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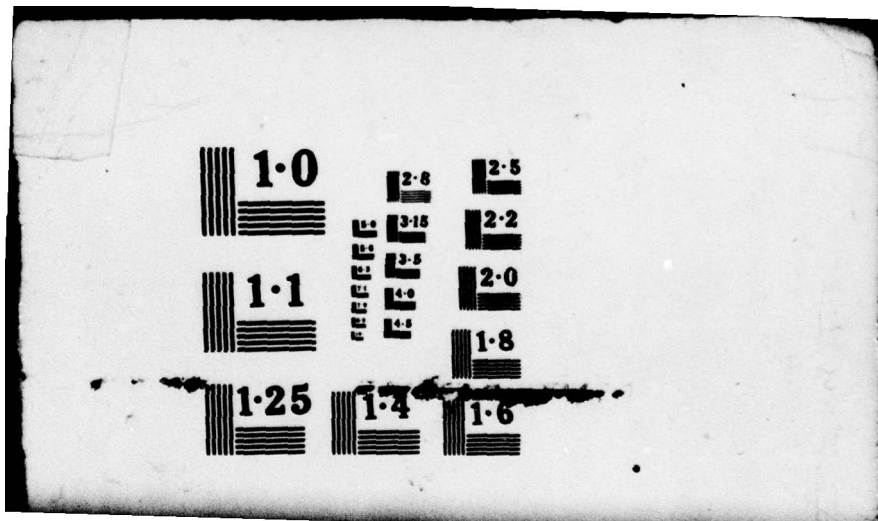
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INCOMPLETE GAMMA FUNCTION

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PREFACE

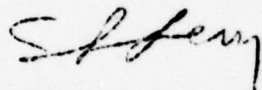
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Y.L.L. and W.F.

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Sheldon L. Levy, Director  
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## INTRODUCTION

→ In a previous report [1] we studied rational approximations to the incomplete gamma function. These were based on the asymptotic expansion of the latter function. In the present study, a similar analysis is done for the same function based on one of its ascending series representations. The work is related to <sup>other</sup> the developments in [2] and [3], but there are numerous improvements and some new features. ←

### I. THE INCOMPLETE GAMMA FUNCTION

We consider

$$\gamma(z, \nu) = \int_0^z t^{\nu-1} e^{-t} dt, \quad R(\nu) > 0, \quad (1.1)$$

$$\Gamma(z, \nu) = \Gamma(\nu) - \gamma(z, \nu) \quad (1.2)$$

where

$$\Gamma(z, \nu) = \int_z^{\infty e^{i\omega}} t^{\nu-1} e^{-t} dt, \quad z = x+iy, \quad x, y \text{ and } \omega \text{ are real,}$$

$$|\omega| < \pi/2, \quad R(\nu) > 0; \quad |\omega| = \pi/2, \quad 0 < R(\nu) < 1. \quad (1.3)$$

In (1.3) the path of integration lies in the  $z$  plane cut along the negative real axis and is the ray  $\eta \exp(i\omega)$ ,  $\eta \rightarrow \infty$  except for an initial finite path. If  $z \neq 0$ ,  $|\omega| < \pi/2$ , the integral in (1.3) exists without the restriction on  $\nu$ .

The connection of these functions to the confluent hypergeometric functions is given by

$$\gamma(z, \nu) = \nu^{-1} z^\nu e^{-z} \phi(1, 1+\nu; z) = \nu^{-1} z^\nu \psi(\nu, 1+\nu; -z), \quad (1.4)$$

$$\phi(a, c; z) = {}_1F_1(a; c; z) \quad , \quad (1.5)$$

$$\Gamma(z, v) = z^v e^{-z} \psi(1, 1+v; z) = e^{-z} \psi(1-v, 1-v; z) \quad , \quad (1.6)$$

$$\psi(a, c; z) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)} \phi(a, c; z) + \frac{\Gamma(c-1)}{\Gamma(a)} z^{1-c} \phi(a-c+1, 2-c; z) \quad . \quad (1.7)$$

The statement (1.4) is

$$\gamma(z, v) = z^v e^{-z} \sum_{k=0}^{\infty} \frac{z^k}{(v)_{k+1}} = z^v \sum_{k=0}^{\infty} \frac{(-)^k z^k}{k!(v+k)} \quad . \quad (1.8)$$

In the above formulas, standard generalized hypergeometric notation is used. Thus

$$\begin{aligned} {}_pF_q \left( \begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} \middle| z \right) &= {}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; z) \\ &= \sum_{k=0}^{\infty} \frac{\prod_{i=1}^p (a_i)_k z^k}{\prod_{i=1}^q (b_i)_k k!} \quad , \quad (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} \quad . \quad (1.9) \end{aligned}$$

For further discussion of the hypergeometric functions, confluent hypergeometric functions and the incomplete gamma function, see [4, Chs. 2, 4, 6], [5, Ch. 9], [6], [7], and [8].

II. RATIONAL APPROXIMATIONS TO  $\gamma(ze^{i\pi}, \nu)$

In [2] we proved that

$$z^{-\nu} e^{-z-i\pi\nu} \gamma(ze^{i\pi}, \nu) = V_n(z, \nu) + R_n(z, \nu) ,$$

$$V_n(z, \nu) = \frac{A_n(z, \nu)}{B_n(z, \nu)} , \quad R_n(z, \nu) = \frac{P_n(z, \nu)}{B_n(z, \nu)} , \quad (2.1)$$

where

$$A_n(z, \nu) = \sum_{k=0}^n \frac{(-n)_k (n+\nu+1)_k}{(\nu+1)_k k!} z^n {}_3F_1 \left( \begin{matrix} -n+k, n+\nu+1+k, 1 \\ 1+k \end{matrix} \middle| -1/z \right) , \quad (2.2)$$

and  $B_n(z, \nu)$  is the  $k = 0$  term in (2.2). In this event, the  ${}_3F_1$  becomes a  ${}_2F_0$ . Thus

$$B_n(z, \nu) = z^n {}_2F_0(-n, n+\nu+1; -1/z) = (n+\nu+1)_n {}_1F_1(-n, -2n-\nu; z) . \quad (2.3)$$

Also,

$$\begin{aligned} P_n(z, \nu) &= \frac{(-)^{n+1} e^{-z}}{z^\nu (\nu+1)_n} \int_0^z (z-t)^n t^{n+\nu} e^t dt \\ &= \frac{(-)^{n+1} n! e^{-z} z^{2n+1}}{(\nu+1)_{2n+1}} {}_1F_1(n+\nu+1; 2n+\nu+2; z) , \end{aligned} \quad (2.4)$$

$$R_n(z, \nu) = \frac{(-)^{n+1} n! \Gamma(\nu+1) \Gamma(n+\nu+1) z^{2n+1} {}_1F_1(n+1; 2n+\nu+2; -z)}{\Gamma(2n+\nu+1) \Gamma(2n+\nu+2) {}_1F_1(-n; -2n-\nu; z)} . \quad (2.5)$$

Both  $A_n(z, v)$  and  $B_n(z, v)$  satisfy the same recurrence equation

$$\frac{(n+v+1)}{(2n+v+1)(2n+v+2)} A_{n+1}(z, v) = \left[ 1 + \frac{vz}{(2n+v)(2n+v+2)} \right] A_n(z, v) + \frac{nz^2}{(2n+v)(2n+v+1)} A_{n-1}(z, v) \quad (2.6)$$

We list some other useful relations:

$$(2n+v)DB_n(z, v) - nB_n(z, v) + nB_{n-1}(z, v) = 0 \quad (2.7)$$

$$(2n+v)DA_n(z, v) + (n+v)A_n(z, v) + nA_{n-1}(z, v) + z^{-1}v(2n+v)[A_n(z, v) - B_n(z, v)] = 0 \quad (2.8)$$

and

$$\left[ zD^2 - (z+2n+v)D + n \right] B_n(z, v) = 0, \quad D = \frac{d}{dz} \quad (2.9)$$

We now develop a useful estimate of the remainder  $R_n(z, v)$ . Consider

$$y = e^{-z/2} {}_1F_1(a, c; z), \quad zy'' + cy' + (K-z/4)y = 0, \quad K = c/2 - a \quad (2.10)$$

Assume

$$y = \sum_{k=0}^{\infty} \frac{a_k(z)}{u^k}, \quad a_0(z) = 1, \quad u = \frac{1}{2}(c-1) \quad (2.11)$$

Then the  $a_k$ 's can be generated by the expression

$$a_{k+1}(z) = -\frac{1}{2}z a_k'(z) + \frac{1}{2} \int_0^z (t/4 - K) a_k(t) dt \quad (2.12)$$

Thus, for example,

$$a_1(z) = z^2/16 - Kz/2, \quad a_2(z) = z^4/512 - Kz^3/32 + z^2(2K^2-1)/16 + Kz/4, \quad \text{and}$$

$$a_3(z) = \frac{z^6}{24576} - \frac{Kz^5}{1024} + \frac{z^4(4K^2-3)}{512} - \frac{K(4K^2-13)z^3}{192} - \frac{z^2(3K^2-1)}{16} - \frac{Kz}{8}. \quad (2.13)$$

Clearly the series (2.11) is absolutely convergent for all  $z$  as it can be found by multiplication of the series  ${}_1F_1(a;c;z)$  by the series for  $e^{-z/2}$ . However, if  $z$  is fixed,  $c$  is large and  $K \ll c$ , then (2.11) is a useful representation of  $y$  for large  $c$ . The expressions (2.6)-(2.8) can also be deduced from some general results of Olver, see the discussion in [8, p. 76].

