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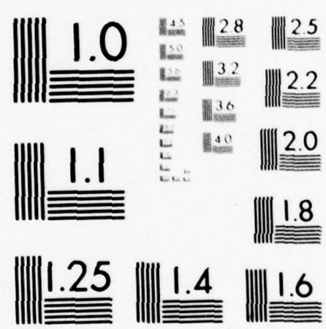

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LIMIT DISTRIBUTIONS OF KOLMOGOROV-SMIRNOV TYPE STATISTICS UNDER A FIXED ALTERNATIVE WITH ESTIMATED LOCATION AND SCALE PARAMETERS

by Constance L. Wood[1] and R. J. Serfling

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Department of Statistics
The Florida State University
Tallahassee, Florida 32306

[1]Assistant Professor, Department of Statistics, University of Kentucky, Lexington, Kentucky 40502

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ABSTRACT

LIMIT DISTRIBUTIONS OF KOLMOGOROV-SMIRNOV TYPE STATISTICS UNDER A FIXED ALTERNATIVE WITH ESTIMATED LOCATION AND SCALE PARAMETERS

Much attention has been devoted to Monte Carlo simulations of the power of Kolmogorov-Smirnov type goodness-of-fit statistics when nuisance parameters of the hypothesized distribution are estimated. Here we consider the asymptotic behavior of such statistics at fixed alternatives when location and scale parameters are estimated. It is shown that suitably normalized Kolmogorov-Smirnov statistics converge in distribution to Gaussian-related random variables depending on the alternative distribution and the maximum deviation between the null and alternative distribution functions. The work of Raghavachari (1973) is thus extended from simple hypotheses to the case of composite hypotheses with estimated nuisance parameters.

Key words and phrases: Limit distributions; Kolmogorov-Smirnov statistics; Fixed alternative; estimated parameters.

1. Introduction. Let $X_1, \ldots, X_n$ be a sample of independent observations on a continuous distribution function F. Consider the *composite* null hypothesis

$$H_0: F(x) = G(x; \theta), \text{ for some } \theta \in \Theta,$$

where G is a specified cdf and Θ is a specified parameter space for the nuisance parameter θ. A test of H_0 may readily be constructed from the sample distribution function and a consistent estimator $\hat{\theta}_n$ of θ. In particular, with

$$U_n(t; G, \hat{\theta}_n) = n^{-1} \sum_{i=1}^{n} I[G(X_i; \hat{\theta}_n) \le t], \quad 0 \le t \le 1,$$

where I[E] denotes the indicator of the event E, a measure of discrepancy between the observations and the hypothesized distribution is given by the modified empirical stochastic process

$$V_n(t; G) = n^{\frac{1}{2}}[U_n(t; G, \hat{\theta}_n) - t], \quad 0 \le t \le 1.$$

Many test statistics, including the Kolmogorov-Smirnov statistics with estimated parameters, can be written as functionals of the stochastic process $V_n(\cdot; G)$. Specifically, the one-sided Kolmogorov-Smirnov statistics are

$$D_n^+ = \sup_{0 \le t \le 1} V_n(t; G), \qquad D_n^- = \inf_{0 \le t \le 1} V_n(t; G),$$

and the two-sided K-S statistic is

$$D_n = \sup_{0 \le t \le 1} |V_n(t; G)|.$$

Stephens (1974) gives a comprehensive treatment of statistics which can be given similar representations. A more extensive class of test statistics can be shown to be asymptotically equivalent to such functionals. See Green and Hegazy (1976) and Kumar and Pathak (1977).

The weak convergence properties of the modified empirical process and corresponding test statistics have been extensively investigated both under the null hypothesis and under sequences of *contiguous* alternatives by Durbin (1973), Neuhaus (1976), and Wood (1978). However, power studies of statistics based on $V_n(\cdot; G)$ have been restricted mainly to Monte Carlo simulations.

