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CONVERGENCE PROOF OF ECONOMIC REPRESENTATION
OF TRANSCENDENTAL FUNCTIONS

TECHNICAL REPORT
July 15, 1960

M.R.I. Project No. 2229-P

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For

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Applied Mathematics Laboratory
David W. Taylor Model Basin
Washington, D. C.

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CONVERGENCE PROOF OF ECONOMIC REPRESENTATION OF TRANSCENDENTAL FUNCTIONS

by

(10) Jerry L. Fields
Yudell L. Luke

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For

Applied Mathematics Laboratory
David W. Taylor Model Basin
Washington, D. C.

PREFACE

This report covers research initiated by the Applied Mathematics Laboratory, David W. Taylor Model Basin, Washington, D. C. The research work upon which this report is based was accomplished by Midwest Research Institute under Contract No. Nonr-2638(00)(X).

The numerics in this report are related to work previously done in earlier reports, and the authors wish to take this opportunity to again thank those members of the Midwest Research Institute staff who materially contributed. They are Geraldine Coombs, Betty Kahn, Anna Lee Samuels and Wanda Shelp.

J.L.F. and Y.L.L.

Approved for:

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July 15, 1960

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CONVERGENCE PROOF OF ECONOMIC REPRESENTATION OF TRANSCENDENTAL FUNCTIONS

by

Jerry L. Fields

and

Yudell L. Luke

Summary and Introduction

In a previous report [1][1], it was shown that a large class of transcendental functions could be represented by a rational function, the ratio of two polynomials, together with a remainder term. More specifically, if $E(z)$ is defined by a Laplace integral, then

$$E(z) = \varphi_n/f_n + R_n , \quad R_n = F_n/f_n$$

where φ_n and F_n are polynomials of degree n and R_n is the remainder. In Section I it is shown that this can be considered a regular summation technique.

By proper choice of some parameters, f_n is a polynomial of hypergeometric form. If, except for a multiplicative factor, $E(z)$ has a generalized hypergeometric series representation, then F_n is a sum of n generalized hypergeometric series.

Only in certain cases could it be shown that the rational representation converges, ~~i.e., for z fixed, $\lim_{n\to\infty} R_n(z) = 0$~~. One of the principal difficulties lies in the fact that asymptotic expansions of a certain class of hypergeometric polynomials were known only for a few special cases, the classical Jacobi polynomials, for example. Asymptotic expansions of this class of hypergeometric polynomials now exist, ~~see [2]~~ and the convergence question can be further explored.

1/ Numbers in square brackets pertain to references at end of report.

The structure of the numerator of the error term, i.e., F_n is complicated and only in a few special cases has it been possible to represent it in a useful form. This drawback is partly alleviated by the fact that one of the above-mentioned special cases includes the Whittaker functions and their natural generalizations. For those cases where F_n can be put into a useful form, we prove convergence of the rational representation.

In the case of particular interest, i.e., the Whittaker functions, we represent F_n as an integral and then find a suitable bound for it. This coupled with an asymptotic estimate of f_n enables us to prove that $\lim_{n \to \infty} R_n = 0$ and also to obtain an asymptotic bound for the error. This is the essence of Section I. In Section II, numerics are presented to display the effectiveness of this bound.

1. Convergence of the Rational Approximations

In [1], it was shown how to formally generate a rather general sequence of rational approximations for the generalized hypergeometric function [3]

$$E(z) = {}_{p+1}F_q\left(\begin{matrix}\sigma, \alpha_1, \ldots, \alpha_p \\ \rho_1, \ldots, \rho_q\end{matrix}\middle| \frac{\lambda}{z}\right), \tag{1.1}$$

where

$$\left.\begin{aligned} {}_mF_n(x) &= {}_mF_n\left(\begin{matrix}\alpha_1, \ldots, \alpha_m \\ \rho_1, \ldots, \rho_n\end{matrix}\middle| x\right) = \sum_{k=0}^{\infty} \frac{\prod_{t=1}^{m} (\alpha_t)_k}{\prod_{t=1}^{n} (\rho_t)_k} \frac{(x)^k}{k!} \\ (\omega)_k &= \frac{\Gamma(\omega+k)}{\Gamma(\omega)} \end{aligned}\right\} . \tag{1.2}$$

