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TREND-FREE BLOCK DESIGNS: EXISTENCE
AND CONSTRUCTION RESULTS¹

by

Ching-Ming Yeh and Ralph A. Bradley
Florida State University

Designs for which treatment and block contrasts are orthogonal to specified common trend components within blocks have been called trend-free block designs. Necessary and sufficient conditions for the existence of such designs were given in a reference. A matrix sum was required to have certain properties. It is shown now that the existence of a matrix with the required properties assures the existence of the necessary component matrices.

The existence of trend-free block designs for specified trends in one or more dimensions is examined in a number of theorems and corollaries. Initial results are general and then trends in one dimension are considered. An alternative formulation of the necessary and sufficient conditions for a trend-free block design is given when each treatment has the same number of replications. Some special results are obtained for complete and balanced incomplete block designs. Youden Square designs are trend-free balanced incomplete block designs for up to p orthonormal, one-dimensional trend components, $p < k$, the block size.

Some design construction methods are developed to establish the existence of certain trend-free block designs. But, apart from the matter of treatment labelling, trend-free block designs may not be unique. The construction methods used may lead to designs particularly vulnerable to incorrect modelling of trend components.

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1. Introduction and summary. Bradley and Yeh (1980) introduced the concept of trend-free block (TFB) designs. They gave a necessary and sufficient condition for the existence of a TFB design and the appropriate analysis of variance. It was shown that a TFB design has important optimality properties among the class of connected designs with the same incidence matrix used in covariance analyses in the presence of within-block trends. The existence and construction of TFB designs are investigated in more detail in this paper.

In review, consider an experimental situation with v treatments applied to plots arranged in b blocks of size $k \leq v$. Each plot receives only one treatment and each treatment occurs at most once in a block. Plots in each block are arranged in the same m -dimensional array, indexed by vectors of positive integers, $\underline{t} = (t_1, \dots, t_m)$, $t_u = 1, \dots, s_u$, $u = 1, \dots, m$, where s_u is the number of plot positions in the u^{th} dimension, $\prod_u s_u = k$. A polynomial trend, common to all blocks, is assumed to exist over the plots in a block and to be a function of the plot position \underline{t} .

The classical model for general block designs is extended through the addition of trend terms. The trend function is expressed as a linear combination of m -dimensional, orthogonal polynomials of the form,

$$(1.1) \quad \phi_{\underline{g}}(\underline{t}) \equiv \phi_{(\alpha_1, \dots, \alpha_m)}(t_1, \dots, t_m) = \prod_{u=1}^m \phi_{\alpha_u}(t_u),$$

where $\phi_{\alpha_u}(t_u)$ is a one-dimensional, orthogonal polynomial of degree α_u on the integers, $1, \dots, s_u$. The model is

$$(1.2) \quad y_{j\underline{t}} = \mu + \sum_{i=1}^v \delta_{j\underline{t}}^i \tau_i + \beta_j + \sum_{\underline{g} \in \mathcal{A}} \theta_{\underline{g}} \phi_{\underline{g}}(\underline{t}) + \epsilon_{j\underline{t}},$$

$j = 1, \dots, b$, $\underline{t} = (t_1, \dots, t_m)$, $t_u = 1, \dots, s_u$, $u = 1, \dots, m$, where $y_{j\underline{t}}$ is the observation on plot position \underline{t} of block j , μ , τ_i and β_j are respectively the usual mean, treatment and block parameters, $\sum_{g \in A} \theta_g \phi_g(\underline{t})$ is the trend effect on plot \underline{t} , not dependent on the particular block j , with θ_g being the regression coefficient of $\phi_g(\underline{t})$ and A , an index set of p , m -dimensional, non-zero vectors of the form g , $p < k$, and the $\epsilon_{j\underline{t}}$ are random errors assumed to be i.i.d. with zero means. Designation of the treatment applied to plot (j, \underline{t}) is effected through indicator variables, $\delta_{j\underline{t}}^i = 1$ or 0 as treatment i is or is not on plot (j, \underline{t}) , $i = 1, \dots, v$. A TFB design exists when it is possible to choose the $\delta_{j\underline{t}}^i$ so that the estimators of treatment and block contrasts are unaffected by the presence or absence of the trend terms in (1.2). For a TFB design, the appropriate treatment and block sums of squares in analysis of variance have the same algebraic forms in the presence or absence of the trend terms.

Let Δ_j , $j = 1, \dots, b$, be the $k \times v$ matrix with $\delta_{j\underline{t}}^i$ in row \underline{t} and column i and let ϕ be the $k \times p$ matrix with $\phi_g(\underline{t})$ in row \underline{t} and column g . Further, let $X_r' = (\Delta_1', \dots, \Delta_b')$ and $X_0 = \mathbf{1}_b \otimes \phi$, where $\mathbf{1}_b$ is the b -dimensional column vector with unit elements and $\beta \otimes \zeta$ is the Kronecker product of β and ζ . Bradley and Yeh (1980, Theorem 3.1) show that a block design is trend-free (to the trend terms specified in the model) if and only if

$$(1.3) \quad X_r' X_0 = 0.$$

or equivalently, if and only if

$$(1.4) \quad \Delta_r' \phi = 0.$$

where $\Delta_r = \sum_{j=1}^b \Delta_j$. Note that

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$$(1.5) \quad \lambda_k' \phi = 0$$

and

$$(1.6) \quad \phi' \phi = I_p,$$

I_p being the p -dimensional identity matrix, from the orthogonality properties of the $\phi_g(\underline{x})$. Note also that A_j , $j = 1, \dots, b$, is a permutation matrix; the elements are 0 or 1 and $A_j A_j' = I_k$. It is seen that A_+ has row sums b and i^{th} column sum r_i , the number of replications of treatment i , $i = 1, \dots, v$. If $X_\beta = I_b \otimes I_k$, $N = X_\tau' X_\beta$ is the $v \times b$ incidence matrix with elements $n_{ij} = 1$ or 0 as treatment i is or is not in block j .