Here we consider the asymptotic behavior of D_n^+, D_n^- and D_n under a *fixed* alternative. In particular, restricting attention to location and scale families of distributions under both the null and the alternative hypotheses, we extend the results of Raghavachari (1973), who treated the case of *simple* fixed alternatives, to the case of estimated nuisance parameters. Section 2 presents the main results, Section 3 the proofs, and Section 4 an illustration testing normal versus Cauchy.

2. <u>Results</u>. Consider the goodness-of-fit problem of testing

$$H_0: F(x) = G((x - \alpha_G)/\beta_G),\ -\infty < x < \infty, \text{ for some } (\alpha_G, \beta_G),\ \beta_G > 0,$$

versus

$$H_1: F(x) = H((x - \alpha_H)/\beta_H),\ -\infty < x < \infty, \text{ for some } (\alpha_H, \beta_H),\ \beta_H > 0.$$

The approach in Section 1 is motivated by the well-known result that under H_0 the random variables $G(X_i; (\alpha_G, \beta_G))$, $1 \le i \le n$, are independent uniform (0, 1) variates. In this case the sample distribution function

$$U_n(t;\, G,\, (\alpha_G,\, \beta_G)) = n^{-1}\sum_{i=1}^{n} I[G(X_i;\, (\alpha_G,\, \beta_G)) \le t],\ 0 \le t \le 1,$$

is strongly consistent estimator of the uniform distribution function $U(t) = t,\ 0 \le t \le 1$. The K-S statistics can then be written as functionals of the empirical stochastic process

$$W_n(t;\, G) = n^{\frac{1}{2}}[U_n(t;\, G,\, (\alpha_G,\, \beta_G)) - t],\ 0 \le t \le 1.$$

That is, $\sup_{0\le t\le 1} W_n(t;\, G)$, $\inf_{0\le t\le 1} W_n(t;\, G)$, and $\sup_{0\le t\le 1} |W_n(t;\, G)|$. Then weak convergence of $W_n(\cdot;\, G)$ to the tied-down Wiener process W^o yields the limit distributions of the K-S statistics. (See Billingsley (1968) for details.) Here W^o is the Gaussian process determined by $E\{W^o(t)\} = 0,\ 0 \le t \le 1$, and $E\{W^o(s)W^o(t)\} = \min(s,\, t) - st,\ 0 \le s,\, t \le 1$.

Similarly, to test H_0 as given above, we replace $W_n(\cdot;\, G)$ by its analogue $V_n(\cdot;\, G)$ based on estimates $(\hat\alpha_n,\, \hat\beta_n)$ of the nuisance parameters $(\alpha_G,\, \beta_G)$. We require that $(\hat\alpha_n,\, \hat\beta_n)$ be consistent estimators satisfying conditions which insure the weak convergence of $V_n(\cdot;\, G)$ under H_0. See Assumptions A and B(i). These conditions also insure that $G(X_1;\, (\hat\alpha_n,\, \hat\beta_n))$ converges in probability to a uniform random variable and that $U_n(t;\, G,\, (\hat\alpha_n,\, \hat\beta_n))$ is a consistent estimator of $U(t),\ 0 \le t \le 1$.

However, under H_1, $(\hat\alpha_n,\, \hat\beta_n)$ need not converge to t he appropriate location and scale parameters of H but we assume the existence of α_K and $\beta_K > 0$ such that $n^{\frac{1}{2}}(\hat\alpha_n - \alpha_K) = O_p(1)$ and $n^{\frac{1}{2}}(\hat\beta_n - \beta_K) = O_p(1)$. (In the sequel, all probability statements will refer to the alternative hypothesis, unless otherwise indicated.) Then $H((X_1 - \hat\alpha_n)/\hat\beta_n)$ will converge in probability to $H((X_1 - \alpha_K)/\beta_K)$, which may not be uniform. In order to produce asymptotically uniform variates, the probability integral transform $K((X_1 - \hat{\alpha}_n)/\hat{\beta}_n)$ is needed, where

$$K(x) = H[(\beta_K x + (\alpha_K - \alpha_H))/\beta_K], -\infty < x < \infty.$$

In terms of this transformation

$$D_n^+ = \sup_{0 \le t \le 1} \{V_n(t; K) + n^{\frac{1}{2}}[t - GK^{-1}(t)]\},$$

with D_n^- and D_n represented analogously.