Throughout this paper we employ a contracted notation and write

$$ {}_mF_n(x) = {}_mF_n\left(\begin{matrix} \alpha_m \\ \rho_n \end{matrix} \middle| x \right) = \sum_{k=0}^{\infty} \frac{(\alpha_m)_k}{(\rho_n)_k} \frac{(x)^k}{k!} \quad . \tag{1.3} $$

Thus $(\alpha_m)_k$ is interpreted as $\prod_{t=1}^{m} (\alpha_t)_k$, and a similar remark holds for $(\rho_n)_k$. We assume that none of the α_t's , or ρ_t's are zero or a negative integer. Also, it is assumed that the difference of any numerator parameter α_j , and any denominator parameter ρ_t , is never equal to zero. Empty terms in expression such as (1.3) are replaced by unity. Two distinct sequences of rational approximations were developed in the above paper [1]. They are as follows.

Type I or Homogeneous Case

$$ E(z) = \frac{\varphi_n(z,\gamma)}{f_n(\gamma)} + R_n(z,\gamma) \ , \ R_n(z,\gamma) = \frac{F_n(a_k,z,\gamma)}{f(\gamma)} \quad . \tag{1.4} $$

$$ \varphi_n(z,\gamma) = \sum_{r=0}^{n} \frac{(\sigma)_r(\alpha_p)_r}{(1)_r(\rho_q)_r} \left(\frac{\lambda}{z}\right)^r f_n^{[r]}(\gamma) \quad . \tag{1.5} $$

$$ f_n^{[r]}(\gamma) = \sum_{k=r}^{n} a_{n-k}\gamma^k \ , \ f_n^{[0]} = f_n(\gamma) \quad . \tag{1.6} $$

$$F_n(a_k, z, \gamma) = \frac{z^\sigma}{\Gamma(\sigma)} \int_0^\infty e^{-zt} t^{\sigma-1} \sum_{k=0}^{n} a_{n-k} (\lambda \gamma t)^k \sum_{r=0}^{\infty} \frac{(\alpha_p)_{r+k+1} (\lambda t)^{r+1}}{(\rho_q)_{r+k+1} (r+k+1)!} dt \tag{1.7}$$

$$= \frac{z^\sigma}{\Gamma(\sigma)} \int_0^\infty e^{-zt} t^{\sigma-1} \sum_{k=0}^{n} a_{n-k} \left(\frac{\lambda \gamma}{z}\right)^k \sum_{r=0}^{\infty} \frac{(\alpha_p)_{r+k+1} (\sigma+r+1)_k (\lambda t)^{r+1}}{(\rho_q)_{r+k+1} (r+k+1)!} dt \tag{1.8}$$

$$= \frac{\sigma \lambda \alpha_p}{z \rho_q} \sum_{r=0}^{\infty} \frac{(\sigma+1)_r (\alpha_p+1)_r}{(2)_r (\rho_q+1)_r} \left(\frac{\lambda}{z}\right)^r \sum_{k=0}^{n} \frac{a_{n-k} (\sigma+r+1)_k (\alpha_p+r+1)_k}{(2+r)_k (\rho_q+r+1)_k} \left(\frac{\lambda \gamma}{z}\right)^k . \tag{1.9}$$

where $\alpha_p = \prod_{t=1}^{p} \alpha_t$, and $\rho_q = \prod_{t=1}^{q} \rho_t$. Here the coefficients a_k are as yet arbitrary, and γ is unrestricted except that $|\gamma/z| \leq 1$.

