps its greatest value lies in making possible the identification of various quantities with the physical causes for the presence of various types of vorticity, a point emphasized by Farokhi.[22]

In (3.3) $\hat{H}(\alpha) \equiv \frac{2\pi}{B} H(\alpha)$, $\alpha \equiv \theta - f(r,z,t)$, where $H(\alpha)$ is the unit "staircase function" with jumps of +1 at the blade surfaces $\alpha = 0, \pm\frac{2\pi}{B}, \pm\frac{4\pi}{B}, \ldots,$; B is the number of blades in the blade row to be described. At $\alpha = 0\pm$, we have $H\pm = \pm 1/2$, or $\hat{H}\pm = \pm\frac{\pi}{B}$. $\hat{H}'(\alpha)$ is then $(2\pi/B)\delta_p(\alpha)$, where $\delta_p(\alpha)$ is the "periodic delta function", zero except at the blades. For purposes of convenience below we note that the "sawtooth function", $S(\alpha)$, (familiar to propeller theory) is identically $(\hat{H}(\alpha) - \alpha)$ and hence $S'(\alpha) = \frac{2\pi}{B}\delta_p(\alpha) - 1$. Note, then, that both $S(\alpha)$ and $S'(\alpha)$ have zero azimuthal average around the annulus and that, unlike $\hat{H}(\alpha)$, $S(\alpha)$ is non-secular in θ, i.e., any "cyclic constant" introduced by a function like $\hat{H}(\alpha)$ is removed through the use instead of $S \equiv \hat{H}(\alpha) - \alpha$. Finally, note that all these (singular) functions are to be understood in the sense of generalized functions. Thus, for example, the second term on the rhs of (3.3) is identically $\nabla H(\alpha) \times \nabla G$.

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* The Clebsch method (see Ref. 24) can be thought of as a particular choice of "gauge" for the vector potential, $\underset{\sim}{A}$. Its special effectiveness for flow in turbomachinery was emphasized by Hawthorne.[23]

Substitution of (3.3) in (3.1), together with expansion of the vector triple products in $\underset{\sim}{V} \times \underset{\sim}{\Omega}$ yields,* after some rearrangement:

$$\frac{D\sigma}{Dt}\nabla\tau - \frac{D\tau}{Dt}\nabla\sigma + \frac{D\hat{H}}{Dt}\nabla G - \frac{DG}{Dt}\nabla\hat{H}(\alpha)$$

$$= -\nabla\left(\frac{p_o}{\rho} + \frac{\partial\phi}{\partial t} + \sigma\frac{\partial\tau}{\partial t} + \hat{H}\frac{\partial G}{\partial t}\right) + \underset{\sim}{F}(\underset{\sim}{x},t). \tag{3.4}$$

where $D/Dt \equiv \partial/\partial t + \underset{\sim}{V} \cdot (\partial/\partial\underset{\sim}{x})$. We note first that $D\hat{H}/Dt = (2\pi/B)\,\delta_p(\alpha)\,(D\alpha/Dt)_{be,w}$, and this term vanishes because of the boundary condition at the blades (or wakes) $(D\alpha/Dt)_{bl,w} = 0$. ($\delta_p(\alpha)$ serves the role of "picking out" those values of α = ct. which define the blade (wake) surfaces. It is on such surfaces that the concentrated (bound and "shed") vorticity associated with the blade forces and their time - and spanwise - variation appears. (The extent to which this "concentrated" vorticity actually becomes diffuse of course depends on the relative importance of viscous and turbulent effects.)

Further, on physical grounds within the present model, we expect $\underset{\sim}{F}(\underset{\sim}{x},t)$ to be of the form $a\nabla H(\alpha) = a(\underset{\sim}{x},t)\,\delta_p(\alpha)\nabla\alpha$, i.e., concentrated at the blade surfaces (and -- except for viscous stresses -- directed normal to those surfaces at any instant). We should also expect $\underset{\sim}{F}$ to be related to the acceleration of an identifiable physical quantity. Thus, we can reasonably choose G and $\underset{\sim}{F}$ so as to put

$$\underset{\sim}{F} = -(DG/Dt)\,\nabla\hat{H}(\alpha). \tag{3.4a}$$

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* Note that, with (3.3), $\underset{\sim}{V} = \nabla\phi + \sigma\nabla\tau + \hat{H}\nabla G$, (plus any gradient, as determined by boundary conditions). Alternatively, defining $\hat{\phi} \equiv \phi - \alpha G$, we can write $\underset{\sim}{V} = \nabla\hat{\phi} + \sigma\nabla\tau - G\nabla\alpha + (H(\alpha) - \alpha)\nabla G$, reflecting the required non-cyclic nature of $\underset{\sim}{V}$.

As we shall see below this is a key to finding a useful solution of (3.4), since it turns out to define "G" in a natural way, and also makes contact with classical results for unsteady thin airfoil and linearized cascade theories.*

Interestingly, the interpretation of "G" and "a" (in $\underset{\sim}{F} = a\nabla H$) turns out to be relatively straightforward in turbomachinery: $\frac{2\pi}{B}a$ is the incremental local specific torque introduced at each blade, while $(-G)$ is the mean (azimuthally - averaged) specific angular momentum of the fluid introduced by all the blades up to any axial station.

In any case, with the above use of the blade boundary condition and the definition (3.4a) of the relation between $\underset{\sim}{F}$ and DG/Dt, (3.4) reduces to

$$\frac{D\sigma}{Dt}\nabla\tau - \frac{D\tau}{Dt}\nabla\sigma = -\nabla\left(\frac{p}{\rho} + \frac{\underset{\sim}{V}\cdot\underset{\sim}{V}}{2} + \frac{\partial\phi}{\partial t} + \sigma\frac{\partial t}{\partial t} + \hat{H}(\alpha)\frac{\partial G}{\partial t}\right) \tag{3.5}$$

of the form: $D\tau/Dt = 1$, $\nabla\sigma = \left(\frac{P_o}{\rho} + \frac{\partial\phi}{\partial t} + \sigma\frac{\partial\tau}{\partial t} + \hat{H}\frac{\partial G}{\partial t}\right)$ and

$$\frac{D\sigma}{Dt} = 0. \tag{3.6}$$

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* From the point of view of classical potential theory, "G" plays the role of a cyclic constant which must be specified (because of the annular geometry) to determine the flow uniquely. In fact, just behind a blade row, $(2\pi/B)G$ is the instantaneous, individual blade circulation at the corresponding spanwise location. As will be pointed out later, the requirements needed for defining a self-consistent mean flow imply an additional constraint on G (beyond the definition (3.4.a)) which is similar to a Kutta condition.

Actually, one can show that this is the general solution of (3.4), once it has been reduced to (3.5). Consistent with earlier remarks, we can also write the conserved quantity as

$$(1 - \frac{\partial \tau}{\partial t})\sigma = (\frac{P_o}{\rho} + \frac{\partial \hat{\phi}}{\partial t} - G\frac{\partial \hat{H}}{\partial t}(\alpha)) \qquad (3.6a)$$

The conserved quantity, σ, in (3.6) or (3.6a) is the generalization, for unsteady incompressible flow including vorticity, of the more familiar Bernoulli constant of potential flows. Note that σ reflects the presence of bound vorticity (non-zero blade forces) and its time variation (varying blade circulation) through the $(\partial G/\partial t)$ term. More specifically, it reflects the specific power put into or extracted from the fluid by the blade forces $\underset{\sim}{F}$ at the moving surfaces, $\alpha(t)$. (It can be shown that $\underset{\sim}{F} \cdot \underset{\sim}{V} = -a\frac{\partial \hat{H}}{\partial t} = -a\,\delta_p(\alpha)\frac{\partial \alpha}{\partial t}$ for this case.)

The requirement

$$\frac{D\tau}{Dt} = 1 \qquad (3.7)$$

identifies "τ" as the fluid-particle drift time (see, for example, the discussion given by Hawthorne, Ref. 23). The general solution of (3.6) can be found with the help of τ.

To illustrate, consider the situation for which the unsteadiness of the flow is introduced by the blade row (or rows) itself, so that for the present example, the flow is steady "far upstream" (e.g., in the inlet). Then, from (3.6) and the definition of σ

$$\sigma(\underset{\sim}{x},t) = p_o^{-\infty}\ (\underset{\sim}{x}_{-\infty}\ (\underset{\sim}{x},t))\rho^{-1} + \text{const.} \tag{3.8}$$

where $\underset{\sim}{x}_{-\infty}$ in the upstream position from which the fluid particle at $(\underset{\sim}{x},t)$ emanated earlier. Considering the meaning of τ, as implied by (3.7), we can write

$$\underset{\sim}{x}_{-\infty}\ (\underset{\sim}{x},t) = \underset{\sim}{x} - \int_{-\infty}^{\tau(\underset{\sim}{x},t)} d\tau'\ \underset{\sim}{V}\ (\underset{\sim}{x}',\tau') \tag{3.9}$$

where

$$\underset{\sim}{x}'(\underset{\sim}{x}_{-\infty}\ ,\ \tau') \equiv \underset{\sim}{x}_{-\infty} + \int_{-\infty}^{\tau'} d\tau''\ \underset{\sim}{V}(\underset{\sim}{x}'',\tau'')$$

Thus, while $\nabla\tau = (\partial \underset{\sim}{x}_{-\infty}/\partial \underset{\sim}{x}) \cdot \nabla p_o^{-\infty}(\underset{\sim}{x}_{-\infty})\ (\rho^{-1})$, and $\rho\frac{\partial \varpi}{\partial t} = \frac{\partial \underset{\sim}{x}_{-\infty}}{\partial t} \cdot \nabla p_o^{-\infty}$

$$\frac{\partial \underset{\sim}{x}_{-\infty}}{\partial \underset{\sim}{x}} = \underset{\approx}{I} - \nabla\tau \underset{\sim}{V}(\underset{\sim}{x},t);\ \frac{\partial \underset{\sim}{x}_{-\infty}}{\partial t} = -\underset{\sim}{V}\frac{\partial \tau}{\partial t} \tag{3.10}$$

so that, from (3.8) with (3.7), we have $\rho\frac{\partial \sigma}{\partial t} = -\frac{\partial \tau}{\partial t}\underset{\sim}{V} \cdot \nabla p_o^{-\infty}$ and

$$\rho\underset{\sim}{V} \cdot \nabla\sigma = \underset{\sim}{V} \cdot \left(\frac{\partial \underset{\sim}{x}_{-\infty}}{\partial \underset{\sim}{x}}\right) \cdot \nabla p_o^{-\infty};\ \rho\underset{\sim}{V} \cdot \nabla\sigma = (1 - \underset{\sim}{V} \cdot \nabla\tau)\ \underset{\sim}{V} \cdot \nabla p_o^{-\infty} = +\frac{\partial \tau}{\partial t}\underset{\sim}{V} \cdot \nabla p_o^{-\infty}$$

whence $D\sigma/Dt = 0$ for the circumstances stated. In that case

$$\nabla\sigma = \frac{1}{\rho}\nabla p_o^{-\infty}(\underset{\sim}{x}_{-\infty}(\underset{\sim}{x},t)) \tag{3.11}$$

and that part of the vorticity in (3.3) which depends on $\nabla\sigma$ is absent whenever the inlet flow is uniform.

These basic concepts are familiar from the theory of secondary flow[23]; for this reason, we refer to the $(\nabla\tau \times \nabla\sigma)$ - term in (3.3) as "secondary vorticity", and denote it by $\underset{\sim}{\Omega}_\tau$.

Using the requirement (3.2) and the forms of $\underset{\sim}{V}$ appropriate to (3.3) (see footnote following Eq. (3.4)) we have

$$\nabla^2\phi = -\nabla\sigma \cdot \nabla\tau - \sigma\nabla^2\tau - \nabla\hat{H} \cdot \nabla G - \hat{H}\nabla^2 G$$

or (3.12)

$$\nabla^2\hat{\phi} = -\nabla\sigma \cdot \nabla\tau - \sigma\nabla^2\tau - \nabla S \cdot \nabla G - S\nabla^2 G + \nabla\alpha\cdot\nabla G + G\nabla^2\alpha$$

where $\hat{\phi} \equiv \phi - \alpha G$ and $S(\alpha) \equiv \hat{H}(\alpha) - 1$. Since σ itself depends on ϕ (or $\hat{\phi}$), Eqs. (3.12) are non-linear. They are, however, in a form appropriate to relatively straightforward perturbative methods with regard to a sufficiently well-chosen "reference flow."

In the following, for illustrational purposes, we limit ourselves to the case of "undistorted" inlet flow, i.e., $(\nabla\bar{p}_o^{\,\infty})_{\underset{\sim}{x}_{-\infty}} = 0$. In that situation the only vorticity present is that due to the blade forces and their time - and spanwise - variations. These latter effects lead to "shed" and/or "trailing" vorticity in the wakes of the kind discussed, for example, in Refs. 7, 21, 20, 13 and in the Introduction.

III. A. UNIFORM INLET, INCOMPRESSIBLE FLOW

In this particular case $D\sigma/Dt = 0 = \nabla\sigma$; thus, throughout the flow, $\sigma = \text{const.} = \overline{\sigma}_o$, and

$$\overline{\sigma}_o = \frac{P_o}{\rho} + \frac{\partial\phi}{\partial t} + \overline{\sigma}_o\frac{\partial\tau}{\partial t} + \hat{H}\frac{\partial G}{\partial t}$$

or

$$\overline{\sigma}_o\,(1 - \frac{\partial\tau}{\partial t}) = \overline{\sigma}_o\,(\underset{\sim}{V}\cdot\nabla\tau) = \frac{P_o}{\rho} + \frac{\partial\phi}{\partial t} + \hat{H}\frac{\partial G}{\partial t}\quad . \tag{3.A.1}$$

Alternatively, when $\sigma = \text{const.}$, the term $\sigma\nabla\tau$ in $\underset{\sim}{V}$ and the term $\sigma\frac{\partial\tau}{\partial t}$ in the Lamb definition of σ can simply be included in the corresponding contributions of the potential, ϕ. Thus, with $\sigma = \overline{\sigma}_o = ct$, let

$$\hat{\hat{\phi}} \equiv \phi + \overline{\sigma}_o\tau\ ,$$

whence

$$\underset{\sim}{V} = \nabla\hat{\hat{\phi}} + \hat{H}\nabla G$$

and

$$\overline{\overline{\sigma}}_o = \frac{P_o}{\rho} + \frac{\partial\hat{\hat{\phi}}}{\partial t} + \hat{H}\frac{\partial G}{\partial t} \tag{3.A.1a}$$

Here, $\overline{\overline{\sigma}}_o$ is simply a "renormalized Bernoulli constant", reflecting the absence of inlet distortion effects.

The formulations discussed above are exact (for the stated physical model) but are not particularly amenable to further useful physical interpretation. Before turning to perturbation techniques, however, we can extract at least one additional item of significant information. If we choose the reference flow (or "primary flow" -- see Ref. 23) to be the axisymmetric equivalent (azimuthal average) of the actual flow, we must recognize that such a "reference state" cannot be time-independent in general. For turbomachinery of practical loadings any significant time dependence of the actual flow is reflected strongly in its (axisymmetric) azimuthal average.* The time average of any flow quantity is not, in general, equal to its θ-average, and vice-versa.