To insure the weak convergence of $V_n(\cdot; K)$ under H_1, we assume that K satisfies Assumptions A and $(\hat{\alpha}_n, \hat{\beta}_n)$ satisfies Assumptions B, as follows.

Assumptions A.

(i) K' is positive on the support of K;

(ii) K" is bounded and $xK'(x) \to 0$ as $|x| \to 0$. □

Assumptions B.

(i) There exist random variables Y and Z such that for each $k = 1, 2, \ldots$ and $0 < t_1, \ldots, t_k < 1$, the vector $W^o(t_1), \ldots, W^o(t_k), Y, Z)$ is multivariate normal with mean vector 0 and $(k+2)\times(k+2)$ covariance matrix $\sum_{t_1, \ldots, t_k}$ and is the weak convergence limit of the random vector

$$(W_n(t_1; K), \ldots, W_n(t_k; K), n^{\frac{1}{2}}(\hat{\alpha}_n - \alpha_K)/\beta_K, n^{\frac{1}{2}}(\hat{\beta}_n - \beta_K)/\beta_K);$$

(ii) There exist constants c_1, c_2, δ_1 and δ_2 with $\min(\delta_1, \delta_2) > 0$ such that for all n sufficiently large

$$P\{n^{\frac{1}{2}}|\hat{\alpha}_n - \alpha_K|/\beta_K > \lambda\} < c_1\lambda^{-(1+\delta_1)}$$

and

$$P\{n^{\frac{1}{2}}|\hat{\beta}_n - \beta_K|/\beta_K > \lambda\} < c_2\lambda^{-(1+\delta_2)}. \quad \square$$

With these assumptions, $V_n(\cdot; K)$ converges weakly to the Gaussian stochastic process

$$V^o(t) = W^o(t) + K'(K^{-1}(t))Y + K'(K^{-1}(t))K^{-1}(t)Z, \; 0 \le t \le 1,$$

(see Wood (1978), (2.2)), while $n^{\frac{1}{2}}[t - GK^{-1}(t)]$ is unbounded for $G \neq K$ and is dominated by $n^{\frac{1}{2}}\sup_{0\le t\le 1} |t - GK^{-1}(t)|$. Setting

$$\theta^+ = \sup_{0\le t\le 1} [t - GK^{-1}(t)], \; \theta^- = \inf_{0\le t\le 1} [t - GK^{-1}(t)]$$

and

$$\Theta^+ = \{t: t - GK^{-1}(t) = \theta^+\}, \; \Theta^- = \{t: t - GK^{-1}(t) = \theta^-\},$$

if we can show that

$$D_n^+ = \sup_{t\in\Theta^+}\{V_n(t; K) + n^{\frac{1}{2}}[t - GK^{-1}(t)]\} + o_p(1)$$

(2.1) $$= \sup_{t\in\Theta^+} V_n(t; K) + n^{\frac{1}{2}}\theta^+ + o_p(1)$$

and similarly

(2.2) $$D_n^- = \inf_{t\in\Theta^-} V_n(t; K) + n^{\frac{1}{2}}\theta^- + o_p(1),$$

we will have from the almost sure continuity of the sample paths of V^o that

THEOREM 1. *Under Assumptions A and B, for every* α,

$$\lim_{n\to\infty} P\{D_n^+ - n^{\frac{1}{2}}\theta^+ \le \alpha\} = P\{\sup_{t\in\Theta^+} V^o(t) \le \alpha\}$$

and

$$\lim_{n\to\infty} P\{D_n^- - n^{\frac{1}{2}}\theta^- \le \alpha\} = P\{\inf_{t\in\Theta^-} V^o(t) \le \alpha\}.$$

Typically, Θ^+ and Θ^- consist of one point each, in which case the limit distributions are normal.

