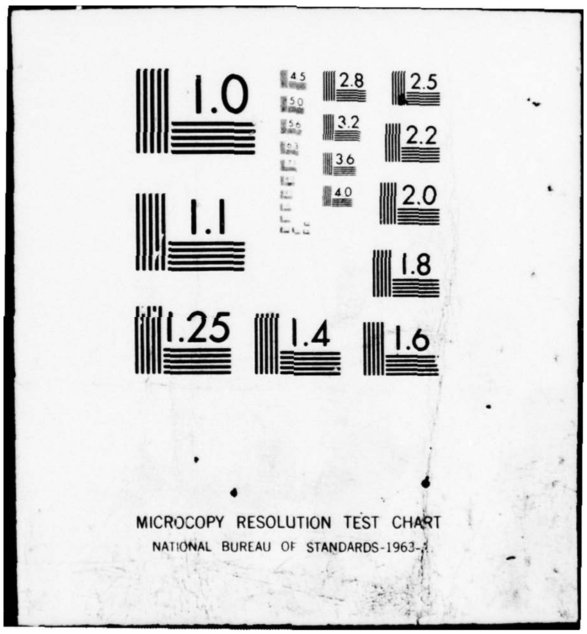
AD-A072 005 NAVAL RESEARCH LAB WASHINGTON DC F/G 20/5
RIGOROUS DERIVATION OF GENERAL DISPERSION RELATIONSHIIP FOR GYR--ETC(U)
JUL 79 S AHN, J Y CHOE
UNCLASSIFIED NRL-MR-4041 NL

| OF |

AD
A072005

END
DATE
FILMED
8--79
DDC


1.0
2.8
2.5
3.2
2.2
3.6
4.0
2.0
1.1
1.8
1.25
1.4
1.6
MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

NRL Memorandum Report 4041

Rigorous Derivation of General Dispersion Relationship for Gyrotron

Joon Y. Choe

Department of Physics and Astronomy
University of Maryland
College Park, Maryland

AND

Saeyoung Ahn

Microwave Tube Group
Electronics Technology Division

July 20, 1979

NAVAL RESEARCH LABORATORY
Washington, D.C.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: NRL Memorandum Report 4041	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): RIGOROUS DERIVATION OF GENERAL DISPERSION RELATIONSHIP FOR GYROTRON	5. TYPE OF REPORT & PERIOD COVERED: Interim report on a continuing NRL problem.
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): Saeyoung Ahn and Joon Y. Choe*	8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Naval Research Laboratory, Washington, DC 20375	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: NRL Problem R18-35
11. CONTROLLING OFFICE NAME AND ADDRESS	12. REPORT DATE: July 20, 1979
	13. NUMBER OF PAGES: 41
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

79 07 30 106

18. SUPPLEMENTARY NOTES

*Department of Physics and Astronomy, University of Maryland, College Park, Maryland

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Gyrotron
Cyclotron maser instability
Coaxial waveguide
Electron density distribution function
Dispersion relation
High harmonic mode

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

General dispersion equation for the electron cyclotron maser instability in co-axial cylindrical geometry is derived in the framework of linearized Vlasov-Maxwell system. While previous works have been carried out for the azimuthally symmetric mode and for the specified equilibrium electron distribution function in simple waveguide structure, we have extended our analysis to include the azimuthally asymmetric mode with general equilibrium distribution function. After selfconsistent analysis of the equilibrium distribution function, we have solved the orbit integral

(Continues)

DD FORM 1473, 1 JAN 73 — EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-014-6601

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

20. Abstract (Continued)

for the perturbed distribution function closely following the previous analysis by Chu, and have derived a new rigorous formal dispersion equation for arbitrary equilibrium distribution function. The resulting equation is useful to find the dispersion relation of electron beam with spread in energy and momenta. A closed form of dispersion relation for the cold beam electrons is given as an example.

TABLE OF CONTENTS

RIGOROUS DERIVATION OF GENERAL DISPERSION RELATIONSHIP FOR GYROTRON

I. Introduction

As a high power microwave and mm wave source the gyrotron[1-3] is gaining much interest recently for its applications to RF heating of fusion plasma, to radar transmitter and to amplifier. Gyrotron makes use of the interaction between transverse electric field and motion of relativistic beam electrons via the electron cyclotron maser instability. This instability is originated by the asymetry (bunching) on the electron beam distribution over gyration orbit, caused by either the signal electromagnetic wave (amplifier) or the initial perturbation in electron distribution (oscillator), which enhances the magnitude of EM wave at the expense of the transverse energy of beam electrons, and which in turn further fuels the bunching mechanism.

Most of the previous theoretical analyses[4-11] have been carried out within the framework of linearized Vlasov-Maxwell system under the tenuous beam assumptions, and some of the results have been shown to agree well with those of the experiments.[12-14] There were, however, some limitations of previous analyses, in that they are limited to the azimuthally symmetric modes and to the specific equilibrium distribution function. In this paper we extend our analysis to include the azimuthally asymmetric mode with general equilibrium profiles. We further extend the geometry of the system from simple waveguide to coaxial waveguide in order to examine the contribution of the center rod in the gyrotron.

The derivation of the dispersion relation consists of two major parts: (1) the orbit integral to calculate the perturbed distribution function and (2) the dispersion equation itself as moment integral of the perturbed distribution function. In the first part (the orbit

Note: Manuscript submitted May 16, 1979.

integral) we follow closely the analysis by Chu,[10] which makes use of the Bessel sumamtion theorem, and in the second part (the dispersion equation) we use a new approach in order to retain arbitrariness of the beam distribution function.

After careful examination of self-consistent distribution function, it is found that the distribution function is determined entirely by three constants of electron motion, namely γ (total relativistic factor), p_z (axial linear momentum) and P_θ (canonical angular momentum). With these constants as independent variables for the distribution function, we can obtain the dispersion equation for general equilibrium beam profiles. We add that, during the course of the orbit integral, we include the contribution of the term that is proportion to $\partial f_0/\partial P_\theta$ (see eq (33)), which is previously ignored by other authors.

In this paper the authors attempted to present as much mathematical detail as possible for future reference at the risk of repetition of some previous works. We emphasize that this work has been primarily the ground work of our current and further developments on this subject, such as the effect due to spread in the energy and momenta, and due to the presence of inner conductor.

In Chapter II, the basic equations and assumptions are described. The orbit integral for the perturbed distribution function is obtained in Chapter III. In Chapter IV the dispersion equation for arbitrary distribution function is derived. The closed form of the dispersion relation for cold beam electrons is given in Chapter V as an example.

II. Basic Equations and Assumptions

The basic equations which govern the combined system of the relativistic electron beam with electric charge (-e) and mass m and the electromagnetic wave are as follows:

$$\nabla \times \underline{B} = \frac{4\pi}{c} \underline{J} + \frac{1}{c} \frac{\partial \underline{E}}{\partial t} , \qquad (1)$$

$$\nabla \cdot \underline{B} = 0 , \qquad (2)$$

$$\nabla \cdot \underline{E} = -4\pi e n , \qquad (3)$$

$$\nabla \times \underline{E} = -\frac{1}{c} \frac{\partial \underline{B}}{\partial t} , \qquad (4)$$

$$\left[\frac{\partial}{\partial t} + \underline{v} \cdot \frac{\partial}{\partial \underline{x}} - e\left(\underline{E} + \frac{1}{c} \underline{v} \times \underline{B}\right) \cdot \frac{\partial}{\partial \underline{p}}\right] f = 0 , \qquad (5)$$

$$n = \int d^3\underline{p}\, f , \qquad (6)$$

$$\underline{J} = -e \int d^3\underline{p}\, \underline{v}\, f , \qquad (7)$$

where the quantities $\underline{B}(\underline{x}, t)$, $E(\underline{x}, t)$, $f(\underline{x}, \underline{p}, t)$, $n(\underline{x}, t)$ and $\underline{J}(\underline{x}, t)$ represent the magnetic field, the electric field, the distribution function of the electron beam, the beam density, and the beam current density, respectively. Here $\underline{x}$ and $\underline{p}$ denote the coordinates in real and momentum spaces, respectively. The electromagnetic wave ($\underline{E}$ and $\underline{B}$) is determined by the Maxwell equations (1) to (4), the electron beam by the relativistic Vlasov equation (5), and the source equations (6) to (7) combine these two systems. The Maxwell equations (1) to (4) can be combined into two wave equations.

$$\left[-\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \nabla^2\right] \underline{B} = -\frac{4\pi}{c} \nabla \times \underline{J} \qquad (8)$$

$$[-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2]\ \underline{E} = -4\pi\{e\nabla n - \frac{1}{c^2}\frac{\partial \underline{J}}{\partial t}\} \tag{9}$$

Generally, different boundary conditions are imposed on $\underline{B}$ and $\underline{E}$ in equations (8) and (9), so that there exist two separate branches of the electromagnetic wave modes, namely transverse electric (TE) and transverse magnetic (TM) modes, which satisfy equations (8) and (9) respectively. In this chapter, we will examine TE modes in the cylindrical waveguide which are commonly used in gyrotron devices. Hence, the cylindrical coordinate system (r, θ, z) is employed throughout this paper.

The geometry in this paper is assumed to be coaxial cylinder whose conductors are located at $r = \delta R$ and $r = R$ (Fig. 1). The parameter δ is the ratio of inner conductor radius to that of outer conductor, and note that when $\delta = 0$, the geometry becomes single cylindrical.