Apply (2.10)-(2.13) to the confluent functions in (2.5). Then, with the aid of the duplication formula for gamma functions, we get

$$R_n(z, \nu) = \frac{(-)^{n+1} n! \pi \Gamma(\nu+1) \Gamma(n+\nu+1) z^{2n+1} e^{-z} \sum_{k=0}^{\infty} \frac{a_k(z)}{u^k}}{2^{4n+2\nu} (2n+\nu+1) \left[ \Gamma\left(n + \frac{\nu+1}{2}\right) \Gamma\left(n + \frac{\nu}{2} + 1\right) \right]^2 \sum_{k=0}^{\infty} \frac{(-)^k a_k(z)}{u^k}}, \quad (2.14)$$

or

$$R_n(z, \nu) = \frac{(-)^{n+1} n! \pi \Gamma(\nu+1) \Gamma(n+\nu+1) z^{2n+1} e^{-z}}{2^{4n+2\nu} (2n+\nu+1) \left[ \Gamma\left(n + \frac{\nu+1}{2}\right) \Gamma\left(n + \frac{\nu}{2} + 1\right) \right]^2} e^{\frac{2a_1(z)}{u}} \times \left\{ 1 + o(1/n^3) \right\}, \quad (2.15)$$

where  $u = n + \frac{1}{2}(\nu+1)$ ,  $K = -\nu/2$  and the  $a_k$ 's are given by (2.12). It follows that for  $z$  and  $\nu$  fixed,

$$\lim_{n \rightarrow \infty} R_n(z, \nu) = 0. \quad (2.16)$$

Since  $\Gamma(n+a)/\Gamma(n+b) = n^{a-b} [1+O(1/n)]$ , it is convenient to write

$$R_n(z, v) = \frac{(-)^{n+1} \pi \Gamma(v+1) z^{2n+1} e^{-z}}{2^{4n+2v+1} n^v (n!)^2} \{1+O(1/n)\} . \quad (2.17)$$

Concerning the use of (2.1), it is important to know the nature of  $z_{n,i}^{(v)}$ ,  $i = 1, 2, \dots, n$ , the zeros of  $B_n(z, v)$ . It is clear from (2.9) that the zeros are simple. If  $v = 0$ ,  $B_n(z, v)$  is essentially the modified Bessel function of half-odd order (see (4.2)). In this instance the zeros can be deduced from the work of Olver [9]. See also Grosswald [12]. The zeros lie in the left half plane and are always complex, except when  $n$  is odd in which case there is a single real root. If  $z_n^{(v)}$  is the magnitude of the smallest root(s) of  $B_n(z, v)$ , then from the work of Olver [9]  $z_n^{(0)} \sim 1.32548n$ . Here  $2(t_0^2 - 1)^{\frac{1}{2}} = 1.32548$  where  $t_0$  is the real zero of  $t = \coth t$ . Thomson [10] and Kublanovskaia and Smirnova [11] have tabulated  $\frac{1}{2}z_{1,0}^{(0)}$ , the former for  $n = 1(1)9$  to  $4d$ , and the latter for  $n = 1(1)21$  to  $5d$ . Salzer [13] has tabulated  $z_{n,i}^{(-1)}$  for  $n = 1(1)16$  to  $15d$ . We have prepared some tables of  $z_{n,i}^{(v)}$  for  $v = -3/2, 1/2, 1$ . For all the  $v$  values mentioned above, the roots lie in the left half plane. If  $n$  is odd, there is a single real root, otherwise the roots are complex. On the basis of the data, we conjecture that  $z_n^{(v)} \sim 1.32548n + v + 1 - 1/\pi$ . An alternative conjecture is that  $z_{2n}^{(v)} \sim \frac{1}{2}\pi^{\frac{1}{2}}(3n+v+1)$ ,  $z_{2n+1}^{(v)} \sim 3/2 n\pi^{\frac{1}{2}} + v + 2$ . Thus as  $n$  increases, the magnitude of the smallest zero(s) of  $B_n(z, v)$  increases linearly with  $n$ . We see from (2.17) that to achieve high accuracy,  $n$  must be considerably larger than  $z$ , and so the value of  $z_n^{(v)}$  is not critical.

Another rational approximation for  $\gamma(ze^{i\pi}, v)$  is

$$vz^{-v} e^{-z-1v\pi} \gamma(ze^{i\pi}, v) = S_n(z, v) + T_n(z, v) ,$$

$$S_n(z, v) = \frac{C_n(z, v)}{D_n(z, v)} , \quad T_n(z, v) = \frac{Q_n(z, v)}{D_n(z, v)} , \quad (2.18)$$

where

$$C_n(z, v) = -\frac{v}{z} \sum_{k=0}^{n-1} \frac{(-n)_{k+1} (n+v)_{k+1}}{(v)_{k+1} (k+1)!} z^n {}_3F_1 \left( \begin{matrix} -n+1+k, n+v+1+k, 1 \\ k+2 \end{matrix} \middle| -\frac{1}{z} \right), \quad (2.19)$$

and  $D_n(z, v)$  is  $z^n$  times the  ${}_3F_1$  in (2.19) with  $k+1$  set equal to zero. Thus  $D_n(z, v)$  is  $B_n(z, v)$  if in the latter we replace  $v$  by  $v-1$ . Also

$$\begin{aligned} Q_n(z, v) &= \frac{(-)^n v z^{-v} e^{-z}}{(v)_n} \int_0^z (z-t)^n t^{n+v-1} e^t dt \\ &= \frac{(-)^n n! e^{-z} z^{2n}}{(v+1)_{2n}} {}_1F_1(n+v; 2n+v+1; z). \end{aligned} \quad (2.20)$$

Both  $C_n(z, v)$  and  $D_n(z, v)$  satisfy (2.6) if  $v$  is replaced by  $v-1$ . Also,  $D_n(z, v)$  satisfies (2.7) and (2.9) with  $v$  replaced by  $v-1$ . The relation analogous to (2.8) reads

$$\begin{aligned} (2n+v-1)DC_n(z, v) + (n+v-1)C_n(z, v) + n z C_{n-1}(z, v) \\ + z^{-1} v(2n+v-1) [C_n(z, v) - D_n(z, v)] = 0. \end{aligned} \quad (2.21)$$

After the manner of deriving (2.15) and (2.17) we have

$$\begin{aligned} T_n(z, v) &= \frac{(-)^n n! \Gamma(v+1) \Gamma(n+v) z^{2n} e^{-z} e^{2a_1^*(z)/u^*}}{2^{4n+2v-2} (2n+v) \left[ \Gamma\left(n + \frac{v}{2}\right) \Gamma\left(n + \frac{v+1}{2}\right) \right]^2} \left\{ 1 + O(1/n^3) \right\} \\ &= \frac{(-)^n n! \Gamma(v+1) z^{2n} e^{-z} e^{2a_1^*(z)/u^*}}{2^{4n+2v-1} n^{v-1} (n!)^2} \left\{ 1 + O(1/n) \right\}, \end{aligned} \quad (2.22)$$

where  $u^* = n+v/2$ ,  $a_1^*(z) = z(z+4v-4)/16$ , whence for  $z$  and  $v$  fixed,  $\lim_{n \rightarrow \infty} T_n(z, v) = 0$ .

If  $z$  and  $v$  are positive and fixed, then for a given  $n$ , it is clear that  $R_n(z, v)$  and  $T_n(z, v)$  are of opposite sign. This leads to the inequalities

$$\frac{C_n(z, v)}{D_n(z, v)} \begin{matrix} > < \\ < > \end{matrix} v z^{-v} e^{-z} \int_0^z t^{v-1} e^t dt \begin{matrix} > < \\ < > \end{matrix} \frac{A_n(z, v)}{B_n(z, v)}, \quad (2.23)$$

where  $>$  or  $<$  sign is chosen according as  $n$  is odd or even, respectively. For example,

$$\frac{(v+1)(v+2)-z}{(v+1)(v+2)+z} < v z^{-v} e^{-z} \int_0^z t^{v-1} e^t dt < \frac{v+1}{v+1+z}, \quad z > 0, \quad v > 0, \quad (2.24)$$

$$\frac{2-z}{2+z} < e^{-z} < \frac{1}{z+1}, \quad z > 0, \quad (2.25)$$

$$\frac{15-4z^2}{15+6z^2} < z^{-1} e^{-z^2} \int_0^z e^{t^2} dt < \frac{3}{2z^2+3}, \quad z > 0. \quad (2.26)$$

In the last three equations, equality prevails if  $z \rightarrow 0$ .

At this juncture, we present a summary of the material in the remaining sections of the report. If  $v = \frac{1}{2}$ , (2.1) yields approximations for the error function and related integrals which is the subject of Section III. Approximations for the exponential function and the circular functions which come from (2.1) when  $v = 0$  are discussed in Section IV. Tables of the polynomials in the approximations of the functions discussed in Sections III and IV, and related data, are found in Appendix A. If  $v = 0$ , (1.3) is the exponential integral and in [1], we developed rational approximations for this function based on its asymptotic representation. Though  $\Gamma(z, 0)$  is defined by (1.3), clearly  $\gamma(z, 0)$  is not. However, we can give an ascending series representation for a function closely related to  $\Gamma(z, 0)$ . We show in Section V how rational approximations to this related function may be derived, and tables of the polynomials in these approximations and related data are given in Appendix B. FORTRAN codes for computing rational approximations to the incomplete gamma function and its special cases are presented in Appendices C, D and E.

### III. ERROR FUNCTION AND RELATED INTEGRALS

We list below formulae useful for the approximation of the error functions and the Fresnel integrals. These are based on (2.1) for  $\nu = \frac{1}{2}$ . It is clear from (2.1) and (2.18) that  $V_n(z, \frac{1}{2})$  and  $R_n(z, \frac{1}{2})$  can be replaced by  $S_n(z, \frac{1}{2})$  and  $T_n(z, \frac{1}{2})$ , respectively.

$$\gamma(z, \frac{1}{2}) = \int_0^z t^{-\frac{1}{2}} e^{-t} dt = 2z^{\frac{1}{2}} e^{-z} \left\{ V_n(ze^{-i\pi}, \frac{1}{2}) + R_n(ze^{-i\pi}, \frac{1}{2}) \right\} . \quad (3.1)$$

$$\text{Erf}(z) = \frac{1}{2} \gamma(z^2, \frac{1}{2}) = \int_0^z e^{-t^2} dt = (\pi/4)^{\frac{1}{2}} - \text{Erfc}(z) , \quad (3.2)$$

$$\text{Erfc}(z) = \int_z^{\infty} e^{-t^2} dt , \quad (3.3)$$

$$\text{Erf}(z) = \int_0^z e^{-t^2} dt = ze^{-z^2} \left\{ V_n(z^2 e^{-i\pi}, \frac{1}{2}) + R_n(z^2 e^{-i\pi}, \frac{1}{2}) \right\} , \quad (3.4)$$

$$\text{Erfi}(z) = \int_0^z e^{t^2} dt = -i \text{Erf}(iz) , \quad (3.5)$$

$$\gamma(ze^{\frac{i3\pi}{2}}, \frac{1}{2}) = e^{\frac{i3\pi}{4}} \int_0^z t^{-\frac{1}{2}} e^{it} dt = 2z^{\frac{1}{2}} e^{iz + i3\pi/4} \left\{ V_n(ze^{\frac{i\pi}{2}}, \frac{1}{2}) + R_n(ze^{\frac{i\pi}{2}}, \frac{1}{2}) \right\} , \quad (3.6)$$

$$C(z) + iS(z) = (2\pi)^{-\frac{1}{2}} \int_0^z t^{-\frac{1}{2}} e^{it} dt . \quad (3.7)$$

The exact coefficients of polynomials simply related to  $A_n(z, \frac{1}{2})$ ,  $B_n(z, \frac{1}{2})$ ,  $C_n(z, \frac{1}{2})$  and  $D_n(z, \frac{1}{2})$  are given in Appendix A for  $n = 0(1)10$ . For  $n = 2(1)10$ , we also record values of  $R_n^*(z, \frac{1}{2}) = |2z^{\frac{1}{2}} e^z R_n(z, \frac{1}{2})|$ , see (2.15), for  $z = re^{i\theta}$ ,  $r = 1(1)10$ ,  $\theta = 0, \pi/2, \pi$ . Error tables for other values of  $\nu$  and for  $T_n(z, \nu)$  are easily derived from the latter table. See the discussion in Appendix A. See Appendix C for FORTRAN codes.