Similarly, setting $\theta = \max(\theta^+, -\theta^-)$, $\Theta_1 = \{t: t - GK^{-1}(t) = \theta\}$, $\Theta_2 = \{t: t - GK^{-1}(t) = -\theta\}$, and $\Theta = \Theta_1 \cup \Theta_2$, if we can show that

(2.3) $$D_n = \sup_{t\in\Theta}|V_n(t;\, K)| + n^{\frac{1}{2}}\theta + o_p(1),$$

we will have

THEOREM 2. *Under Assumptions A and B, for every* α,

$$\lim_{n\to\infty} P\{D_n - n^{\frac{1}{2}}\theta \le \alpha\} = P\{\max[\sup_{t\in\Theta_1} V^o(t),\, -\inf_{t\in\Theta_2} V^o(t)] \le \alpha\}.$$

(Note that one of Θ_1 or Θ_2 may be empty. In such a case, we adopt the convention that the sup over the empty set is $-\infty$.)

3. <u>Proofs</u>. In order to complete the proof of Theorem 1, we need only show (2.1) and (2.2). Since the arguments in each case are identical, we will consider only (2.1).

By construction of K and by Assumptions A and B(i), it follows from Wood (1978), formulas (3.3) and 3.14), that

$$\sup_{0\le t\le 1} |V_n(t;\, K) - \Delta_n(t;\, K)| = o_p(1),$$

where

$$\Delta_n(t;\, K) = n^{\frac{1}{2}}\{W_n(t;K) + K'(K^{-1}(t))[(\hat{\alpha}_n - \alpha_K)/\beta_K + K^{-1}(t)(\hat{\beta}_n - \beta_K)/\beta_K]\},$$

for $0 \le t \le 1$. Thus (2.1) is equivalent to

$$\sup_{0\le t\le 1}\{\Delta_n(t;K) + n^{\frac{1}{2}}[t - GK^{-1}(t)]\} = \sup_{t\in\theta^+}\Delta_n(t;K) + n^{\frac{1}{2}}\theta^+ + o_p(1).$$

PROOF OF THEOREM 1. For brevity, we put

$$E_n^+ = \sup_{t\in\theta^+}\{\Delta_n(t;K) + n^{\frac{1}{2}}[t - GK^{-1}(t)]\}$$

$$= \sup_{t\in\theta^+}\{\Delta_n(t;K) + n^{\frac{1}{2}}\theta.$$

Next for every integer ℓ, $\delta = \min\{\delta_1, \delta_2\}$, and $0 \le y \le 1$, define

(3.1) $\quad S(y, \ell) = \{t: 0 \le t \le 1, |GK^{-1}(y) - GK^{-1}(t)| < e^{-\frac{1}{2}\delta}\}$

and

(3.2) $\quad \mathcal{S} = \{S(t, \ell): t \in \theta^+\}.$

Since GK^{-1} is continuous, θ^+ is compact and $\mathcal{S}$, being an open cover for θ^+, has a finite cover $\mathcal{T}(\mathcal{T} \subset \mathcal{S})$. By the lemma given at the conclusion of this section, $\mathcal{T}$ has cardinality γ not exceeding $2[[2\ell^{\frac{1}{2}\delta}]]$, where $[[\cdot]]$ denotes greatest integer part, i.e., $\mathcal{T} = S(t_i, \ell), 1 \le i \le \gamma\}$.

Also, let

$$M = \bigcup_{i=1}^{\gamma} S(t_i, \ell),$$

with closure $\bar{M}$ and complement M^c (with respect to [0, 1]).