Present analysis is carred out in the framework of the linearized Maxwell-Vlasov system, where any quantity $\psi(\underline{x}, t)$ is expanded into

$$\psi(\underline{x}, t) = \psi_0(r) + \psi^1(r) \exp[i(kz + \ell\theta - \omega t)] \quad . \tag{10}$$

Here ψ_0 is the equilibrium quantity, whose symmetry properties (viz. $\frac{\partial}{\partial z} = 0 = \frac{\partial}{\partial t}$) are explicitly assumed, and ψ^1 is the small perturbed quantity which is already Fourier decomposed into the integer azimuthal mode number ℓ, and the axial wave number k. With eq (10), TE mode equation (8) becomes

$$[\nabla_t^{\ 2} + \alpha^2]\ B_z^1 = -\frac{4\pi}{c}\ [\frac{1}{r}\frac{\partial}{\partial r}(rJ_\theta^1) - \frac{i\ell}{r}J_r^1] \quad , \tag{11}$$

where $\nabla_t^2 = \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial}{\partial r}) - \frac{\ell^2}{r^2}$, $\alpha^2 \equiv \frac{\omega^2}{c^2} - k^2$

Once B_z^1 is obtained, other components easily follow in terms of B_z^1

$$B_r^1 = \frac{ik}{\alpha^2} [\frac{\partial B_z^1}{\partial r} + \frac{4\pi}{c} J_\theta^1]$$

$$B_\theta^1 = \frac{ik}{\alpha^2} [\frac{i\ell}{r} B_z^1 - \frac{4\pi}{c} J_r^1] \tag{12}$$

$$E_r^1 = \frac{\omega}{ck} B_\theta^1, \quad E_\theta^1 = - \frac{\omega}{ck} B_r^1 \quad .$$

The boundary conditions to be imposed on eq (11) are

$$\left. \frac{\partial B_z^1}{\partial r} \right|_{r=R} = 0 \quad , \text{ and} \tag{13}$$

$$\left. \frac{\partial B_z^1}{\partial r} \right|_{r=\delta R} = 0 \text{ or } \left. \frac{\partial B_z^1}{\partial r} \right|_{r=0} : \text{ finite if } \delta = 0 \quad .$$

The perturbed current density $\underline{J}^1$ is given by

$$\underline{J}^1 = - e \int d^3\underline{p}\, \underline{v}\, f^1 \tag{14}$$

The perturbed distribution function f^1 is given by orbital equation using characteristic method

$$f^1 = e \int_{-\infty}^{0} a(\Delta t) \exp[i(k\Delta z+\ell\Delta\theta - \omega\Delta t)] (\underline{E}^{1'} + \frac{1}{c} \underline{v}' \times \underline{B}^{1'}) \cdot \frac{\partial f_0}{\partial \underline{p}'} \tag{15}$$

where primed (') coordinates represent the equilibrium location of the single electron at time $t = t'$, and $\Delta t = t'-t$, $\Delta z = z'-z$, $\Delta\theta = \theta' - \theta$.

It is evident that the TE differential equation system (eq. 11, 14, 15) is self-consistent for $\underline{E}^1$ and $\underline{B}^1$. A fully self-consistent problem is difficult to solve, and some assumptions should be made to overcome this difficulty. First, we assume that the electron beam is very tenuous. Specifically, we assume

$$\nu/\gamma_0 <<< 1 \quad , \tag{16}$$

where the Budker parameter ν is defined by

$$\nu \equiv \frac{Ne^2}{mc^2} \quad , \tag{17}$$

and the number of electrons per unit axial length by

$$N = 2\pi \int rdr \int d^3\underline{p}\; f_0 \quad . \tag{18}$$

In Eq. (16), γ_0 is the total relativistic mass factor. In presently available gyrotron devices, the Budker parameter ν is in the order of 10^{-3}, so that assumption (16) is valid. Noting that the beam correction term in the right hand side of the TE dispersion equation (11) is proportional to ν, under the tenuous beam approximation (eq (16)) we conclude that the zeroth order solution of TE waveguide is free wave guide mode. We will use this mode to derive the perturbed distribution function f^1 (eq. 15). With the boundary condition (13), the free waveguide mode is given by

$$\begin{aligned}
B_z^1 &= Z_\ell\,(\alpha_{\ell n}^{\;2}\, r) \\
B_r^1 &= i\,\frac{k}{\alpha_{\ell n}}\, Z_\ell'\,(\alpha_{\ell n} r) \\
B_\theta^1 &= -\,\frac{k\ell}{\alpha_{\ell n}^2\, r}\; Z_\ell(\alpha_{\ell n} r) \\
E_r^1 &= \frac{\omega}{ck}\; B_\theta^1, \quad E_\theta^1 = -\,\frac{\omega}{ck}\, B_r^1, \quad E_z^1 \equiv 0 \quad ,
\end{aligned} \tag{19}$$

where we define

$$Z_m(x) \equiv \begin{cases} J_m(x) - \dfrac{Y_\ell'\,(\delta x_{\ell n})}{J_\ell'\,(\delta x_{\ell n})}\; Y_m(x), \text{if } \delta > 0 \\ J_m(x), \;\; \text{if } \delta = 0 \end{cases}$$

$$Z_\ell'\;(x_{\ell n}) = 0 \quad , \qquad \alpha_{\ell n} = \frac{x_{\ell n}}{R} \tag{20}$$

Here unsigned integer n denotes the radial mode number, and J_ℓ and Y_ℓ represent first and second kind Bessel functions of order ℓ. The amplitude of B_z^1 is already normalized to 1. If we combine eqs (11), (14), and (15) with those in (19), and by making use of orthogonality of Bessel function (see Appendix A), the dispersion equation for $TE_{\ell n}$ mode results. Namely,

$$\frac{\omega^2}{c^2} - k^2 - \alpha_{\ell n}^2 = - \frac{2\pi e \alpha_{\ell n}}{c} \quad \frac{1}{N_{\ell n}}$$

$$\times \int_{\delta R}^{R} rdr \quad d^3\underline{p}\ f^1 \ [Z_{\ell-1}(\alpha_{\ell n} r)\ (iv_r + v_\theta) + Z_{\ell+1}(\alpha_{\ell n} r)(iv_r - v_\theta)] , \quad (21)$$

where the perturbed distribution function f^1 is given by the orbit integral (eq. 15), and

$$N_{\ell n} \equiv \ (x_{\ell n}^2 - \ell^2)\ Z^2(x_{\ell n}) - (\delta^2 x_{\ell n}^2 - \ell^2)\ Z^2(\delta x_{\ell n})\ /2\alpha_{\ell n}^2 \quad . \quad (21a)$$

Next, we assume that, at equilibrium,

$$\underline{E}^0 \equiv 0 \quad ,$$
$$\underline{B}^0 = (0,\ 0,\ B_0) \quad , \quad (22)$$

where B_0 is constant axial magnetic field. These assumptions are consistent with the tenuous beam assumption (eq. 16) in that we neglect the self-fields. Therefore, the equilibrium particle orbit is composition of circular gyration motion in transverse direction with constant longitudinal speed. The orbit of electron is discussed in detail in Appendix B.

Another assumption we make is the symmetry of the problem at equilibrium in θ - and z- direction, which is previously made use of in eq. (10). This symmetry enables us to use three constants of motion in describing the equilibrium distribution function f_0.

We also assume that the growth rate ω_i (= Imω) is small;

$$|\omega_i| << \frac{\omega_c}{\gamma_0} \quad , \tag{23a}$$

where the classical cyclotron frequency ω_c is given by

$$\omega_c = \frac{eB_0}{mc} \quad . \tag{23b}$$

This assumption leads to single mode calculation and non-linear interactions between the different modes can be neglected.

Under these assumptions, we will find the dispersion relationship by eq (21). This requires finding the perturbed distribution function f^1 in eq (15) via the orbital integral in the first place.

III. Orbit Integral

The perturbed distribution function f^1 is given by the orbit integral equation (15), which can be written as

$$f^1 = e \int_{-\infty}^{0} d(\Delta t) \exp[i(k\Delta z + \ell\Delta\theta - \omega\Delta t)] \; I \; , \tag{24}$$

where

$$I \equiv (\underline{E}^1 + \frac{1}{c} \underline{v}' \times \underline{B}^{1'}) \cdot \frac{\partial f_0}{\partial \underline{p}'} \quad . \tag{25}$$

Here prime denotes the phase space coordinates of the particle at time t' = t + t. The integral is taken over the equilibrium path of a single electron under constant axial magnetic field $B_0 \hat{e}_z$ (eq. 22). The symmetry in θ - and z - directions and time independent nature of equilibrium enables us to use three constants of motion for momentum space $\underline{p}$ (Appendix B3-B5). Namely,

$$\underline{p} = \underline{p}(\gamma, p_z, P_\theta), \tag{26}$$

where

$$\gamma \equiv [1 + \underline{p} \cdot \underline{p}/m^2c^2]^{\frac{1}{2}} \tag{27}$$

$$p_z \equiv \underline{p} \cdot \hat{e}_z \tag{28}$$

$$P_\theta \equiv r \, \underline{p} \cdot e_\theta - \tfrac{1}{2} m \, \omega_c r^2 \quad . \tag{29}$$

Here three constants of motion, γ, p_z and P_θ represent the total relativistic factor, the axial linear momentum, and the canonical angular momentum, respectively. In Fig. 2, the transverse motion of a single particle is shown. The electron will rotate from position A to A' in time Δt with guiding center located at point G. (For detailed discussions of the particle motion, see Appendix B.) The associated constants of motion, i.e. transverse linear momentum, Larmor radius and guiding center radius are defined in order (see Appendix B6-B8) by

$$p_\perp \equiv [\, m^2c^2(\gamma^2 - 1) - p_z{}^2]^{1/2} \quad , \tag{30}$$

$$r_L \equiv \frac{p_\perp}{m\omega_c} \quad , \tag{31}$$

and

$$r_0 \equiv [r_L^2 - \frac{2P_\theta}{m\omega_c}]^{1/2} \quad . \tag{32}$$

Now let us make a canonical transformation in the momentum space $\underline{p}$ to the variables γ, and P_θ in eq (26). Then the integral I has the form below:

$$I \equiv I_\gamma \frac{\partial f_0}{\partial \gamma} + I_z \frac{\partial f_0}{\partial p_z} + I_\theta \frac{\partial f_0}{\partial P_\theta} \quad . \tag{33}$$

With the substitution of the perturbed TE fields in eq (19),

$$I_\gamma = (-i) \frac{1}{\gamma m^2 c^2} \frac{p_\perp}{\alpha_{\ell n}} \frac{\omega}{c} I_1$$

$$I_z = (-i) \frac{1}{\gamma mc} \frac{p}{\alpha_{\ell n}} \; k \; I_1 \quad \text{and} \tag{34}$$

$$I_\theta = (-i)[\frac{1}{\alpha_{\ell n} c} (\omega - \frac{kp_z}{\gamma m}) I_2 - \frac{p_\perp}{\gamma mc} I_3]$$

where

$$I_1 \equiv \tfrac{1}{2}[\exp(-i\xi') \, Z_{\ell-1}(\alpha_{\ell n} r') - \exp(+i\xi') \, Z_{\ell+1}(\alpha_{\ell n} r')] \quad ,$$

$$I_2 \equiv r' \, Z_\ell' \, (\alpha_{\ell n} r') \quad , \text{and}$$

$$I_3 \equiv ir' \sin \xi' \, Z_\ell(\alpha_{\ell n} r') = i \, r_0 \sin \chi' \, Z_\ell(\alpha_{\ell n} r') \tag{35}$$

Here $Z_\ell'(x) \equiv \frac{d}{dx} Z_\ell(x)$, and α', χ' and ξ' are angles of the triangle OGA' in Fig. 3.