III. A. i. THE MEAN (θ-AVERAGED) FLOW AS REFERENCE

In the uniform-inlet, incompressible case the mean flow is defined by

$$\frac{\partial \overline{\underset{\sim}{V}}}{\partial t} - \overline{\underset{\sim}{V}} \times \overline{\underset{\sim}{\Omega}} = \nabla\left(\frac{\overline{p}_o}{\rho}\right) + \overline{\underset{\sim}{F}}_o \quad , \tag{3.A.2}$$

where $\overline{(\ )} \equiv (1/2\pi) \int_o^{2\pi} d\theta\ (\ )$, and

$$\nabla \cdot \overline{\underset{\sim}{V}} = 0 \ . \tag{3.A.3}$$

The mean-flow absolute vorticity is

$$\overline{\underset{\sim}{\Omega}}_s \equiv \nabla\alpha \times \nabla G \tag{3.A.4}$$

where, with $\alpha \equiv \theta - f(r,z,t)$, we must require $G \equiv -\overline{rv}_\theta$. (Since we expect

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* The important exception to this statement, of course, is for situations in in which the time dependence can be transformed away with a simple Galilean transformation. For the "isolated rotor" problem, for example, quantities depend only on $\theta' = \theta - \omega t$.

no secondary vorticity in this example, (3.A.4) contains, as a key part of the definition of the mean flow, simply the θ- average of the last term in (3.3).) The meaning of $\overline{\underset{\sim}{F}}_o$ will be discussed below.

Inserting (3.A.4) in (3.A.2) and expanding as before, we find

$$\frac{\overline{D\alpha}}{Dt} \nabla G - \frac{\overline{DG}}{Dt}\nabla\alpha = -\nabla\left(\frac{p_o}{\rho} + \frac{\partial\overline{\hat{\phi}}}{\partial t} - G\,\frac{\partial\alpha}{\partial t}\right) + \overline{\underset{\sim}{F}}_o \qquad (3.A.5)$$

where now $(\overline{D}/Dt) \equiv (\partial/\partial t + \overline{\underset{\sim}{V}} \cdot \nabla)$ and where we have used, consistently with (3.A.4),

$$\overline{\underset{\sim}{V}} = \nabla\overline{\hat{\phi}} - G\nabla\alpha \equiv \nabla\overline{\phi} + \alpha\nabla G. \qquad (3.A.6)$$

Correspondingly, in view of (3.A.3), (cf. (3.12)):

$$\nabla^2\overline{\phi} = -\nabla\alpha \cdot \nabla G - \alpha\nabla^2 G$$

or (3.A.7)

$$\nabla^2\overline{\hat{\phi}} = +\nabla G \cdot \nabla\alpha + G\nabla^2\alpha$$

The θ-component of (3.A.5) reveals

$$\overline{(rF_\theta)}_o = -\frac{\overline{DG}}{Dt} \equiv \frac{\partial(\overline{rv}_\theta)}{\partial t} + \overline{\underset{\sim}{V}} \cdot \nabla\,(\overline{rv}_\theta) \qquad (3.A.8)$$

Thus, recalling the form of the mean angular momentum conservation equation, we see that $(r\overline{F}_\theta)_o$ is identically the mean specific torque applied to the fluid as the result of the concerted action of all the blades in a given

blade row. This does not yet imply, however, that $\overline{\underset{\sim}{F}}_o$ is the θ-average of $\underset{\sim}{F}$ in (3.1) or (3.4). As (3.4.a) shows, in fact, (with $\alpha = \theta - f(r,z,t)$)

$$\overline{\underset{\sim}{F}} = - \left(\frac{\partial G}{\partial t} + (\overline{\underset{\sim}{V}} + \underset{\sim}{\tilde{v}}) \cdot \nabla G\right)_{bl,w} \nabla\alpha \qquad (3.A.9)$$

or

$$\overline{rF}_\theta = - \frac{\overline{DG}}{Dt} - \langle\underset{\sim}{\tilde{v}}\rangle \cdot \nabla G; \; (G = -\overline{rv}_\theta) \qquad (3.A.9a)$$

Here, $\underset{\sim}{\tilde{v}}$ is defined through $\underset{\sim}{V} \equiv \overline{\underset{\sim}{V}} + \underset{\sim}{\tilde{v}}$, and $\langle\underset{\sim}{\tilde{v}}\rangle_{bl,w}$ is its local value at each blade or wake surface. We then see that $\overline{rF}_\theta$ and $(\overline{rF}_\theta)_o$ differ unless

$$\langle\underset{\sim}{\tilde{v}}\rangle_{bl,w} \cdot \nabla G = 0 \qquad (3.A.10)$$

i.e., unless the meridional component of the variation of $\underset{\sim}{V}$ about its θ-average, $\overline{\underset{\sim}{V}}$, either vanishes at each blade (or wake) surface or itself lies in surfaces of constant mean angular momentum (constant G). (Such "G-surfaces" are surfaces of revolution about the hub of the annulus; it is sometimes helpful to think of G as a "Stokes stream function" for the mean vorticity, together with the corresponding physical meaning of such quantities. Thus, G measures the flux of vorticity through surfaces enclosed by G = const., and is then the circulation for corresponding paths enclosing the hub.)

Consider now the θ-average of Eq. (3.1):

$$\frac{\partial \overline{\underset{\sim}{V}}}{\partial t} + \overline{\underset{\sim}{V} \times \underset{\sim}{\Omega}} = - \overline{\nabla\left(\frac{P_o}{\rho}\right)} + \overline{\underset{\sim}{F}} \qquad (3.A.11)$$

In the present example, this averaging process can be carried out without approximation, even for the non-linear term, because of the concentrated ("singular") nature of $\underset{\sim}{\Omega}$. The result is

$$\frac{\partial \overline{\underset{\sim}{V}}}{\partial t} + \overline{\underset{\sim}{V}} \times \overline{\underset{\sim}{\Omega}} + \langle \tilde{\underset{\sim}{v}} \rangle_{bl,w} \times \overline{\underset{\sim}{\Omega}} = -\nabla\left(\frac{\overline{p_o}}{\rho}\right) + \overline{\underset{\sim}{F}} \tag{3.A.12}$$

Comparing this exact* average with the definition (3.A.2), we see that

$$\overline{\underset{\sim}{F}} - \overline{\underset{\sim}{F}}_o = \langle \underset{\sim}{v} \rangle_{bl,w} \times \overline{\Omega} = (\langle \underset{\sim}{v} \rangle_{bl,w} \cdot \nabla G)\nabla\alpha - (\langle v \rangle_{bl,w} \cdot \nabla\alpha)\nabla G \tag{3.A.13}$$

When (3.A.10) holds, the θ-components of $\overline{\underset{\sim}{F}}_o$ and $\overline{\underset{\sim}{F}}$ are the same, consistent with their both producing the same mean meridional vorticity [of the form $(\frac{1}{r}\frac{\partial G}{\partial z}, \ldots, -\frac{1}{r}\frac{\partial G}{\partial r})$] and the same mean specific angular momentum, $\overline{rv_\theta} \equiv -G$.

Thus, (3.A.10) must hold for the sake of self-consistency. This is tantamount to specifying the required cyclic constant for the potential part of the mean flow, as mentioned earlier.

Then, if there is a remaining difference between $\overline{\underset{\sim}{F}}$ and $\overline{\underset{\sim}{F}}_o$, it lies in meridional planes and (locally) along ∇G. The coefficient is $(\langle v \rangle_{bl,w} \cdot \nabla\alpha)$, and this is related to the boundary conditions at each blade (wake) surface. At such surfaces we recall that $(D\alpha/Dt)_{bl,w} = 0$, i.e.,

$$\frac{\partial \alpha}{\partial t} + \overline{\underset{\sim}{V}} \cdot \nabla\alpha + \langle \underset{\sim}{v} \rangle_{bl,w} \cdot \nabla\alpha = 0 \tag{3.A.14}$$

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* "Exact" within the present physical model.

On the other hand, the requirement (3.A.3) allows us to define, even though the flow may be unsteady, a set of surfaces (α_o, ψ) such that, instantaneously,*

$$\underset{\sim}{\overline{V}} = \nabla\alpha_o \times \nabla\psi + \omega r \hat{e}_\theta \tag{3.A.15}$$

with $\alpha_o \equiv \theta - f_o(r,z,t)$, $\psi = \psi(r,z,t)$. When the blade-row solidity is sufficiently large (i.e., when a sufficiently large number, B, of blades are doing the required job of a given blade row) we can expect the "swirl" surfaces, α_o, to coincide, very nearly, with the surfaces, α, in which the vorticity is concentrated. (In the limit $B \to \infty$, the α and α_o surfaces must merge, a notion which forms the basis for the use of axisymmetric through-flow theory.) Let us then define

$$C(r,z,t) \equiv \alpha - \alpha_o = f_o(r,z,t) - f(r,z,t) \tag{3.A.16}$$

with the expectation that $C \to 0$ as $B \to \infty$.

Clearly, from (3.A.15), at any instant

$$\underset{\sim}{\overline{V}} \cdot \nabla\alpha_o \equiv \omega \tag{3.A.15a}$$

whence, using (3.A.14), the blade (wake) boundary condition yields,

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* We include explicitly the divergence-free vector, $\omega r \hat{e}_\theta$, to allow the possibility that the α- surfaces, and hence α_o-surfaces, may be rotating at angular speed ω, relative to "absolute", i.e., duct-fixed, coordinates.

$$\omega + \frac{\partial \alpha_o}{\partial t} + \frac{\partial C}{\partial t} + \overline{\underset{\sim}{V}} \cdot \nabla C = -\langle \tilde{\underset{\sim}{v}} \rangle_{bl,w} \cdot \nabla (\alpha_o + C)$$

or, (3.A.16a)

$$\frac{\overline{DC}}{Dt} = -\langle \tilde{\underset{\sim}{v}} \rangle_{bl,w} \cdot \nabla (\alpha) \quad -(\frac{\partial \alpha_o}{\partial t} + \omega)$$

Now, returning to (3.A.5) and using (3.A.9), (3.A.13), and the constraint (3.A.10) together with (3.A.16), we can write: [recall, a = - $\overline{DG}/Dt$ (= -DG/Dt, from (3.A.10))]

$$\overline{\underset{\sim}{F}}_o = a\nabla\alpha + (\frac{\overline{DC}}{Dt})\nabla G \tag{3.A.17}$$

so that

$$\frac{\overline{D\alpha_o}}{Dt} \nabla G + \frac{\overline{DC}}{Dt} \nabla G - 0 = -\nabla(\frac{\overline{P_o}}{\rho} + \frac{\overline{\partial \hat{\phi}}}{\partial t} - G\frac{\partial \alpha}{\partial t}) + (\omega + \frac{\partial \alpha_o}{\partial t} + \frac{\overline{DC}}{Dt})\nabla G$$

and, hence, since $\overline{\underset{\sim}{V}} \cdot \nabla \alpha_o \equiv \omega$,

$$0 = -\nabla(\frac{\overline{P_o}}{\rho} + \frac{\overline{\partial \hat{\phi}}}{\partial t} - G\frac{\partial \alpha}{\partial t}) \; . \tag{3.A.18}$$

Thus, in fact, the mean "Bernoulli pressure" is modified significantly by unsteadiness in highly-loaded machines. Note that (3.A.18) implies

$$\frac{\overline{P_o}}{\rho} + \frac{\overline{\partial \hat{\phi}}}{\partial t} - G\frac{\partial \alpha}{\alpha t} = P(t) \; ; \tag{3.A.18a}$$

indeed, $P(t) = (\overline{p}_o - \infty)/\rho$ = const. when there is no unsteadiness in the inlet.

(Note also that if the only time-dependence is that associated with steady rotation of the blade surfaces, so that $\partial\alpha/\partial t = -\omega$, the above conserved quantity becomes

$$\overline{\frac{p_o}{\rho}} + G\,\omega \equiv \overline{\frac{p_o}{\rho}} - \omega\overline{rv_\theta} = \text{const.}, \qquad (3.A.18b)$$

i.e., it is then precisely the "rotary stagnation pressure", as expected.)

The equations determining the (α_o, ψ) surfaces provide a useful alternative formulation of the mean flow problem, which can be used, for example, as a complement to Eqs. (3.A.6,7) for $\overline{\phi}$. Given that $\overline{\underset{\sim}{V}}$ is described in the present case by (3.A.15) at any instant, while curl $\overline{\underset{\sim}{V}} \equiv \overline{\underset{\sim}{\Omega}}_s = \nabla\alpha \times \nabla G$ from (3.A.4), we have

$$\overline{\underset{\sim}{\Omega}} \cdot \nabla\alpha = 0 \; ; \quad \overline{\underset{\sim}{\Omega}} \equiv (\overline{\xi}, \overline{\eta}, \overline{\zeta})$$

or

$$\therefore \overline{\eta}/r = \overline{\underset{\sim}{\Omega}} \cdot \nabla f = \frac{1}{r}\left(\frac{\partial G}{\partial z}\frac{\partial f}{\partial r} - \frac{\partial G}{\partial r}\frac{\partial f}{\partial z}\right) = \nabla\theta \cdot (\nabla G \times \nabla f)$$

But $\overline{\eta} = (\partial\overline{u}/\partial z - \partial\overline{w}/\partial r)$, while the mean meridional velocity, $(\overline{u}, \overline{w})$, = $\left(\frac{1}{r}\frac{\partial\psi}{\partial z}, -\frac{1}{r}\frac{\partial\psi}{\partial r}\right)$ from (3.A.15). Then

$$\Delta^*\psi = r^2\nabla\theta \cdot (\nabla G \times (\nabla f_o - \nabla C)) \qquad (3.A.19)$$

where $\Delta^*\psi \equiv \nabla^2\psi - \frac{2}{r}\frac{\partial\psi}{\partial r}$ in cylindrical coordinates.

This is the appropriate equation[6,19] (for the incompressible case) determining the Stokes stream function in terms of the mean vorticity generated at the blades. This equation is equivalent to any of a variety of formulations of "axisymmetric throughflow theory". In practical situations it is solved numerically (exact analytic solutions are very rare); for our present purposes, we regard its solution as known or readily at hand, given the rhs or sufficient information to determine it: (torque schedule, mean swirl, ...).

At the same time, the mean swirl surfaces, $\alpha_o = \theta - f_o(r,z,t)$, are determined, at each instant, from the requirement $\overline{\underset{\sim}{V}} \cdot \nabla\alpha_o = \omega$; that is,

$$r\overline{\underset{\sim}{V}} \cdot \nabla f_o = \overline{v}_\theta - \omega r$$

or by characteristic integration at constant ψ(and t):

$$f_o = \int_{\text{const. } \psi}^{z} \frac{dz'}{\overline{w}} \left(\frac{-G}{r^2} - \omega \right) + f_o^o(t) \qquad (3.A.20)$$

In order that the relative camber displacement, C(r,z,t) in (3.A.16), remain bounded for all time, even when the α-surfaces are rotating, we must choose $f_o^o(t) = \omega t$. Then

$$\alpha_o = \theta - \omega t + \int^{z} \frac{dz'}{\overline{w}} \left(\frac{-G}{r^2} - \omega \right) . \qquad (3.A.21)$$

When there is no rotation, of course, $\omega \equiv 0$.