Since

$$\sup_{0\le t\le 1}\{\Delta_n(t;K) + n^{\frac{1}{2}}[t - GK^{-1}(t)]\} \ge E_n^+,$$

it suffices to show that, for any $\epsilon > 0$,

$$(3.3)\quad P(\sup_{0\le t\le 1}\{\Delta_n(t;K) + n^{\frac{1}{2}}[t - GK^{-1}(t)]\} - E_n^{+} > \epsilon) = o(1).$$

We deal with the sup in (3.3) in two parts $\sup_{t\in\overline{M}}$ and $\sup_{t\in M^c}$. First,

$$P(\sup_{t\in\overline{M}}\{\Delta_n(t;K) + n^{\frac{1}{2}}[t - GK^{-1}(t)\} - E_n^{+} > \epsilon)$$

$$\le \sum_{i=1}^{\gamma} P(\sup_{t\in\overline{S}(t_i,\ell)}\{\Delta_n(t;K) - \Delta_n(t_i;K) + n^{\frac{1}{2}}[t - GK^{-1}(t) - \theta^{+}]\} > \epsilon)$$

$$\le \sum_{i=1}^{\gamma} P(\sup_{t\in\overline{S}(t_i,\ell)}|W_n(t;K) - W_n(t_i;K)| > \epsilon)$$

$$+ \sum_{i=1}^{\gamma} P(\sup_{t\in\overline{S}(t_i,\ell)}|K'(K^{-1}(t)) - K'(K^{-1}(t_i))|\cdot|\delta_n - \alpha_K||\beta_K > \epsilon)$$

$$+ \sum_{i=1}^{\gamma} P(\sup_{t\in\overline{S}(t_i,\ell)}|K'(k^{-1}(t))K^{-1}(t) - K'(K^{-1}(t_i))K^{-1}(t_i)||\hat{\beta}_n - \beta_K||\beta_K > \epsilon).$$

By an argument identical to that of Raghavachari (1973) for the uniform distribution, it follows that the final summation in the last inequality may be made arbitrarily small for appropriate choice of ℓ. We now derive similar results for the other two summations.

First consider $z(t) = K'(K^{-1}(t))K^{-1}(t)$, $0 \le t \le 1$. Recalling (3.1) and (3.2), define

$$Z(y, \ell) = \{t: 0 \le t \le 1, |z(y) - z(t)| < \epsilon^{(1+\delta_2)/\delta_2}\ell^{-1}\}$$

and

$$Z_i = \{Z(y, \ell): y \in \overline{S}(t_i, \ell)\},\ 1 \le i \le \gamma.$$

By assumption B, $\sup_{0\le t\le 1}|z(t)| < c$ for some $c > 0$. Also, since GK^{-1} is nondecreasing, $S(t_i, \ell)$ is an interval and $\overline{S}(t_i, \ell)$ is compact.

Further, $\mathcal{Z}_i$ is an open cover for $\bar{S}(t_i, \ell)$, $1 \le i \le \gamma$. Therefore, there exists a finite subcover of $\mathcal{Z}_i$ with cardinality $\gamma_i \le [[2c\ell\epsilon^{-(1+\delta_2)/\delta_2}]]$, say $\{Z(t_{ij}, \ell), i \le j \le \gamma_i\}$. Then, for n sufficiently large,

$$P(n^{\frac{1}{2}}|(\hat{\beta}_n - \beta_K)/\beta_K| \cdot \sup_{t\in\bar{S}(t_i,\ell)} |z(t) - z(t_i)| > \epsilon)$$

$$\le \sum_{j=1}^{\gamma_i} P(n^{\frac{1}{2}}|(\hat{\beta}_n - \beta_K)/\beta_K| \sup_{t\in\bar{Z}(t_{ij},\ell)} |z(t) - z(t_i)| > \epsilon)$$

$$\le \sum_{j=1}^{\gamma_i} P(n^{\frac{1}{2}}|(\hat{\beta}_n - \beta_K)/\beta_K| \, \epsilon^{(1+\delta_2)/\delta_2} \cdot \ell^{-1} > \epsilon/4)$$

$$\le \sum_{j=1}^{\gamma_i} P(n^{\frac{1}{2}}|(\hat{\beta}_n - \beta_K)/\beta_K| > \ell/4\epsilon^{(1/\gamma_2)})$$

$$\le c_2 \sum_{j=1}^{\gamma_i} [4\epsilon^{(1/\delta_2)} \cdot \ell^{-1}]^{(1+\delta_2)}$$