To illustrate the utility of (2.15), we present the numerics below. The tables give  $e^{-i\pi/2}\gamma(z e^{i\pi}, \frac{1}{2})$ , its rational approximation (2.1), the exact error, and the approximate error according to (2.15). The data are developed for  $z = 2e^{i\theta}$ ,  $n = 4$ .

$\theta$	$\int_0^z t^{-\frac{1}{2}} e^t dt$ <u>Exact</u>	$2z^{\frac{1}{2}} e^z V_n(z, \frac{1}{2})$
0	6.6876855	6.6877002
$\pi/2$	$3.3756999 \times 10^{-1} + 2.3328174i$	$3.3756168 \times 10^{-1} + 2.3328241i$
$\pi$	$1.0324115 \times 10^{-8} + 1.6918067i$	$1.0317960 \times 10^{-8} + 1.6917947i$

$\theta$	<u>Exact Error</u>	$2z^{\frac{1}{2}} e^z R_n(z, \frac{1}{2})$ <u>Approximate</u>
0	$-1.47 \times 10^{-5}$	$-1.47 \times 10^{-5}$
$\pi/2$	$8.31 \times 10^{-6} - 6.70i \times 10^{-6}$	$8.35 \times 10^{-6} - 6.74i \times 10^{-6}$
$\pi$	$6.16 \times 10^{-12} + 1.20i \times 10^{-5}$	$1.05 \times 10^{-11} + 1.19i \times 10^{-5}$

#### IV. THE EXPONENTIAL FUNCTION

If  $\nu \rightarrow 0$ , (2.1) becomes

$$e^{-z} = \frac{G_n(-z)}{G_n(z)} + R_n(z, 0) \quad , \quad (4.1)$$

$$G_n(z) = z^n {}_2F_0(-n, n+1; -\frac{1}{2}) = (2/\pi)^{\frac{1}{2}} z^n K_{n+\frac{1}{2}}(z/2) \quad , \quad (4.2)$$

$$\begin{aligned}
 R_n(z, 0) &= \frac{(-)^{n+1} {}_nI_{n+\frac{1}{2}}(z/2)}{e^z K_{n+\frac{1}{2}}(z/2)} = \frac{2(-)^{n+1} (z/4)^{2n+1} e^{-z+z^2/(8n+4)}}{n \left[ (\frac{1}{2})_n \right]^2} \left\{ 1 + O(1/n^3) \right\} \\
 &= \frac{(-)^{n+1} n z^{2n+1} e^{-z}}{2^{4n+1} (n!)^2} \left\{ 1 + O(1/n) \right\} \quad . \quad (4.3)
 \end{aligned}$$

Here  $I_n(z)$  and  $K_n(z)$  are the familiar notations for the modified Bessel functions.

It is of interest to show how (4.1) can be used to compute the exponential and circular functions in an efficient manner. Let

$$G_n(z) = M_n(z^2) + zN_n(z^2) \quad (4.4)$$

where  $M_n(z^2)$  and  $N_n(z^2)$  are even polynomials in  $z$ . Clearly

$$e_n^{-z} = \frac{G_n(-z)}{G_n(z)} = \frac{M_n(z^2) - zN_n(z^2)}{M_n(z^2) + zN_n(z^2)}, \quad (4.5)$$

and one readily verifies that  $G_n(z)$ ,  $M_n(z^2)$  and  $N_n(z^2)$  all satisfy the recurrence formula

$$G_{n+1}(z) = 2(2n+1)G_n(z) + z^2G_{n-1}(z) \quad (4.6)$$

Let the approximation to the circular functions be denoted by

$$e_n^{-iz} = \cos_n z - i \sin_n z \quad (4.7)$$

Then

$$\cos_n z = \frac{U_n(z^2)}{W_n(z^2)}, \quad \sin_n z = \frac{zV_n(z^2)}{W_n(z^2)},$$

$$U_n(z^2) = [M_n(-z^2)]^2 - z^2[N_n(-z^2)]^2, \quad V_n(z^2) = 2M_n(-z^2)N_n(-z^2),$$

$$W_n(z^2) = [M_n(-z^2)]^2 + z^2[N_n(-z^2)]^2 \quad (4.8)$$

It can be shown that\*  $U_n(z^2)$ ,  $V_n(z^2)$  and  $W_n(z^2)$  all satisfy the recurrence formula

$$(2n-1)U_{n+1}(z^2) = [-z^2+4(4n^2-1)] [(2n+1)U_n(z^2)-z^2(2n-1)U_{n-1}(z^2)] \\ + z^6(2n+1)U_{n-2}(z^2) \quad . \quad (4.9)$$

The exact coefficients of the polynomials  $G_n(z)$ ,  $U_n(z^2)$ ,  $V_n(z^2)$  and  $W_n(z^2)$  are given in Appendix A for  $n = 1(1)10$ . See also the introduction to Table A.III in Appendix A for the evaluation of  $R_n(z,0)$ . Appendix D gives FORTRAN codes for the computation of  $e_n^{-z}$ ,  $\cos_n z$ ,  $\sin_n z$  and  $\tan_n z$  for  $z$  real only. We conclude this section with an example to indicate the effectiveness of (4.1)-(4.3).

If  $n = 4$  and  $z = 1$ , (4.5) gives the value  $e_4^{-1} = 0.36787\ 94561$ . The error is  $-1.5 \times 10^{-8}$  whereas the last of (4.3) gives the value  $-1.6 \times 10^{-8}$ . For  $n = 4$  and  $z = i$ , (4.5) yields  $e_4^{-i} = 0.54030\ 23380 + 0.8414709642i$ . The true error is  $-3.2 \times 10^{-8} - i2.1 \times 10^{-8}$  as compared with the estimated error from the last of (4.3),  $-3.4 \times 10^{-8} - i2.2 \times 10^{-8}$ .

#### V. RATIONAL APPROXIMATIONS FOR $E(z) = z^{-1} \int_0^z t^{-1}(1-e^{-t})dt$

In this section we develop some rational approximations for  $E(z)$ . It is of interest to first list some functions related to  $E(z)$ . We have

$$E(z) = z^{-1} \int_0^z t^{-1}(1-e^{-t})dt = z^{-1} [-Ei(-z) + \gamma + \ln z] , \quad -Ei(-z) = \Gamma(z,0) , \quad (5.1)$$

where  $\gamma$  is Euler's constant.

\* We are indebted to Mr. Jet Wimp for proof of this statement and other helpful suggestions.

$$\begin{aligned} \frac{1}{2} [E(ze^{i\pi/2}) + E(ze^{-i\pi/2})] &= z^{-1} \text{Si}(z) = z^{-1} \int_0^z t^{-1} \sin t \, dt \\ &= z^{-1} [\pi/2 + \text{si}(z)], \quad \text{si}(z) = \int_{\infty}^z t^{-1} \sin t \, dt, \quad (5.2) \end{aligned}$$

$$\begin{aligned} 2i [E(ze^{i\pi/2}) - E(ze^{-i\pi/2})] &= 4z^{-2} \text{H}(z) = 4z^{-2} \int_0^z t^{-1} (1 - \cos t) \, dt \\ &= 4z^{-2} [\text{Ci}(z) - \gamma - \ln z], \quad \text{Ci}(z) = \int_{\infty}^z t^{-1} \cos t \, dt. \quad (5.3) \end{aligned}$$

Approximations for  $E(z)$  can be derived on the basis of the results given in Section II. Clearly from (1.1) and (2.1),

$$\begin{aligned} zE(z) &= \lim_{\nu \rightarrow 0} \int_0^z t^{\nu-1} (1 - e^{-t}) \, dt = \lim_{\nu \rightarrow 0} \left[ \frac{z^{\nu}}{\nu} - \gamma(z, \nu) \right] \\ &= \frac{\partial}{\partial \nu} \left\{ z^{\nu} e^{-z} z^{\nu} [V_n(ze^{-i\pi}, \nu) + R_n(ze^{-i\pi}, \nu)] \right\}_{\nu=0} \\ &= \frac{-e^{-z}}{[B_n(ze^{-i\pi}, 0)]^2} \left\{ B_n(ze^{-i\pi}, 0) \frac{\partial A_n(ze^{-i\pi}, 0)}{\partial \nu} - A_n(ze^{-i\pi}, 0) \frac{\partial B_n(ze^{-i\pi}, 0)}{\partial \nu} \right\} \\ &\quad - e^{-z} \frac{\partial R_n(ze^{-i\pi}, 0)}{\partial \nu}. \quad (5.4) \end{aligned}$$

This approach is not satisfactory since the numerator and denominator polynomials of the rational approximation are now each of degree  $2n$ .

It calls for remark that the rational approximations given in Section II are of the Padé type. Let

$$E(z) = \sum_{k=0}^{\infty} a_k z^k \quad (5.5)$$

be approximated by

$$E_{p,q}(z) = \frac{f_p(z)}{g_q(z)} \quad , \quad (5.6)$$

where  $f_p(z)$  and  $g_q(z)$  are polynomials in  $z$  of degree  $p$  and  $q$ , respectively. If

$$g_q(z)E(z) - f_p(z) = z^{p+q+1}h(z) \quad , \quad h(0) \neq 0 \quad , \quad (5.7)$$

then (5.6) is the Padé approximant of  $E(z)$ . Thus (2.1) is of the type (5.6) where  $p = q = n$ , and (2.18) is also of the same type with  $q = p+1 = n$ . The approximation (5.6) is called the main diagonal Padé approximation if  $p = q = n$ .