We now see explicitly, either from the formulation in (3.A.19) or from that in (3.A.7), that the mean flow "reference" problem is coupled with the perturbation problem (at least) through the displacement, C, relating the α- and α_o - surfaces. The function C(r,z,t) must satisfy, according to (3.A.16a),

$$-\frac{DC}{Dt} = \omega + \frac{\partial \alpha_o}{\partial t} + \langle \tilde{\underset{\sim}{v}} \rangle_{bl,w} \cdot \nabla \alpha_o \ . \qquad (3.A.22)$$

While α_o depends only on mean-flow quantities, C(r,z,t) can be determined only by solving for the perturbation velocity $\tilde{\underset{\sim}{v}}$.

As anticipated, these results show that any unsteady effects, even though they may be "small" in some sense, directly affect the mean-flow itself, to <u>first order</u> in the perturbations. They do so in two ways:

1) kinematically, they enter into the "source" terms for either $\overline{\phi}$(3.A.7) or ψ (3.A.19) as a result of the displacement between α- and α_o- surfaces;

2) dynamically, they have a direct effect on the mean "total pressure" (see (3.A.18a)) and hence on the instantaneous θ-averaged thermodynamic properties.

In the incompressible case these two effects of time dependence can be analyzed separately, as we have shown above. For compressible flow, however, the dynamical and kinematic features are intertwined in a very interesting way. In either case, important aspects of the mean flow remain undetermined until the three-dimensional perturbation problem itself is solved. This can only be done in conjunction with self-consistent perturbations of the mean-flow problem as indicated above, and in the following.

III. A ii. THE PERTURBATION PROBLEM FOR UNSTEADY FLOW IN HIGHLY-LOADED TURBOMACHINES; UNIFORM INLET FLOW.

With uniform inlet flow the vorticity appropriate to the present model is, from (3.3),

$$\underset{\sim}{\Omega} = \nabla\hat{H}(\alpha) \times \nabla G \tag{3.A.23}$$

while

$$\overline{\underset{\sim}{\Omega}} = \nabla\alpha \times \nabla G \tag{3.A.24}$$

Hence the peturbation vorticity, $\underset{\sim}{\tilde{\Omega}} \equiv \underset{\sim}{\Omega} - \overline{\underset{\sim}{\Omega}}$, is simply

$$\underset{\sim}{\tilde{\Omega}} = \nabla S(\alpha) \times \nabla G \tag{3.A.25}$$

where $S(\alpha) \equiv \hat{H}(\alpha) - \alpha$ is the sawtooth function introduced briefly below Eq. (3.3). $S(\alpha)$ has the values $\pm\pi/B$ on the pressure (+) and suction (-) sides of each of the blades, respectively. Also, from the point of view of generalized functions, $S(0) \equiv 0$. That is, operationally, it is zero "in" each blade or wake surface. It is also periodic, $S(\alpha) = S\left(\alpha + \frac{2\pi n}{B}\right)$, and has zero θ-average over a blade-passage interval.

The perturbation velocity, $\underset{\sim}{\tilde{v}}$, corresponding to the vorticity in (3.A.25), is

$$\underset{\sim}{\tilde{v}} = \nabla\tilde{\phi} + S\nabla G, \tag{3.A.26}$$

and, since div $\underset{\sim}{v}$ = div $\overline{\underset{\sim}{v}}$ = 0, we require div $\underset{\sim}{\tilde{v}}$ = 0. Then

$$\nabla^2\tilde{\phi} = -\nabla S\cdot\nabla G - S\nabla^2 G \qquad (3.A.26a)$$

As we have seen earlier, with $\underset{\sim}{V} = \overline{\underset{\sim}{V}} + \underset{\sim}{v}$ and $\nabla\cdot\underset{\sim}{V} = \nabla\cdot\overline{\underset{\sim}{V}} = \nabla\cdot\tilde{\underset{\sim}{v}} = 0$, while $\underset{\sim}{V} = \nabla\hat{\phi} - G\nabla\alpha + S(\alpha)\nabla G$, and $\overline{\underset{\sim}{V}} = \nabla\overline{\hat{\phi}} - G\nabla\alpha$, we have both for the complete and for the reference flow:

$$\text{a) } \nabla^2\hat{\phi} = -\nabla S\cdot\nabla G - S\nabla^2 G + \nabla\alpha\cdot\nabla G + G\nabla^2\alpha$$

$$\text{b) } \nabla^2\overline{\hat{\phi}} = \nabla\alpha\cdot\nabla G + G\nabla^2\alpha, \qquad (3.A.27)$$

consistent with $\tilde{\phi} \equiv \hat{\phi} - \overline{\hat{\phi}}$. Note that actually these Eqs. are not themselves more difficult than (3.A.26). However, the applicable boundary conditions are:

$$\text{a) } \nabla\hat{\phi}+\ldots;\ \text{(Complete flow):}\ \frac{D\alpha}{Dt} = 0\ \left(\begin{matrix}\text{blade,}\\ \text{wake}\end{matrix}\right);\ \underset{\sim}{V}\cdot\nabla\psi = 0\ \left(\begin{matrix}\text{hub,}\\ \text{shroud}\end{matrix}\right) \qquad \text{a)}$$

$$\text{b) } \nabla\tilde{\phi}+\ldots;\ \text{(Mean flow):}\ \overline{\underset{\sim}{V}}\cdot\nabla\alpha_o = \omega\ \left(\begin{matrix}\text{Swirl}\\ \text{surfaces}\end{matrix}\right);\ \overline{\underset{\sim}{V}}\cdot\nabla\psi \equiv 0\ \left(\begin{matrix}\text{hub,}\\ \text{shroud}\end{matrix}\right) \qquad \text{b)}$$

$$\text{c) } \nabla\phi+\ldots;\ \text{(perturbation):}\ \frac{DC}{Dt} = -\left[\left(\omega + \frac{\partial\alpha_o}{\partial t}\right) + \underset{\substack{\text{bl,w}\\ \text{blade,}\\ \text{wake}}}{\langle\tilde{\underset{\sim}{v}}\rangle}\cdot\nabla\alpha_o\right];$$

$$\text{and, } \tilde{\underset{\sim}{v}}\cdot\nabla\psi = 0\ \left(\begin{matrix}\text{hub,}\\ \text{shroud}\end{matrix}\right) \qquad \text{c)}$$

(3.A.28 a,b,c)

where, by definition, $\alpha \equiv \alpha_o + C(r,z,t)$, and is still to be determined.

The presumption is, of course, that if a perturbation technique is at all applicable, then not only are the swirl surfaces, α_o, determined entirely from the mean flow, but also that $|\nabla C| << |\nabla \alpha_o|$; in fact, we expect $|\nabla C| / |\nabla \alpha_o| = 0\ (1/B^n)$, where $n > 1$. [One should not be misled by the presence of the term $(\omega = \frac{\partial \alpha_o}{\partial t})$ in (3.A.28c); e.g., $\omega \equiv 0$ if the blade forces are not on rotating surfaces, while, generally, (3.A.21) shows that

$$(\omega + \frac{\partial \alpha_o}{\partial t}) = \frac{\partial}{\partial t} \overline{\int^{z} \frac{dz'}{\bar{w}} (-\frac{G}{r^2} - \omega)} \quad ,$$

$$= ct.$$

so $(\omega + \frac{\partial \alpha_o}{\partial t})$ is actually "small" (more precisely, perturbative) and represents the extent to which α_o-surfaces are "rippled" by inherent unsteady effects.]

On the basis of these equations the perturbation procedure can be conceptualized readily. Equation (3.A.27b) contains no singular functions, but the rhs still involves the unknown displacement C(r,z,t). Let us then define a "zeroth-order" mean velocity

$$\bar{\underset{\sim}{v}}_o \equiv \nabla \bar{\phi}_o - G \nabla \alpha_o \quad . \tag{3.A.29}$$

and corresponding perturbation mean velocity, $\bar{\underset{\sim}{v}}_o$ such that

$$\bar{\underset{\sim}{v}}_o \equiv \bar{\underset{\sim}{v}} - \bar{\underset{\sim}{v}}_o = \nabla(\hat{\phi} - \bar{\phi}_o) - G \nabla C \equiv \nabla(\phi_o) - G \nabla C \tag{3.A.30}$$

(Clearly, $\overline{\underset{\sim}{V}}_o$ is a purely meridional vector, has no θ-dependence and is its own θ-average.) Thus, if the net perturbation, relative to $\overline{\underset{\sim}{V}}_o$, is "$\underset{\sim}{v}$", then

$$\underset{\sim}{v} \equiv \tilde{\underset{\sim}{v}} + \overline{\underset{\sim}{v}}_o = \nabla\phi_o - G\nabla C + \nabla\tilde{\phi} + S\nabla G \equiv \nabla\phi + S\nabla G - G\nabla C \qquad (3.A.31)$$

while

$$\underset{\sim}{\omega} \equiv \text{curl } \underset{\sim}{v} = \nabla C \times \nabla G + \nabla S \times \nabla G \qquad (a)$$

and (3.A.32)

$$\overline{\underset{\sim}{\Omega}}_o \equiv \text{curl } \overline{\underset{\sim}{V}}_o = \nabla\alpha_o \times \nabla G \qquad (b)$$

Note that $\overline{\underset{\sim}{V}}_o$, as defined by (3.A.29), is keyed to the same value of G (cf. (3.A.6)) as is the actual mean flow, $\overline{\underset{\sim}{V}}$. In addition to having identical θ-components (= $-G/r$), each is referred to the same meridional vorticity:

$$\overline{\underset{\sim}{\Omega}}_{mo} = \overline{\underset{\sim}{\Omega}}_m = \left(\frac{1}{r}\frac{\partial G}{\partial z}\,,\; -\frac{1}{r}\frac{\partial G}{\partial r}\right) \equiv \nabla\theta \times \nabla G \qquad (3.A.33)$$

However, if $\overline{\underset{\sim}{V}}_o$ is to be useful as a first approximation (zeroth iterate) for the mean reference flow, we must be able to use it to generate the α_o - surfaces, to a good approximation, without prior knowledge of $C(r,z,t)$. Thus, we need also $\overline{\underset{\sim}{V}}_o \cdot \nabla\alpha_o = \overline{\underset{\sim}{V}} \cdot \nabla\alpha_o \equiv \omega$, or

$$\overline{\underset{\sim}{V}}_{mo} \cdot \nabla f_o = \overline{\underset{\sim}{V}}_{m} \cdot \nabla f_o = -(\omega + G/r^2) \ . \qquad (3.A.32c)$$

These requirements are met adequately if we choose, as a complement to (3.A.29),

$$\overline{\underset{\sim}{V}}_{o} = -\frac{G}{r}\hat{e}_\theta + \nabla\theta \times \nabla\psi_o \qquad (3.A.34)$$

whence $\overline{\underset{\sim}{V}}_{o} \cdot \nabla f_o = -(\nabla f_o \times \nabla\psi_o) \cdot \nabla\psi$, and to fulfill (3.A.32c):

$$f_o(r,z,t) = -\int_{\substack{-\infty \\ \text{constant } \psi_o}}^{z} \frac{dz'}{\overline{w}_o}\left(\frac{G}{r^2} + \omega\right) \qquad (3.A.35)$$

(compare (3.A.20), (3.A.21)). Here, $\overline{w}_o \equiv -(1/r)(\partial\psi_o/\partial r)$. Note that, in this initial approximation, the ψ_o- and ψ- surfaces do not generally coincide*. However, we can assure that they do so, more and more as the perturbation procedure is carried out, by requiring $\psi_o\big|_{hub} = \psi\big|_{hub}$ and $\psi_o\big|_{shroud} = \psi\big|_{shroud}$ as boundary conditions in the equation for ψ_o described below.

As with ψ, the equation for ψ_o is obtained by requiring consistency with the vorticity condition, in this case (3.A.32). Thus, with $\overline{\underset{\sim}{\Omega}}_{o} \cdot \nabla f_o = \overline{\eta}_o/r$, we find

$$\Delta^*\psi_o = r^2\, \nabla\theta \cdot (\nabla G \times \nabla f_o) \qquad (3.A.36)$$

---

* An exception occurs when $C(r,z,t) = C(G)$.

ψ_o can then be determined from this equation, subject to the hub-shroud boundary conditions stated above. This allows one, in turn, to determine the first approximation to the swirl surfaces, $\alpha_o^{(o)}$, through (3.A.35).

We are now in a position to return to (3.A.26a) to determine, to lowest order, the blade-to-blade perturbations, given appropriate boundary conditions at the blades, as well as at the hub and shroud. (The blade boundary conditions can be rather elaborate, depending on the problem at hand, but in any case must include (3.A.10) -- or its generalization for a given blade thickness profile -- as a matter of consistency.) In treating the blade-to-blade problem, it is essential, on the rhs of (3.A.26a) or at any stage (n) of an iterative or perturbative procedure, to use:

$$S(\alpha_o^{(o)}), \ \ldots \ , S(\alpha_o^{(n)}); \lim_{n\to\infty} \alpha_o^{(n)}$$

rather than $S(\alpha)\big|_{\alpha=\alpha_n}$. (Otherwise, the perturbative method will diverge.) On solution of this problem, $C(r,z,t)$ ... or its corresponding iterate, $C^{(n)}(r,z,t)$ --- is available (see (3.A.28c) with (3.A.26a)) and $\alpha = \alpha_o + C$ (or $\alpha_o^{(n)} \equiv \alpha_o^{(n-1)} + C^{(n)}$) can be determined.

Finally, returning to (3.A.20) and (3.A.19), improved versions of the (α_o, ψ) surfaces can be obtained; these are then to be used to find improved solutions of (3.A.26a), and so on, ...,. Providing due care is taken to maintain internal consistency, and the boundary conditions are appropriately monitored, this procedure converges extremely rapidly. (For analytic purposes much of the above can be handled by implicit methods.)

There is a useful physical picture underlying starting the above perturbative solution of the unsteady flow problem at high loadings in the manner outlined above. By initially concentrating distributed vorticity of (relatively well-known) strength on the mean-flow swirl (α_o) surfaces in high-deflection, turbomachinery flows, one avoids the difficulties, encountered in earlier studies, of allowing any shed or trailing vorticity to be convected "through" the regions of concentrated bound vorticity, i.e. "through" what amount to solid surfaces!

III. B. CLEBSCH-HAWTHORNE METHOD FOR THIN AIRFOIL EXAMPLE

In this section we offer, very briefly, an example, taken from the classical theory of incompressible unsteady two-dimensional flow past thin airfoils, for which the above analysis makes contact with well-known for a simpler but related problem.