In this article, attention is directed first to the existence of TFB designs and then, to a lesser extent, to the construction of such designs when they do exist. Some general results are given first. These are followed by more specialized investigations for complete block and balanced incomplete block designs. Our investigations are not exhaustive, but they do lead to new insights into trend-free block designs.

2. Some general results. The necessary and sufficient condition (1.3) for a block design to be trend-free under model (1.2) may be simplified. It is true, in fact, that the existence of permutation matrices A_1, \dots, A_b satisfying (1.4) is guaranteed by the existence of a suitable matrix A_+ , as shown in the following theorem.

Theorem 2.1. Under model (1.2), a TFB design exists if and only if there exists a $k \times v$ matrix A_+ of nonnegative integers such that

- (i) each row sum of A_+ is b ,
- (ii) the i^{th} column sum of A_+ is r_i , $i = 1, \dots, v$, and
- (iii) $A_+ \phi = 0$.

The proof of Theorem 2.1 is given in Appendix A.1. It is based on the concept of systems of distinct representatives (SDR). Applications of SDR to experimental design are given by Raghavarao (1971, Chapter 6).

A general block design for the experimental situation of Section 1 may be represented by the design parameters, v, b, k, r_1, \dots, r_v . If the δ_{jt}^i may be chosen to produce a TFB design relative to the trend components of model (1.2), we designate such a design as $TF_{\phi} B(v, b, k; r_1, \dots, r_v)$. If $r_i = r, i = 1, \dots, v$, we abbreviate to $TF_{\phi} B(v, b, k, r)$. We use only $TF_{\phi} B$ when design parameter specifications are clear.²

Some general results follow immediately from (1.3) or (1.4).

Theorem 2.2. If a $TF_{\phi} B$ design exists, then it is also a $TF_{\psi} B$ design for $\psi = \phi \Lambda'$, where Λ' is any $p^* \times p$ permutation matrix, $p^* \leq p$. Equivalently, if a $TF_{\psi} B$ design does not exist, then no $TF_{\phi} B$ design exists.

Proof. The orthogonality relationships, $\mathbf{1}'_k \psi = 0$ and $\psi' \psi = \mathbf{I}_{p^*}$, follow from the definition of ψ and (1.5) and (1.6). Similarly, $\Lambda'_+ \psi = 0$ from (1.4) for ϕ . The theorem is proved.

Theorem 2.2 may be restated. When a TFB design exists for a specified set of orthogonal trend components, it is also a TFB design for any subset of those components. If a TFB design does not exist for a specified set of orthogonal trend components, a TFB design does not exist for any larger set of components containing the specified set.

²Bradley and Yeh (1980) suggested a less general notation useful in listing designs. If the trend in dimension u is to degree $p_u, u = 1, \dots, m$, and $\sum_{u=1}^m p_u = p$, they designated a TFB design as $TF_{p_1, \dots, p_m} B(v, b, k; r_1, \dots, r_v)$.

Theorem 2.3. If $TF_{\phi} B(v, b, k; r_1, \dots, r_v)$ and $TF_{\phi} B(v, b^*, k; r_1^*, \dots, r_v^*)$ designs exist, then a $TF_{\phi} B(v, b + b^*, k; r_1 + r_1^*, \dots, r_v + r_v^*)$ design exists.

Proof. The existence of the two initial designs implies the existence of two $k \times v$ matrices \underline{A}_+ and \underline{A}_+^* with row sums b and b^* and i^{th} column sums r_i and r_i^* , $i = 1, \dots, v$, respectively. Then $\underline{A}_+^{**} = \underline{A}_+ + \underline{A}_+^*$ has the properties required in Theorem 2.1 and the theorem follows.

The following corollary is a direct consequence of Theorem 2.3.

Corollary 2.1. The existence of a $TF_{\phi} B(v, b, k; r_1, \dots, r_v)$ design implies the existence of a $TF_{\phi} B(v, nb, k; nr_1, \dots, nr_v)$ design for any positive integer n .

3. Trends in one dimension. Some special results are obtained for trends in one dimension. Limited implications for m -dimensional trends are noted.

Let $\phi_{\alpha}' = [\phi_{\alpha}(1), \dots, \phi_{\alpha}(k)]$ represent a one-dimensional orthogonal polynomial of degree α . When α is odd, $\phi_{\alpha}(t) = -\phi_{\alpha}(k + 1 - t)$ and, if k is also odd, $\phi_{\alpha}(\frac{k+1}{2}) = 0$.

Theorem 3.1. If $\phi = \phi_1$, no $TF_{\phi} B$ design exists when k is even and r_i is odd for some $i = 1, \dots, v$.

Proof. Suppose that r_1 is odd without loss of generality. Consider $S = \sum_{j=1}^b \sum_{t=1}^k \delta_{jt}^1 \phi_1(t)$, an element of $\underline{A}_+ \phi_1$ in (1.4). When k is even, each $\phi_1(t)$, $t = 1, \dots, k$, is a common multiple of an odd integer, positive or negative. When r_1 is odd, S is a non-zero multiple of the sum of an odd number of odd integers and hence $S \neq 0$, (1.4) does not hold, and the theorem follows.

Corollary 3.1. If ϕ_1 is a column of ϕ , no $TF_{\phi} B$ design exists when k is even and r_i is odd for some $i = 1, \dots, v$.

Corollary 3.1 follows from Theorems 3.1 and 2.2.

Corollary 3.2. For model (1.2), if ϕ has a column ϕ_{α_u} such that $\alpha_{u^*} = 1, \alpha_u = 0, u \neq u^*, u, u^* = 1, \dots, m, s_{u^*}$ is even, r_i is odd for some $i = 1, \dots, v$, then no $TF_{\phi} B$ design exists.

Proof of Corollary 3.2 follows in the same way as proofs of Theorem 3.1 and Corollary 3.1.