Two things deserve to be mentioned in the orbit integral eq (24) with its integrand I in eq (33). First, we note that $\frac{\partial f_0}{\partial \gamma}$, $\frac{\partial f_0}{\partial p_z}$, $\frac{\partial f_0}{\partial P_\theta}$

are constant along the equilibrium orbit path and can be factored out of the integral. Due to this reason we introduced the set (γ, p_z, P_θ) as our independent variables. Secondly, the integrands I_1, I_2, and I_3 are functions of r', which is in turn a function of t. The orbit equation $r' = r'(\Delta t)$ is not only complicated, but makes it extremely difficult to perform the integral. We note that the orbit is best described in angle variations. They are from Fig. 3,

$$\Delta\phi = \frac{\omega_c}{\gamma} t \quad , \tag{36}$$

$$\Delta z = \frac{p_z}{\gamma m} \Delta t \quad , \tag{37}$$

where
$$\chi' - \chi = \Delta\phi \quad , \tag{38}$$

$$\alpha - \alpha' = \Delta\theta \quad . \tag{39}$$

It is, therefore, desirable to convert r' (Δt) into r' (Δφ) through angle variations. This can be done with summation theorem of Bessel function (Appendix A4). Use is made of results of the theorem, i.e. eqs (A-9), (A-7) and (A-10) to obtain

$$I_1 = (-1)^\ell \sum_{s=-\infty}^{\infty} \begin{Bmatrix} Z_{s-\ell}(x_0)\, J_s'(x_L) \\ \\ J_{s-\ell}(x_0)\, Z_s'(x_L) \end{Bmatrix} \exp(is\chi' + i\ell\alpha') \tag{40}$$

$$I_2 = (-1) \sum_{s=-\infty}^{\infty} \frac{1}{\alpha_{\ell n}} \begin{Bmatrix} x_0\, Z_{s-\ell}'(x_0)\, J_s(x_L) + x_L Z_{s-\ell}(x_0) J_s'(x_L) \\ \\ x_0\, J_{s-\ell}'(x_0)\, Z_s(x_L) + X_L\, J_{s-\ell}(x_0)\, Z_s'(X_L) \end{Bmatrix} \cdot \exp(is\chi' + i\ell\alpha') \tag{41}$$

and

$$I_3 = (-1)^\ell \sum_{s=\infty}^{\infty} \frac{1}{\alpha_{\ell n} x_L} \begin{Bmatrix} sx_0\, Z_{s-\ell}'(x_0)\, J_s(x_2) + (s-\ell)x_L Z_{s-\ell}(x_0) J_s'(x_L) \\ sx_0\, J_{s-\ell}'(x_0)\, Z_s(x_L) + (s-\ell)x_L J_{s-\ell}(x_0) Z_s'(x_L) \end{Bmatrix} \cdot \exp(is\chi' + i\ell\alpha') \quad , \tag{42}$$

where $x_0 \equiv \alpha_{\ell n} r_0$, $x_L \equiv \alpha_{\ell n} r_L$ (43)

and $\alpha_{\ell n} \equiv \frac{x_{\ell n}}{R}$ was previously defined. Here upper (lower) expressions are for those electrons with the guiding center radius to greater (less) than the Larmor radius r_L, that is, with the canonical angular momentum P_θ less (greater) than zero. Note that when $\delta = 0$ (viz. simple waveguide geometry) the combination Bessel function Z is replaced by the first order Bessel function J, and the expressions remain the same regardless of P_θ. Now that the Δt dependence in the orbit integral appears only through exponential function with angle variables χ', α', θ', we can easily perform the integral. The perturbed distribution function f^1 is proportional to the angle integral. That is,

$$f^1 \propto \int_{-\infty}^{0} d(\Delta t) \exp[i(k\Delta z + \ell\Delta\theta - \omega\Delta t) + is\chi' + i\ell\alpha']$$

$$= \exp(is\chi + i\ell\alpha) \int_{-\infty}^{0} d(\Delta t) \exp[-i(\omega - \frac{kp_z}{\gamma m} - s\frac{\omega_c}{\gamma})\Delta t]$$

$$= \frac{1}{(-i)} \exp(is\chi + i\ell\alpha) \frac{1}{\omega - \Omega_s} \qquad (44)$$

where

$$\Omega_s \equiv \frac{kp_z}{\gamma m} + s\frac{\omega_c}{\gamma} \qquad (45)$$

represents the Doppler shifted cyclotron frequency of harmonic number s. With eq (44) the full expression for f^1 is given by

$$f^1 \equiv \sum_{s=-\infty}^{\infty} f^1_s \quad , \qquad (46)$$

$$f^1_s = \frac{?}{\alpha_{\ell n}^2 c} (-1)^\ell \exp(is\chi + i\ell\alpha)$$

$$\times \left[\hat{I}_\gamma \frac{\partial f_0}{\partial \gamma} + \hat{I}_z \frac{\partial f_0}{\partial p_z} + \hat{I}_\theta \frac{\partial f_0}{\partial P_\theta} \right] \qquad (47)$$

where

$$\hat{I}_{\gamma} \equiv \frac{\omega_c}{mc^2} \, \omega \, \frac{x_L}{\gamma} \left| \begin{matrix} Z_{s-\ell}(x_0) J'_s(x_L) \\ \\ J_{s-\ell}(x_0) Z'_s(x_L) \end{matrix} \right| \frac{1}{(\omega - \Omega_s)} \, , \tag{48}$$

$$\hat{I}_z \equiv \omega_c \, k \, \frac{x_L}{\gamma} \left| \begin{matrix} Z_{s-\ell}(x_0) J'_s(x_L) \\ \\ J_{s-\ell}(x_0) Z'_s(x_L) \end{matrix} \right| \frac{1}{(\omega - \Omega_s)} \, , \tag{49}$$

$$\hat{I}_\theta \equiv x_0 \begin{Bmatrix} Z'_{s-\ell}(x_0) J_s(x_L) \\ J'_{s-\ell}(x_0) Z_s(x_L) \end{Bmatrix}$$

$$+ \, x_L \begin{Bmatrix} Z_{s-\ell}(x_0) J'_s(x_L) \\ \\ J_{s-\ell}(x_0) Z'_s(x_L) \end{Bmatrix} \left[1 + \frac{\frac{\ell}{\gamma} \omega_c}{(\omega - \Omega_s)} \right] \tag{50}$$

Again the upper (lower) expressions are for $P_\theta < 0$ ($P_\theta > 0$), and x_0, x_L are defined by equation (43). The angles α and χ are shown in Fig. 4.

It should be noted here that another wave mode number s is introduced to overcome mathematical difficulty which turns out to possess physical meaning of magnetic harmonic number (see eq. 45). Also note that the response of the perturbed distribution function gives rise to resonance of ω at Ω_s with the dependence of $(\omega-\Omega_s)^{-1}$. The discreteness of Ω_s with respect to s makes it possible to treat the problems of different mode numbers individually (see eq. 46). We will, therefore, designate the transverse electric (TE) mode with $TE_{\ell ns}$ notation where ℓ denotes the azimuthal mode number, n the radial mode number, and s the magnetic harmonic number of cyclotron frequency.

It is interesting to observe the responses of f^1 to the independent momentum variables γ, p_z and P_θ. While $\hat{I}_\gamma$ and $\hat{I}_z$ (eqs. 48 and 49) are proportional to $(\omega-\Omega_s)^{-1}$, $\hat{I}_\theta$ has a constant term with respect to $(\omega-\Omega_s)$. Furthermore, for the azimuthally symmetric ($\ell=0$) mode, the resonance term $(\omega-\Omega_s)^{-1}$ completely disappears in $\hat{I}_\theta$. We conclude, therefore, the contribution of $\partial f_0/\partial P_\theta$ to $TE_{\ell ns}$ mode will remain significant only for the azimuthally asymmetric mode ($\ell \neq 0$).