Returning to (3.4), we recall that a key step in dissecting that relation was the identification of the "boundary-condition"-term (3rd term on l.h.s. of (3.4) vanishes) and, also, the "body-force" term leading to (3.4a).

For the thin airfoil case (steady or unsteady) the reference flow is a uniform, steady potential flow $\overline{\underset{\sim}{V}}_o \equiv (U,0,0)$ in an (x,y,z) cartesian system. [Conventionally, $x \rightarrow$ (streamwise), $y \rightarrow$ (spanwise), $z \rightarrow$ (normal - to airfoil)]. Then, on linearizing

$$\frac{D}{Dt} \rightarrow \frac{\overline{D}}{Dt} \rightarrow \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right) \equiv \frac{D_o}{Dt} \qquad (3.B.1)$$

Again, the controlling equations are, for this example,

$$\frac{\partial \underset{\sim}{V}}{\partial t} - \underset{\sim}{V} \times \underset{\sim}{\Omega} = -\nabla\left(\frac{p_o}{\rho}\right) + \underset{\sim}{F} \; ; \; \text{div } \underset{\sim}{V} = 0$$

In this case, $\nabla p_o = 0$, except for perturbations. The flow is, correspondingly, potential (curl $\underset{\sim}{V} = 0$) aside from perturbations introduced by the airfoil and in its wake.

Thus, if $\underset{\sim}{V} = \nabla\phi + \underset{\sim}{A}$ as before, and if the vorticity is written

$$\underset{\sim}{\Omega} = \nabla H(z) \times \nabla G \equiv \delta(z)(\hat{e}_z \times \nabla G) \tag{3.B.3}$$

we find $\underset{\sim}{\Omega}$ to be concentrated on the plane $z = 0$, and "G" is, itself, a perturbation quantity -- still to be determined. This is precisely the picture offered in thin airfoil theory.

Making the same expansions as in section III, we have

$$\frac{DH(z)}{Dt} \nabla G - \frac{DG}{Dt} \nabla H(z) = -\nabla\left(\frac{p_o}{\rho} + \frac{\partial\phi}{\partial t} + H \frac{\partial G}{\partial t}\right) + \underset{\sim}{F}(\underset{\sim}{x},t) \tag{3.B.4}$$

But, from the (exact) boundary condition, the first term vanishes. Identifying $\underset{\sim}{F}(\underset{\sim}{x},t)$ as before, viz.,

$$\underset{\sim}{F} = -\frac{DG}{Dt} \nabla H(z) \equiv -\frac{DG}{Dt} \delta(z) \triangleq -\left[\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)G\right] \delta(z) \equiv -\frac{D_o G}{\nabla t} \delta(z)$$

(3.B.5)

we are left with: (recall $\nabla p_o = 0$ for upstream)

$$\left(\frac{P_o}{\rho} + \frac{\partial \phi}{\partial t} + H(z)\,\frac{\partial G}{\partial t}\right) = \text{ct.} \tag{3.B.6}$$

as the "Bernoulli constant". (Note the jump in the $\frac{\partial G}{\partial t}$ term across $z = 0$.)

On the other hand, the physical meaning of $\underset{\sim}{F}$ is, in this case, such that it must balance the instantaneous jump in static pressure ("loading") at the airfoil surface:

$$\underset{\sim}{F} = \frac{\nabla p}{\rho}\,(x,t)\;\partial(z) \equiv \frac{\Delta p}{\rho}\,\nabla H \tag{3.B.7}$$

Thus, from (3.B.5),

$$-\frac{D_o G}{Dt} = -\frac{\partial G}{\partial t} - U\,\frac{\partial G}{\partial x} = \frac{\Delta p(x,t)}{\rho} \tag{3.B.8}$$

or, if $\Delta p(x,t) = \Delta p_\omega(x) e^{i\omega t}$ and $G(x,t) = G_\omega(x) e^{i\omega t}$

$$-i\omega G_\omega = U\,\frac{\partial G_\omega}{\partial x} + \frac{\Delta p\,\omega}{\rho}\,(x) \tag{3.B.9}$$

and, if the leading edge of the airfoil is at $x = 0$,

$$G_\omega(x) = -\left[e^{+i\,\omega x/u} \int_o^x dx' \left(\frac{\Delta p \omega}{U}\right) e^{-i\omega\, x'/U}\right]$$

Then,

$$\nabla G_{\omega}(x) = \hat{e}_x \left[-\frac{\Delta p_\omega}{\rho U} - \frac{i\omega}{U} \int_o^x dx' \left(\frac{\nabla p_\omega}{\rho U}\right) e^{\frac{i\omega(x-x')}{U}} \right]$$

and, identifying $\gamma_B = \gamma_{BOUND} \equiv (\Delta p(x,t)/\rho U)$, we have

$$\underset{\sim}{\Omega}(x,t) = \hat{e}_y \, \delta(z) \left[-\gamma_{b\omega} e^{i\omega t} - \frac{i\omega}{U} \int_o^x dx' \, \gamma_{B\omega} \, e^{i\omega(x-x')/U} \right]$$

This is the classical thin-airfoil result, distinguishing the bound vorticity on the airfoil and the free, "shed" vorticity in the wake, x > CHORD.

With (3.B.10), one can determine the relationship between the instantaneous blade loading, the vorticity on and off the blades, the corresponding velocity field, and using (3.B.6), the pressure field. These results agree precisely with those obtained from "classical" procedures (including for example, use of the Biot-Savart Law - which is, unfortunately, extremely awkward in applications to internal flows such as those typical to turbomachinery).

FUTURE STUDIES

As a result of the work accomplished so far, there are many lines open to future research. Examples are extension of present work to the case of the behavior of blade-wakes (consisting of the Beltrami vorticity as well as that from boundary layers swept downstream of the blades) in flows with the reference flow departing substantially from free-vortex swirl, and of more importance, extensions through highly-loaded blade rows of finite chord length, c. Comparison of the present theory with available experimental data (obtained by work sponsored by the AFOSR in UTRC and at M.I.T. Blow-Down Compressor Facility) are currently under way. Such comparisons will undoubtedly help to further our understanding of the three-dimensional aspects of turbomachinery flow, and aid us in assessing the importance of the finiteness of the blade-chord, thus leading to improvements and extensions of our analytical work.

Another area in which a potential exists for a major contribution is in the combination of the analytical approach described above with the newly developing time-marching numerical schemes for studying turbomachinery flows - steady flow and unsteady flow in future. In the numerical solution of turbomachinery flow, for practical purposes, it is almost always the case that uniform boundary conditions are imposed at approximately one chord upstream and downstream of the blade-row. Theoretical studies have shown that upstream acoustic radiation occurs in transonic blade-rows, and the existence of oscillating disturbances downstream of the blade-row. Thus, the upstream and downstream boundary conditions are never actually uniform,

except in the truly "incompressible" limit. Thereby, it emerges that our analytic work can be used to provide the near upstream and downstream boundary conditions, thus allowing more precise numerical study of blade-passage regions. Such a numerical-analytical "hybrid" scheme helps to put numerical solutions of turbomachinery flows on sound mathematical and scientific ground. Future studies will certainly include analysis of unsteady flows in turbomachines including, now, of that in highly loaded blade rows.

In contrast to our earlier works where lifting line models and activator disc models are used (see Introduction), the present and future emphasis will be on the more realistic models of highly-loaded blade rows of finite chord length. It is expected that the later extensions will cover both steady and unsteady problems, and with particular emphasis on "three-dimensional design" of highly loaded rotors. As illustrated in Sections III and III.A., some of these extensions are already in progress. In what follows, a brief account is given of the three-dimensional problems of steady and unsteady flows through highly-loaded rotor of finite chord length, with possible application to the rotor design problem.

In the context of the design problem, we seek the determination of a blade shape, given a specified loading (i.e., $\underset{\sim}{F}$). The basic notion then involves the finding of appropriate blade shapes by distributing vorticity on the α_o-surfaces so as to create the required local $\underset{\sim}{F}$. The specification of $\underset{\sim}{F}$, aside from allowing the determination of the stream surfaces ψ and the α_o surfaces from the mean flow equations, enables one to compute the strength of the distributed bound vorticity on the α_o surfaces. This then permits the actual camber lines relative to the α_o surfaces to be calculated.

The advantages of such an approach for the design problem are: (i) large deflections are included in the mean, azimuthally-averaged flow, and (ii) any shed vorticity emanating from the bound vortex system will be carried along the α_o surfaces, thus eliminating the problem of earlier linearized theories of having such free vortices crossing the airfoil surfaces. Furthermore, such an approach makes possible the interpretation of the physical phenomena associated with the flow. It can also be adapted to the 'analysis' problem, and thus its use for predictive and interpretive purposes. As shown in Section III, the general formulations of the blade-to-blade equations (or rotor passage equations) in terms of the multiple Glebesch-Hawthorne transformation may turn out to be an elegant and practical way to study the unsteady internal flow in turbomachines. Two possible approaches to the solution are available: (i) an iterative method, and (ii) an implicit perturbative scheme. This arises because the mean reference flow (time-average and/or azimuthal average) is coupled with the perturbation flow through the displacement, C (a relative camber), relating the α- and α_o surfaces, together with the various constraints on the flow. Section III.B shows how the Clebesch-Hawthorne approach recovers the results of the classical thin-airfoil theory elegantly.

Preliminary works on unsteady flows in turbomachines using the present approach has indicated the significance of three-dimensional effects with regard to blade flutter boundaries, even though, up til now, only lightly-loaded rectilinear cascades are being considered. As mentioned, earlier, future studies will attempt to treat the unsteady flow through highly-loaded blade-rows, and to make possible combined numerical analytical study of unsteady flow phenomena in turbomachines.

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FIGURE CAPTIONS

Fig. 1 Radial distribution of flow angle deflection given by the free vortex solution, the axisymmetric activator disc solution and the present three-dimensional Beltrami flow solution.

Fig. 2 Comparison of the blade-to-blade radial Mach number as given by the Beltrami flow theory with experimental data obtained at M.I.T. Blow-Down Compressor Facility.

Fig. 3 Comparison of the blade-to-blade pitchwise Mach number as given by the Beltrami flow theory with the experimental data obtained at M.I.T. Blow-Down Compressor Facility.

Fig. 4 Comparison of the blade-to-blade axial Mach number as given by the Beltrami flow theory with the experimental data obtained at M.I.T. Blow-Down Compressor Facility.

Fig. 5 Comparison of the blade-to-blade radial Mach number as given by the Beltrami flow theory with the experimental data obtained at M.I.T. Blow-Down Compressor Facility.

Fig. 6 Comparison of the blade-to-blade pitchwise Mach number as given by the Beltrami flow theory with the experimental data obtained at M.I.T. Blow-Down Compressor Facility.

Fig. 7 Comparison of the blade-to-blade axial Mach number as given by the Beltrami flow theory with the experimental data obtained at M.I.T. Blow-Down Compressor Facility.

Fig. 8 Radial distribution of flow angle deflection given by the radial equilibrium solution, the axisymmetric activator disc solution, the three-dimensional solution with the circumferential non-uniformity due to the presence of wakes with total pressure defect.

Fig. 9 Downstream convection of stagnation pressure distortion at $r/r_T = 0.83$ as given by the linearized theory and experimental data.

Fig. 10 Axial velocity distribution at different axial stations for radius $r/r_T = 0.83$.

Fig. 11 Comparison of downstream static pressure perturbations at radius $r/r_T = 0.83$: Theory vs. Experiment.

Fig. 12 Comparison of downstream flow angle perturbations at radius $r/r_T = 0.83$: Theory vs. Experiment.

Fig. 13 Downstream variation of static pressure perturbation at the tip.

Fig. 14 Downstream variation of radial velocity at radius $r/r_T = 0.55$.

Fig. 15 Downstream variation of absolute flow angle at the hub.

Fig. 16 Downstream variation of static pressure perturbation at the tip for the case with five loop type of stagnation pressure distortion introduced far upstream.

Fig. 17 Pitchwise variation of axial velocity at various radial stations as given by the strong Beltrami flow theory and the M.I.T. Blow-Down Compressor Facility.

Fig. 18 Pitchwise variation of tangential velocity at various radial stations as given by the strong Beltrami flow theory and the M.I.T. Blow-Down Compressor Facility.

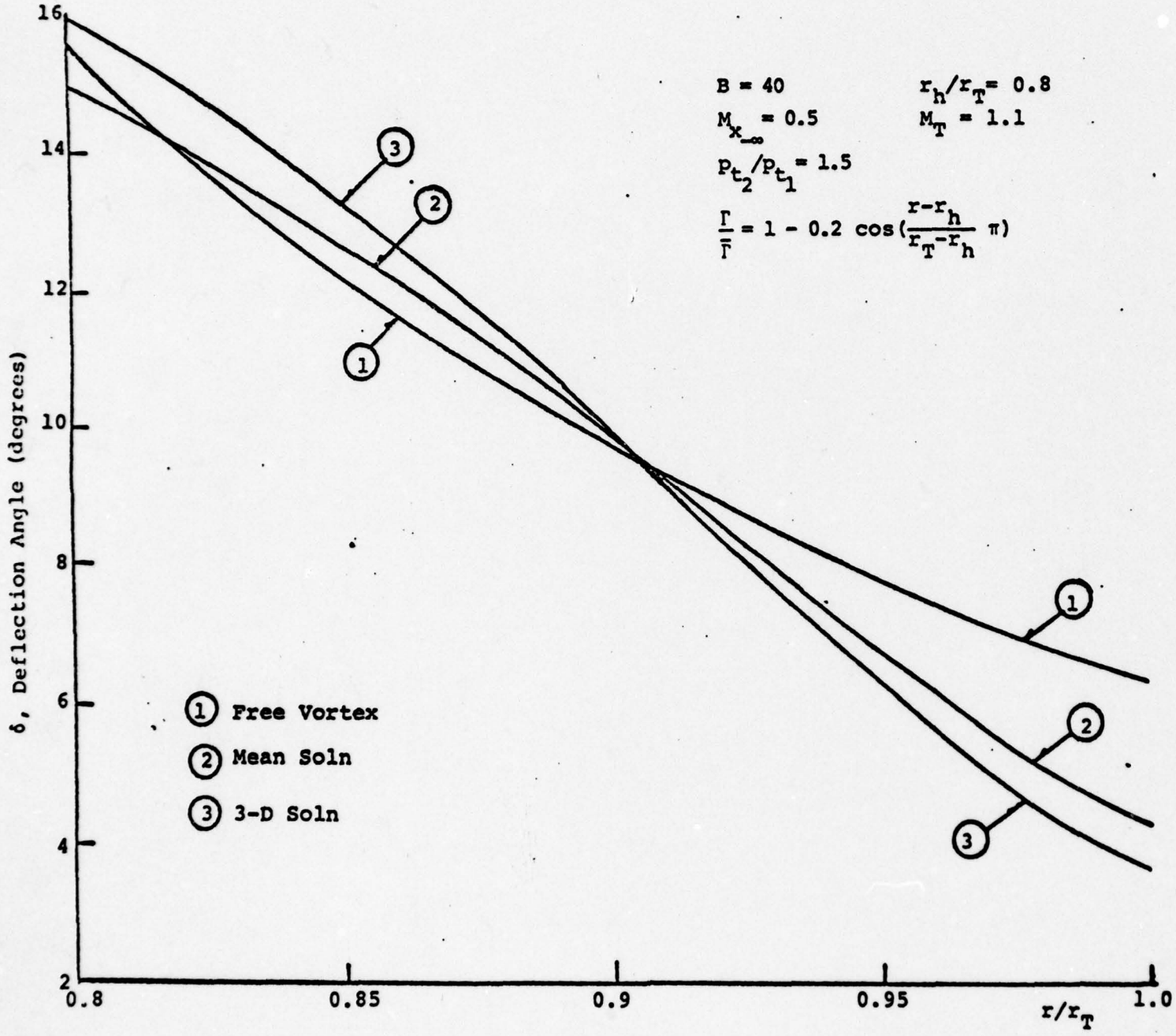

B = 40
r_h/r_T = 0.8
M_{x_∞} = 0.5
M_T = 1.1
p_{t_2}/p_{t_1} = 1.5
$\frac{\Gamma}{\bar{\Gamma}} = 1 - 0.2 \cos(\frac{r-r_h}{r_T-r_h}\pi)$
δ, Deflection Angle (degrees)
16
14
12
10
8
6
4
2
0.8
0.85
0.9
0.95
1.0
r/r_T
1 Free Vortex
2 Mean Soln
3 3-D Soln