Let $r_1 = \dots = r_v = r$ and define \underline{A}^R to be a matrix such that its j^{th} column is the $(n + 1 - j)^{\text{th}}$ column of \underline{A} , where n is the number of columns of \underline{A} and $j = 1, \dots, n$. The columns of \underline{A} and \underline{A}^R are in reverse order. Also, let $\underline{J}_{m \times n}$ be an $m \times n$ matrix of unit elements.

Theorem 3.2. If $\phi = (\phi_{\alpha_1}, \dots, \phi_{\alpha_p}), \alpha_1, \dots, \alpha_p$ odd, $p < k$, then a $TF_{\phi} B(v, b, k, r)$ design exists when

- (i) both k and r are even,
- (ii) k is odd and r is even, or
- (iii) both k and r are odd and $b \geq v$.

Proof. The proof takes the form of specification of a \underline{A}_+ for each of (i), (ii) and (iii). Verification of the conditions of Theorem 2.1 is then trivial. In effect, the proof provides a first step in design construction.

All integers in this proof are positive.

(i). Let $k = 2m$ and $r = 2n$, m and n integers; then $b = vr/k = vn/m$. Since b is an integer, there exist integers s and t , $m = st$, such that $v = cs$, $n = dt$, and $b = cd$ for integers c and d . A choice of \underline{A}_+ follows from

$$\underline{A}'_+ = (J_{c \times t} \otimes d I_s, J_{c \times t} \otimes d I_s^R).$$

(ii), (iii). Since $b = vr/k$ is an integer and k is odd, there exist odd integers s and t , $k = st$, such that $v = cs$, $r = dt$, and $b = cd$ for integers c and d . As r is even or odd, d is even or odd. We define an s -square matrix \underline{A} to be

$$\underline{A} = \frac{d}{2}(I_s + I_s^R), \text{ when } r \text{ is even,}$$

and

$$\underline{A} = \begin{bmatrix} w I_u & I_u & w I_u^R \\ I_u & a & I_u \\ w I_u^R & I_u & w I_u \end{bmatrix}, \text{ when } r \text{ is odd,}$$

$s = 2u + 1$, $d = 2w + 1$, $a = 2(w - u) + 1$, the last positive when $b \geq v$. Choices for \underline{A}_+ for (ii) and (iii) follow from

$$\underline{A}'_+ = J_{c \times t} \otimes \underline{A}.$$

Theorem 3.2 is proved.

If both k and r are odd and $b < v$, $TF_B(v, b, k, r)$ designs may or may not exist.

Corollary 3.3. For model (1.2), if ϕ consists of p column vectors of the form ϕ_{α} , α such that α_{u^*} is odd for some u^* , $\alpha_u = 0$, $u \neq u^*$, $u, u^* = 1, \dots, m$, a $TF_{\phi}B$ design exists under the conditions of Theorem 3.2.

Proof. The matrices Δ_{α} defined in Theorem 3.2 for the three conditions apply. It is only necessary to note that condition (iii) of Theorem 2.1 applies for ϕ as now defined.

A combinatorial property of $TF_{\phi}B(v, b, k, r)$ designs is obtained when $\phi = (\phi_1, \dots, \phi_p)$, yielding a one-dimensional trend of degree p in (1.2). Define $\tilde{A}'_{\alpha} = (1^{\alpha}, \dots, k^{\alpha})$ and $S_{\alpha}(k, r) = \frac{r}{k} \sum_k \tilde{A}'_{\alpha}$, $\alpha = 1, \dots, p$. Set $\phi_0 = \mathbf{1}_k$ and note that each \tilde{A}'_{α} is a linear combination of $\phi_0, \phi_1, \dots, \phi_p$, say

$$\tilde{A}'_{\alpha} = c_{\alpha\alpha} \phi_{\alpha} + \dots + c_{\alpha 0} \phi_0, \quad c_{\alpha\alpha} \neq 0.$$

Then, from (1.5), $\sum_k \tilde{A}'_{\alpha} = k c_{\alpha 0}$ and $rc_{\alpha 0} = S_{\alpha}(k, r)$.

Theorem 3.3. If $\phi = (\phi_1, \dots, \phi_p)$ and $r_1 = \dots = r_v = r$ under model (1.2), then a $TF_{\phi}B(v, b, k, r)$ design exists if and only if there exists a $v \times b$ matrix \underline{W} with non-negative elements such that

- (i) each column of \underline{W} has the integers $1, \dots, k$ as elements along with $(v - k)$ zero elements, and,
- (ii) for any $\alpha = 1, \dots, p$, the sum of the α^{th} powers of elements for any single row of \underline{W} is $S_{\alpha}(k, r)$.

Proof. (Necessity). Given a $TF_{\phi}B(v, b, k, r)$ design, $\Delta_1, \dots, \Delta_b$ exist such that (1.4) holds. Let

$$(3.1) \quad w_{ij} = \sum_{t=1}^k \delta_{jt}^i t, \quad i = 1, \dots, v, \quad j = 1, \dots, b.$$

Since Δ_j is a permutation matrix, condition (i) follows. Condition (ii) is verified directly:

$$\begin{aligned} \sum_{j=1}^b w_{ij}^\alpha &= \sum_{j=1}^b \left(\sum_{t=1}^k \delta_{jt}^i t \right)^\alpha = \sum_j \sum_t \delta_{jt}^i t^\alpha \\ &= \sum_j \sum_t \delta_{jt}^i [c_{\alpha\alpha} \phi_\alpha(t) + \dots + c_{\alpha 0} \phi_0(t)] \\ &= r c_{\alpha 0} = S_\alpha(k, r), \end{aligned}$$

since $\sum_j \sum_t \delta_{jt}^i \phi_\alpha(t) = 0$, $\alpha = 1, \dots, p$, from (1.4), $\phi_0(t) = 1$, and $\sum_j \sum_t \delta_{jt}^i = r$.

(Sufficiency). Given W satisfying conditions (i) and (ii) of the theorem, define $\delta_{jt}^i = 1$, if $t = w_{ij}$, and zero otherwise. Then condition (i) is sufficient to demonstrate that each Δ_j is a permutation matrix.