IV. Dispersion Equation for $TE_{\ell ns}$ Mode

The dispersion equation for $TE_{\ell ns}$ mode is obtained by substituting the perturbed function f^1 with f_s^1 into the original $TE_{\ell n}$ equation (21). Namely,

$$\frac{\omega^2}{c^2} - k^2 - \alpha_{\ell n}^{\;2} = -\frac{2\pi e}{c}\,\alpha_{\ell n}\,\frac{1}{N_{\ell n}}$$

$$\times \int r dr \int d^r \underline{p}\; f_s^1 [\, Z_{\ell-1}(\alpha_{\ell n} r)\,(iv_r + v_\theta) + Z_{\ell+1}(\alpha_{\ell n} r)(iv_r - v_\theta)] . \quad (51)$$

From Appendix B [eqs (B-9), (B-15), and (B-7)], we have

$$i\, v_r \mp v_\theta = \pm \frac{\omega_c r_L}{\gamma} \exp[\pm\, i(\alpha + \chi)] \quad , \quad (52)$$

$$d^3 \underline{p} = \frac{mc^2}{\omega_c} \frac{\gamma}{r_0 r_L} \frac{1}{\sin\chi}\, d\gamma\, dp_z\, dP_\theta \quad . \quad (53)$$

Here the angles α, χ, and ξ of the triangle OGA are as shown in Fig. 4. With eqs (52) and (53) substituted into (47), the dispersion eq. (51) can be rewritten as follows:

$$\frac{\omega^2}{c^2} - k^2 - \alpha_{\ell n}^{\;2} = -\frac{2\pi m e^2 (-1)^\ell}{\alpha_{\ell n} N_{\ell n}} \int dr \int d\gamma \int dp_z \int dP_\theta$$

$$\times \frac{r}{r_0} \frac{1}{\sin\chi} (\Phi^+ - \Phi^-) [\, \hat{I}_\gamma \frac{\partial f_0}{\partial \gamma} + \hat{I}_z \frac{\partial f_0}{\partial p_z} + \hat{I}_\theta \frac{\partial f_0}{\partial P_\theta}] \quad , \quad (54)$$

where

$$\Phi^{\pm} \equiv Z_{\ell \pm 1}(\alpha_{\ell n} r) \exp[\, i(s \pm 1)\chi + i(\ell \pm 1)\alpha] \quad . \quad (55)$$

and $\hat{I}$'s are given in eqs. (48) to (50). We note here that I's and f_0 is independent of r (see discussion in Appendix C), so that the terms in [ ] bracket can be factored out of r-integration. Thus,

$$\frac{\omega^2}{c^2} - k^2 - \alpha_{\ell n}^2$$

$$= - \frac{2\pi m e^2 (-1)^{\ell}}{\alpha_{\ell n}^2 N_{\ell n}} \int d\gamma \int dp_z \int dP_\theta \, X_L \, [\hat{I}_\gamma \frac{\partial f_0}{\partial \gamma} + \hat{I}_z \frac{\partial f_0}{\partial p_z} + \hat{I}_\theta \frac{\partial f_0}{\partial P_\theta}]$$

$$\times \int \frac{r dr}{r_o r_L \sin\chi} \quad (\Phi^+ - \Phi^-) \quad , \qquad (56)$$

when X_L ($\equiv \alpha_{\ell n} r_L$) is previously defined. The r-integration in eq. (56) is difficult to perform through complicated dependence of angles χ and α on r (see Appendix B), and again we invoke Bessel summation theorem (Appendix A) to overcome this difficulty. That is, we expand $Z_{\ell \pm 1}$ $(\alpha_{\ell n} r)$ in $\Phi^{\pm}$ (eq. 55) using running index s' on triangle OGA in Fig. 4. By eq. (A-6),

$$Z_{\ell \pm 1}(\alpha_{\ell n} r) = (-1)^{\ell+1} \sum_{s'=-\infty}^{+\infty} \left\{ \begin{matrix} Z_{s'-\ell}(x_0) & J_{s' \pm 1}(x_L) \\ J_{s'-\ell}(x_0) & Z_{s' \pm 1}(x_L) \end{matrix} \right\}$$

$$\times \exp\,[-i(s' \pm 1)\chi - i\,(\ell \pm 1)\alpha] \quad , \qquad (57)$$

where upper (lower) expression is for P_θ <0 (P_θ >0), and $x_0 \equiv \alpha_{\ell n} r_o$. With the expansion (57), the term $(\Phi^+ - \Phi^-)$ in eq. (56) becomes

$$\Phi^+ - \Phi^- = 2(-1)^{\ell} \sum_{s'} \left\{ \begin{matrix} Z_{s'-\ell}(x_0)\, J'_{s'}(x_L) \\ J_{s'-\ell}(x_0)\, Z'_{s'}(x_L) \end{matrix} \right\}$$

$$\times \exp\,[i(s-s')\chi] \quad , \qquad (58)$$

The upper (lower) expression is for $P_\theta < 0$ ($P_\theta > 0$) in eqs. (58) and (59). Using eq. (58) in eq. (56) leads to

$$\frac{\omega^2}{c^2} - k^2 - \alpha_{\ell n}{}^2 = - \frac{4\pi me^2}{\alpha_{\ell n}{}^2 N_{\ell n}} \int d\gamma \int dp_z \int dP_\theta \sum_{s'=-\infty}^{\infty} x_L$$

$$\times \left\{ \begin{matrix} Z_{s'-\ell}(x_0) J'_{s'}(x_L) \\ J_{s'-\ell}(x_0) Z'_{s'}(x_L) \end{matrix} \right\} \left[\hat{I}_\gamma \frac{\partial f_0}{\partial \gamma} + \hat{I}_z \frac{\partial f_0}{\partial p_z} + \hat{I}_\theta \frac{\partial f_0}{\partial P_\theta} \right] \hat{\Phi}_{ss'} \quad , \tag{59}$$

where $\hat{\Phi}_{ss'}$ stands for the integral by r, i.e.,

$$\hat{\Phi}_{ss'} \equiv \int \frac{rdr}{r_o r_L \sin\chi} \exp\,[i(s-s')\chi] \quad . \tag{60}$$

Since the angle χ in Fig. 4 is function of r for given γ, p_z and P_θ (i.e. r_o and r_L), we transform $\chi = \chi(r)$ in eq. (60) via

$$\cos\chi = \frac{r_0{}^2 + r_L{}^2 - r^2}{2 r_o r_L} \quad , \tag{61}$$

and noting

$$\frac{rdr}{r_o r_L \sin\chi} = d\chi \tag{62}$$

we can rewrite $\hat{\Phi}_{ss'}$ as integral over angle χ ;

$$\hat{\Phi}_{ss'} = \int_{-\pi}^{\pi} d\chi \exp\,[i(s-s')\chi\,] \tag{63}$$

It is easy to show that

$$\hat{\Phi}_{ss'} = 2\pi\delta_{ss'} \quad , \tag{64}$$

where $\delta_{ss'}$ is Kronecker δ, which eliminates summation over s' except for s'=s. Finally, with eq. (64), the dispersion equation (59) becomes as follows:

$$\frac{\omega^2}{c^2} - k^2 - \frac{x_{\ell n}{}^2}{R^2} = - \frac{8\pi^2 me^2}{N_{\ell n}} \int d\gamma \int dp_z \int dP_\theta$$

$$\times \left(T_r \frac{\partial f_0}{\partial \gamma} + T_z + \frac{\partial f_0}{\partial \gamma} + T_\theta + \frac{\partial f_0}{\partial P_\theta} \right) \tag{65}$$

Here, from eq (21a),

$$\hat{N}_{\ell n} \equiv \alpha_{\ell n}{}^2 N_{\ell n} = \tfrac{1}{2} \{(x_{\ell n}{}^2 - \ell^2)\, Z_\ell{}^2(x_{\ell n}) - (\delta^2 x_{\ell n}{}^2 - \ell^2) Z_\ell{}^2(\delta x_{\ell n})\} .$$

When $P_\theta < 0$ (or $r_0 > r_L$), we use the following T_i's in eq. (65);

$$T_r \equiv \frac{\omega_c}{mc^2}\ \omega\, \frac{x_L{}^2}{\gamma}\, [Z_{s-\ell}(x_0) J_s'(x_L)\,]^2\ \frac{1}{(\omega-\Omega_s)} ,$$

$$T_z \equiv k\omega_c\, \frac{x_L{}^2}{\gamma}\, [Z_{s-\ell}(x_0) J_s'(x_L)\,]^2\ \frac{1}{(\omega-\Omega_s)} ,$$

$$T_\theta \equiv x_L Z_{s-\ell}(x_0) J_s'(x_L) \Big[\ x_0 Z_{s-\ell}'(x_0) J_s(x_L) \qquad (67)$$

$$+ x_L\, Z_{s-\ell}(x_0) J_s'(x_L) \left\{ 1 + \frac{\frac{\ell}{\gamma}\omega_c}{(\omega-\Omega_s)} \right\} \Big] ,$$

when $P_\theta > 0$ (i.e. $r_0 < r_L$), $Z_{s-\ell}(x_0)$ and $J_s(x_L)$ are replaced by $J_{s-\ell}(x_0)$ and $Z_s(x_L)$ respectively in eqs (66) and (67). The dispersion relation eq. (65) can be further reduced by integration by part. However, caution must be taken in the process especially when the distribution function f_0 has spread in γ or p_z. In that case the integral over p_z, for example, has a form of

$$\int dp_z \frac{T}{\omega - \frac{kp_z}{\gamma m} - s\frac{\omega_c}{\gamma}}\ \frac{\partial f_0}{\partial p_z} , \qquad (68)$$

and the denominator $(\omega-\Omega_s)$ has a simple pole at $p_z = \frac{\gamma m}{k}(\omega - s\frac{\omega_c}{\gamma})$, which contributes to residue at the same value p_z, thereby resulting in Landau type damping. The same is true for γ-integration. We have to bear in mind to include this residue contribution whenever need may arise. We will here ignore this contribution, by noting that we can always add the residue contribution to the results of integration by part. By integration by part, and assuming that f_0 vanishes at the limiting values of integration, we get

$$\frac{\omega^2}{c^2} - k^2 - \frac{x_{\ell n}^2}{R^2} = \frac{8\pi^2 me^2}{\hat{N}_{\ell n}} \int d\gamma \int dp_z \int dP_\theta$$

$$\times \left[f_0 \left(\frac{\partial T_\gamma}{\partial \gamma} + \frac{\partial T_z}{\partial p_z} + \frac{\partial T_\theta}{\partial P_\theta} \right) \right] \quad . \tag{69}$$