Fig. 1

MACH NUMBER COMPONENTS
R/RT = 0.950
0.1 AXIAL CHORDS DOWNSTREAM OF ROTOR

× EXPERIMENT
— THEORY

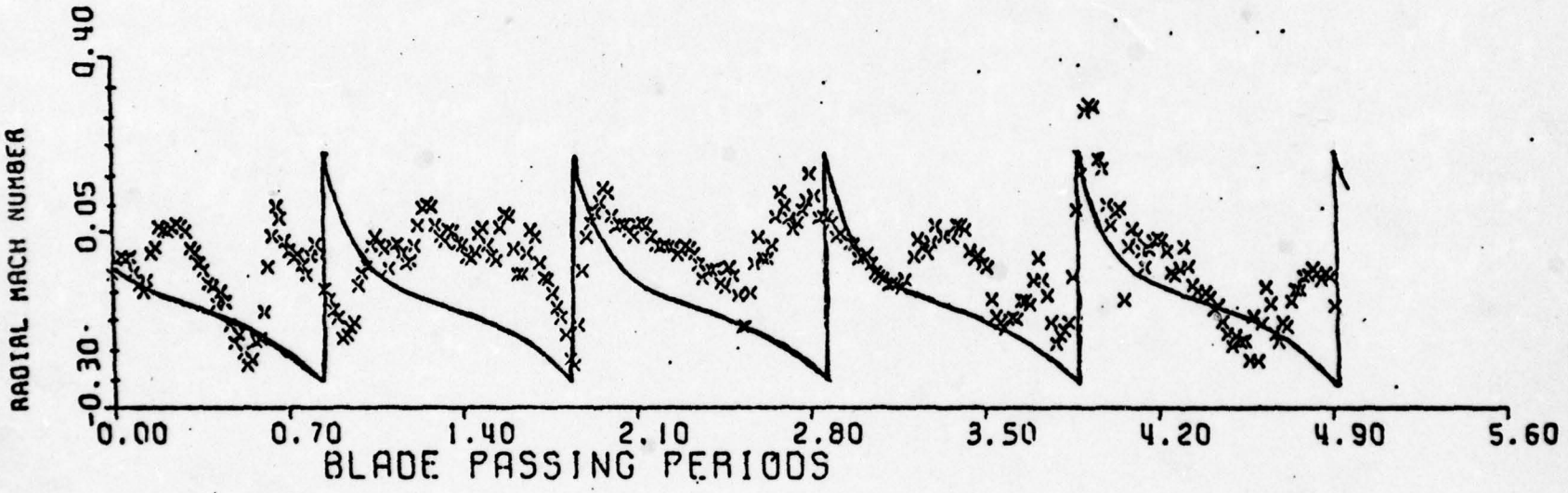


Fig. 2

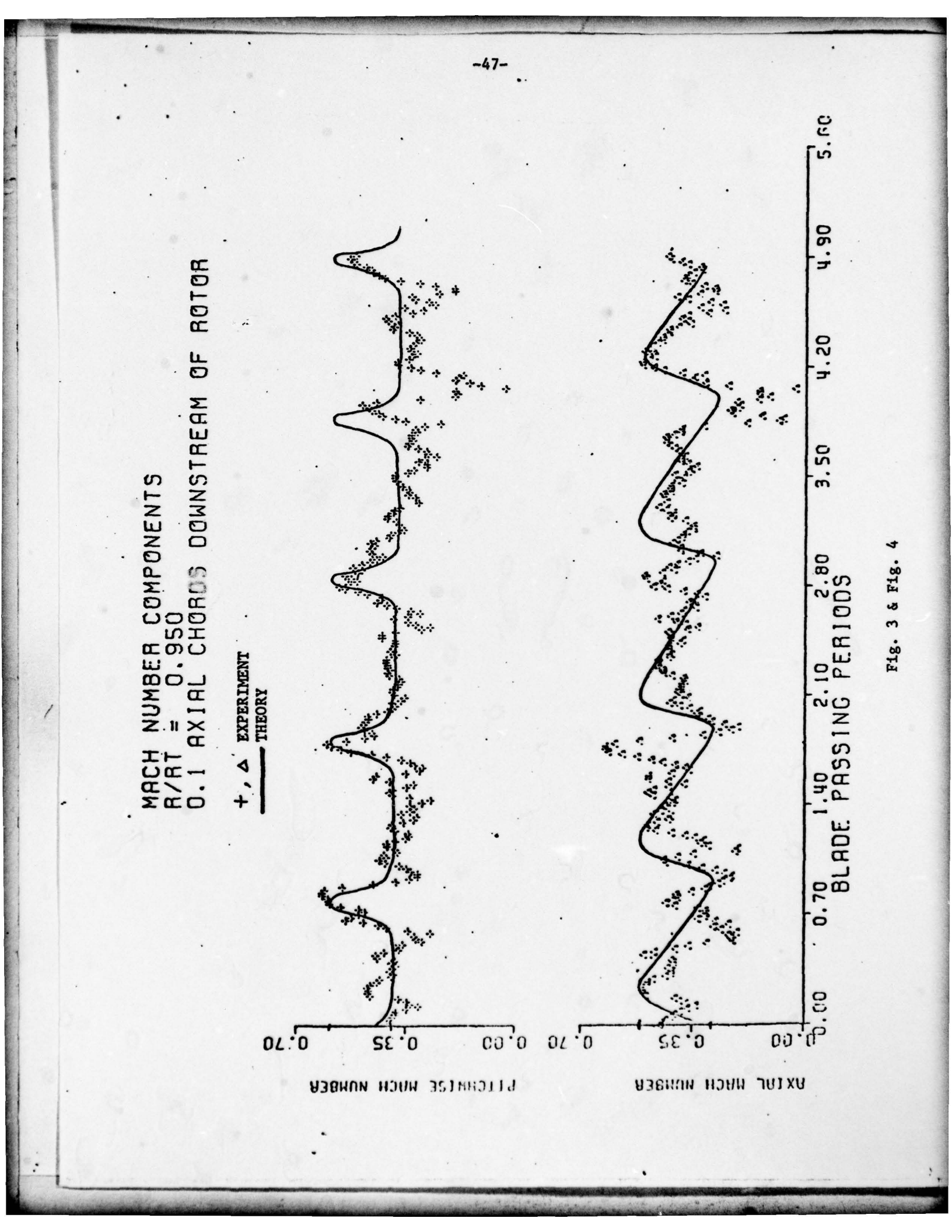

MACH NUMBER COMPONENTS
R/RT = 0.950
0.1 AXIAL CHORDS DOWNSTREAM OF ROTOR
+, Δ EXPERIMENT
—— THEORY
AXIAL MACH NUMBER
0.00
0.35
0.70
TANGENTIAL MACH NUMBER
0.00
0.35
0.70
0.00
0.70
1.40
2.10
2.80
3.50
4.20
4.90
5.60
BLADE PASSING PERIODS

Fig. 3 & Fig. 4

MACH NUMBER COMPONENTS
R/RT = 0.700
0.1 AXIAL CHORDS DOWNSTREAM OF ROTOR

× EXPERIMENT
— THEORY

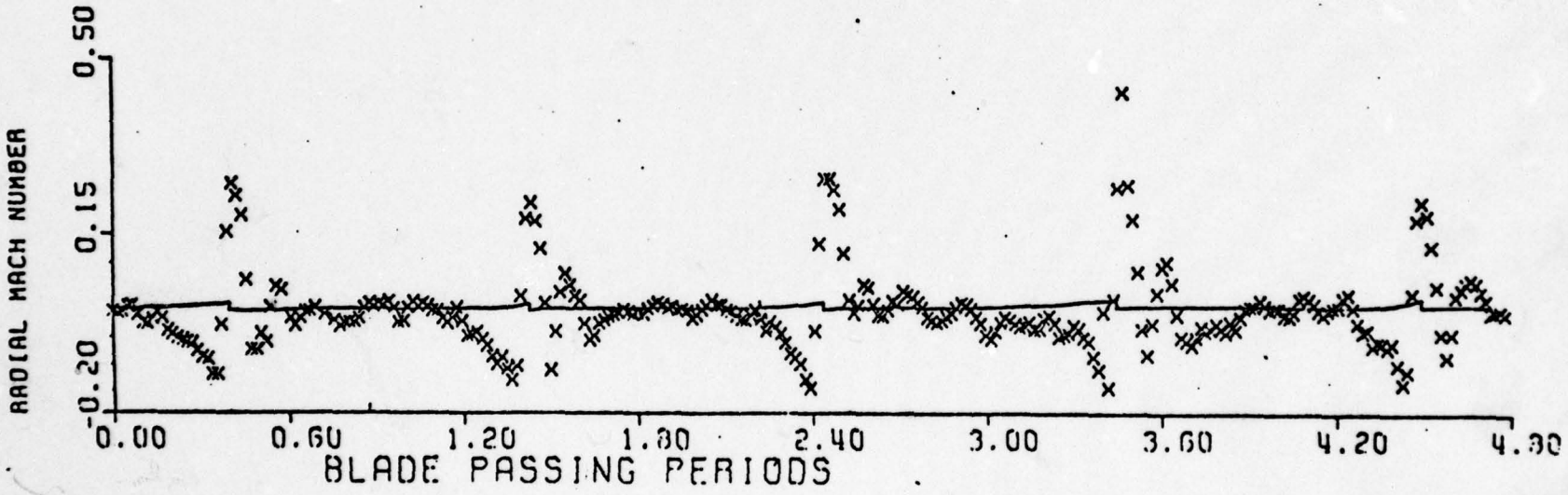


Fig. 5

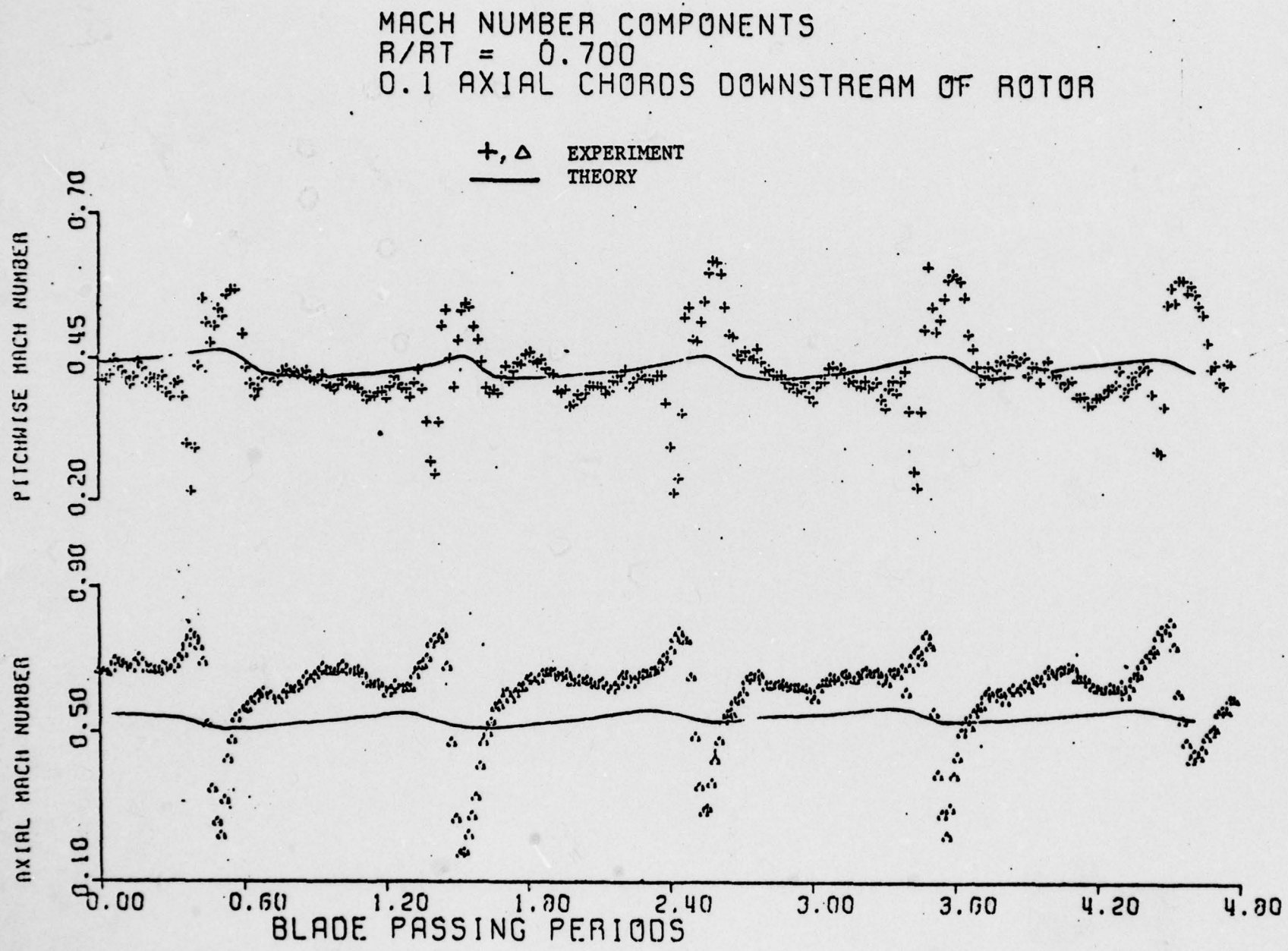

MACH NUMBER COMPONENTS
R/RT = 0.700
0.1 AXIAL CHORDS DOWNSTREAM OF ROTOR
+, Δ EXPERIMENT
THEORY
PITCHWISE MACH NUMBER
0.70
0.45
0.20
AXIAL MACH NUMBER
0.90
0.50
0.10
0.00
0.60
1.20
1.80
2.40
3.00
3.60
4.20
4.80
BLADE PASSING PERIODS

Fig. 6 & Fig. 7

FLOW ANGLE DEFLECTION VS. RADIUS FOR THE CASE WHERE MINIMUM OF STAGNATION PRESSURE COINCIDES WITH TRAILING EDGE OF THE BLADES.