We verify (1.4).

$$S_\alpha(k, r) = \sum_j w_{ij}^\alpha = \sum_j \sum_t \delta_{jt}^i t^\alpha$$

and

$$\begin{aligned} S_\alpha(k, r) \mathbf{l}_v &= \sum_j \Delta_j' \mathbf{A}_\alpha = \Delta_\alpha' \mathbf{A}_\alpha \\ &= \Delta_\alpha' (c_{\alpha\alpha} \phi_\alpha + \dots + c_{\alpha 0} \phi_0) \\ &= c_{\alpha\alpha} \Delta_\alpha' \phi_\alpha + \dots + c_{\alpha 1} \Delta_\alpha' \phi_1 + S_\alpha(k, r) \mathbf{l}_v, \end{aligned}$$

$\alpha = 1, \dots, p$. It is seen that $\Delta_\alpha' \phi_\alpha = \mathbf{0}$, $\alpha = 1, \dots, p$, through selection of successive values of α , and (1.4) follows.

The theorem follows with use of Theorem 3.1 of Bradley and Yeh (1980).

Interpretation of \underline{W} is easy. From (3.1), it is seen that w_{ij} is the plot position of treatment i in block j if $w_{ij} \neq 0$, and, if $w_{ij} = 0$, treatment i does not occur in block j . Thus \underline{W} completely specifies the TFB design. As an example, consider the complete block design with $v = k = 7$, $b = r = 6$ and take $p = 2$ under the conditions of Theorem 3.3.

Take

$$\underline{W} = \begin{bmatrix} 1 & 2 & 4 & 5 & 5 & 7 \\ 2 & 6 & 6 & 6 & 2 & 2 \\ 3 & 3 & 7 & 1 & 6 & 4 \\ 4 & 5 & 2 & 7 & 1 & 5 \\ 5 & 7 & 5 & 2 & 4 & 1 \\ 6 & 1 & 3 & 4 & 7 & 3 \\ 7 & 4 & 1 & 3 & 3 & 6 \end{bmatrix}.$$

Note that the sum of the elements in each row of \underline{W} is 24 and the sum of squares of the same elements is 120. The corresponding design matrix is

$$\underline{D} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 1 & 3 & 7 & 4 & 2 & 5 \\ 7 & 4 & 6 & 1 & 5 & 2 & 3 \\ 3 & 5 & 7 & 6 & 1 & 2 & 4 \\ 4 & 2 & 7 & 5 & 1 & 3 & 6 \\ 5 & 2 & 6 & 3 & 4 & 7 & 1 \end{bmatrix}$$

where d_{jt} is the treatment applied to plot t of block j .

4. Trend-free complete block designs. We use $TF_{\underline{\phi}}CB(v, b)$ to designate a complete block design with v treatments and b blocks free of a trend described by $\underline{\phi}$. The existence of $TF_{\underline{\phi}}CB$ designs is explored.

Some preliminary results are noted. It is obvious that a Latin Square design of order v , with rows regarded as blocks, is a $TF_{\underline{\phi}}CB(v, v)$ design for $\underline{\phi} = (\phi_1, \dots, \phi_{v-1})$. Corollary 4.1 follows at once from Corollary 2.1 and Theorem 2.2.

Corollary 4.1. If $\underline{\phi} = (\phi_1, \dots, \phi_p)$, $p < v$, p and n positive integers, then a $TF_{\underline{\phi}}(v, nv)$ design exists.

If $b = 2$ and $\underline{\phi} = (\phi_{\alpha_1}, \dots, \phi_{\alpha_p})$, $p < v$, $\alpha_1, \dots, \alpha_p$ odd positive integers, it is clear that a $TF_{\underline{\phi}}CB(v, 2)$ design exists, since one may order the treatments in any random way in a first block and in reverse order in a second block. Corollary 4.2 follows, again from Corollary 2.1 and Theorem 2.2.

Corollary 4.2. If $\underline{\phi} = (\phi_{\alpha_1}, \dots, \phi_{\alpha_p})$, $p < v$, $\alpha_1, \dots, \alpha_p$ odd positive integers, n , a positive integer, a $TF_{\underline{\phi}}CB(v, 2n)$ design exists.

The matrix \underline{W} of Theorem 3.3 has columns that are permutations of the integers, $1, \dots, v$. The following theorem is proved in Appendix A.2.

Theorem 4.1. If $\underline{\phi} = (\phi_1, \dots, \phi_p)$, $p < v$, then a $TF_{\underline{\phi}}CB(v, b)$ design exists if and only if there exist t_{ij} , $i = 1, \dots, v$, $j = 1, \dots, b$, such that

(i) among the bv values of t_{ij} , each integer, $1, \dots, v$, appears b times and

$$(ii) \sum_{j=1}^b t_{ij}^{\alpha} = S_{\alpha}(v, b) \text{ for all } i = 1, \dots, v \text{ and } \alpha = 1, \dots, p.$$

To construct a TF _{ϕ} CB design, $\phi = (\phi_1, \dots, \phi_p)$, $p < v$, when it exists, one may find first a set of t_{ij} 's satisfying the conditions of Theorem 4.1 and then rearrange the orders of the t_{ij} 's for fixed values of i to obtain the elements of \underline{W} in Theorem 3.3. A TF _{ϕ} CB design is thus obtained.

Corollary 3.1 and Theorem 3.2 apply to complete block designs with $k = v$, $r = b$. The requirement in condition (iii) of Theorem 3.2 that $b \geq v$ may be dropped when $\phi = \phi_1$ for complete block designs.

Theorem 4.2. A TF _{ϕ_1} CB design exists if and only if

- (i) b is even, or
- (ii) both b and v are odd, $b \geq 3$.