The numerator and denominator polynomials in the Padé approximants to transcendental functions are known in closed form, as for the functions in Section II, only in very few cases. However, the Padé approximant to a Taylor series expansion can always be found by solving systems of linear equations. Thus, if

$$f(z) = \sum_{k=0}^p f_k z^k \quad , \quad g(z) = \sum_{k=0}^q g_k z^k \quad , \quad (5.8)$$

then

$$\sum_{j=0}^r a_j g_{r-j} = 0 \quad , \quad r = p+1, p+2, \dots, p+q \quad ,$$

$$f_k = \sum_{j=0}^k a_j g_{k-j} \quad , \quad k = 0, 1, \dots, p \quad ,$$

$$h(z) = \sum_{k=0}^{\infty} h_k z^k \quad , \quad h_k = \sum_{j=0}^{p+q+1+k} a_j g_{p+q+1+k-j} \quad . \quad (5.9)$$

The coefficients  $f_k$ ,  $g_k$  and  $h_k$  must be determined anew for each choice of  $p$  and  $q$ .

As remarked previously, it is inconvenient to use (5.4) as a rational approximation for  $E(z)$ . To circumvent this difficulty, we have computed the coefficients of the main diagonal Padé approximants of the functions  $E(z)$  for  $n = 0(1)10$  and  $z^{-1}Si(z)$  and  $4z^{-2}H(z)$  for  $n = 0(2)10$ . These are tabulated in Appendix B.

We also include tables of the absolute values of the errors incurred by using the Padé approximation of  $E(z)$  for  $n = 2(1)10$ ,  $r = 1(1)10$ , and  $\theta = 0, \pi/2, \pi$ , where  $z = re^{i\theta}$ , and the Padé approximants of  $Si(z)$  and  $H(z)$  for  $z$  real,  $z = 1(1)10$  and for  $n = 4(2)10$ . These tables may be used as a guide in selecting the order of approximation necessary to obtain a desired accuracy. For example, if six decimal accuracy is desired for  $E(z)$  for  $z = 2\frac{1}{2}e^{i\pi/4}$ , i.e.,  $|\text{error}| < 0.5 \times 10^{-6}$ , interpolation of Table B.II indicates that a third order approximation should be sufficient. The third order approximation gives  $0.76507\ 22371 - 0.15899\ 83867i$  and the true value is  $0.76507\ 22539 - 0.15899\ 86256i$ . Thus  $|\text{error}| < 2.41 \times 10^{-7}$ .

Now

$$E(iz) = z^{-1}Si(z) - iz^{-1}H(z) \quad . \quad (5.10)$$

Select  $n$ . It is readily deduced from the error tables that the Padé values for  $E(iz)$  are better than the values deduced from (5.10) by using the Padé approximants for  $S(z)$  and  $H(z)$  for  $z$  real. Thus, if both  $Si(z)$  and  $H(z)$  are needed, it is better to use the Padé for  $E(iz)$ . However, if only  $H(z)$ , say, is needed, it is more economical to use the Padé for  $H(z)$ .

An examination of the zeros of the denominators of the Padé approximations of  $E(z)$ ,  $Si(z)$  and  $H(z)$  indicates that the magnitudes of the smallest zeros of the denominator increase linearly with  $n$ . Since  $n$  must be significantly larger than the argument to attain good accuracy, the location of these zeros is not important.

As noted above, the Padé approximants for  $E(z)$  are not known in closed form, and it is necessary to solve systems of linear equations for each selection of  $p$  and  $q$  (see (5.8)). We now give another rational approximation of  $E(z)$  which is of the form (5.8) with  $p = q = n$ . In over-all accuracy, it is somewhat inferior to the corresponding Padé approximant, see Tables B.II and B.VIII. However, it has the desirable advantage that the numerator and denominator polynomials can be computed by recurrence formulas. Following [14], it can be shown that

$$E(z) = {}_2F_2\left(\begin{matrix} 1, 1 \\ 2, 2 \end{matrix} \middle| -z\right) = \varphi_n(z)/f_n(z) + \epsilon_n(z), \quad \epsilon_n(z) = F_n(z)/f_n(z), \quad (5.11)$$

$$\varphi_n(z) = \sum_{r=0}^m \frac{(-)^r \binom{n}{r} (n+1)_r}{(2)_r} {}_4F_2\left(\begin{matrix} -n+r, n+1+r, 1, 2+r \\ 1+r, 1+r \end{matrix} \middle| -1/z\right), \quad (5.12)$$

and  $f_n(z)$  is the  ${}_4F_2$  in (5.12) with  $r = 0$  whence it becomes a  ${}_3F_1$ , see (5.13). The nature of  $F_n(z)$  will not be discussed here in detail. Suffice it to remark, it will be shown elsewhere that for  $z$  fixed  $\lim_{n \rightarrow \infty} F_n(z) = 0$ .

Now

$$\begin{aligned} f_n(z) &= {}_3F_1\left(\begin{matrix} -n, n+1, 2 \\ 1 \end{matrix} \middle| -1/z\right) \\ &= \frac{(n+1)(2n)!}{n!z^n} {}_2F_2\left(\begin{matrix} -n, -n \\ -2n, -n-1 \end{matrix} \middle| z\right), \end{aligned} \quad (5.13)$$

where it is to be understood that in the  ${}_2F_2$  only the first  $(n+1)$  terms of the series are retained. Thus

$$f_n(z) = \frac{(n+1)(2n)!}{n!z^n} \exp\left\{\frac{nz}{2(n+1)}\right\} \left\{1 - \frac{z^2(n^2+2n-2)}{8(n+1)^2(2n-1)} + o(z^3/n^3)\right\} \quad (5.14)$$

and so for  $z$  fixed,

$$\lim_{n \rightarrow \infty} \epsilon_n(z) = 0, \quad (5.15)$$

whence the rational approximants converge. It will also be demonstrated elsewhere that both  $\varphi_n(z)$  and  $f_n(z)$  satisfy the recurrence formula

$$f_n(z) = (B_1 z + A_1) f_{n-1}(z) + (B_2 z + A_2) f_{n-2}(z) + A_3 f_{n-3}(z) ,$$

$$A_1 = -A_3 = \frac{(n-2)(2n-1)}{n(2n-3)} , A_2 = 1 ,$$

$$B_1 = \frac{2(2n-1)(n+1)}{n} , \text{ and } B_2 = \frac{-2(2n-1)(n-3)}{n} . \quad (5.16)$$

To illustrate the power of (5.14), take  $z = \frac{1}{2}$  and  $n = 4$ . The true value (from (5.12)) is 1101 and the value using (5.14) is 1098.5.

From the preceding development and the last remark, it is obvious that all that is needed for a useful expression for the error  $|\epsilon_n(z)|$  is an approximation of  $F_n(z)$  (see (5.11)) for large  $n$ . This is not available at the present time. If  $z$  is real and positive, we can show that

$$|\epsilon_n(z)| \leq \left| \frac{5(2z)^{\frac{1}{2}}}{f_n(z)} \right| . \quad (5.17)$$

This bound, however, is very conservative.

The coefficients of the polynomials  $\varphi_n(z)$  and  $f_n(z)$  are given in Appendix B for  $n = 0(1)10$ . Also presented are values of  $|\epsilon_n(z)|$  for  $n = 3(1)10$  and for  $z = re^{i\theta}$  where  $r = 1(1)10$ ,  $\theta = 0, \pi/2, \pi$ .

The approximation (5.11) has the same properties as the Padé' approximation in that the magnitude of the smallest zeros of the denominator polynomials increase linearly with  $n$ , the order of approximation. Again since the order of approximation must be significantly larger than the argument, the location of zeros of the denominator polynomials is not critical.

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APPENDIX A

COEFFICIENTS FOR RATIONAL APPROXIMATIONS TO THE ERROR FUNCTION,  
EXPONENTIAL FUNCTION AND CIRCULAR FUNCTIONS

We first show how the exact coefficients in the approximations (2.1) and (2.18) can be generated when  $\nu$  is rational. Suppose  $\nu = p/q$ ,  $q \neq 0$  and  $p, q$  are co-prime integers.

Let

$$\left. \begin{aligned} A_n^*(z, \nu) &= q^{2n} \frac{\Gamma(\nu+n+1)}{\Gamma(\nu+1)} A_n(z, \nu) = \sum_{k=0}^n a_{n,k} z^k, \\ B_n^*(z, \nu) &= q^{2n} \frac{\Gamma(\nu+n+1)}{\Gamma(\nu+1)} B_n(z, \nu) = \sum_{k=0}^n b_{n,k} z^k, \end{aligned} \right\} \quad (\text{A.1})$$

where  $A_n(z, \nu)$  and  $B_n(z, \nu)$  are defined in (2.2) and (2.3), respectively. Note that

$$\frac{A_n^*(z, \nu)}{B_n^*(z, \nu)} = \frac{A_n(z, \nu)}{B_n(z, \nu)}, \quad (\text{A.2})$$

and the transformation (A.1) insures that the coefficients  $a_{n,k}$  and  $b_{n,k}$  are integers if  $p$  and  $q$  are co-prime integers. If  $\nu = 0$ , set  $p = 0$  and  $q = 1$ . From (2.6) it is seen that

$$\Lambda_{n+1} = \frac{(2nq+p+q)}{(2nq+p)} \left[ (2nq+p)(2nq+p+2q)+pqz \right] \Lambda_n + \frac{q^3 n z^2 (2nq+p+2q)(nq+p)}{(2nq+p)} \Lambda_{n-1},$$

$$\Lambda_n = A_n^*(z, \nu) \text{ or } B_n^*(z, \nu), \quad (\text{A.3})$$

and

$$\left. \begin{aligned} A_0^*(z, v) = 1, \quad A_1^*(z, v) &= (p+q)(p+2q) - q^2z, \\ B_0^*(z, v) = 1, \quad B_1^*(z, v) &= (p+q)(p+2q) + q(p+q)z. \end{aligned} \right\} \quad (\text{A.4})$$

Using (A.3) and (A.1), we get the recurrence formula

$$\left. \begin{aligned} \lambda_{n+1, k} &= (2nq+p+q)(2nq+p+2q)\lambda_{n, k} + \frac{pq(2nq+p+q)}{(2nq+p)} \lambda_{n, k-1} \\ &\quad + \frac{q^3n(2nq+p+2q)(nq+p)}{(2nq+p)} \lambda_{n-1, k-2}, \\ \lambda_{n, k} &= a_{n, k} \quad \text{or} \quad b_{n, k}; \quad k = 0, 1, 2, \dots, n \\ \text{and } \lambda_{n, k} &= 0 \quad \text{if } k < 0 \quad \text{or} \quad k > n. \end{aligned} \right\} \quad (\text{A.5})$$

The initial values are

$$\left. \begin{aligned} a_{0,0} &= 1, \\ a_{1,0} &= (p+q)(p+2q), \quad a_{1,1} = -q^2, \\ b_{0,0} &= 1, \\ b_{1,0} &= (p+q)(p+2q), \quad b_{1,1} = q(p+q). \end{aligned} \right\} \quad (\text{A.6})$$