<u>Type II or Non-Homogeneous</u>

$$E(z) = \frac{\Phi_n(z, \gamma)}{f_n(\gamma)} + R_n(z, \gamma) \quad , \quad R_n(z, \gamma) = \frac{F_n(a_k, z, \gamma)}{f_n(\gamma)} \quad . \tag{1.10}$$

$$\Phi_n(z, \gamma) = \sum_{r=0}^{n-1} \frac{(\sigma)_r (\alpha_p)_r}{(1)_r (\rho_q)_r} \left(\frac{\lambda}{z}\right)^r f_n^{[r+1]}(\gamma) \quad . \tag{1.11}$$

$$f_n^{[r]}(\gamma) = \sum_{k=r}^{n} a_{n-k} \gamma^k \quad , \quad f_n^{[0]}(\gamma) = f(\gamma) \quad . \tag{1.12}$$

$$F_n(a, z, \gamma) = \frac{z^\sigma}{\Gamma(\sigma)} \int_0^\infty e^{-zt} t^{\sigma-1} \sum_{k=0}^{n} a_{n-k} (\lambda \gamma t)^k \sum_{r=0}^{\infty} \frac{(\alpha_p)_{r+k} (\lambda t)^r}{(\rho_q)_{r+k} (r+k)!} \, dt \tag{1.13}$$

$$= \frac{z^\sigma}{\Gamma(\sigma)} \int_0^\infty e^{-zt} t^{\sigma-1} \sum_{k=0}^{n} a_{n-k} \left(\frac{\lambda \gamma}{z}\right)^k \sum_{r=0}^{\infty} \frac{(\alpha_p)_{r+k} (\sigma+r)_k (\lambda t)^r}{(\rho_q)_{r+k} (r+k)!} \, dt \tag{1.14}$$

$$= \sum_{r=0}^{\infty} \frac{(\sigma)_r (\alpha_p)_r}{(1)_r (\rho_q)_r} \left(\frac{\lambda}{z}\right)^r \sum_{k=0}^{n} \frac{a_{n-k} (\sigma+r)_k (\alpha_p+r)_k}{(1+r)_k (\rho_q+r)_k} \left(\frac{\lambda \gamma}{z}\right)^k \tag{1.15}$$

where, as before, the a_k's are unspecified and γ is subject only to the restriction $|\gamma/z| \leq 1$.

The difference between the two sequences is that when $z = \gamma$, in the Type I representation, $\varphi_n(0) \neq 0$, whereas for the Type II representation under the same condition, $\varphi_n(0) = 0$. For some particular case one of the representations may be more desirable than the other. Also, it sometimes happens that one sequence approaches the desired limit monotonically from above, and the other monotonically from below, thus yielding rational inequalities for the functions in question. The above formal sequences were derived, strictly speaking, with the restrictions $p \leq q$ and $Re(z) > 0$ if $p < q$, while $Re(z) > Re(\lambda)$ if $p = q$. However, these restrictions may be weakened whenever the resulting expressions are defined and make sense.

The merit of [1] lies in the fact that it motivates how the arbitrary coefficients a_k should be chosen and gives a closed form expression for the error. The a_k's were chosen in [1] to agree under certain circumstances, at least, with the rational approximations given by application of Lanczos' τ-method (see [4]) to the linear differential equation satisfied by $E(z)$ (see [3]). The choices made for the a_k were

<u>Type I - Homogeneous</u>

$$a_{n-k} = \frac{C_{nk} (\rho_q)_k}{(\alpha_p+1)_k \lambda^k} , \qquad \text{if } \sigma = 1 \tag{1.16}$$

$$a_{n-k} = \frac{C_{nk}(\rho_q)_k(1)_k}{(\alpha_p+1)_k(\sigma+1)_k\,\lambda^k}\,, \qquad \text{if } \sigma \neq 1 \tag{1.17}$$

where

$$C_{nk} = \frac{(-n)_k(n+\alpha+\beta+1)_k}{(\beta+1)_k\,k!}\,.$$

Type II - Non-Homogeneous

$$a_{n-k} = \frac{C_{nk}(\rho_{q-1})_k(1)_k}{(\alpha_p)_k(\sigma)_k\,\lambda^k} \tag{1.18}$$