Proof. Part (i) follows directly from Theorem 3.2. To prove part (ii), it suffices to show that a TF _{ϕ_1} CB(v , 3) design, exists for v odd, since the desired TF _{ϕ_1} CB(v , b) design with both v and b odd may be constructed from a TF _{ϕ_1} CB(v , 3) design and a TF _{ϕ_1} CB(v , $b - 3$) design, the latter known to exist, since $b - 3$ is even, by Theorem 2.3. It may be verified that the following three blocks (rows) constitute a TF _{ϕ_1} CB(v , 3) design for odd v :

$$(4.1) \begin{array}{cccccccc} 1 & 2 & \dots & (v-1)/2 & (v+1)/2 & (v+3)/2 & \dots & v-1 & v \\ v-1 & v-3 & \dots & 2 & v & v-2 & \dots & 3 & 1 \\ v & v-2 & \dots & 3 & 1 & v-1 & \dots & 4 & 2. \end{array}$$

Consider a quadratic trend in one dimension. Now $\phi = (\phi_1, \phi_2)$ and we must take $v \geq 3$. Three preliminary lemmas are given and the existence of designs for $3 \leq v \leq 16$ is examined.

Lemma 4.1. When $\phi = (\phi_1, \phi_2)$, a necessary condition for the existence of a $TF_{\phi}CB$ design is that $b(v+1)/2$ and $b(v+1)(2v+1)/6$ be integers.

Proof. The lemma is an immediate consequence of Theorem 3.3 in which $S_1(v, b) = b(v+1)/2$ and $S_2(v, b) = b(v+1)(2v+1)/6$ must be integers.

Lemma 4.2. When $\phi = (\phi_1, \phi_2)$, a $TF_{\phi}CB(v, 2)$ design does not exist for any v .

Proof. By Theorem 2.2, if a $TF_{\phi}CB$ design exists, $\phi = (\phi_1, \phi_2)$, it must also be a $TF_{\phi_1}CB$ design. All possible $TF_{\phi_1}(v, 2)$ designs must have the treatments with two blocks in opposite orders, violating condition (ii) of Theorem 3.3 when $\alpha = 2$. Thus a $TF_{\phi}CB(v, 2)$ design does not exist when $\phi = (\phi_1, \phi_2)$.

Lemma 4.3. When $\phi = (\phi_1, \phi_2)$, a $TF_{\phi}CB(v, 3)$ design exists if and only if $v = 3$.

Proof. Throughout the proof, $\phi = (\phi_1, \phi_2)$. Corollary 3.1 is sufficient to demonstrate that a $TF_{\phi}CB(v, 3)$ design does not exist when v is even. A Latin Square design of order 3 is a $TF_{\phi}CB(3, 3)$ design. It suffices to show that no $TF_{\phi}CB(2m+1, 3)$ design exists for any integer $m \geq 2$.

Consider any row of N of Theorem 3.3, now having only three elements, say w_1, w_2 and w_3 . Condition (ii) of Theorem 3.3 requires that

$$(4.2) \quad \begin{aligned} w_1 + w_2 + w_3 &= 3(m+1) \\ w_1^2 + w_2^2 + w_3^2 &= (m+1)(4m+3). \end{aligned}$$

If a $TF_{\phi}CB(2m+1, 3)$ design exists, $m \geq 2$, there must be a solution of (4.2) with some $w_j = 2m+1$, say $j = 3$. Then (4.2) reduces to

$$(4.3) \quad \begin{aligned} w_1 + w_2 &= m + 2 \\ w_1^2 + w_2^2 &= 3m + 2 \end{aligned}$$

and integer solutions are required for w_1 and w_2 . It is easily checked that m cannot exceed 2. But, when $m = 2$, (4.2) cannot have 3 as a solution in violation of condition (i) of Theorem 3.3. Hence the lemma follows.

In general, the existence of $TF_{\phi}CB$ designs, $\phi = (\phi_1, \phi_2)$, may be judged through use of Theorem 4.1 and examination of integer solutions, t_1, \dots, t_b , $1 \leq t_j \leq v$, $j = 1, \dots, b$, of the system of equations,

$$(4.4) \quad \begin{aligned} t_1 + \dots + t_b &= S_1(v, b) \\ t_1^2 + \dots + t_b^2 &= S_2(v, b). \end{aligned}$$

Such examination may be difficult when b or v is large due to the large number of solutions. The following two propositions are helpful in determining the non-existence of a design.

Proposition 4.1. If there is an integer c , $1 \leq c \leq v$, that can never be a solution of (4.4), a $TF_{\phi}CB$ design does not exist when $\phi = (\phi_1, \phi_2)$.

Examples of integers c for various values of v and b are as follows:

v	5	5	7	7	10	11	13	14	16
b	4	6	4	5	4	4	4	4	4
c	4	3	7	6	7	11	12	10	16

Given solutions to (4.4), consider possible sets of v solutions, not necessarily distinct.

Proposition 4.2. No $TF_{\phi}CB(v, b)$ design exists for $\phi = (\phi_1, \phi_2)$ unless there is a set of v solutions to (4.4) with each of the integers, $1, \dots, v$, occurring b times as required by Theorem 4.1.

As an example, consider $(v, b) = (11, 5)$, $S_1(11, 5) = 30$, $S_2(11, 5) = 230$. Any solution of the first equation of (4.4) must have (i) all t_j 's even, (ii) three even t_j 's, or (iii) one even t_j , $j = 1, \dots, 5$. But (i) and (iii) cannot yield solutions to the second equation of (4.4), since the left-hand side would be a multiple of 4 and cannot sum to 230. On the other hand, (ii) may yield a solution to (4.4), but no $TF_{\phi}CB(11, 5)$ design exists, $\phi = (\phi_1, \phi_2)$, by Proposition 4.2. Similar arguments demonstrate the non-existence of $TF_{\phi}CB(v, b)$ designs when $\phi = (\phi_1, \phi_2)$ for $(v, b) = (5, 7), (8, 6), (11, 6), (13, 5),$ and $(13, 6)$.