Using a similar argument we find the following relations.

$$\left. \begin{aligned}
C_n^*(z, v) &= \frac{q^{2n}\Gamma(v+n+1)}{\Gamma(v+1)} C_n(z, v) = \sum_{k=0}^n c_{n,k} z^k, \\
D_n^*(z, v) &= \frac{q^{2n}\Gamma(v+n+1)}{\Gamma(v+1)} D_n(z, v) = \sum_{k=0}^n d_{n,k} z^k,
\end{aligned} \right\} \quad (A.7)$$

where  $C_n(z, v)$  and  $D_n(z, v)$  are defined in (2.19).

$$\left. \begin{aligned}
\Omega_{n+1} &= \frac{(2nq+p)}{(2nq+p-q)} [(2nq+p-q)(2nq+p+q) + q(p-q)z] \Omega_n \\
&\quad + \frac{q^3 n z^2 (2nq+p+q)(p+nq-q)}{(2nq+p-q)} \Omega_{n-1}, \\
\Omega_n &= C_n^*(z, v) \text{ or } D_n^*(z, v),
\end{aligned} \right\} \quad (A.8)$$

$$\left. \begin{aligned}
C_0^*(z, v) &= 0, \quad C_1^*(z, v) = (p+q), \\
D_0^*(z, v) &= 1/p, \quad D_1^*(z, v) = (p+q) + qz,
\end{aligned} \right\} \quad (A.9)$$

$$\left. \begin{aligned}
\gamma_{n+1,k} &= (2nq+p)(2nq+p+q)\gamma_{n,k} + \frac{q(p-q)(2nq+p)}{(2nq+p-q)} \gamma_{n,k-1} \\
&\quad + \frac{q^3 n (2nq+p+q)(p+nq-q)}{(2nq+p-q)} \gamma_{n-1,k-2}, \\
\gamma_{n,k} &= c_{n,k} \text{ or } d_{n,k}; \quad k = 0, 1, \dots, n \\
\text{and } \gamma_{n,k} &= 0 \text{ if } k < 0 \text{ or } k > n,
\end{aligned} \right\} \quad (A.10)$$

$$\left. \begin{aligned}
 c_{0,0} &= 0 \quad , \\
 c_{1,1} &= 0 \quad , \quad c_{1,0} = (p+q) \quad , \\
 d_{0,0} &= 1/p \quad , \\
 d_{1,0} &= (p+q) \quad , \quad d_{1,1} = q \quad .
 \end{aligned} \right\} \quad (A.11)$$

Tables A.I and A.II give the coefficients of the polynomials in the rational approximations to the error function, see (2.1), (2.18), (3.2) and (3.3). Table A.III gives the error associated with the approximation (2.1) for  $\nu = \frac{1}{2}$ . The introduction to Table A.III shows how the table may be used to get  $R_n(z, \nu)$  for other values of  $\nu$  and  $T_n(z, \nu)$ . Table A.IV gives coefficients of the polynomials in the rational approximation to the exponential function, see (4.1). Tables A.V, A.VI and A.VII list the coefficients of  $U_n(z^2)$ ,  $W_n(z^2)$  and  $V_n(z^2)$ , respectively, see (4.7) and (4.8). These coefficients are pertinent to the evaluation of the circular functions.

Most of the tables in the Appendices were typed by the IBM 1620 computer directly on stencils, while the balance of the report was done on an ordinary typewriter. The typewriters have different type sizes. The computer has no lower case characters, etc., and so a slight variance in notation <sup>N</sup> is introduced. For example, in Table A.I,  $AN*(Z, 1/2)$  corresponds to  $A_n(z, \frac{1}{2})$ , etc. Also in Table A.V,  $UN(Z**2)$  corresponds to  $U_n(z^2)$ , etc.

TABLE A.I

TABLES OF THE COEFFICIENTS OF THE POLYNOMIALS

$A_n^*(z, \frac{1}{2})$  AND  $B_n^*(z, \frac{1}{2})$

Note: For the definition of  $A_n^*(z, \frac{1}{2})$  and  $B_n^*(z, \frac{1}{2})$ , see (2.1) and (A.1). The sequence of numbers given is for the highest power to the lowest power, respectively.

e.g.,  $A_3^*(z, \frac{1}{2}) = -128z^3 + 1932z^2 - 9240z + 45045$

	<u><math>AN^*(Z, 1/2)</math></u>	<u><math>BN^*(Z, 1/2)</math></u>
N = 00	1	1
N = 01	- 1 15	6 15
N = 02	32 -210 945	60 420 945
N = 03	-128 1932 -9240 45045	280 3780 20790 45045

TABLE A.I (Continued)

	<u>AN*(Z, 1/2)</u>	<u>BN*(Z, 1/2)</u>
N = 04		
	2048	5040
	-43560	1 10880
	5 40540	10 81080
	-22 52250	54 05400
	114 86475	114 86475
N = 05		
	-8192	22176
	2 81424	7 20720
	-38 91888	108 10800
	453 33288	918 91800
	-1745 94420	4364 86050
	9166 20705	9166 20705
N = 06		
	65536	1 92192
	-28 85792	86 48640
	699 73904	1837 83600
	-7914 94704	23279 25600
	89625 13560	1 83324 14100
	-3 27946 51890	8 43291 04860
	17 56856 35125	17 56856 35125
N = 07		
	-2 62144	8 23680
	161 40480	490 08960
	-4353 52320	13967 55360
	91454 22000	2 44432 18800
	-9 17867 80800	28 10970 16200
	102 39962 73300	210 82276 21500
	-361 41044 94000	948 70242 96750
	1965 16931 86125	1965 16931 86125

TABLE A.I (Concluded)

AN\*(Z, 1/2)

BN\*(Z, 1/2)

N = 08

	83	88608		280	05120		
	-6277	25952		21283	89120		
2	51360	15040	7	82183	00160		
-53	38931	08320	179	90209	03680		
1038	33795	78000	2810	97016	20000		
-9618	68096	04600	30358	47774	96000		
1	06381	16578	08900	2	20098	96368	46000
-3	65521	49326	19250	9	74723	98203	18000
20	10368	21294	05875	20	10368	21294	05875

N = 09

	-335	54432		1182	43840		
	32467	92960	1	11740	42880		
-14	34070	38720	51	40059	72480		
473	05494	72000	1499	18408	64000		
-8681	49968	40000	30358	47774	96000		
1	61017	72510	82400	4	40197	92736	92000
-14	07934	64071	26000	45	48711	91614	84000
154	87281	04783	86000	321	65891	40704	94000
-521	20657	37253	37500	1407	25774	90584	11250
2892	69648	41756	23125	2892	69648	41756	23125

N = 10

	2684	35456		9932	48256				
-3	05076	71040	11	42235	49400				
182	46049	11200	642	50746	50000				
-6185	20520	21760	23130	26876	16000				
1	82600	62172	35200	58693	05698	25600			
-30	32474	61076	56000	109	16908	59875	61600		
544	83460	50675	52800	1501	07493	23289	72000		
-4560	40860	39327	81600	15010	74932	32897	20000		
49985	79524	65547	67600	1	04137	07343	03224	32500	
-1	65462	23889	48456	42750	4	51260	65153	13972	07500
9	25084	33563	93642	75375	9	25084	33563	93642	75375

TABLE A.II

TABLES OF THE COEFFICIENTS OF THE POLYNOMIALS

$C_n^*(z, \frac{1}{2})$  AND  $D_n^*(z, \frac{1}{2})$

Note: For the definition of  $C_n^*(z, \frac{1}{2})$  and  $D_n^*(z, \frac{1}{2})$ , see (2.18) and (A.7). The sequence of numbers given is for the highest power to the lowest power, respectively.

e.g.,  $C_3^*(z, \frac{1}{2}) = 84z^2 - 420z + 3465$

	<u>CN*(Z, 1/2)</u>	<u>DN*(Z, 1/2)</u>
N = 00	0	1
N = 01	0 3	2 3
N = 02	0 -10 105	12 60 105
N = 03	0 84 -420 3465	40 420 1890 3465

TABLE A.II (Continued)

	<u>CN*(Z, 1/2)</u>	<u>DN*(Z, 1/2)</u>
N = 04	0 -744 23100 -90090 6 75675	560 10080 83160 3 60360 6 75675
N = 05	0 5104 -82368 17 29728 -61 26120 436 48605	2016 55440 7 20720 54 05400 229 72950 436 48605
N = 06	0 -25376 15 53552 -171 53136 3026 30328 -10184 67450 70274 25405	14784 5 76576 108 10800 1225 22400 8729 72100 36664 82820 70274 25405
N = 07	0 1 58528 -53 60576 2061 87696 -19288 52640 3 07868 16060 -10 03917 91500 67 76445 92625	54912 28 82880 735 13440 11639 62800 1 22216 09400 8 43291 04860 35 13712 70250 67 76445 92625

TABLE A.II (Concluded)

	<u>CN*(Z, 1/2)</u>	<u>DN*(Z, 1/2)</u>
N = 08		
	0	16 47360
	-33 70624	1120 20480
	3395 37600	37246 80960
	-73102 91040	7 82183 00160
	22 56425 02800	112 43880 64800
	-192 17857 23000	1124 38806 48000
	2873 21307 27300	7589 61943 74000
	-9170 79015 35250	31442 70909 78000
	60920 24887 69875	60920 24887 69875
N = 09		
	0	62 23360
	198 88896	5320 97280
	-11584 58880	2 23480 85760
	7 01345 14560	59 96736 34560
	-122 38237 44000	1124 38806 48000
	3311 97300 72000	15179 23887 48000
	-26551 62101 59200	1 46732 64245 64000
	3 79059 32634 57000	9 74723 98203 18000
	-11 91329 31137 22000	40 20736 42588 11750
	78 18098 60588 00625	78 18098 60588 00625
N = 10		
	0	472 97536
	-1105 68960	49662 41280
	1 65206 97600	25 70029 86240
	-58 55525 91360	856 67662 80000
	2759 72254 27200	20238 98516 64000
	-42544 70268 04800	3 52158 34189 53600
	10 54145 93612 32800	45 48711 91614 84000
	-81 01039 31733 09600	428 87855 20939 92000
	1117 46689 40671 23600	2814 51549 81168 22500
	-3471 23578 10107 47750	11570 78593 67024 92500
	22563 03257 65698 60375	22563 03257 65698 60375

TABLE A.III

TABLES OF THE ERROR FOR THE ERROR FUNCTION

Here we give the values of  $R_n^*(z, \frac{1}{2}) = |2z^{\frac{1}{2}}e^z R_n(z, \frac{1}{2})|$ , see (2.15), for  $n = 2(1)10$ ,  $r = 1(1)10$  and  $\theta = 0, \pi/2, \pi$  where  $z = re^{i\theta}$ .