NUMBER OF BLADES	B = 40
INLET MACH NUMBER	= 0.65
RELATIVE TIP MACH NUMBER	= 1.3
PRESSURE RATIO	= 1.8
HUB-TO-TIP RATIO	= 0.8

CURVES:

1. RADIAL EQUILIBRIUM SOLUTION
2. ACTUATOR DISC SOLUTION
3. CIRCUMFERENTIALLY NON-UNIFORM FLOW DUE TO PRESENCE OF WAKES WITH TOTAL PRESSURE LOSS-COEFFICIENT OF 0.025.

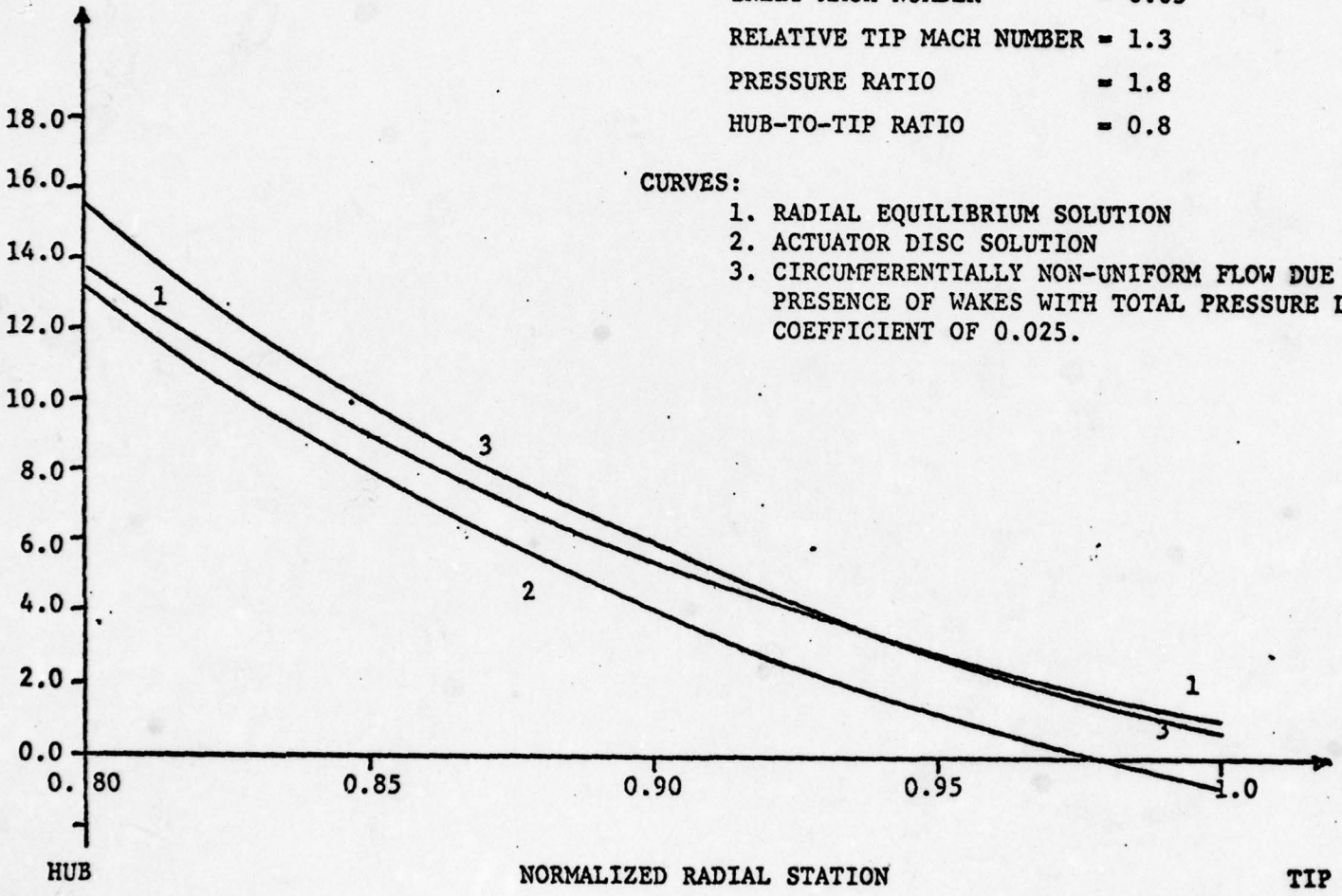
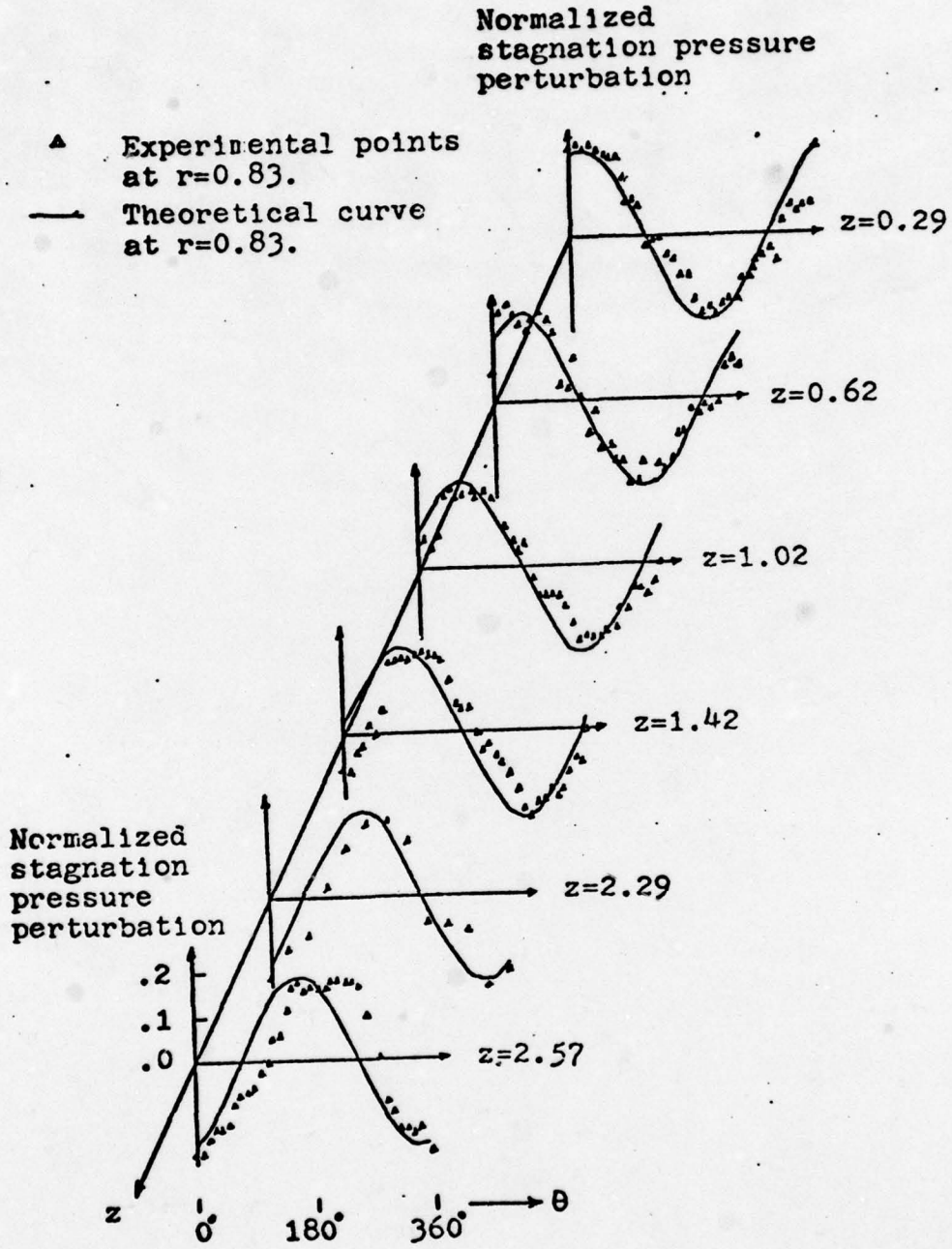


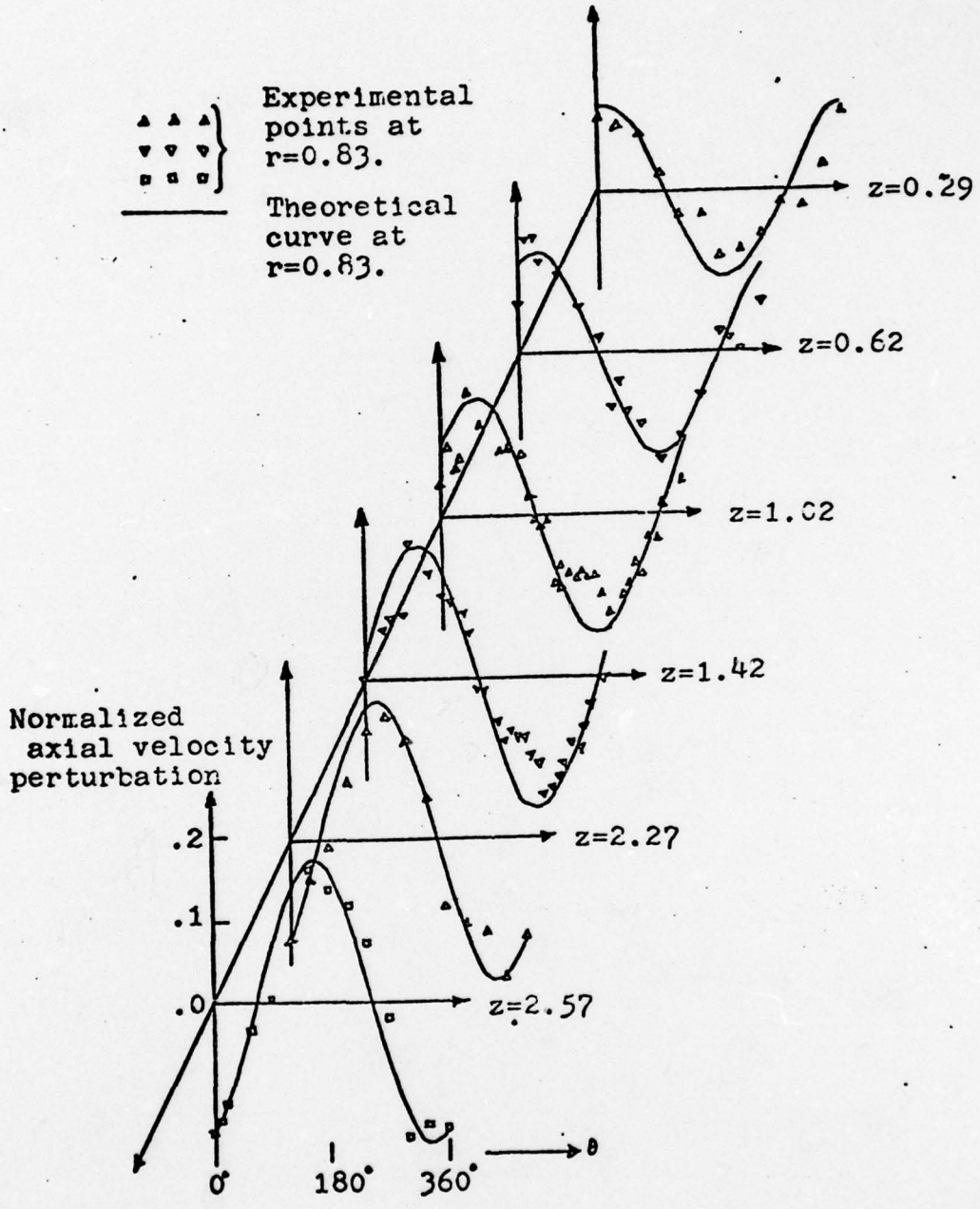
Fig. 8



DOWNSTREAM CONVECTION OF STAGNATION PRESSURE DISTORTION AT r=0.83.