In Table I, we list the parameter combinations (v, b) , $v \leq 16$, for which a $TF_{\phi}CB(v, b)$ design exists, $\phi = (\phi_1, \phi_2)$.

TABLE I

Parametric combinations (v, b) with $TF_{\phi}CB$ designs, $v \leq 16$, m , a positive integer, $\phi = (\phi_1, \phi_2)$.

v	b	v	b	v	b	v	b
3	$3m$	7	$b \geq 6$	11	$b \geq 7$	15	$3m, m > 1$
4	$2m, m > 1$	8	$2m, m \neq 1, 3$	12	$6m$	16	$2m, m > 2$
5	5 or $b \geq 8$	9	$3m, m > 1$	13	$b \geq 7$		
6	$6m$	10	$2m, m > 2$	14	$2m, m > 2$		

Existence theorems for TFCB designs become complex in both notation and combinatorics for m -dimensional trends. Corollary 3.2 is useful with $r_i = b$, $i = 1, \dots, v$, in establishment of non-existence of certain designs. Theorem 4.2 may be generalized and this is done in Theorem 4.3 below. Theorem 4.3 is particularly useful since TFCB designs for m -dimensional trends will usually include in practice linear trend components in each of the m dimensions.

Consider model (1.2) with ϕ representing m linear trends, one for each of the m dimensions. Then ϕ_{α_u} , the u^{th} column of ϕ , has $\alpha_u = (\alpha_{u1}, \dots, \alpha_{um})$ such that $\alpha_{uu'} = 1$ or 0 as $u' = u$ or $u' \neq u$, $u, u' = 1, \dots, m$.

Theorem 4.3. Given model (1.2) with ϕ representing m linear trends, one for each of the m dimensions, a $\text{TF}_{\phi}\text{CB}(v, b)$ design exists if and only if

- (i) b is even, or
- (ii) both v and b are odd, $b \geq 3$.

Proof. Given v even, since $v = \prod_{u=1}^m s_u$, some s_u is even. Corollary 3.2 is sufficient to state that no $\text{TF}_{\phi}\text{CB}(v, b)$ design exists when v is even and b is odd.

If b is even, a $\text{TF}_{\phi}\text{CB}(v, b)$ design may be constructed. Let the typical plot be designated by (t_1, \dots, t_m) , $t_u = 1, \dots, s_u$, $u = 1, \dots, m$. When $b = 2$, apply the treatments to plots in the first block in random order and, in the second block, place the treatment on plot (t_1, \dots, t_m) in the first block on plot $(s_1 + 1 - t_1, \dots, s_m + 1 - t_m)$. It is easy to check that this design is a $\text{TF}_{\phi}\text{CB}(v, 2)$ design for the ϕ of the theorem. Designs for b even, $b > 2$, may be constructed through the combination of $\text{TF}_{\phi}\text{CB}(v, 2)$ designs.

If v is odd, each s_u is odd, $u = 1, \dots, m$. Consider $b = 3$. Apply the treatments to plots in the first block in random order and place the treatment on plot (t_1, \dots, t_m) in the first block on plots (j_1, \dots, j_m) and (j'_1, \dots, j'_m) in the second and third blocks, where

$$j_u = a_u \left(s_u - \frac{t_u - 1}{2} \right) + (1 - a_u) \left(\frac{s_u}{2} - \frac{t_u - 1}{2} \right),$$

$$j'_u = a_u \left(\frac{s_u}{2} - \frac{t_u - 2}{2} \right) + (1 - a_u) \left(s_u - \frac{t_u - 2}{2} \right),$$

and

$$a_u = 1 \text{ or } 0 \text{ as } t_u \text{ is odd or even,}$$

$u = 1, \dots, m$; the transfers described by the arrays of (4.1) have been used in each dimension. It is easy to check again that this design is a $TF_{\phi}CB(v, 3)$ design. Designs for v and b odd, $b > 3$, exist since they may be constructed from the combination of a $TF_{\phi}CB(v, 3)$ design and one or more $TF_{\phi}CB(v, 2)$ designs.

Some further generalization of Theorem 4.3 is possible. If ϕ represents m' linear trends, $0 < m' < m$, one for each of m' distinct dimensions, a $TF_{\phi}CB$ design exists if and only if b is even, or b is odd and s_u is odd for all u with values corresponding to the m' distinct specified dimensions, $b \geq 3$.

5. Trend-free incomplete block designs. Trend-free block designs have been treated sufficiently generally in Sections 2 and 3 to include incomplete block designs with special balance properties. Results obtained may be specialized easily. It is not clear that balance properties associated with classes of balanced or partially balanced incomplete block

designs are of particular assistance in determination of the existence or in generation of their trend-free counterparts. It does seem that trend-free designs are easier to construct when block sizes are small.

Since the Latin Square design has trend-free properties among complete block designs, Youden Squares are considered below. Two additional theorems for very special incomplete block designs are given also.

Symmetric balanced incomplete block designs, for which $v = b$ and hence $k = r$, free of a trend described by ϕ , may be designated as $TF_{\phi}SBIB(v, k, \lambda)$ designs, where λ has the usual meaning. A Youden Square design is a special case of an SBIB design in a rectangular array with the columns forming a CB design and the rows an SBIB design. It is obvious that a Youden Square design is a $TF_{\phi}SBIB(v, k, \lambda)$ design for any ϕ with p one-dimensional components. But a Youden Square design may always be constructed from an SBIB design.

Theorem 5.1. Given ϕ with p one-dimensional components, $p < k$, and an $SBIB(v, k, \lambda)$ design, a $TF_{\phi}SBIB(v, k, \lambda)$ design exists. A Youden Square design corresponding to the $SBIB(v, k, \lambda)$ design is such a $TF_{\phi}SBIB(v, k, \lambda)$ design.

Let a balanced incomplete block (BIB) design free of a trend described by ϕ be designated as a $TF_{\phi}BIB(v, b, k, r, \lambda)$ design. An irreducible BIB design is one for which the blocks consist of all possible subsets of size k of the v treatments. When $k = 2$ and $\phi = \phi_1$, the one-dimensional linear trend, an existence theorem results.