The various derivatives in [ ] bracket can be computed by using the table I given in Appendix B. Then we obtain for eq(69)

$$\frac{\omega^2}{c^2} - k^2 - \frac{x_{\ell n}^2}{R^2} = \frac{8\pi^2 e^2 x_{\ell n}^2}{\omega_c \hat{N}_{\ell n} R^2} \int d\gamma \int dp_z \int dP_\theta f_0(\gamma, p_z, P_\theta)$$

$$\times \left[-\beta_\perp^2 H_{s\ell}(x_0,x_L) \frac{(\Omega_s^2 - c^2 k^2)}{(\omega-\Omega_s)^2} + \frac{2}{(\omega-\Omega_s)} \left\{ \frac{\omega_c}{\gamma} Q_{s\ell}(x_0,x_L) - \beta^2 \Omega_s H_{s\ell}(x_0,x_L) \right\} + U_{s\ell}(x_0,x_L) - \beta^2 H_{s\ell}(x_0,x_L) \right] \tag{70}$$

Here $\hat{N}_{\ell n}$ is defined previously in eq (66), and

$$H_{s\ell}(x_0,x_L) \equiv [\, Z_{s-\ell}(x_0) J'_s(x_L)]^2 \; ,$$

$$Q_{s\ell}(x_0,x_L) \equiv Z_{s-\ell}(x_0) J'_s(x_L) \left[\frac{s(s^2 - x_L^2)}{x_L} Z_{s-\ell}(x_0) J_s(x_L) + \frac{x_L^2 (s-\ell)}{x_0} Z'_{s-\ell}(x_0)\, J'_s(x_L) \right] \; , \tag{71}$$

$$U_{s\ell}(x_0, x_L) \equiv J_s(x_L)\, J_s'(x_L)\, [\left\{ \frac{(2s^2 - x_L^2)}{x_L} - \frac{x_L}{x_0^2}(s-\ell)^2 \right\} Z_{s-\ell}^2(x_0)$$

$$- x_L \{ Z_{s-\ell}'(x_0) \}^2 \quad] .$$

We define the following quantities:

$$\Omega_s \equiv \frac{kp_z}{\gamma m} + \frac{s\omega_c}{\gamma} \quad ,$$

$$x_0 \equiv \frac{x_{\ell n}}{R} r_0, \quad x_L \equiv \frac{x_{\ell n}}{R} r_L \ , \quad \beta \equiv \frac{\omega_c r_L}{\gamma c} \qquad (72)$$

$$r_L \equiv \frac{1}{m\omega_c} [m^2 c^2 (\gamma^2 - 1) - p_z^2]^{\frac{1}{2}} \ ,$$

$$r_0 \equiv \frac{1}{m\omega_c} [m^2 c^2 (\gamma^2 - 1) - p_z^2 - 2m\omega_c P_\theta]^{\frac{1}{2}} \ .$$

The expressions given in eq (71) hold when $P_\theta < 0$. When $P_\theta > 0$, $Z_{s-\ell}(x_0)$ and $J_s(x_L)$ are replaced by $J_{s-\ell}(x_o)$ and $Z_s(x_L)$ respectively as before.

Once the distribution function $f_0(\gamma, p_z, P_\theta)$ is given, the eigenfrequency $\omega(\equiv \omega_r + i\omega_i)$ with its real frequency ω_r and its growth rate $|\omega_i|$ can be determined from the dispersion relation eq (70) with terms defined in eqs (71) and (72). It should be emphasized that the dispersion relationship eq (70) is, unlike previous works, valid for the completely general distribution function of the beam electrons as long as the cylindrical symmetry exists and the self-field can be neglected. The left hand side of the dispersion relation (70) is the free waveguide equation with cutoff frequency $\frac{cx_{\ell n}}{R}$, while the right hand side is the correction to the waveguide equation due to the presence of the beam. Although the right hand side is small and proportional to ν (see eqs. (17) and (18)), it is this term indeed which gives rise to the instability. Careful examination shows that while the term proportional to $(\omega-\Omega_s)^{-2}$ is responsible for instability, the terms proportional to $(\omega-\Omega_s)^{-1}$ and $(\omega-\Omega_s)^0$ are stabilizing ones. More detailed analysis of dispersion equations will be carried out in a separate paper.[15]

V. Zero-Temperature Beam Dispersion Relationship

In this chapter we will derive the complete dispersion relationship for a specific distribution function, given by

$$f_0 = \frac{\omega_c N}{4\pi^2 mc^2 \bar{\gamma}} \delta(\gamma-\bar{\gamma})\ \delta(p_z-\bar{p}_z)\ \delta(P_\theta-\bar{P}_\theta)\ , \tag{73}$$

where $\bar{\gamma}$, $\bar{p}_z$, and $\bar{P}_\theta$ are constants. This electron distribution function describes the electrons which have constant energy $\bar{\gamma}mc^2$, constant guiding center radii $\bar{r}_0$, and constant Larmor radii $\bar{r}_L$. Here the constants $\bar{r}_L$, and $\bar{r}_0$ are defined as follows:

$$\begin{aligned} \bar{r}_L &\equiv \frac{1}{m\omega c}\left[m^2c^2(\bar{\gamma}^2 - 1) - \bar{p}_z^{\,2}\right]^{1/2} , \\ \bar{r}_0 &\equiv \frac{1}{m\omega c}\left[m^2c^2 (\bar{\gamma}^2-1) - \bar{p}_z^2 - 2m\omega c\bar{P}_\theta\right]^{1/2} . \end{aligned} \tag{74}$$

Since there is no spread in γ, p_z or P_θ, this distribution is often called a zero-temperature distribution function. It can be easily verified that

$$2\pi\int r dr \int d^3\underline{p}\ f_0 = N \quad , \tag{75}$$

where N is the total number of electrons per unit axial length. After some mathematical manipulation with eq. (73), the zero-temperature beam dispersion relation is derived below.

$$\frac{\omega^2}{c^2} - k^2 - \frac{x_{\ell n}^2}{R^2} = \frac{2\nu\, x_{\ell n}^2}{\bar{\gamma} N_{\ell n} R^2} \times \left[\begin{aligned} &- \bar{\beta}_\perp^2\ H_{s\ell}(\bar{x}_0, \bar{x}_L)\ \frac{(\bar{\Omega}_s^{\,2} - c^2k^2)}{(\omega-\bar{\Omega}_x)^2} \\ &+ \frac{2}{(\omega-\bar{\Omega}_s)}\left\{ \frac{\omega_c}{\gamma}\ Q_{s\ell}(\bar{x}_0, \bar{x}_L) - \bar{\beta}_\perp^{\,2}\ \bar{\Omega}_s\ H_{s\ell}\ (\bar{x}_0,\bar{x}_L)\right\} \\ &+ U_{s\ell}\ (\bar{x}_0, \bar{x}_L) - \bar{\beta}^2\ H_{s\ell}\ (\bar{x}_0, X_L) \end{aligned} \right] . \tag{76}$$

Here barred quantities are those which are obtained after $\gamma = \bar{\gamma}$, $p_z = \bar{p}_z$, $P_\theta = \bar{P}_\theta$ are inserted in eqs (71) and (72). It is this dispersion relation we will examine extensively both analytically and numerically, the results of which will be reported in separate papers.

VI. Conclusions

A detailed procedure of obtaining a general dispersion equation for the electron cyclotron maser instability in the waveguide has been presented with and without a center rod. The analysis has been carried out in the framework of linearized Vlasov-Maxwell system under the tenuous beam assumption for the azimuthally asymmetric mode and for the aribrary equilibrium distribution function.

The orbit integral for the calculation of the perturbed distribution function is solved with the aid of the Bessel summation theorem without neglecting any term, and the moment integral for the dispersion equation for an arbitrary equilibrium beam profile is obtained with the help of self-consistent properties of the distribution function, namely the fact that the distribution function is entirely described by three constants of motion. As an example, the closed form of dispersion function for cold electron beam is given by eq (76).

The resulting general dispersion equation ((69) or (70)) is useful for further development of the theory, such as the effect of the spread in energy and momenta. The analysis of the cold beam dispersion relation is given in accompanying memorandum paper.[15]

APPENDIX A

Some Properties of Bessel Function

In this appendix some properties of general Bessel function, which are used in the text are summarized. General Bessel function $Z_m(x)$ is defined by

$$Z_m(x) \equiv C_1 J_m(x) + C_2 Y_m(x),$$

where C_1 and C_2 are constants, and J_m and Y_m are the first and the second kind Bessel function of integer order m with $x \geq 0$.

1) Recurrence formulae

$$\begin{aligned} Z_{m-1}(x) + Z_{m+1}(x) &= \frac{2m}{x} Z_m(x) \\ Z_{m-1}(x) - Z_{m+1}(x) &= 2 Z'_m(x) \\ Z_{m-1}(x) - \frac{m}{x} Z_m(x) &= Z'_m(x) \\ Z'_m(x) &= \frac{m}{x} Z_m(x) - Z_{m+1}(x) \end{aligned} \tag{A-1}$$

2) Negative integer order

$$Z_{-m}(x) = (-1)^m Z_m(x) \tag{A-2}$$

3) Asymptotic behavior for small arguments ($m \geq 0$)

$$\begin{aligned} J_m(x) &\to \frac{1}{m!} \left(\frac{x}{2}\right)^m \quad , (m \neq 0) \\ J_0(x) &\to 1 - \left(\frac{x}{2}\right)^2 \\ Y_m(x) &\to -\frac{(m-1)!}{\pi} \left(\frac{2}{x}\right)^m \quad , (m \neq 0) \\ Y_0(x) &\to \frac{2}{\pi} \log \left(\frac{x}{2}\right) \end{aligned} \tag{A-3}$$

4) Summation theorem

In a triangle OGA as shown in Fig. 4, first let

$\alpha \equiv \angle GOA$, $\xi \equiv \angle OAG$, $\chi \equiv \angle OGA$, and $r \equiv \overline{OA}$, $r_0 \equiv \overline{OG}$, $r_L \equiv \overline{GA}$. Then we have

$$\xi + \alpha + \chi = \pi \quad . \tag{A-4}$$

Bessel summation theorem states that, when $r_0 > r_L$ (i.e. $\alpha \geq \frac{\pi}{2}$), then for real $a > 0$,

$$\exp(i\ell\alpha) Z_\ell(ar) = \sum_{s=-\infty}^{\infty} Z_{\ell+s}(ar_0)\, J_s(ar_L) \exp(is\chi) \tag{A-5}$$

with integer s. If $r_0 < r_L$ (i.e. $\xi \leq \frac{\pi}{2}$), we have

$$\exp(i\ell\xi)\, Z_\ell(ar) = \sum_{s=-\infty}^{\infty} Z_{\ell+s}(ar_L)\, J_s(ar_0) \exp(is\chi) \tag{A-5'}$$

From here on, we designate unprimed (primed) equations for $r_0 > r_L$ ($r_0 < r_L$).