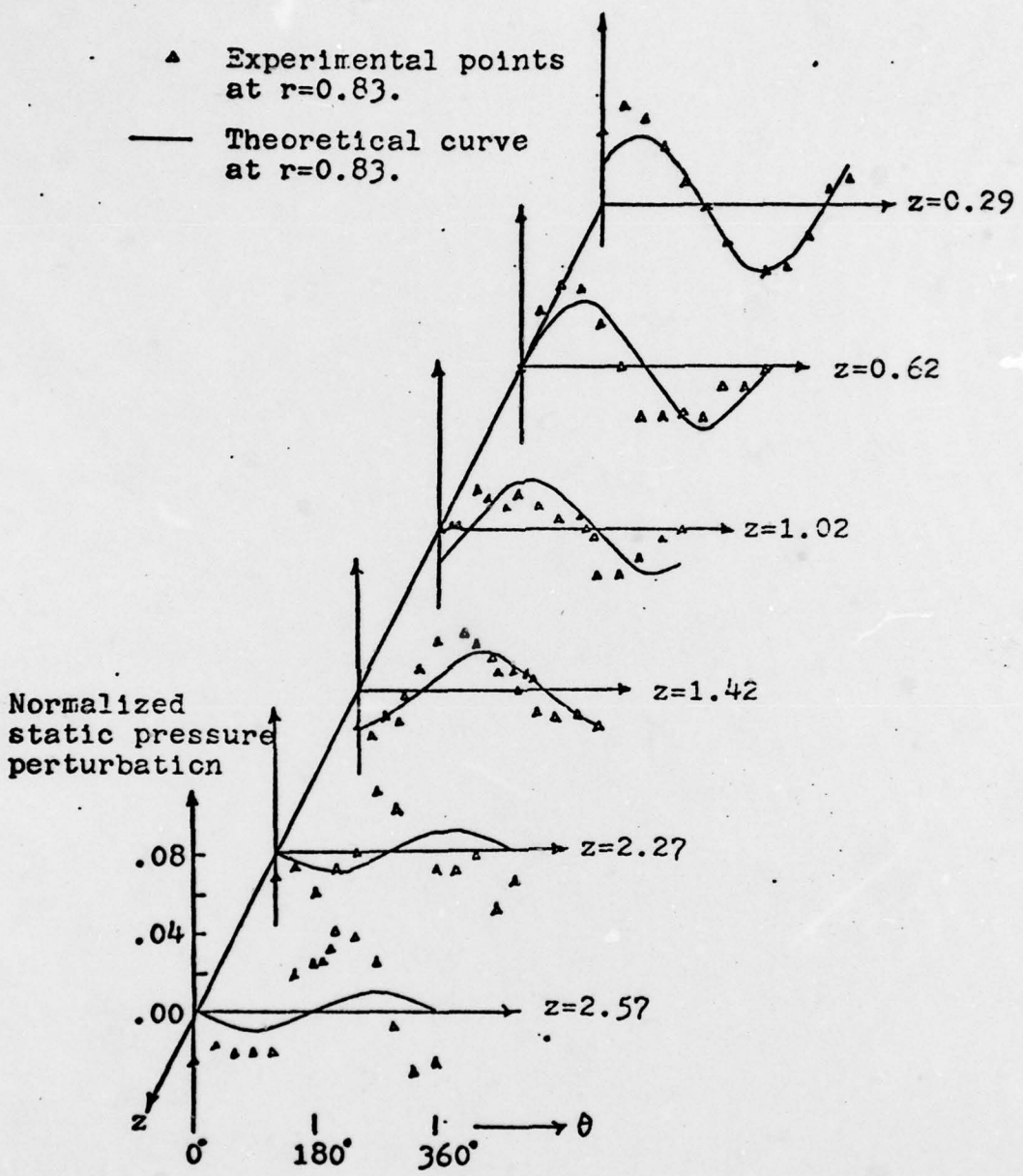
Fig. 9

Stator:
Hub-to-tip ratio=0.43
Mean outflow angle=48°



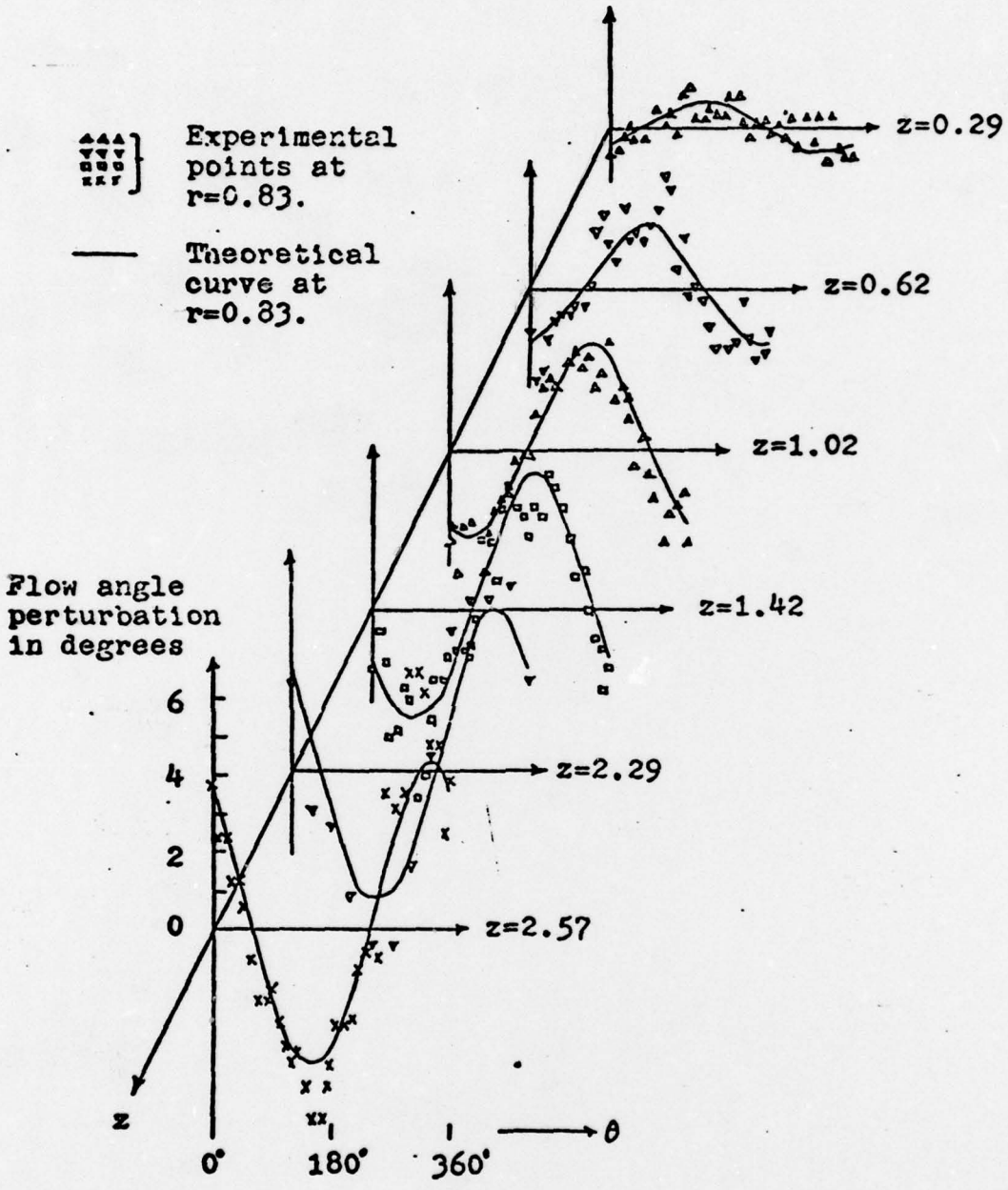
AXIAL VELOCITY DISTRIBUTION AT DIFFERENT AXIAL STATIONS FOR RADIUS r=0.83.

Fig. 10



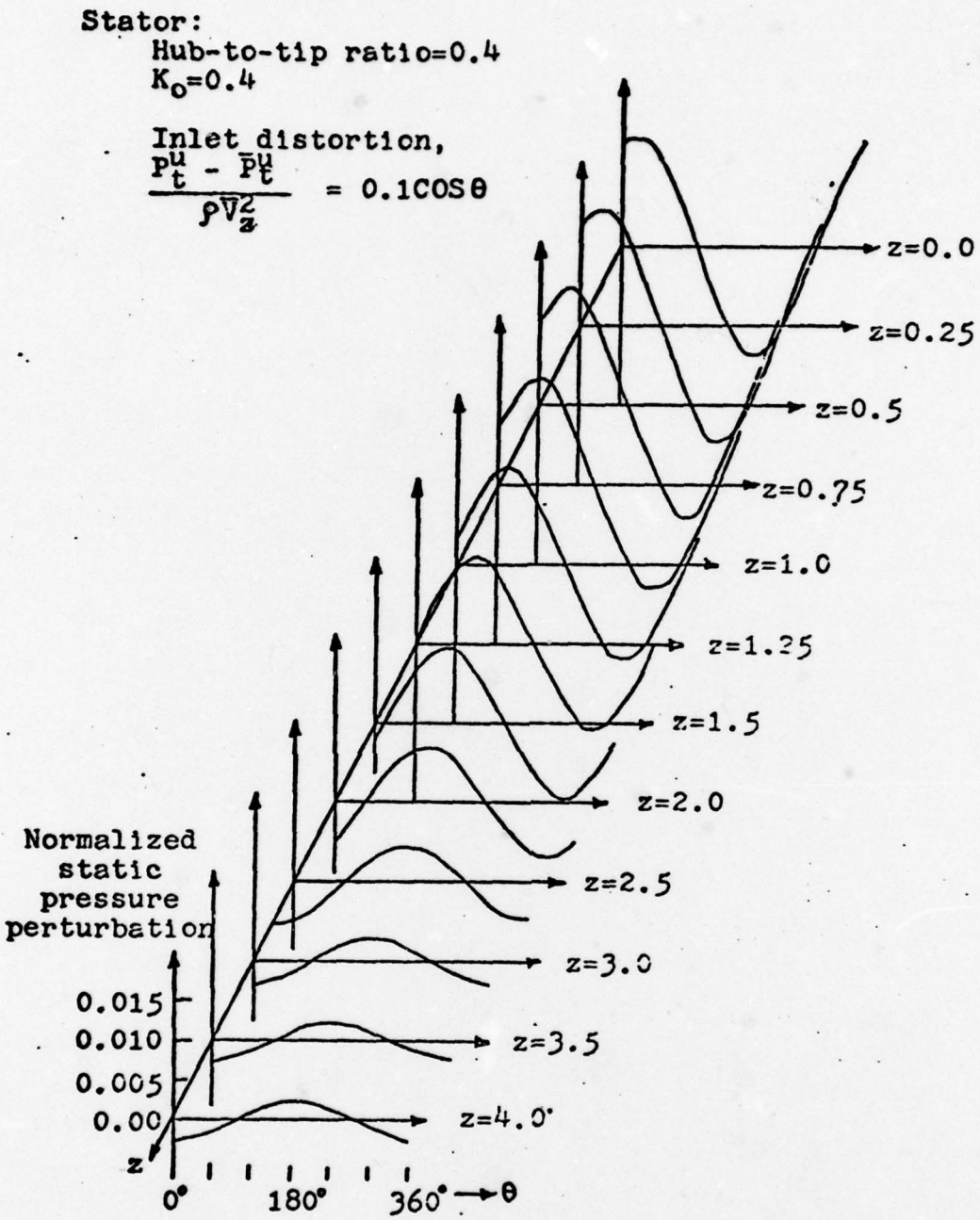
COMPARISON OF DOWNSTREAM STATIC PRESSURE PERTURBATIONS AT r=0.83: THEORY VS. EXPERIMENT.

Fig. 11



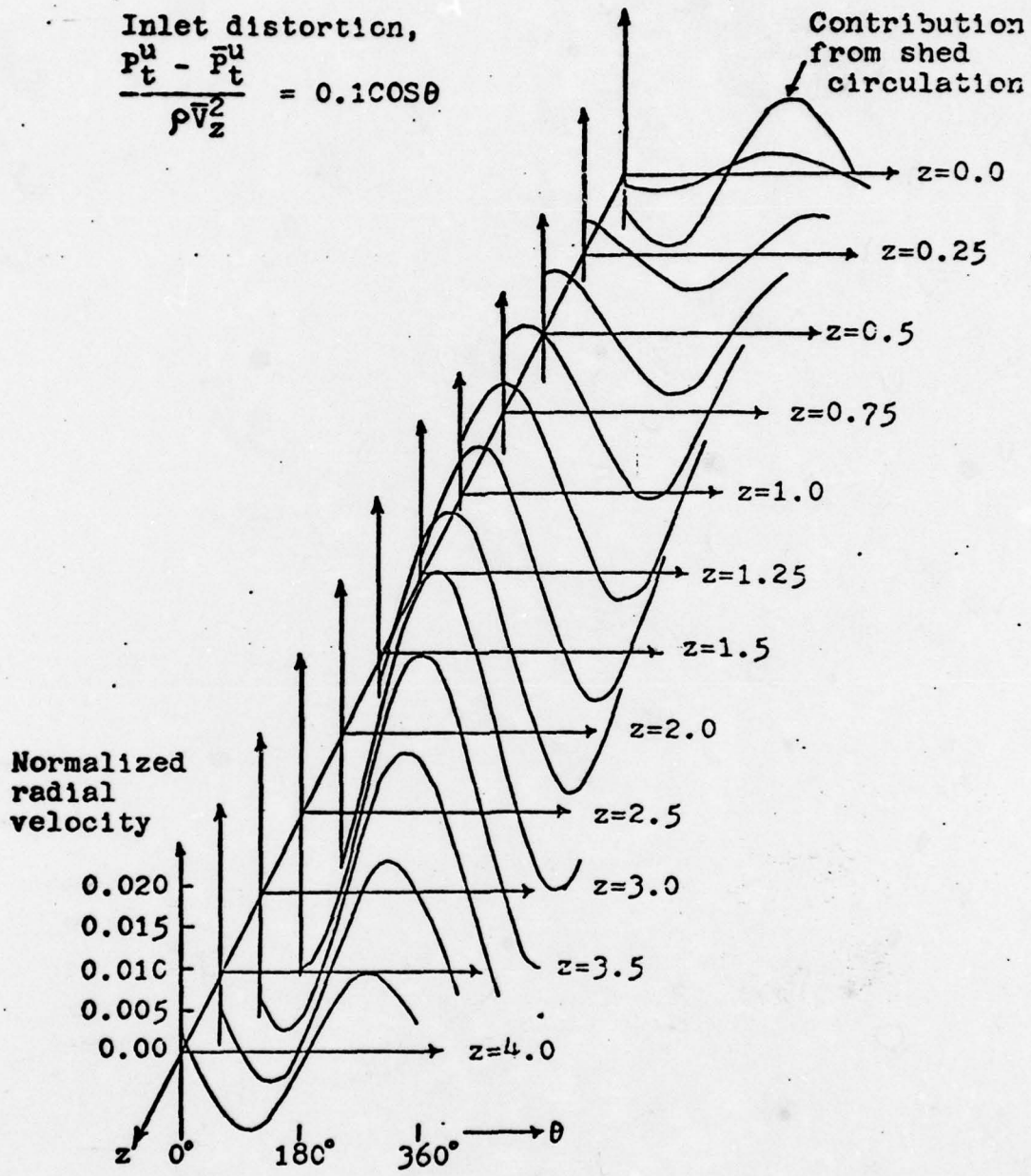
COMPARISON OF DOWNSTREAM FLOW ANGLE PERTURBATIONS AT RADIUS r = 0.83: THEORY VS. EXPERIMENT.