Theorem 5.2. Given $\phi = \phi_1$, a $TF_{\phi}BIB(v, \frac{1}{2}v(v-1), 2, v-1, 1)$ design exists when v is odd.

Proof. The design is a paired comparisons design, the blocks being the complete set of treatment pairs. Treatments within blocks may be ordered. Let the two treatments within a block be in ascending or descending order as their sum is odd or even. The design so obtained is trend-free as desired because r is even when v is odd and each treatment appears $r/2$ times on each of the two plot positions.

No trend-free paired comparisons design exists when $\xi = \xi_1$ and v is even. This follows from Theorem 3.1, now rewritten for BIB designs.

Corollary 5.1. If $\xi = \xi_1$, no $TF_{\xi} BIB(v, b, k, r, \lambda)$ design exists when k is even and r is odd.

A final theorem gives a necessary condition for a TFBIB design free of a quadratic trend where $k = 3$.

Theorem 5.3. If a $TF_{\xi} BIB(v, b, 3, r, \lambda)$ design exists for $\xi = (\xi_1, \xi_2)$, then r is a multiple of 3.

Proof. Except for standardizing constant multipliers, $\xi_1' = (-1, 0, 1)$ and $\xi_2' = (1, -2, 1)$. Let a specified treatment in the TF design occur a , b and c times on plot positions 1, 2, and 3 respectively. Since the design is trend-free, condition (1.4) applies; $-a + c = 0$ and $a - 2b + c = 0$. Thus $a = b = c$ and $r = 3a$, a multiple of 3.

We give two examples, a TF BIB(5, 10, 2, 4, 1) design and a TF _{ξ} BIB(5, 10, 3, 6, 3) design, $\xi = (\xi_1, \xi_2)$, associated respectively with Theorems 5.2 and 5.3. Both are irreducible designs.

TF BIB(5, 10, 2, 4, 1) ξ_1	TF BIB(5, 10, 3, 6, 3) ξ $\xi = (\xi_1, \xi_2)$
1 2	1 4 2
3 1	1 5 2
1 4	2 1 3
5 1	2 3 4
2 3	3 4 1
4 2	3 5 1
2 5	4 1 5
3 4	4 3 5
5 3	5 2 3
4 5	5 2 4

6. Concluding remarks. The emphasis in this article has been on criteria for trend-free block designs and their existence. Model (1.2) applies very generally to block designs with treatments in blocks of equal size. The necessary and sufficient condition (1.4) for the existence of trend-free block designs was given in the cited earlier paper, together with some optimality results. Theorem 2.1 demonstrates that the condition developed earlier depends only on the matrix Δ_+ and its component matrices Δ_j , $j = 1, \dots, b$, need not be considered individually. When $r_i = r$, $i = 1, \dots, v$, the usual design situation with an equal number of replications of each treatment, Theorem 3.3 provides an alternative representation of the necessary and sufficient condition.

A number of theorems and corollaries have been given to assist in the determination of existence of trend-free block designs in more special situations, sometimes with restrictions on the nature of the trend involved. Theorem 3.1 is useful in that it describes situations for which trend-free block designs do not exist, while Theorem 3.2 indicates when such designs do exist for trends with odd-order trend components. Complete block designs are examined in some detail and Theorems 4.1 and 4.2 assist in existence determinations. Design parameters for trend-free complete block designs for a one-dimensional quadratic trend are given in Table I for $3 \leq v \leq 16$. Trend-free incomplete block designs have not been investigated in any detail for design classes; the general model (1.2) applies and the general results of Sections 2 and 3 are helpful in existence determinations. Youden Square designs are trend-free for p one-dimensional trend components, $p < k$, the block size. It is not apparent that the usual balance properties of classes of incomplete block designs play a major role in existence criteria for their trend-free derivative designs.

No elegant design construction algorithms for trend-free block designs have been found. Some of the existence results given do depend on the construction of trend-free block designs and hence design construction has been considered. But the designs constructed to demonstrate existence may not be "good" designs and, since trend-free block designs are not unique in general, "better" designs may exist. Let us illustrate. In consideration of trend-free complete block designs, we have used, as a building block, the two-block design with treatments in random order in a first block and in reverse order in a second block, such a design to be repeated for complete blocks with b even. Such designs are trend-free

for a one-dimensional linear trend; they appear to be particularly vulnerable if the linearity assumption is incorrect and, say, a one-dimensional quadratic trend is present. A "better" design, if it exists, might be one that is trend-free for a trend with both linear and quadratic components, even though only the linear trend is assumed. It is apparent that the price for the benefits of a trend-free design may be increased vulnerability to bias from incorrect trend specification.

Some concept of "nearly trend-free" designs may need to be developed. For an assumed one-dimensional trend of degree p , one might prefer the design that is trend-free, or nearly trend-free, for a one-dimensional trend of degree $(p + 1)$. The authors propose to develop these ideas and to prepare a catalog of useful trend-free block designs.

APPENDIX

A.1. Proof of Theorem 2.1. Proof of Theorem 2.1 follows after the development of some preliminary results.

Let a set S have k elements and let sets, A_1, \dots, A_n , $n \geq k$, be non-empty subsets of S . A set $\{a_1, \dots, a_n\}$, $a_i \in A_i$, $i = 1, \dots, n$, $a_i \neq a_j$ for $i \neq j$, is defined to be a system of distinct representatives (SDR) for the sets A_1, \dots, A_n .

Lemma A.1. Hall (1935). A necessary and sufficient condition for the existence of an SDR for the sets A_1, \dots, A_n is that, for every integer m , $1 \leq m \leq n$, and every set of indices, i_1, \dots, i_m , $1 \leq i_1 < \dots < i_m \leq n$, the inequality, $|A_{i_1} \cup \dots \cup A_{i_m}| \geq m$, holds, where $|A|$ denotes the cardinality of the set A .