It is pertinent to point out that  $R_n^*(z, \nu)$ , see (2.15), and  $T_n^*(z, \nu) = |v^{-1}z^{\nu}e^z T_n(z, \nu)|$ , see (2.22), can both be obtained from  $R_n^*(z, \frac{1}{2})$  since

$$R_n(z, \nu) = 2^{2-\nu} \Gamma(\nu+1) n^{\frac{1}{2}-\nu} \pi^{-\frac{1}{2}} R_n(z, \frac{1}{2}) \left\{ 1 + O(1/n) \right\},$$

and

$$T_n(z, \nu) = -2^{4-2\nu} \Gamma(\nu+1) n^{3/2-\nu} z^{-1} \pi^{-\frac{1}{2}} R_n(z, \frac{1}{2}) \left\{ 1 + O(1/n) \right\}.$$

Hence, the tables given here essentially include the values of  $R_n^*(z, \nu)$  and  $T_n^*(z, \nu)$  for admissible  $\nu$  fixed, but otherwise arbitrary and the values of  $n$ ,  $r$  and  $\theta$  mentioned above.

$\theta$ $r \setminus$	0	$\pi/2$	$\pi$
<u><math>n = 2</math></u>			
1	0.896 (-3)	0.747 (-3)	0.747 (-3)
2	0.509 (-1)	0.295 (-1)	0.354 (-1)
3	0.651 ( )	0.219	0.377
4	0.477 ( 1)	0.774	0.230 ( 1)
5	0.268 ( 2)	0.175 ( 1)	0.108 ( 2)
6	0.132 ( 3)	0.290 ( 1)	0.443 ( 2)
7	0.609 ( 3)	0.375 ( 1)	0.171 ( 3)
8	0.275 ( 4)	0.395 ( 1)	0.642 ( 3)
9	0.125 ( 5)	0.349 ( 1)	0.243 ( 4)
10	0.578 ( 5)	0.262 ( 1)	0.938 ( 4)

TABLE A.III (Continued)

$\frac{\theta}{r}$	$0$	$\frac{\pi}{2}$	$\pi$
<u><math>n = 3</math></u>			
1	0.521 (- 5)	0.456 (- 5)	0.456 (- 5)
2	0.111 (- 2)	0.746 (- 3)	0.853 (- 3)
3	0.294 (- 1)	0.132 (- 1)	0.197 (- 1)
4	0.344	0.905 (- 1)	0.202
5	0.264 ( 1)	0.358	0.136 ( 1)
6	0.160 ( 2)	0.973	0.719 ( 1)
7	0.838 ( 2)	0.200 ( 1)	0.330 ( 2)
8	0.402 ( 3)	0.331 ( 1)	0.138 ( 3)
9	0.183 ( 4)	0.454 ( 1)	0.552 ( 3)
10	0.813 ( 4)	0.531 ( 1)	0.214 ( 4)
<u><math>n = 4</math></u>			
1	0.178 (- 7)	0.160 (- 7)	0.160 (- 7)
2	0.147 (- 4)	0.107 (- 4)	0.119 (- 4)
3	0.833 (- 3)	0.443 (- 3)	0.608 (- 3)
4	0.162 (- 1)	0.567 (- 2)	0.107 (- 1)
5	0.181	0.373 (- 1)	0.108
6	0.144 ( 1)	0.158	0.765
7	0.923 ( 1)	0.484	0.442 ( 1)
8	0.513 ( 2)	0.116 ( 1)	0.221 ( 2)
9	0.259 ( 3)	0.227 ( 1)	0.100 ( 3)
10	0.123 ( 4)	0.375 ( 1)	0.428 ( 3)
<u><math>n = 5</math></u>			
1	0.401 (-10)	0.368 (-10)	0.368 (-10)
2	0.130 (- 6)	0.998 (- 7)	0.109 (- 6)
3	0.160 (- 4)	0.949 (- 5)	0.123 (- 4)
4	0.532 (- 3)	0.223 (- 3)	0.375 (- 3)
5	0.879 (- 2)	0.238 (- 2)	0.569 (- 2)
6	0.949 (- 1)	0.153 (- 1)	0.563 (- 1)
7	0.774	0.678 (- 1)	0.421
8	0.520 ( 1)	0.227	0.259 ( 1)
9	0.305 ( 2)	0.609	0.139 ( 2)
10	0.161 ( 3)	0.135 ( 1)	0.677 ( 2)

TABLE A.III (Continued)

$\frac{\theta}{r}$	$0$	$\frac{\pi}{2}$	$\pi$
<u>n = 6</u>			
1	0.639 (-13)	0.593 (-13)	0.593 (-13)
2	0.812 (- 9)	0.650 (- 9)	0.700 (- 9)
3	0.220 (- 6)	0.141 (- 6)	0.176 (- 6)
4	0.127 (- 4)	0.603 (- 5)	0.941 (- 5)
5	0.316 (- 3)	0.104 (- 3)	0.218 (- 3)
6	0.470 (- 2)	0.993 (- 3)	0.302 (- 2)
7	0.498 (- 1)	0.625 (- 2)	0.296 (- 1)
8	0.414	0.287 (- 1)	0.229
9	0.288 ( 1)	0.103	0.148 ( 1)
10	0.176 ( 2)	0.300	0.841 ( 1)
<u>n = 7</u>			
1	0.757 (-16)	0.710 (-16)	0.710 (-16)
2	0.380 (-11)	0.313 (-11)	0.334 (-11)
3	0.228 (- 8)	0.155 (- 8)	0.188 (- 8)
4	0.228 (- 6)	0.120 (- 6)	0.176 (- 6)
5	0.866 (- 5)	0.329 (- 5)	0.627 (- 5)
6	0.180 (- 3)	0.465 (- 4)	0.122 (- 3)
7	0.250 (- 2)	0.411 (- 3)	0.159 (- 2)
8	0.261 (- 1)	0.256 (- 2)	0.156 (- 1)
9	0.220	0.121 (- 1)	0.123
10	0.158 ( 1)	0.455 (- 1)	0.829
<u>n = 8</u>			
1	0.693 (-19)	0.655 (-19)	0.655 (-19)
2	0.138 (-13)	0.116 (-13)	0.123 (-13)
3	0.184 (-10)	0.131 (-10)	0.155 (-10)
4	0.322 (- 8)	0.182 (- 8)	0.256 (- 8)
5	0.187 (-6)	0.793 (- 7)	0.140 (- 6)
6	0.547 (- 5)	0.165 (- 5)	0.388 (- 5)
7	0.101 (- 3)	0.203 (- 4)	0.674 (- 4)
8	0.133 (- 2)	0.170 (- 3)	0.840 (- 3)
9	0.138 (- 1)	0.104 (- 2)	0.817 (- 2)
10	0.117	0.503 (- 2)	0.659 (- 1)

TABLE A.III (Concluded)

$\frac{\theta}{r}$	<u>0</u>	<u><math>\pi/2</math></u>	<u><math>\pi</math></u>
<u>n = 9</u>			
1	0.505 (-22)	0.480 (-22)	0.480 (-22)
2	0.400 (-16)	0.343 (-16)	0.361 (-16)
3	0.119 (-12)	0.872 (-13)	0.102 (-12)
4	0.364 (-10)	0.218 (-10)	0.296 (-10)
5	0.325 (- 8)	0.151 (- 8)	0.251 (- 8)
6	0.134 (- 6)	0.458 (- 7)	0.987 (- 7)
7	0.329 (- 5)	0.783 (- 6)	0.230 (- 5)
8	0.553 (- 4)	0.873 (- 5)	0.367 (- 4)
9	0.701 (- 3)	0.698 (- 4)	0.442 (- 3)
10	0.716 (- 2)	0.427 (- 3)	0.429 (- 2)
<u>n = 10</u>			
1	0.300 (-25)	0.286 (-25)	0.286 (-25)
2	0.943 (-19)	0.820 (-19)	0.860 (-19)
3	0.625 (-15)	0.473 (-15)	0.544 (-15)
4	0.337 (-12)	0.217 (-12)	0.280 (-12)
5	0.464 (-10)	0.231 (-10)	0.368 (-10)
6	0.272 (- 8)	0.102 (- 8)	0.206 (- 8)
7	0.891 (- 7)	0.242 (- 7)	0.643 (- 7)
8	0.192 (- 5)	0.359 (- 6)	0.132 (- 5)
9	0.301 (- 4)	0.371 (- 5)	0.198 (- 4)
10	0.370 (- 3)	0.287 (- 4)	0.232 (- 3)

TABLE A.IV

TABLE OF THE COEFFICIENTS OF THE POLYNOMIAL  $G_n(z)$

Note: For the definition of  $G_n(z)$ , see (4.1). The sequence of numbers given is for the highest power to the lowest power, respectively.

e.g.,  $G_4(z) = z^4 + 20z^3 + 180z^2 + 840z + 1680$

$G_N(Z)$

N = 00

1

N = 01

1

2

N = 02

1  
12

6

N = 03

1  
60

12  
120

N = 04

1  
180  
1680

20  
840

TABLE A.IV (Continued)

GN(Z)

N = 05

1	30
420	3360
15120	30240

N = 06

1	42
840	10080
75600	3 32640
6 65280	

N = 07

1	56
1512	25200
2 77200	19 95840
86 48640	172 97280

N = 08

1	72
2520	55440
8 31600	86 48640
605 40480	2594 59200
5189 18400	

N = 09

1	90
3960	1 10880
21 62160	302 70240
3027 02400	20756 73600
88216 12800	1 76432 25600

TABLE A.IV (Concluded)

GN(Z)

N = 10

	1		110
	5940		2 05920
50	45040		908 10720
12108	09600		1 17621 50400
7 93945	15200		33 52212 86400
67 04425	72800		

TABLE A.V

TABLE OF THE COEFFICIENTS OF THE POLYNOMIAL  $U_n(z^2)$

Note: For the definition of  $U_n(z^2)$ , see (4.8). The sequence of numbers given is for the lowest power to the highest power, respectively.

e.g.,  $U_3(z) = -z^6 + 264z^4 - 6480z^2 + 14400$

UN(Z\*\*2)

N = 00

.10000 00000 00000 00000 +01

N = 01

.40000 00000 00000 00000 +01      -.10000 00000 00000 00000 +01

N = 02

.14400 00000 00000 00000 +03      -.60000 00000 00000 00000 +02  
 .10000 00000 00000 00000 +01