We can rewrite eq. (A-5) by substituting $(-\ell)$ for ℓ and $(-s)$ for s, and by making use of eqs. (A-2) and (A-4). Namely,

$$Z_\ell(ar) = (-1)^\ell \sum_s Z_{s-\ell}(ar_0)\, J_s(ar_L) \exp[is\chi + i\ell\alpha] . \tag{A-6}$$

Similarly by substituting $-\ell$ for ℓ, and $-s+\ell$ for s in eq. (A-5'),

$$Z_\ell(ar) = (-1)^\ell \sum_s J_{s-\ell}(ar_0)\, Z_s(ar_L) \exp[is\chi + i\ell\alpha] \tag{A-6'}$$

After we differentiate eqs (A-6) and A-6') with respect to a, we get

$$rZ_\ell(ar) = (-1)^\ell \sum_s [r_0 Z'_{s-\ell}(ar_0)\, J_s(ar_L) + r_L Z_{s-\ell}(ar_0) J'_s(ar_L)] \times \exp[is\chi + i\ell\alpha] , \tag{A-7}$$

$$rZ_\ell(ar) = (-1)^\ell \sum_s [r_0 J'_{s-\ell}(ar_0)\, Z_s(ar_L) + r_L\, J_{s-\ell}(ar_0)\, Z'_s(ar_L)] \times \exp[is\chi + i\ell\alpha] . \tag{A-7'}$$

By substituting $\ell \pm 1$ for ℓ, and $s \pm 1$ for s into eqs (A-6) and (A-6'), we obtain with eq (A-4)

$$\exp[\pm i\xi]\, Z_{\ell \pm 1}(ar) = (-1)^\ell \sum_s \left\{ \begin{matrix} Z_{s-\ell}(ar_0)\, J_{s\pm 1}(ar_L) & \text{(A-8)} \\ \\ J_{s-\ell}(ar_0)\, Z_{s\pm 1}(ar_L) & \text{(A-8')} \end{matrix} \right\} \exp[is\chi + i\ell\alpha]$$

Therefore, from eqs (A-8) and (A-8') with eq (A-1), we find

$\frac{1}{2}[\exp(-i\xi)\, Z_{\ell-1}(ar) - \exp(+i\xi)\, Z_{\ell+1}(ar)]$

$$= (-1)^{\ell} \sum_s \left\{ \begin{matrix} Z_{s-\ell}(ar_0)\, J'_s\,(ar_L) \\ \\ J_{s-\ell}(ar_0)\, Z'_s\,(ar_L) \end{matrix} \right\} \exp\,[is\chi + i\ell\alpha\,] \qquad \begin{matrix} \text{(A-9)} \\ \\ \text{(A-9')} \end{matrix}$$

When we substitute ($s \pm 1$) for s in eqs (A-6) and (A-6'), then by making use of eq (A-1), the results are

$$i \sin\chi \;\; Z_{\ell}(ar) = (-1)^{\ell} \sum_s \exp[is\chi + i\ell\alpha\,]$$

$$\times \left\{ \begin{matrix} \frac{(s-\ell)}{ar_o}\;\; Z_{s-\ell}(ar_0)\; J'_s(ar_L) + \frac{s}{ar_L}\;\; Z'_{s-\ell}(ar_0)\; J_s(ar_L) \\ \\ \frac{(s-\ell)}{ar_0}\;\; J_{s-\ell}(ar_0)\; Z'_s(ar_L) + \frac{s}{ar_L}\;\; J'_{s-\ell}(ar_0)\; Z_s(ar_L) \end{matrix} \right\} \qquad \begin{matrix} \text{(A-10)} \\ \\ \text{(A-10')} \end{matrix}$$

5) Orthogonality

Consider Z (x) defined by

$$Z_{\ell}(x) \equiv J_{\ell}(x) - \frac{J'_{\ell}(\,x\;\;_n)}{Y'_{\ell}\;(\delta x_{\ell n})}\;\; Y_{\ell}(x) \quad , \qquad \text{(A-11)}$$

such that

$$Z_{\ell}(\delta X_{\ell n}) = 0, \qquad \text{(A-12)}$$

where $X_{\ell n} \geq 0$ is obtained from

$$Z'_{\ell}\;(X_{\ell n}) = 0\;. \qquad \text{(A-13)}$$

$Z_{\ell}(x)$ thus defined is in the form of eq (A-1) so that all of previous relations are applicable. If we introduce new variable r such that

$$X \equiv \frac{X_{\ell n}}{R}\; r \equiv \alpha_{\ell n} r \quad , \qquad \text{(A-14)}$$

then it is to show the orthogonality of $Z_{\ell}(\alpha_{\ell n} r)$ as follows:

$$\int_{\delta}^{R} rdr\; Z_{\ell}(\alpha_{\ell n} r)\; Z_{\ell}(\alpha_{\ell n'} r) = N_{\ell n}\,\delta_{nn'} \quad . \qquad \text{(A-15)}$$

Here the normalization factor $N_{\ell n}$ is given by

$$N_{\ell n} \equiv \frac{1}{2\alpha^2_{\ell n}} \left[(X_{\ell n}^{\;2} - \ell^2)\; Z^2(X_{\ell n}) - (\delta^2 X_{\ell n}^{\;2} - \ell^2)\; Z^2_{\ell}(\delta X_{\ell n}) \right] \;. \qquad \text{(A-16)}$$

Here $\delta_{nn'}$ is Kronecker δ-notation.

APPENDIX B

Electron Orbit Under Constant Magnetic Field

In this section, we will examine a single electron orbit with charge (- e) under the constant axial magnetic field (eq 22). The electric potential Φ is assumed to be zero and the vector potential for constant magnetic field A = $(0, A_\theta(Cr), 0)$ due to the symmetry in θ and z. Then Lagrangian L of the relativistic particle is given by

$$L = -mc^2 [1 - \frac{1}{c^2} (\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2)]^{1/2} - \frac{e}{c} r\dot{\theta} A_\theta \quad \text{(B-1)}$$

where $A_\theta = \frac{1}{2} B_0 r$. (B-2)

Since the Lagrangian L is cyclic in θ and and time independent, there exist three constants of motion associated. Namely,

$$\frac{\partial L}{\partial \theta} \equiv P_\theta = \gamma mr^2 \theta - \frac{e}{c} r A_\theta = rp_\theta - \frac{m\omega_c}{2} r^2 = \text{const.} \quad \text{(B-3)}$$

$$\frac{\partial L}{\partial \dot{z}} \equiv P_z = \gamma m\dot{z} = p_z = \text{const.} \quad \text{(B-4)}$$

$$H = \sqrt{c^2 p^2 + m^2 c^4} = \gamma mc^2 = \text{const.} \quad \text{(B-5)}$$

If we let

$$p_\perp \equiv [m^2c^2(\gamma^2-1) - p_z^2]^{1/2} \quad \text{(B-6)}$$

$$r_L \equiv \frac{p_\perp}{m\omega_c} \quad \text{(B-7)}$$

$$r_0 \equiv [r_L^2 - \frac{2P_\theta}{m\omega_c}]^{1/2} \quad \text{(B-8)}$$

and let the angle between $\underline{p}_\perp$ and $\underline{p}_\theta$ e_θ as ξ as shown in Fig. 4.

$$\hat{e}_\perp = \sin \xi \, \hat{e}_r + \cos \xi \, \hat{e}_\theta \quad \text{(B-9)}$$

then

$$P_\theta \equiv -\tfrac{1}{2}m\omega_c(r_0^{\,2}-r_L^{\,2}) = m\omega_c r r_L \cos\xi - \tfrac{1}{2}m\omega_c r^2 \qquad \text{(B-10)}$$

That is

$$r_L^{\,2} = r^2 + r_0^{\,2} - 2rr_L\cos\xi \quad . \qquad \text{(B-11)}$$

This cosine equation describes the triangle shown in Fig. 3 with $\overline{OA} = r$, $\overline{OG} = r_0$, $\overline{GA} = r_L$, and the constants r_0 and r_L correspond to the guiding center and the Larmor radii. Some useful properties result from the triangle OGA.

$$\frac{r_0}{\sin\xi} = \frac{r}{\sin\chi} = \frac{r_L}{\sin\alpha} \qquad \text{(B-12)}$$

$$\cos\xi = \frac{r^2+r_L^{\,2}-r_0^{\,2}}{2rr_L}, \quad \sin\xi = \frac{[(r^2-r_1^{\,2})(r_2^{\,2}-r^2)]^{\frac{1}{2}}}{2rr_L}$$

$$\cos\chi = \frac{r_0^{\,2}+r_L^{\,2}-r^2}{2r_0r_L}, \quad \sin\chi = \frac{[(r^2-r_1^{\,2})(r_2^{\,2}-r^2)]^{\frac{1}{2}}}{2r_0r_L}$$

$$\cos\alpha = \frac{r^2+r_0^{\,2}-r_L^{\,2}}{2rr_0}, \quad \sin\alpha = \frac{[(r^2-r_1^{\,2})(r_L^{\,2}-r^2)]^{\frac{1}{2}}}{2rr_0}$$

$$r_1 = |r_0-r_L|, \quad r_2 = r_0 + r_L \; .$$

Note that $r_0\,(P_\theta,\Upsilon,p_z)$, $r_L(\Upsilon,p_z)$. From eq. (B-9), we find that if $P_\theta > 0$ ($P_\theta < 0$), then $r_0 < r_L$ ($r_0 > r_L$). That is, the orbit of electrons with $P_\theta > 0$ ($P_\theta < 0$) encompasses (does not encompass) the origin of axis.