Theorem A.1. If \underline{B} is a v -square matrix of non-negative integers with equal row and column sums b , then \underline{B} may be expressed as the sum of b v -square permutation matrices.

Proof. Let the i^{th} row of \underline{B} have positive elements in columns i_1, \dots, i_t and zeros elsewhere, $t \leq v$. Let $A_i = \{i_1, \dots, i_t\}$, $i = 1, \dots, v$. Then the union of any m , $1 \leq m \leq v$, of the sets contains at least m of the integers $1, \dots, v$, since otherwise the sum of the elements in the m corresponding rows of \underline{B} would be less than mb , contrary to the assumption that each row sum of \underline{B} is b . Therefore, by Lemma A.1, an SDR exists for the sets A_1, \dots, A_v . The SDR consists of the integers $1, \dots, v$ in some order and defines a v -square permutation matrix, say \underline{P}_1 .

Consider the matrix $B - P_1$. It has non-negative integer elements with row and column sums, $b - 1$. A permutation matrix P_2 may be found just as P_1 was determined. Continuation of the process lets us write $B = P_1 + \dots + P_b$ where each P_i , $i = 1, \dots, b$, is a v -square permutation matrix.

Remark. Theorem A.1 is a generalization of a result of Mann and Ryser (1953), who limited B to be a matrix with elements zero or one.

Lemma A.2. If A_+ is a $k \times v$ matrix, $k \leq v$, of non-negative integers satisfying conditions (i) and (ii) of Theorem 2.1 and $r_i \leq b$ for all $i = 1, \dots, v$, then A_+ can be augmented to complete a $v \times v$ matrix of the form of B in Theorem A.1 through the addition of $v - k$ rows of non-negative integers.

Proof. Let $s_j = b - r_j$, $j = 1, \dots, v$. It suffices to show that a $(v - k) \times v$ matrix A of non-negative integers with equal row sums b and j^{th} column sums s_j , $j = 1, \dots, v$, can be found. Note that $\sum_{j=1}^v r_j = bk$ and $\sum_{j=1}^v s_j = (v - k)b$ so that row sums and column sums are compatible.

The elements a_{ij} , $i = 1, \dots, (v - k)$, $j = 1, \dots, v$, of A may be chosen as follows: $a_{ij} = \min(s_j - \sum_{i'=0}^{i-1} a_{i'j}, b - \sum_{j'=0}^{j-1} a_{ij'})$, where $a_{ij} = 0$ if $i = 0$

or $j = 0$. Now $B = \begin{bmatrix} A_+ \\ A \end{bmatrix}$ satisfies the requirements of Theorem A.1.

Necessity in Theorem 2.1 follows from Theorem 3.1 of Bradley and Yeh (1980). Sufficiency follows from Lemma A.2 and Theorem A.1, since they demonstrate that A_+ may be expressed as the sum of b $k \times v$ permutation matrices.

A.2. Proof of Theorem 4.1. If a TF₂CB design exists, the t_{ij} of Theorem 4.1 may be taken to be the w_{ij} of Theorem 3.3. Necessity in Theorem 4.1 is proved.

To prove sufficiency, suppose that there exist t_{ij} , $i = 1, \dots, v$, $j = 1, \dots, b$, satisfying conditions (i) and (ii) of the theorem. Let $A_i = \{t_{ij}, j = 1, \dots, b\}$, $i = 1, \dots, v$. Each set A_i has b elements, not necessarily distinct. The union of any m , $1 \leq m \leq v$, of the sets A_1, \dots, A_v contains at least m of the integers $1, \dots, v$, since condition (i) would be violated otherwise. By Lemma A.1, an SDR exists for the sets A_1, \dots, A_v . Take $w_{i1} \in A_i$, $i = 1, \dots, v$, to be the distinct representatives of A_1, \dots, A_v ; the set (w_{11}, \dots, w_{v1}) is a permutation of the integers, $1, \dots, v$.

The remaining $(b - 1)v$ t_{ij} 's are integers such that each of the integers, $1, \dots, v$, is included $(b - 1)$ times. Repetitions of the argument above lead to \underline{w} of Theorem 3.3. It is clear that we have found elements w_{ij} of \underline{w} such that condition (i) of Theorem 3.3 holds. Condition (ii) of that theorem also holds, since (w_{i1}, \dots, w_{ib}) is only a permutation of (t_{i1}, \dots, t_{ib}) , $i = 1, \dots, v$, assured by condition (ii) of Theorem 4.1.

The proof of Theorem 4.1 is complete.

REFERENCES

- Bradley, R. A. and Yeh, C. M. (1980). Trend-free block designs: theory. Ann. Statist., 8, (To appear).
- Hall, P. (1935). On representatives of subsets. J. London Math. Soc., 10, 26-30.
- Mann, H. B. and Ryser, H. J. (1953). Systems of distinct representatives. Am. Math. Monthly, 60, 397-401.
- Raghavarao, D. (1971). Constructions and Combinatorial Problems in Design of Experiments. John Wiley & Sons, Inc., New York.

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20. ABSTRACT Designs for which treatment and block contrasts are orthogonal to specified common trend components within blocks have been called trend-free block designs. Necessary and sufficient conditions for the existence of such designs were given in a reference. A matrix sum was required to have certain properties. It is shown now that the existence of a matrix with the required properties assures the existence of the necessary component matrices. The existence of trend-free block designs for specified trends in one or more dimensions is examined in a number of theorems and corollaries. Initial results are general and then trends in one dimension are considered. An alternative formulation of the necessary and sufficient conditions for a trend-free block design is given when each treatment has the same number of replications. Some special results are obtained for complete and balanced incomplete block designs. Youden Square designs are trend-free balanced incomplete block designs for up to porthonormal, one-dimensional trend components, $p < k$, the block size. Some design construction methods are developed to establish the existence of certain trend-free block designs. But, apart from the matter of treatment labelling, trend-free block designs may not be unique. The construction methods used may lead to designs particularly vulnerable to incorrect modelling of trend components.		