$$\le c_2 \, 2[2c\ell\epsilon^{-(1+\delta_2)/\delta_2} + 1][4\epsilon^{(1/\delta_2)}\ell^{-1}]^{(1+\delta_2)}$$

$$= o(\ell^{-\frac{1}{2}\delta_2}).$$

Note that the fourth inequality above follows from Assumption B(ii). Since $\gamma = O(\ell^{\frac{1}{2}\delta_2})$, for n sufficiently large, the sum

$$\sum_{i=1}^{\gamma} P(n^{\frac{1}{2}}|(\hat{\beta}_n - \beta_K)/\beta_K| \sup_{t\in\bar{S}(t_i,\ell)} |z(t) - z(t_i)| > \epsilon)$$

is $\le \gamma \cdot o(\ell^{-\frac{1}{2}\delta_2})$ and can be made arbitrarily small for appropriate choice of ℓ. A similar argument with $z(t) = K'(K^{-1}(t))$, $0 \le t \le 1$, completes the proof that

$$\text{(3.4)} \qquad P(\sup_{t\in\bar{M}}\{\Delta_n(t; K) + n^{\frac{1}{2}}[t - GK^{-1}(t)]\} - E_n^+ > \epsilon) = o(1).$$

Next we deal with the $\sup_{t\in M^c}$ part. Since GK^{-1} is continuous, $\Theta^+ \subset M$ for every choice of $\ell \geq 1$ and $\sup_{t\in M^c}[t - GK^{-1}(t)]$ is bounded above by a number ρ with $0 < \rho < \theta^+$. Choose $\eta < \theta^+ - \rho$. Then

$$\sup_{t\in M^c}\{\Delta_n(t; K) + n^{\frac{1}{2}}[t - GK^{-1}(t)]\} \cdot E_n^+$$

$$\leq \sup_{t\in M^c}\{\Delta_n(t; K) - \sup_{t\in\Theta^+}\Delta_n(t) + n^{\frac{1}{2}}(\rho - \theta^+)$$

$$\leq 2 \sup_{0\leq t\leq 1}|\Delta_n(t; K)| + n^{\frac{1}{2}}(\rho - \theta^+).$$

Since $\sup_{0\leq t\leq 1}|\Delta_n(t; K)| = O_p(1)$ and $n^{\frac{1}{2}}(\rho - \theta^+) < -n^{\frac{1}{2}}\eta$,
we have

$$\text{(3.5)} \quad P(\sup_{t\in M^c}\{\Delta_n(t; K) + n^{\frac{1}{2}}(t - GK^{-1}(t)]\} - E_n^+ > \epsilon) = o(1).$$

Thus (3.3) follows, completing the proof of Theorem 1. □

The proof of Theorem 2 is now given. The method also provides a simpler proof for Theorem 2 of Raghavachari (1973).

PROOF OF THEOREM 2. Write $D_n = \max\{D_n^-, D_n^+\}$. If Θ_1 and Θ_2 are nonempty, then $\theta^+ = \theta = \theta^-$, $\Theta_1 = \Theta^+$, and $\Theta_2 = \Theta^-$. Therefore, for every $\alpha > 0$,

$$P(D_n - n^{\frac{1}{2}}\theta \leq \alpha) = P(\max\{D_n^+ - n^{\frac{1}{2}}\theta^+, -(D_n^- - n^{\frac{1}{2}}\bar{\theta})\} \leq \alpha)$$

$$\rightarrow P(\max\{\sup_{t\in\Theta^+}V^o(t), -\inf_{t\in\Theta^-}V^o(t)\} \leq \alpha),$$

as $n \rightarrow \infty$.