and C_{nk} is defined as above. With this selection of the coefficients a_k, we show that the rational sequences defined by (1.4) - (1.9) and (1.10) - (1.15) converge for two general classes of functions defined as follows: Class I is composed of those hypergeometric functions such that $p \leq q$ in (1.1) and Class II is made up of the hypergeometric functions such that $p = q+1$ in (1.1). Class II includes the Whittaker functions, modified Bessel functions, the Weber Parabolic Cylinder functions, and various special cases thereof. For completeness, we define a Class III composed of all hypergeometric functions such that $p \geq q+1$ in (1.1).

Considering $E(z)$ as a formal infinite series, we define the partial sum of this series by

$$P_n(z) = \sum_{r=0}^{n-1} \frac{(\sigma)_r(\alpha_p)_r}{(\rho_q)_r}\,\frac{(\lambda/z)^r}{r!}\,. \tag{1.19}$$

Rearrangement of (1.5) and (1.11) gives

$$\varphi_n(z,\gamma) = \sum_{k=a}^{n} a_{n-k}\,\gamma^k\,P_k(z) \tag{1.20}$$

where $a = 0$ for the Type I or Homogeneous Case,

$a = 1$ for the Type II or Non-Homogeneous Case.

Thus our rational approximations may be considered as a summation technique. A method of summability is said to be regular if it sums a convergent series to its ordinary sum. We now show that our economization process is a regular summability process.

<u>Theorem</u>. If

(1) $\varphi_n(z,\gamma) = \sum_{k=a}^{n} a_{n-k}\gamma^k P_k(z) \quad , f_n(\gamma) = \sum_{k=0}^{n} a_{n-k}\gamma^k \quad ,$

(2) $\lim_{k\to\infty} P_k(z) = E(z)$, for fixed z ,

(3) $a_{n-k} > 0$, $\gamma > 0$, and

(4) $\lim_{n\to\infty} f_n(\gamma) = \infty$, $f_{n+1}(\gamma) > f_n(\gamma)$ for fixed γ ,

then $\lim_{n\to\infty} \left| \varphi_n(z,\gamma)/f_n(\gamma) - E(z) \right| = 0$ for fixed γ and z .

<u>Proof</u>.

Given $\epsilon > 0$, there exists a positive integer N (usually dependent on z) such that $n > N$ implies $|P_k(z) - E(z)| < \epsilon/3$. Then

$$\varphi_n(z,\gamma)/f_n(\gamma) = \sum_{k=a}^{N} \left\{ a_{n-k}\gamma^k \left[P_k(z) - E(z) - \epsilon/3 \right] \right\} + \epsilon/3 \sum_{k=0}^{N} a_{n-k}\gamma^k$$

$$+ \frac{\sum_{k=N}^{n} a_{n-k}\gamma^k \left[P_k(z) - E(z) \right] - a\, a_n \left[\epsilon/3 + E(z) \right]}{f_n(\gamma)} .$$

Thus, for $n > N$,

$$\left|\varphi_n(z,\gamma)/f_n(\gamma)-E(z)\right| \le \frac{\sum_{k=a}^{N}\left\{a_{n-k}\,\gamma^k\left|P_k(z)-E(z)-\epsilon/3\right|\right\} + a\,a_n\left(\epsilon/3+\left|E(z)\right|\right)}{f_n(\gamma)}$$

$$+ \epsilon/3\,\frac{f_n(\gamma)}{f_n(\gamma)} \quad .$$

Since N is fixed, n can be chosen large enough such that the right hand side of the above is less than ϵ . This proves the theorem. From the results of [2] , and (1.16) - (1.18), we see that convergence for Class I of the hypergeometric functions is proved, when $\lambda \ge 0$.