N = 03

.14400 00000 00000 00000 +05      -.64800 00000 00000 00000 +04  
 .26400 00000 00000 00000 +03      -.10000 00000 00000 00000 +01

N = 04

.28224 00000 00000 00000 +07      -.13104 00000 00000 00000 +07  
 .69360 00000 00000 00000 +05      -.76000 00000 00000 00000 +03  
 .10000 00000 00000 00000 +01

TABLE A.V (Continued)

UN(2\*\*2)

N = 05

.91445	76000	00000	00000	+09	-.43182	72000	00000	00000	+09
.25804	80000	00000	00000	+08	-.40824	00000	00000	00000	+06
.17400	00000	00000	00000	+04	-.10000	00000	00000	00000	+01

N = 06

.44259	74784	00000	00000	+12	-.21123	97056	00000	00000	+12
.13539	05280	00000	00000	+11	-.25788	67200	00000	00000	+09
.17035	20000	00000	00000	+07	-.34440	00000	00000	00000	+04
.10000	00000	00000	00000	+01					

N = 07

.29919	58953	98400	00000	+15	-.14384	41804	80000	00000	+15
.96499	66233	60000	00000	+13	-.20552	09587	20000	00000	+12
.17141	24160	00000	00000	+10	-.56629	44000	00000	00000	+07
.61600	00000	00000	00000	+04	-.10000	00000	00000	00000	+01

N = 08

.26927	63058	58560	00000	+18	-.13015	02144	98304	00000	+18
.90161	53232	48640	00000	+16	-.20687	40850	17600	00000	+15
.19940	43744	00000	00000	+13	-.86313	42720	00000	00000	+10
.15996	96000	00000	00000	+08	-.10224	00000	00000	00000	+05
.10000	00000	00000	00000	+01					

N = 09

.31128	34095	72495	36000	+21	-.15106	40075	86652	16000	+21
.10717	19697	31706	88000	+20	-.25935	10574	05440	00000	+18
.27586	13803	54560	00000	+16	-.14176	33176	96000	00000	+14
.35472	72960	00000	00000	+11	-.39964	32000	00000	00000	+08
.16020	00000	00000	00000	+05	-.10000	00000	00000	00000	+01

TABLE A.V (Concluded)

UN(7\*\*2)

N = 10

.44949	32434	22683	29984	+24	-.21883	22369	29464	23808	+24
.15812	89202	64694	57920	+23	-.39825	96563	64810	24000	+21
.45494	37982	88179	20000	+19	-.26326	17302	51520	00000	+17
.79982	79569	28000	00000	+14	-.12473	80992	00000	00000	+12
.90676	08000	00000	00000	+08	-.23980	00000	00000	00000	+05
.10000	00000	00000	00000	+01					

TABLE A.VI

TABLES OF THE COEFFICIENTS OF THE POLYNOMIAL  $W_n(z^2)$

Note: For the definition of  $W_n(z^2)$ , see (4.8). The sequence of numbers given is for the lowest power to the highest power, respectively.

e.g.,  $W_3(z^2) = z^6 + 24z^4 + 720z^2 + 14400$

WN(Z\*\*2)

N = 00

. 10000 00000 00000 00000 +01

N = 01

. 40000 00000 00000 00000 +01

. 10000 00000 00000 00000 +01

N = 02

. 14400 00000 00000 00000 +03  
 . 10000 00000 00000 00000 +01

. 12000 00000 00000 00000 +02

N = 03

. 14400 00000 00000 00000 +05  
 . 24000 00000 00000 00000 +02

. 72000 00000 00000 00000 +03  
 . 10000 00000 00000 00000 +01

N = 04

. 28224 00000 00000 00000 +07  
 . 21600 00000 00000 00000 +04  
 . 10000 00000 00000 00000 +01

. 10080 00000 00000 00000 +06  
 . 40000 00000 00000 00000 +02

TABLE A.VI (Continued)

WN(Z\*\*2)

N = 05

.91445	76000	00000	00000	+09	.25401	60000	00000	00000	+08
.40320	00000	00000	00000	+06	.50400	00000	00000	00000	+04
.60000	00000	00000	00000	+02	.10000	00000	00000	00000	+01

N = 06

.44259	74784	00000	00000	+12	.10059	03360	00000	00000	+11
.12700	80000	00000	00000	+09	.12096	00000	00000	00000	+07
.10080	00000	00000	00000	+05	.84000	00000	00000	00000	+02
.10000	00000	00000	00000	+01					

N = 07

.29919	58953	98400	00000	+15	.57537	67219	20000	00000	+13
.60354	20160	00000	00000	+11	.46569	60000	00000	00000	+09
.30240	00000	00000	00000	+07	.18144	00000	00000	00000	+05
.11200	00000	00000	00000	+03	.10000	00000	00000	00000	+01

N = 08

.26927	63058	58560	00000	+18	.44879	38430	97600	00000	+16
.40276	37053	44000	00000	+14	.26153	48736	00000	00000	+12
.13970	88000	00000	00000	+10	.66528	00000	00000	00000	+07
.30240	00000	00000	00000	+05	.14400	00000	00000	00000	+03
.10000	00000	00000	00000	+01					

N = 09

.31128	34095	72495	36000	+21	.45776	97199	59552	00000	+19
.35903	50744	78080	00000	+17	.20138	18526	72000	00000	+15
.91537	20576	00000	00000	+12	.36324	28800	00000	00000	+10
.13305	60000	00000	00000	+08	.47520	00000	00000	00000	+05
.18000	00000	00000	00000	+03	.10000	00000	00000	00000	+01

TABLE A.VI (Concluded)

WN(Z\*\*2)

N = 10

.44949	32434	22683	29984	+24	.59143	84781	87741	18400	+22
.41199	27479	63596	80000	+20	.20345	32088	70912	00000	+18
.80552	74106	88000	00000	+15	.27461	16172	80000	00000	+13
.84756	67200	00000	00000	+10	.24710	40000	00000	00000	+08
.71280	00000	00000	00000	+05	.22000	00000	00000	00000	+03
.10000	00000	00000	00000	+01					

TABLE A.VII

TABLES OF THE COEFFICIENTS OF THE POLYNOMIAL  $V_n(z^2)$ 

Note: For the definition of  $V_n(z^2)$ , see (4.8). The sequence of numbers given is for the lowest power to the highest power, respectively.

$$\text{e.g., } V_3(z^2) = 24z^4 - 1680z^2 + 14400$$

 $V_N(Z^{**2})$ 

N = 00

.00000 00000 00000 00000 00

N = 01

.40000 00000 00000 00000 01

N = 02

.14400 00000 00000 00000 03      -.12000 00000 00000 00000 02

N = 03

.14400 00000 00000 00000 +05      -.16800 00000 00000 00000 +04  
.24000 00000 00000 00000 +02

N = 04

.28224 00000 00000 00000 +07      -.36960 00000 00000 00000 +06  
.88800 00000 00000 00000 +04      -.40000 00000 00000 00000 +02

TABLE A.VII (Continued)

N = 05

.91445	76000	00000	00000	+09	-.12700	80000	00000	00000	+09
.37900	80000	00000	00000	+07	-.31920	00000	00000	00000	+05
.60000	00000	00000	00000	+02					

N = 06

.44259	74784	00000	00000	+12	-.63707	21280	00000	00000	+11
.21388	14720	00000	00000	+10	-.23950	08000	00000	00000	+08
.90720	00000	00000	00000	+05	-.84000	00000	00000	00000	+02

N = 07

.29919	58953	98400	00000	+15	-.44112	21534	72000	00000	+14
.15946	92126	72000	00000	+13	-.21009	54240	00000	00000	+11
.11124	28800	00000	00000	+09	-.21974	40000	00000	00000	+06
.11200	00000	00000	00000	+03					

N = 08

.26927	63058	58560	00000	+18	-.40391	44587	87840	00000	+17
.15362	55847	52640	00000	+16	-.22479	54508	80000	00000	+14
.14503	37011	20000	00000	+12	-.41646	52800	00000	00000	+09
.47376	00000	00000	00000	+06	-.14400	00000	00000	00000	+03

N = 09

.31128	34095	72495	36000	+21	-.47302	87106	24870	40000	+20
.18669	82387	28601	60000	+19	-.29397	64065	63840	00000	+17
.21608	80001	28000	00000	+15	-.77785	86816	00000	00000	+12
.13278	98880	00000	00000	+10	-.93456	00000	00000	00000	+06
.18000	00000	00000	00000	+03					

TABLE A.VII (Concluded)

N = 10

.44949	32434	22683	29984	+24	-.69001	15578	85698	04800	+23
.28012	45506	33915	18720	+22	-.46561	72008	73144	32000	+20
.37541	77850	14272	00000	+18	-.15728	18837	99040	00000	+16
.34464	83040	00000	00000	+13	-.37378	59840	00000	00000	+10
.17186	40000	00000	00000	+07	-.22000	00000	00000	00000	+03

APPENDIX B

COEFFICIENTS OF RATIONAL APPROXIMATIONS TO  $E(z)$   
AND RELATED INTEGRALS

Table B.I gives the coefficients of the polynomials of the main diagonal Padé approximants of  $E(z)$ , see (5.1). Displayed in Table B.II is the absolute value of the errors associated with the approximations listed in B.I. Similar coefficients for the main diagonal Padé approximants to  $z^{-1}Si(z)$ , see (5.2), are given for  $n$  even in Table B.III. Table B.IV gives the error associated with the approximations listed in B.III. Table B.V lists the coefficients of the polynomials of the main diagonal Padé approximants to  $4z^{-2}H(z)$ , see (5.3), for even  $n$ . Table B.VI lists the errors associated with the approximants in B.V. Tables B.VII and B.VIII give the coefficients which define the approximations (5.11) to  $E(z)$ , and the corresponding error tables, respectively.