Fig. 12



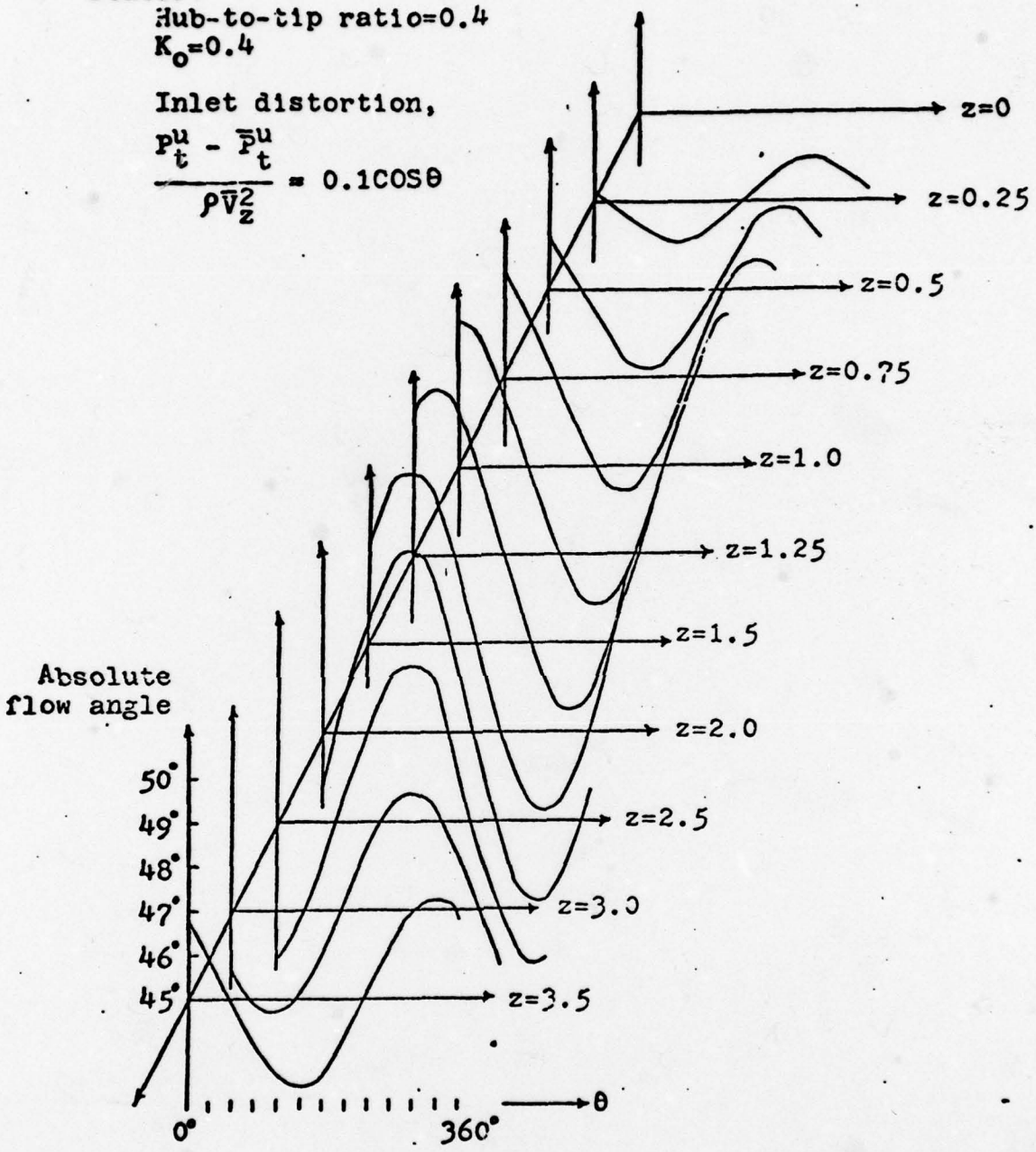
DOWNSTREAM VARIATION OF STATIC PRESSURE PERTURBATION AT THE TIP.

Fig. 13



DOWNSTREAM VARIATION OF RADIAL VELOCITY AT r/r_t=0.55.

Fig. 14



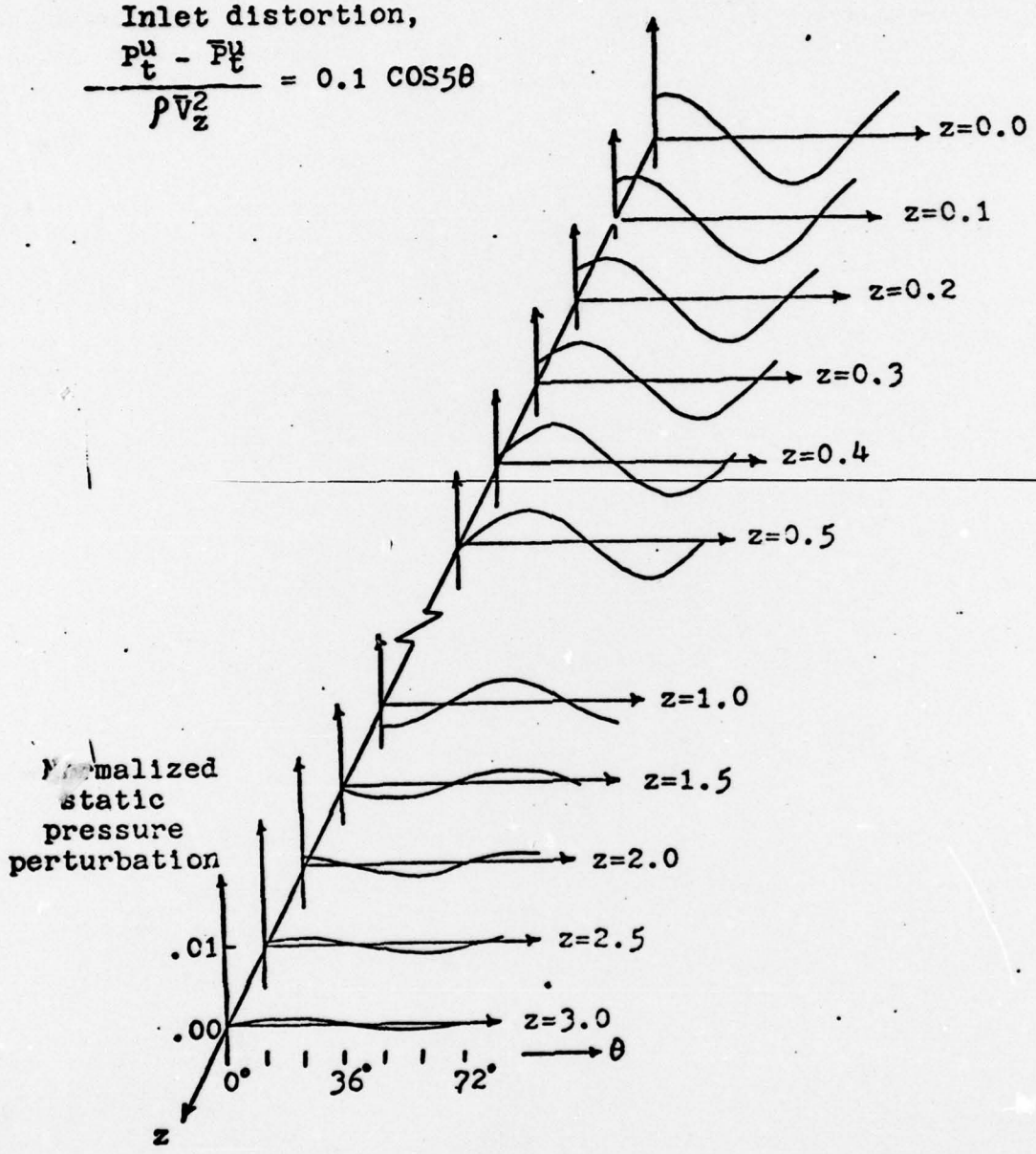
DOWNSTREAM VARIATION OF ABSOLUTE FLOW ANGLE AT THE HUB.

Fig. 15

Stator:
Hub-to-tip ratio=0.4
K_o=0.4

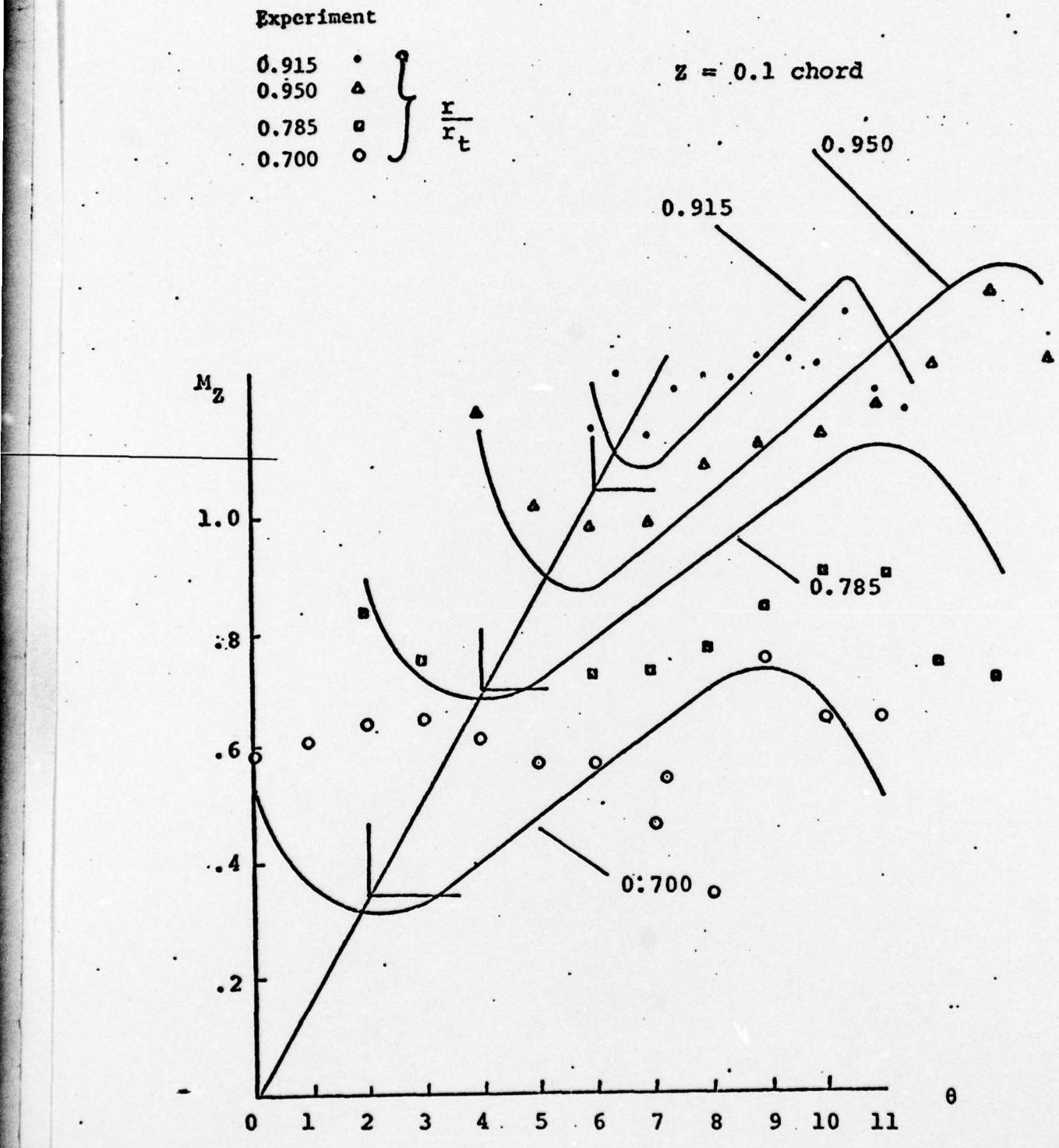
Inlet distortion,

$$\frac{p_t^u - \bar{p}_t^u}{\rho \bar{V}_z^2} = 0.1 \cos 5\theta$$



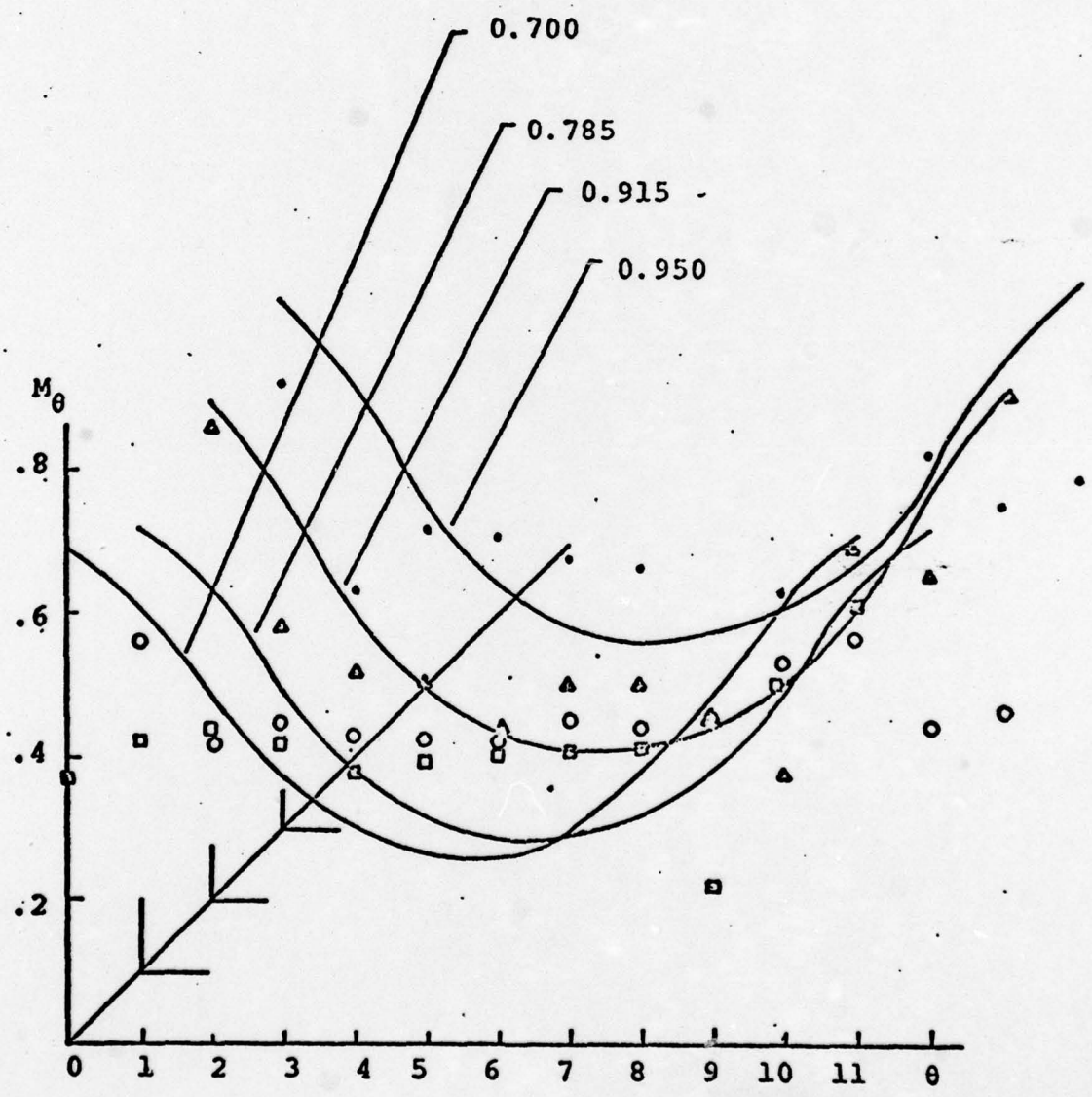
DOWNSTREAM VARIATION OF STATIC PRESSURE PERTURBATION AT THE TIP.

Fig. 16



Axial velocity as a function of θ at various radial locations

Fig. 17



Azimuthal velocity as a function of θ at several radial locations

Fig. 18

Unclassified

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A method of describing three dimensional flows in highly-loaded turbomachines was discussed by using the azimuthally-average flow as a model of a rotor. The overall turning of the flow caused by the cascade was assumed to be large and the spanwise variation of the loading small. In this model the "axisymmetric" flow was replaced by the pitchwise average of the rectilinear flow. Under these circumstances the reference flow could be correlated with the highly loaded model of rectilinear "actuator disc", having a weak spanwise variation of the specified mean turning. They developed methods describing the three-dimensional flow as a perturbation of the aforementioned nonlinear two-dimensional "reference flow".

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