Now suppose Θ_1 is empty. Then $\theta^+ < \theta$. Since Θ_2 cannot simultaneously be empty, $\theta^- = -\theta$ and $\Theta_2 = \Theta^-$. Therefore,

$$D_n - n^{\frac{1}{2}}\theta = \max\{(D_n^+ - n^{\frac{1}{2}}\theta^+) + n^{\frac{1}{2}}(\theta^+ - \theta), \ -(D_n^- - n^{\frac{1}{2}}\theta^-)\}.$$

Since $D_n - n^{\frac{1}{2}}\theta^+ = O_p(1)$ and $\theta^+ - \theta < 0$,

$$P(D_n - n^{\frac{1}{2}}\theta \le \alpha) \to P(\inf_{t\in\Theta^-} V^o(t) \le \alpha), \ n \to \infty.$$

But for Θ_1 empty, by convention,

$$\max\{\sup_{t\in\Theta_1} V^o(t), -\inf_{t\in\Theta_2} V^o(t)\} = \inf_{t\in\Theta} V^o(t).$$

For Θ_2 empty, a similar argument gives the desired result. □

We now prove the basic lemma which was used at several points in the proof of Theorem 1. For any continuous function f on [0, 1] and subset A of [0, 1], and any $\epsilon > 0$, define

$$S'(x, \epsilon) = \{y: 0 \le t \le 1, |f(x) - f(y)| < \epsilon\}$$

and

$$S' = \{S(x, \epsilon): x \in A\}.$$

LEMMA. *Let* A *be a subset of* [0, 1] *and a continuous function on* [0, 1]. *Then there exists a finite open cover of* A, T(⊂ S'), *with cardinality* $\le 2[[2m/\epsilon]]$, *where* $|f| \le M$.

PROOF. Let

$$D_i = \{x: \tfrac{1}{2}i\epsilon < f(x) \le \tfrac{1}{2}(i + 1)\epsilon\}, \ 0 \le i \le [[2M/\epsilon]]$$

and

$$D_{i+[[2M/\epsilon]]} = \{x: -\tfrac{1}{2}(i + 1)\epsilon < f(x) \le -\tfrac{1}{2}i\epsilon\}, \ 1 \le i \le [[2M/\epsilon]].$$

Without loss of generality, it may be assumed that $D_i \cap A \neq \phi$, $0 \le i \le 2[[2M/\epsilon]]$. Now choose $x_i \in D_i \cap A$, $0 \le i \le 2[[2M/\epsilon]]$, and consider $S'(x_i, \epsilon)$, $0 \le i \le 2[[2M/\epsilon]]$. Suppose $y \in D_i$. Then, if $f(x_i) \le 0$,

$$\tfrac{1}{2}i\epsilon < f(x) \le \quad (i + 1)\epsilon \text{ and } -\tfrac{1}{2}(i + 1)\epsilon < -f(y) \le -\tfrac{1}{2}i\epsilon.$$

Therefore, $|f(x_i) - f(y)| \le \frac{1}{2}\epsilon$ and $D_i \subset S'(x_i, \epsilon)$. Similarly, this latter result also holds if $f(x_i) \le 0$. Thus

$$\cup_{0\le i\le 2[[2M/\epsilon]]} D_i \subset \cup_{1\le i\le 2[[2M/\epsilon]]} S'(x_i, \epsilon).$$

But $A \subset \cup_{0\le i\le 2[[2M/\epsilon]]} D_i$. Therefore, by construction, we have demonstrated the existence of a finite subcover with the required properties. □

4. Testing normal versus Cauchy. Suppose we want to test

$$H_0\text{: } F(x) = \Phi((x - \mu)/\sigma), \ -\infty < x < \infty, \text{ for some } \mu \text{ and } \sigma > 0,$$

against the fixed alternative

$$H_1\text{: } F(x) = C((x - \alpha)/\beta), \ -\infty < x < \infty, \text{ for some } \alpha \text{ and } \beta > 0,$$

where $C(\cdot)$ is the standard Cauchy cdf. In this case μ and σ will be estimated by the sample median and a linear function of the sample interquantile range, respectively. In particular, if we define, for any $0 < p < 1$, $\xi_p = \alpha + \beta C^{-1}(p)$ and $\xi_p = F_n^{-1}(p)$, where F_n is the sample cdf, then (μ, σ) will be estimated by $(\hat{\mu}_n, \hat{\sigma}_n)$, where $\hat{\mu}_n = \hat{\xi}_{1/2}$ and $\hat{\sigma}_n = (\hat{\xi}_{3/4} - \hat{\xi}_{1/4})/\ 1.349$. (Note that the interquartile range of the standard normal distribution is $\Phi^{-1}(3/4) - \Phi^{-1}(1/4) = 1.349$.)