We now consider Class II, i.e., the situation where $p = q+1$. Treating the non-homogeneous or Type II case only, for $p = 1$ and $q = 0$, from (1.15) and (1.18) we get

$$F_n(z,\gamma) = \frac{z^\sigma}{\Gamma(\sigma)}\int_0^\infty e^{-zt}t^{\sigma-1}\sum_{k=0}^{n} C_{nk}(\gamma/z)^k\;{}_3F_2\left(\begin{matrix}\alpha_1+k,\sigma+k,1\\ k+1,\sigma\end{matrix}\middle|\lambda t\right)dt\;. \tag{1.21}$$

By use of the relationships, see [3]

$${}_{p+1}F_{q+1}\left(\begin{matrix}\alpha_p,\sigma_1\\ \rho_q,\sigma_1+\sigma_2\end{matrix}\middle|z\right) = \frac{\Gamma(\sigma_1+\sigma_2)}{\Gamma(\sigma_1)\Gamma(\sigma_2)}\int_0^1 u^{\sigma_1-1}(1-u)^{\sigma_2-1}\,{}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle|zu\right)du\;,$$

$$\operatorname{Re}\sigma_1 > 0\;,\quad \operatorname{Re}\sigma_2 > 0 \tag{1.22}$$

and

$${}_2F_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix}\middle|z\right) = (1-z)^{-\beta}\,{}_2F_1\left(\begin{matrix}\gamma-\alpha,\beta\\ \gamma\end{matrix}\middle|\frac{z}{z-1}\right)\quad , \tag{1.23}$$

we can write

$$F_n(z,\gamma) = \frac{z^{\sigma}}{\Gamma(\sigma-1)\Gamma(1-\alpha_1)\Gamma(\alpha_1)} \int_0^{\infty}\int_0^1\int_0^1 \frac{e^{-zt}t^{\sigma-1}u^{\alpha_1-1}(1-u)^{-\alpha_1}v^{\sigma-2}}{(1-t\lambda u)}$$

$$\times\, {}_3F_2\left(\begin{matrix}-n, n+\alpha+\beta+1, 1\\ \alpha_1, 1+\beta\end{matrix}\middle|\, u\left(\frac{\gamma}{z}\right)\left(\frac{1-t\lambda uv}{1-t\lambda u}\right)\right) dv\,du\,dt \quad ,$$

$$\text{Re}\,\lambda < 0,\ \text{Re}(z) > 0,\ \ 0 < \text{Re}\,\alpha_1 < 1,\ \text{Re}\,\sigma > 1 \quad . \tag{1.24}$$

Defining

$$M_1^{II} = \max_{0\le w\le 1}\left|{}_3F_2\left(\begin{matrix}-n, n+\alpha+\beta+1, 1\\ \beta+1, \alpha_1\end{matrix}\middle|\, w\right)\right| \quad , \tag{1.25}$$

and noticing that

$$\left.\begin{aligned}&|1-t\lambda u| \ge 1,\ 0\le t\le\infty,\ 0\le u\le 1\ ,\\ &0 \le \left|u\left(\frac{\gamma}{z}\right)\left(\frac{1-t\lambda uv}{1-t\lambda u}\right)\right| \le 1\ ,\quad |\gamma/z| \le 1\ ,\ 0\le v\le 1\end{aligned}\right\} \tag{1.26}$$

it is easily seen by direct computation that

$$F_n(z,\gamma) \le M_1^{II}\ . \tag{1.27}$$

By repeated use of (1.22) and the same method of proof as above, it can be shown that

$$\left.\begin{aligned}&F_n(z,\gamma) \le M_{q+1}^{II}\\ &M_{q+1}^{II} = \max_{0\le w\le 1}\left|{}_{q+3}F_{q+2}\left(\begin{matrix}-n, n+\alpha+\beta+1, \rho_q-1, 1\\ \beta+1,\ \alpha_{q+1}\end{matrix}\middle|\, w\right)\right.\end{aligned}\right\}\ . \tag{1.28}$$