For future references, the following table for the derivatives is useful (see Table I).

It is our intention to use the set (γ, p_z, P_θ) as the independent momentum space variables $\underline{p}$, provided that the Jacobian $\left| \frac{\partial(p_x, p_\theta)}{\partial(\gamma, P_\theta)} \right| \neq 0$. We now proceed to complete the Jacobian. The Cartesian momentum coordinates p_x, p_y can be shown as (from Fig. 4)

$$p_x = p_\perp (\sin \xi \cos \theta - \cos \xi \sin \theta) \qquad \text{(B-13)}$$

$$p_y = p_\perp (\sin \xi \sin \theta + \cos \xi \cos \theta).$$

Then the Jacobian is found by

$$\left| \frac{\partial(p_x, p_y)}{\partial(\gamma, P_\theta)} \right| = \left| \frac{\partial p_x}{\partial \gamma} \frac{\partial p_y}{\partial P_\theta} - \frac{\partial p_y}{\partial \gamma} \frac{\partial p_x}{\partial P_\theta} \right| = \frac{mc^2}{\omega_c} \frac{\gamma}{r_0 r_L} \frac{1}{\sin\chi} \,. \qquad \text{(B-14)}$$

Therefore the volume element in the momentum space d^3p is now written as

$$d^3\underline{p} = \frac{mc^2}{\omega_c} \frac{\gamma}{r_0 r_L} \frac{1}{\sin\chi} \, d\gamma \, dp_z \, dP_\theta \qquad \text{(B-15)}$$

by making use of the table I and the relations (B-12).

APPENDIX C

Self-consistent Electron Distribution Function

In this appendix we will describe characteristics of the self-consistent distribution function of beam electrons, f_0 under the influence of constant magnetic field $B_0 e_z$. Using the constants of motion, γ, p_z and P_θ as the independent variables in momentum space, the equilibrium distribution function f_0 is generally written in the cylindrical coordinate system as follows:

$$f_0 = f_0(r, \gamma, p_z, P_\theta), \qquad \text{(C-1)}$$

where

$$r \equiv \underline{x} \cdot \hat{e}_r,$$

$$\gamma \equiv \left[1 + \frac{\underline{p}\cdot\underline{p}}{m^2c^2}\right]^{1/2}, \qquad \text{(C-2)}$$

$$p_z \equiv \underline{p}\cdot\hat{e}_z \;,$$

$$P_\theta \equiv (\underline{x}\cdot\hat{e}_r)(\underline{p}\cdot\hat{e}_\theta) - \tfrac{1}{2} m\omega_c(\underline{x}\cdot\hat{e}_r)^2 \;.$$

Here, the symmetry of equilibrum in θ- and z- coordinates (viz. $\frac{\partial}{\partial\theta} = 0 = \frac{\partial}{\partial z}$) is explictly assumed. It is our objective to show that $\frac{\partial f_0}{\partial r} = 0$, from equilibrium Vlasov equation. The time independent Vlasov equation is given by

$$\underline{v} \cdot \frac{\partial f_0}{\partial \underline{x}} - \frac{e}{c}\,\underline{v} \times \underline{B} \cdot \frac{\partial f_0}{\partial \underline{p}} = 0 \;. \qquad \text{(C-3)}$$

Now we will obtain the expressions of $\frac{\partial f_0}{\partial \underline{x}}$ and $\frac{\partial f_0}{\partial \underline{p}}$ when f_0 is given by eq (C-1).

$$\frac{\partial f_0}{\partial \underline{x}} = \frac{\partial r}{\partial \underline{x}} \frac{\partial f_0}{\partial r} + \frac{\partial \gamma}{\partial \underline{x}} \frac{\partial f_0}{\partial \gamma} + \frac{\partial p_z}{\partial \underline{x}} \frac{\partial f_0}{\partial p_z} + \frac{\partial P_\theta}{\partial \underline{x}} \frac{\partial f_0}{\partial P_\theta} \tag{C-4}$$

and

$$\frac{\partial f_0}{\partial \underline{p}} = \frac{\partial r}{\partial \underline{p}} \frac{\partial f_0}{\partial r} + \frac{\partial \gamma}{\partial \underline{p}} \frac{\partial f_0}{\partial \gamma} + \frac{\partial p_z}{\partial \underline{p}} \frac{\partial f_0}{\partial p_z} + \frac{\partial P_\theta}{\partial \underline{p}} \frac{\partial f_0}{\partial P_\theta} \tag{C-5}$$

It can be easily shown from eq (C-2) that

$$\frac{\partial r}{\partial \underline{x}} = \hat{e}_r, \quad \frac{\partial \gamma}{\partial \underline{x}} = \frac{\partial p_z}{\partial \underline{x}} = 0$$

$$\frac{\partial P_\theta}{\partial \underline{x}} = - m \omega_c r \hat{e}_r \tag{C-6}$$

and

$$\frac{\partial r}{\partial \underline{p}} = 0 \ , \quad \frac{\partial \gamma}{\partial \underline{p}} = \frac{\underline{p}}{\gamma m^2 c^2} \quad .$$

$$\frac{\partial p_z}{\partial \underline{p}} = \hat{e}_z, \quad \frac{\partial P_\theta}{\partial \underline{p}} = r \ \hat{e}_\theta \quad . \tag{C-7}$$

Here, the relations, $\frac{\partial}{\partial \theta} \hat{e}_r = \hat{e}_\theta$ and $\frac{\partial}{\partial \theta} \hat{e}_\theta = - \hat{e}_r$, are made use of.

By substituting eqs (C-6) and (C-7) into (C-4) and (C-5), respectively, we obtain

$$\frac{\partial f_0}{\partial \underline{x}} = \hat{e}_r \left[\frac{\partial f_0}{\partial r} - m\omega_c r \frac{\partial f_0}{\partial P_\theta} \right] \tag{C-8}$$

and

$$\frac{\partial f_0}{\partial \underline{p}} = \frac{\underline{p}}{\gamma m^2 c^2} \frac{\partial f_0}{\partial \gamma} + \hat{e}_z \frac{\partial f_0}{\partial p_z} + r \ \hat{e}_\theta \frac{\partial f_0}{\partial P_\theta} \tag{C-9}$$

With eqs (C-8) and (C-9), the Vlasov equation (C-3) becomes

$$V_r \frac{\partial f_0}{\partial r} = 0 \tag{C-10}$$

Since $V_r \equiv 0$, the equilibrium constraint on the equilibrium distribution function f_0 specified with eq (C-1) now states that

$$\frac{\partial f_0}{\partial r} = 0 \quad . \qquad \text{(C-11)}$$

The condition (C-11) is expected from simple argument. We note Vlasov equation is the statement of Liouville theorem,

$$\frac{df_0}{dt} = 0 \quad . \qquad \text{(C-12)}$$

If the functional dependence of f_0 is assumed as eq (C-1), then

$$\frac{df_0}{dt} = \frac{\partial f_0}{\partial r}\frac{dr}{dt} + \frac{\partial f_0}{\partial \gamma}\frac{d\gamma}{dt} + \frac{\partial f_0}{\partial p_z}\frac{dp_z}{dt} + \frac{\partial f_0}{\partial P_\theta}\frac{dP_\theta}{dt} \quad . \qquad \text{(C-13)}$$

As pointed out in Appendix B, however, the variables γ, p_z and P_θ are constants of motion, so that $\frac{d\gamma}{dt} = \frac{dp_z}{dt} = \frac{dP_\theta}{dt} = 0$. Then Liouville theorem eq (C-12) now becomes

$$\frac{\partial f_0}{\partial r}\frac{dr}{dt} = 0 \quad , \qquad \text{(C-14)}$$

which is identical with previous result eq (C-10).

We see the significance of the equilibrium condition $\partial f_0/\partial r = 0$ in that this makes $\frac{\partial f_0}{\partial \gamma}$, $\frac{\partial f_0}{\partial p_z}$ and $\frac{\partial f_0}{\partial P_\theta}$ to be independent of r, thereby independent of gyration angle χ (see chaps. III and IV) so that these terms can be factored out of the orbit integral (eq. 24) and of r integration in the dispersion (eq. 51). The constants of motion (γ, p_z, P_θ) as independent variables were thus adopted in the description of equilibrium distribution function of beam electrons.

REFERENCES

1. V.A. Flyagin, A.V. Gaponov, M.I. Petelin and V.K. Yulpatov, "The Gyrotron," IEEE Trans. Microwave Theory Tech., vol. MTT-25, 514-521 (1977).
2. J.L. Hirshfield and V.L. Granatstein, "The Electron Cyclotron Maser--an historical survey," IEEE Trans. Microwave Theory Tech., vol. MTT-25, pp. 522-527 (1977).
3. Also see the papers referred in Refs. 1 and 2.
4. E. Ott and W.M. Manheimer, "Theory of Microwave Emission by Velocity - Space Instabilities of an Intense Relativistic Electron Beam," IEEE Trans. Plasma Science PS-3, 1 (1975).
5. P. Sprangle and A. Drobot, "The linear and self-consistent nonlinear theory of the electron cyclotron maser instability," Trans. IEEE Microwave Theory and Technique, vol. MTT-25, pp. 528-544, June 1977; also, P. Sprangle and W.M. Manheimer, "Coherent Nonlinear Theory of a Cyclotron Instability," Phys. Fluids, vol. 18, pp. 224-230, February 1975.
6. K.R. Chu, A. T. Drobot, V.L. Granatstein and J.L. Seftor, "Characteristics and Optimum Operating Parameters of a Gyrotron Travelling Wave Amplifier," IEEE Trans. Microwave Theory and Techniques, vol. MTT-27, pp. 178-187, February 1979.
7. K.R. Chu and A.T. Drobot, "Theory and single wave simulation of the gyrotron travelling wave amplifier operating at cyclotron harmonics," Naval Research Laboratory Memorandum Report 3788, August 1978 (unpublished).