Since the Cauchy distribution satisfies Assumptions A with $C'(x) > 0$, $-\infty < x < \infty$, we have, from Bahadur (1966), with probability 1,

$$(4.1) \qquad n^{\frac{1}{2}}(\hat{\xi}_p - \xi_p) = n^{\frac{1}{2}}[p - F_n(\xi_p)]/C'(\xi_p) + O(n^{-\frac{1}{4}} \log n).$$

This implies consistency of $(\hat{\mu}_n, \hat{\sigma}_n)$, which shows that $\alpha_K = \xi_{\frac{1}{2}} = \alpha$ and $\beta_K = [\frac{1}{2}(\xi_{3/4} - \xi_{1/4})]/[\frac{1}{2}(1.349] = \beta/.6745$, since the semi-interquartile range is the scale parameter of the Cauchy. Therefore,

$$K(x) = C(x/.6745), \; -\infty < x < \infty.$$

Also, B(ii) follows from (4.1) and Chebyshev's inequality and B(i) follows from (4.1) and the Cramér-Wold theorem.

Next we need to specify θ_1, θ_2, θ^+ and θ^-. Since the minimum and maximum of $[t - \Phi K^{-1}(t)]$ occur at points $-t = K(y)$, where $K'(y) = \Phi'(y)$, $y \neq 0$, i.e., at $y = \pm .33927$, we obtain $\theta_2 = \theta^- = \{.60407\}$ and $\theta_1 = \theta^+ = \{.39593\}$, and$\theta\theta_L = \theta\theta^+ = -\theta^- = 0.0182$.

Combining this with the fact that for $0 \le s \le 1$, the vector $n^{\frac{1}{2}}(W_n(s; K), (\hat{\mu}_n - \alpha_K)/\beta_K, (\hat{\sigma}_n - \beta_K)/\beta_K)$ converges in distribution to normal with mean vector 0 and covariance matrix

$$\begin{bmatrix} s(1-s) & \dfrac{(.6745)[\frac{1}{2}s - \min(s,\frac{1}{2})]}{C'(C^{-1}(\frac{1}{2}))} & (.6745)[\min(s,1/4) - \min(s,3/4) + s] \\[2ex] * & \dfrac{(.6745)^2}{C'(C^{-1}(\frac{1}{2}))} & 0 \\[2ex] * & * & \dfrac{(.6754)^2}{[4C'(C^{-1}(3/4))]^2} \end{bmatrix}$$

we obtain that the vector $(\sup_{t\in\Theta_1} V^o(t), \sup_{t\in\Theta_2} V^o(t))$ is bivariate normal with mean 0 and covariance matrix

$$\begin{bmatrix} .070 & -.010 \\ -.010 & .070 \end{bmatrix}.$$

Thus D_n^+ and D_n^-, suitably normalized, are asymptotically normally distributed, while D_n, suitably normalized, corresponds to the supremum of two bivariate normal r.v.'s with zero mean.

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20. ABSTRACT
Much attention has been devoted to Monte Carlo simulations of the power of Kolmogorov-Smirnov type goodness-of-fit statistics when nuisance parameters of the hypothesized distribution are estimated. Here we consider the asymptotic behavior of such statistics at fixed alternatives when location and scale parameters are estimated. It is shown that suitably normalized Kolmogorov-Smirnov statistics converge in distribution to Gaussian-related random variables depending on the alternative distribution and the maximum deviation between the null and alternative distribution functions. The work of Raghavachari (1973) is thus extended from simple hypotheses to the case of composite hypotheses with estimated nuisance parameters.