From [2], if $(2\omega+\alpha+\beta+3/2)>0$, where

$$2\omega = \left\{1/2 + \sum_{i=1}^{q} \rho_i - \left(q + \beta + \sum_{i=1}^{q+1} \alpha_i\right)\right\} , \tag{1.29}$$

$$\left.\begin{aligned} M_{q+1}^{II} &= \left| \frac{\Gamma(\beta+1)\Gamma(\alpha_{q+1})\, N^{4\omega+\alpha+\beta+1}}{\Gamma(\rho_q-1)\Gamma(2\omega+\alpha+\beta+3/2)} \left\{1+O(1/N)\right\} \right| \\ N^2 &= n(n+\alpha+\beta+1) \end{aligned}\right\} . \tag{1.30}$$

Thus $F_n(z,\gamma)$ is bounded by a term of algebraic order in n . On the other hand, by (1.18), (1.12) and a result given in [2],

$$\left.\begin{aligned} F_n(\gamma) &= {}_{q+3}F_{q+3}\left(\begin{matrix}-n, n+\alpha+\beta+1, \rho_q-1, 1\\ \alpha_{q+1}, \sigma, 1+\beta\end{matrix}\;\middle|\; \gamma/\lambda\right) \\ &= \frac{\Gamma(\alpha_{q+1})\Gamma(\sigma)\Gamma(1+\beta)}{\Gamma(\rho_q-1)2\pi 3^{1/2}} \left(\frac{N^2\gamma}{\lambda}\right)^{\bar{\omega}} \exp\left\{3\left(N^2\frac{\gamma}{\lambda}\right)^{1/3} - \left(\frac{\gamma}{3\lambda}\right)\right\} \\ &\quad + \frac{\phi\left(\frac{-\gamma}{\lambda}\right)}{\left(N^2\frac{\gamma}{\lambda}\right)^{1/3}} + O\left(\frac{\gamma^{7/3}}{N^{4/3}}\right)\Bigg\} + \text{terms of algebraic order in } N \end{aligned}\right| \tag{1.31}$$

where

$$\left.\begin{aligned}
N^2 &= n(n+\alpha+\beta+1) \\
\bar{\omega} &= \frac{2\omega+1/2-\sigma}{3} \\
\bar{\lambda} &= -\lambda \quad , \ \mathrm{Re}\,(\bar{\lambda}) \geq 0 \\
\varphi(-\gamma/\bar{\lambda}) &= +\frac{1}{15}(\gamma/\bar{\lambda})^2 - (\alpha+\beta+2/3+2\bar{\omega})(\gamma/\bar{\lambda}) \\
&\quad + \frac{(D_1-D_2)}{3}(2D_1+D_2-1) + E_2 - E_1 - 2/9 \\
D_1 &= 1 + \sum_{t=1}^{q} (\rho_i - 1) \quad , \ D_2 = \sum_{i=1}^{q+3} \alpha_i \quad , \ \alpha_{q+2} = \sigma \ , \ \alpha_{q+3} = 1+\beta \\
E_1 &= D_1 - 1 + \sum_{n=2}^{q} \sum_{j=1}^{n-1} (\rho_n - 1)(\rho_j - 1) \\
E_2 &= \sum_{n=2}^{q+3} \sum_{j=1}^{n-1} (\alpha_n)(\alpha_j)
\end{aligned}\right\} \quad . \tag{1.32}$$

Since $f_n(\gamma)$ is of exponential order in n ,

$$\lim_{n\to\infty} R_n(z,\gamma) = \lim_{n\to\infty} \frac{F_n(z,\gamma)}{f_n(z,\gamma)} = 0 \quad . \tag{1.33}$$