8. H. Uhm, R.C. Davidson and K.R. Chu, "Self-Consistent Theory of Cyclotron Maser Instability for Intense Hollow Electron Beams," Phys. Fluids 21, 1866 (1978).

9. H. Uhm, R.C. Davidson, K.R. Chu,."Cyclotron Maser Instability for General Magnetic Harmonic Number," Phys. Fluids 21, 1877 (1978).

10. K.R. Chu, "Theory of Electron Cyclotron Maser Interaction in a Cavity at the Harmonic Frequencies," Phys. Fluids (to be published, December 1978).

11. S. Ahn and J. Choe, "Azimuthally Asymmetric Electron Cyclotron Maser Instability," Bull. Amer. Phys. Soc. Vol. 23, No. 7, p. 749 (Sept. 1978).

12. V.L. Granatstein, P. Sprangle, M. Herndon, R.K. Parker and S.P. Schlesinger, "Microwave amplification with an intense relativistic electron beam," J. Appl. Phys. vol. 46, pp. 3800-3805, September 1975.

13. H. Jory, S. Hegji, J. Shively, R. Symons, "Gyrotron Developments," Microwave Journal, vol. 21, pp. 30-34, August 1978.

14. M. Read, L. Seftor, R. Lucey, K.R. Chu, J.D. Silverstein and V.L. Granatstein, "A High Power 35 GHz Microwave Source for ECRH," 1978 IEEE International Conference on Plasma Science, Monterey, Calif., 15-17 May, 1978, IEEE Conf. Record, 78CH 1357-3 NPS, p. 302.

15. J. Choe and S. Ahn, "Analysis of Linear Growth Rates of Gyrotron for $TE_{\ell ns}$ Mode with Zero-Temperature Beam Electrons," NRL Memo Report No. 4035 (May 1979).

TABLE I

Derivatives of quantities (Q's) in the first column with respect to γ (second column), p_z (third column) and P_θ (fourth column). This table was used in Appendix B.

Q	$\frac{\partial Q}{\partial \gamma} / \frac{\gamma c^2}{mc^2}$	$\frac{\partial Q}{\partial p_z} / \frac{p_z}{m^2 \omega_c^2}$	$\frac{\partial Q}{\partial P_\theta} / \frac{1}{m\omega_c}$
r_L	$\frac{1}{r_L}$	$-\frac{1}{r_L}$	0
r_0	$\frac{1}{r_0}$	$-\frac{1}{r_0}$	$-\frac{1}{r_0}$
ξ	$\frac{1}{r_0 r_L \sin\chi} \times [1 - \frac{r_0}{r_L} \cos\chi]$	$\frac{1}{r_0 r_L \sin\chi} \times [1 - \frac{r_0}{r_L} \cos\chi]$	$-\frac{1}{r_o r_L \sin\chi}$
α	$\frac{1}{r_0 r_L \sin\chi} \times [1 - \frac{r_L}{r_0} \cos\chi]$	$-\frac{1}{r_0 r_L \sin\chi} \times [1 - \frac{r_L}{r_0} \cos\chi]$	$\frac{\cos\chi}{r_0^2 \sin\chi}$
χ	$-\frac{1}{r_0 r_L \sin\chi} \times [2 - \cos\chi (\frac{r_L}{r_0} + \frac{r_0}{r_L})]$	$\frac{1}{r_0 r_L \sin\chi} \times [2 - \cos\chi (\frac{r_L}{r_0} + \frac{r_0}{r_L}]$	$\frac{1}{r_0 r_L \sin\chi} \times [1 - \frac{r_L}{r_0} \cos\chi]$

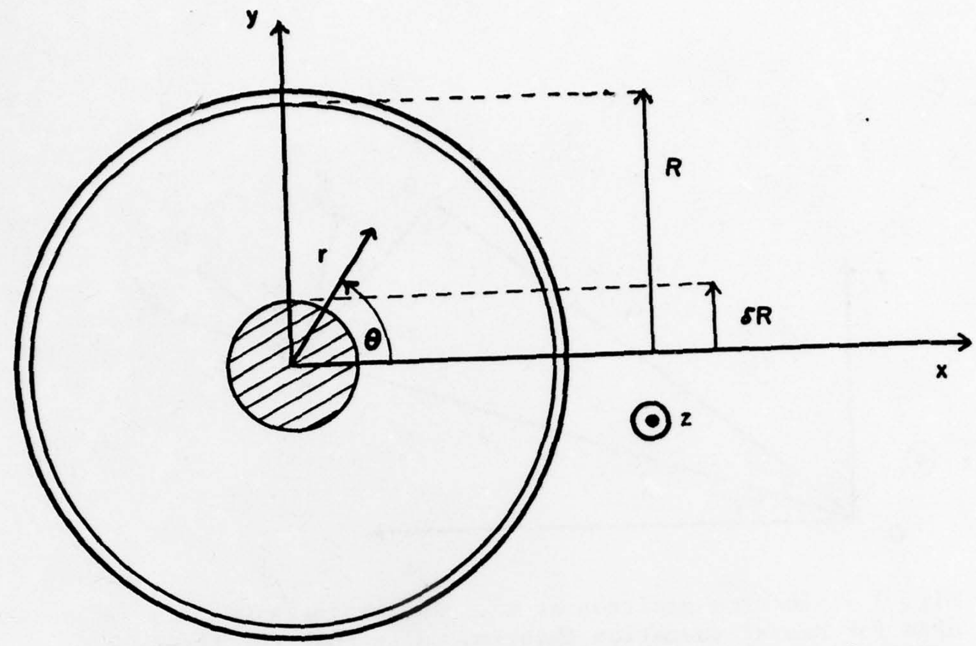


Fig. 1 - Cross section of coaxial gyratron. The conductors are located at r=R and r=δR, and the electron beam is contained between two conductors. The constant equilibrium magnetic field is in $\hat{e}_z$ direction.

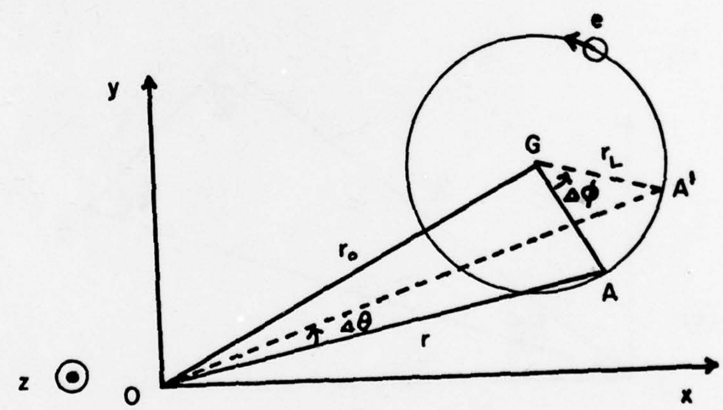


Fig. 2 - Electron orbit. The transverse equilibrium orbit of an electron is shown. The electron is moved from position A at time t to position A' at timet'. The guiding center radius r_0 $(=\overline{OG})$, and the Larmor radius r_L $(= \overline{GA}=\overline{GA'})$ are constants of motion. Here $\angle AOA' = \Delta\theta = \theta' - \theta$, and $\angle AGA' = \Delta\phi$.

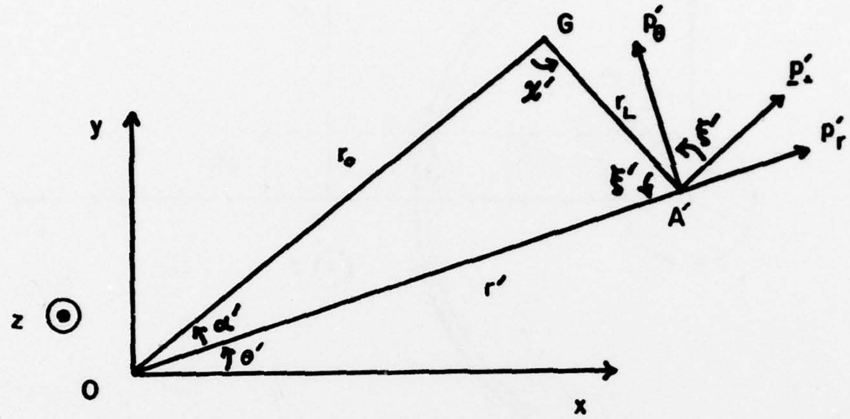


Fig. 3 - Electron position at t'. The triangle OGA' is used for Bessel summation theorem. Note that the transverse linear momentum p' is perpendicular to the line GA'.

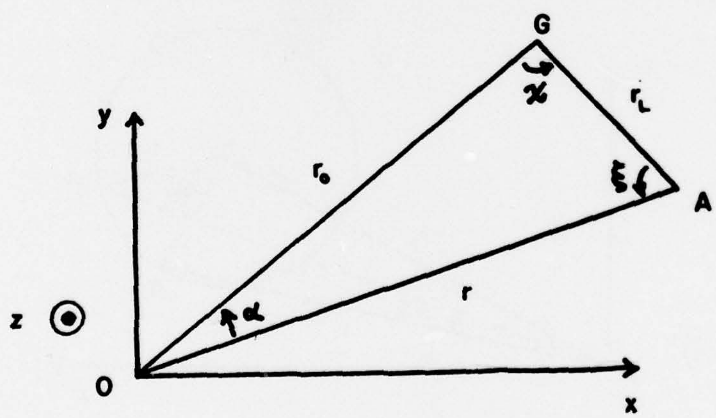


Fig. 4 - Electron position at t. Here we have $\alpha + \xi + \chi = 2\pi$