Similarly in the homogeneous or Type I case, one can show

$$F_n(z,\gamma) \leq \frac{\alpha_{q+1}\lambda\sigma}{\rho_q z} M^{I}_{q+1} \tag{1.34}$$

where

$$
\left.
\begin{aligned}
M^{I}_{q+1} &= \max_{0 \le w \le 1} \left| {}_{q+3}F_{q+2}\left(\begin{matrix} -n, n+\alpha+\beta+1, 1+\rho_q, 1 \\ \beta+1, \alpha_{q+1} \end{matrix} \middle| w \right) \right| \\
&\qquad\qquad\qquad\qquad\qquad\qquad \sigma \neq 1 \\
&= \max_{0 \le w \le 1} \left| {}_{q+3}F_{q+2}\left(\begin{matrix} -n, n+\alpha+\beta+1, \rho_q, 2 \\ \beta+1, 1+\alpha_{q+1} \end{matrix} \middle| w \right) \right| \\
&\qquad\qquad\qquad\qquad\qquad\qquad \sigma = 1
\end{aligned}
\right\} . \tag{1.35}
$$

Again, by [2] , if $(2\Omega+\alpha+\beta+3/2)>0$, where

$$
\left.
\begin{aligned}
2\Omega &= \frac{1}{2} + q - \beta + \sum_{i=1}^{q} \rho_i - \sum_{i=1}^{q+1} \alpha_{q+1} \\
&\qquad\qquad\qquad\qquad\qquad \sigma \neq 1 \\
&= \frac{1}{2} - q - \beta + \sum_{i=1}^{q} \rho_i - \sum_{i=1}^{q+1} \alpha_{q+1} \\
&\qquad\qquad\qquad\qquad\qquad \sigma = 1
\end{aligned}
\right\} \tag{1.36}
$$

$$
\left.
\begin{aligned}
M^{I}_{q+1} &= \left| \frac{\Gamma(\beta+1)\Gamma(\alpha_{q+1})\, N^{4\Omega+\alpha+\beta+1}}{\Gamma(1+\rho_q)\Gamma(2\Omega+\alpha+\beta+3/2)} \left\{ 1 + O\left(\frac{1}{N}\right) \right\} \right| \\
&\qquad\qquad\qquad\qquad\qquad\qquad \sigma \neq 1 \\
&= \left| \frac{\Gamma(\beta+1)\Gamma(1+\alpha_{q+1})\, N^{4\Omega+\alpha+\beta+1}}{\Gamma(\rho_q)\Gamma(2\Omega+\alpha+\beta+3/2)} \left\{ 1 + \Theta\left(\frac{1}{N}\right) \right\} \right| \\
&\qquad\qquad\qquad\qquad\qquad\qquad \sigma = 1 \\
N^2 &= n(n+\alpha+\beta+1)
\end{aligned}
\right\} . \tag{1.37}
$$

Thus, by (1.34) and (1.37), $F_n(z, \gamma)$ is again bounded by a term of algebraic order in n. Combining this with (1.31), we again have

$$\lim_{n \to \infty} R_n(z, \gamma) = \lim_{n \to \infty} \frac{F_n(z, \gamma)}{f_n(\gamma)} = 0 \quad . \qquad (1.38)$$

It calls for remark that the bound for the integral in (1.24) was obtained by ignoring the oscillatory nature of the hypergeometric function in its integrand over the entire region of integration. Thus, the asymptotic bound for the remainder is conservative as the rational approximations converge much more rapidly than indicated by (1.28). Realistic estimates of $F_n(z, \gamma)$ and so of $R_n(z, \gamma)$ seem to be much more difficult. However, though (1.28) is quite rough, it is easy to use and not excessively misleading.

2. Numerics

To give a qualitative idea of the effectiveness of the bound M_{q+1}^{II} for $F_n(z, \gamma)$, we consider an example given in [1] (Table (8.2)). The example is connected with the modified Bessel function $K_\nu(z)$.

$$\left.\begin{aligned} K_\nu(z) &= e^{-z}(\pi/2z)^{1/2}E(z) \\ E(z) &= {}_2F_0(1/2 - \nu, 1/2 + \nu \mid -1/2z) \end{aligned}\right\} \quad . \qquad (2.1)$$

Using the non-homogeneous or Type II approach,

$$\left.\begin{aligned}
f_n(\gamma) &= {}_3F_3\left(\begin{matrix}-n, n+\alpha+\beta+1, 1\\ \beta+1, 1/2-\nu, 1/2+\nu\end{matrix}\,\middle|\,-2\gamma\right)\\
\varphi_n(z,\gamma) &= (1/4-\nu^2)\sum_{r=0}^{n}\frac{C_{n,r}(\gamma/z)^r}{(1/2+\nu+r)(1/2-\nu+r)}\\
&\quad\times\ {}_3F_3\left(\begin{matrix}-n+r, n+\alpha+\beta+1+r, 1\\ \beta+1+r, 3/2+\nu+r, 3/2-\nu+r\end{matrix}\,\middle|\,-2\gamma\right)\\
E(z,\gamma) &= \frac{\varphi_n(z,\gamma)}{f_n(z,\gamma)} + R_n(z,\gamma)\\
&= E_n(z,\gamma) + R_n(z,\gamma)
\end{aligned}\right\} . \tag{2.2}$$

If $\alpha = \beta = -1/2$, $\nu = 0$, $\gamma = z$, and denoting asymptotic values as computed from (1.30) and (1.31) by $\sim$, we have to four significant figures,

z	$f_3(z)$	$\sim f_3(z)$	$R_3(z,\gamma)$
0.5	298.3	295.5	5.598×10^{-4}
1.0	1 265.	1 276.	3.145×10^{-4}
2.0	6 515.	7 007.	5.84×10^{-5}
5.0	72 400.	121 000.	0. --
7.0	184 300.	507 800.	$-1. \times 10^{-6}$
10.0	506 600.	3 994 000.	$-8. \times 10^{-7}$

z	$F_3(z,\gamma)$	M_1^{II}	$\sim M_1^{II}$	$\sim M_1^{II}/\sim f_3(z)$
0.5	0.1670	-9.4	9.42	3.188×10^{-2}
1.0	0.3977	↓	↓	7.382×10^{-3}
2.0	0.3805	↓	↓	1.344×10^{-3}
5.0	0.0--	↓	↓	7.785×10^{-5}
7.0	-0.1843	↓	↓	1.855×10^{-5}
10.0	-0.4053	↓	↓	2.359×10^{-6}

(2.3)

z	$f_4(z)$	$\sim f_4(z)$	$R_4(z,\gamma)$
0.5	1 206.	1 198.	1.170×10^{-4}
1.0	7 169.	7 217.	-6.1×10^{-6}
2.0	55 840.	61 170.	-5.4×10^{-6}
5.0	1 208 000.	1 754 000.	$-2. \times 10^{-7}$
7.0	4 052 000.	8 491 000.	$-1. \times 10^{-7}$
10.0	15 140 000.	68 760 000.	$-1. \times 10^{-7}$

z	$F_4(z,\gamma)$	M_1^{II}	$\sim M_1^{II}$	$\sim M_1^{II} / \sim f_4(z)$
0.5	0.1411	12.58	12.57	1.049×10^{-2}
1.0	-0.04373	↓	↓	1.742×10^{-3}
2.0	-0.3015			2.055×10^{-4}
5.0	-0.2417			7.166×10^{-6}
7.0	-0.4052			1.480×10^{-6}
10.0	-1.514			1.828×10^{-7}

(2.4)

The above example shows that our error analysis is quite conservative, as expected. For the present, a pragmatic view should be taken concerning the error. That is, if more precise information is required, in those cases where convergence is assured, one should compute for successive values of n and accept the common digits as correct. To illustrate, in the above example for $K_0(z)$, if

$$E_3(0.5) = 0.85932$$

$$E_4(0.5) = 0.85976$$

and it is quite reasonable to say

$$E(0.5) = 0.860$$

with a possible error of one unit in the third decimal place.

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