TABLE B.I

COEFFICIENTS OF PADE' APPROXIMATION TO E(z)

Note: For the definition of E(z) and its Pade' approximants, see Section V. The coefficients are given for n = 1(1)10. The expression attached to the numbers on the right indicates the power of 10 by which the number is multiplied.

$$\text{e.g., } E_2(z) = \frac{1 + 0.086z + 0.04555\dots z^2}{1 + 0.336z + 0.0333\dots z^2}$$

	<u>NUMERATOR</u>	<u>DENOMINATOR</u>
N = 01		
	. 10000 00000 00000 00000 +01	. 10000 00000 00000 00000 +01
	- . 27777 77777 77777 77778 -01	. 22222 22222 22222 22222 +00
N = 02		
	. 10000 00000 00000 00000 +01	. 10000 00000 00000 00000 +01
	. 86000 00000 00000 00000 -01	. 33600 00000 00000 00000 +00
	. 45555 55555 55555 55556 -02	. 33000 00000 00000 00000 -01
N = 03		
	. 10000 00000 00000 00000 +01	. 10000 00000 00000 00000 +01
	. 11548 60120 33992 92848 +00	. 36548 60120 33992 92848 +00
	. 14987 35259 04789 47405 -01	. 50803 30004 34216 23969 -01
	- . 85008 16740 06988 81376 -04	. 27277 05063 78843 33065 -02
N = 04		
	. 10000 00000 00000 00000 +01	. 10000 00000 00000 00000 +01
	. 15183 03075 00550 75001 +00	. 40183 03075 00550 75001 +00
	. 22174 10124 06340 46892 -01	. 67076 12256 02161 78840 -01
	. 71679 28213 71348 66118 -03	. 55785 84155 83924 05926 -02
	. 10529 22995 44767 84505 -04	. 19778 97186 99680 64309 -03

TABLE B.I (Continued)

<u>NUMERATOR</u>					<u>DENOMINATOR</u>				
N = 05									
. 10000	00000	00000	00000	+01	. 10000	00000	00000	00000	+01
. 16204	16009	62342	70975	+00	. 41204	16009	62342	70975	+00
. 26595	51845	22923	74506	-01	. 74050	36313	73224	96387	-01
. 12747	83272	48448	67889	-02	. 73128	40670	01829	36777	-02
. 60877	80145	04733	48711	-04	. 40061	22491	28200	19535	-03
. 13654	42296	45193	12376	-06	. 98510	21164	96373	17832	-05
N = 06									
. 10000	00000	00000	00000	+01	. 10000	00000	00000	00000	+01
. 17964	76037	82301	29344	+00	. 42964	76037	82301	29344	+00
. 30906	39014	59549	31654	-01	. 82762	73553	59746	99458	-01
. 19643	87594	46735	53361	-02	. 92024	26823	88873	61206	-02
. 11565	07429	94841	78517	-03	. 62715	69030	34069	87432	-03
. 19803	30657	69150	50495	-05	. 25037	59209	97727	45364	-04
. 12947	26971	80815	10906	-07	. 46180	79516	66610	03704	-06
N = 07									
. 10000	00000	00000	00000	+01	. 10000	00000	00000	00000	+01
. 18480	11227	26475	81023	+00	. 43480	11227	26475	81023	+00
. 33261	21378	35356	62215	-01	. 86405	93890	95990	59216	-01
. 23552	22731	54871	76423	-02	. 10217	75619	63664	92989	-01
. 16377	94153	15153	96938	-03	. 78040	02200	52063	61676	-03
. 43980	12290	60812	99224	-05	. 38719	75252	89419	77095	-04
. 11014	77035	81094	26833	-06	. 11633	17088	02093	24466	-05
. 12839	85657	45606	74048	-09	. 16457	10741	45580	73556	-07

TABLE B. I (Concluded)

<u>NUMERATOR</u>					<u>DENOMINATOR</u>				
N = 08									
. 10000	00000	00000	00000	+01	. 10000	00000	00000	00000	+01
. 19513	36973	48381	21416	+00	. 44513	36973	48381	21416	+00
. 36028	72455	48033	66895	-01	. 91756	59333	63431	14880	-01
. 28530	41423	63713	82221	-02	. 11479	20657	17017	38378	-01
. 21463	38297	93936	23562	-03	. 95698	96347	45946	53974	-03
. 77757	97235	47219	15163	-05	. 54679	34064	85047	10025	-04
. 25482	33856	33725	45105	-06	. 21015	17140	91386	35547	-05
. 27089	12491	61763	77927	-08	. 49960	40804	95845	41259	-07
. 97825	79525	95621	99712	-11	. 56629	36991	72465	57055	-09
N = 09									
. 10000	00000	00000	00000	+01	. 10000	00000	00000	00000	+01
. 19823	84189	19627	19265	+00	. 44823	84189	19627	19265	+00
. 37482	24827	32679	61682	-01	. 93986	29744	76192	04289	-01
. 31215	54261	10534	67154	-02	. 12132	66090	52530	81529	-01
. 25168	68581	42141	13445	-03	. 10658	74645	55601	67575	-02
. 10438	66937	29114	63287	-04	. 66311	99549	82139	94997	-04
. 41779	89027	92798	76366	-06	. 29328	17107	87334	24422	-05
. 72282	10040	34696	94233	-08	. 89307	30009	59419	32977	-07
. 11156	38762	07900	30289	-09	. 17075	36697	26887	18576	-08
. 78469	72732	66716	49614	-13	. 15708	67540	09828	56485	-10
N = 10									
. 10000	00000	00000	00000	+01	. 10000	00000	00000	00000	+01
. 20502	08456	77917	06628	+00	. 45502	08456	77917	06628	+00
. 39390	75193	16296	55245	-01	. 97590	40779	55533	66258	-01
. 34858	23655	29237	91179	-02	. 13021	15639	98519	94778	-01
. 29317	75506	14266	48946	-03	. 11999	11137	74704	76100	-02
. 13754	73570	29922	39386	-04	. 80015	09559	21661	45984	-04
. 60964	46174	77455	80030	-06	. 39222	83073	88575	92254	-05
. 14447	18655	00891	74805	-07	. 14003	62118	96032	45150	-06
. 30430	04327	31332	24684	-09	. 34984	41348	05290	45646	-08
. 22059	38908	74765	26250	-11	. 55465	89453	73869	45817	-10
. 49848	28058	16872	88340	-14	. 42591	33901	24021	43020	-12

TABLE B.II

ERROR OF PADE' APPROXIMATIONS TO E(z)

Let  $E_n(z)$  be the  $n$ th order main diagonal Padé' approximation to  $E(z)$ , see B.I, and define  $\epsilon_n(z) = |E(z) - E_n(z)|$ . The tables give  $\epsilon_n(z)$  for  $n = 2(1)10$ ,  $r = 1(1)10$  and  $\theta = 0, \pi/2, \pi$  where  $z = rei\theta$ . The expression in parentheses to the right of each number indicates the power of 10 by which the number is multiplied.

$\theta$ <u>r</u>	<u>0</u>	<u><math>\pi/2</math></u>	<u><math>\pi</math></u>	<u><math>\pi</math> (Relative Error)</u>
<u>n = 2</u>				
1	0.782 (- 5)	0.107 (- 4)	0.320 (- 4)	0.243 (- 4)
2	0.137 (- 3)	0.418 (- 3)	0.232 (- 2)	0.126 (- 2)
3	0.599 (- 3)	0.267 (- 2)	0.433 (- 1)	0.157 (- 1)
4	0.152 (- 2)	0.849 (- 2)	0.455	0.103
5	0.292 (- 2)	0.202 (- 1)	0.288 ( 1)	0.379
6	0.473 (- 2)	0.361 (- 1)	0.102 ( 1)	0.732
7	0.686 (- 2)	0.542 (- 1)	0.247 ( 2)	0.915
8	0.922 (- 2)	0.713 (- 1)	0.533 ( 2)	0.974
9	0.117 (- 1)	0.845 (- 1)	0.114 ( 3)	0.991
10	0.143 (- 1)	0.825	0.248 ( 3)	0.996
<u>n = 3</u>				
1	0.217 (- 7)	0.443 (- 7)	0.101 (- 6)	0.776 (- 7)
2	0.139 (- 5)	0.523 (- 5)	0.302 (- 4)	0.164 (- 4)
3	0.125 (- 4)	0.780 (- 4)	0.128 (- 2)	0.465 (- 3)
4	0.510 (- 4)	0.482 (- 3)	0.252 (- 1)	0.571 (- 2)
5	0.138 (- 3)	0.179 (- 2)	0.345	0.454 (- 1)
6	0.291 (- 3)	0.474 (- 2)	0.454 ( 1)	0.326
7	0.519 (- 3)	0.985 (- 2)	0.233 ( 3)	0.863
8	0.823 (- 3)	0.170 (- 1)	0.703 ( 2)	0.128 ( 1)
9	0.120 (- 2)	0.255 (- 1)	0.123 ( 3)	0.107 ( 1)
10	0.164 (- 2)	0.340 (- 1)	0.254 ( 3)	0.102 ( 1)

TABLE B.II (Continued)

$r \setminus \theta$	$0$	$\pi/2$	$\pi$	$\pi$ (Relative Error)
<u>n = 4</u>				
1	0.593 (-10)	0.128 (- 9)	0.302 (- 9)	0.229 (- 9)
2	0.144 (- 7)	0.610 (- 7)	0.375 (- 6)	0.204 (- 6)
3	0.273 (- 6)	0.209 (- 5)	0.366 (- 4)	0.133 (- 4)
4	0.186 (- 5)	0.236 (- 4)	0.130 (- 2)	0.294 (- 3)
5	0.738 (- 5)	0.141 (- 3)	0.271 (- 1)	0.366 (- 2)
6	0.209 (- 4)	0.558 (- 3)	0.401	0.288 (- 1)
7	0.474 (- 4)	0.162 (- 2)	0.428 ( 1)	0.159
8	0.920 (- 4)	0.375 (- 2)	0.273 ( 2)	0.499
9	0.159 (- 3)	0.720 (- 2)	0.945 ( 2)	0.822
10	0.251 (- 3)	0.119 (- 1)	0.237 ( 3)	0.952
<u>n = 5</u>				
1	0.837 (-13)	0.187 (-12)	0.451 (-12)	0.342 (-12)
2	0.781 (-10)	0.362 (- 9)	0.227 (- 8)	0.123 (- 8)
3	0.319 (- 8)	0.284 (- 7)	0.502 (- 6)	0.182 (- 6)
4	0.368 (- 7)	0.588 (- 6)	0.316 (- 4)	0.715 (- 5)
5	0.206 (- 6)	0.574 (- 5)	0.102 (- 2)	0.134 (- 3)
6	0.834 (- 6)	0.342 (- 4)	0.217 (- 1)	0.156 (- 2)
7	0.243 (- 5)	0.143 (- 3)	0.355	0.131 (- 1)
8	0.579 (- 5)	0.457 (- 3)	0.513 ( 1)	0.937 (- 1)
9	0.119 (- 4)	0.117 (- 2)	0.104 ( 3)	0.904
10	0.218 (- 4)	0.250 (- 2)	0.442 ( 3)	0.178 ( 1)
<u>n = 6</u>				
1	0.117 (-15)	0.268 (-15)	0.658 (-15)	0.499 (-15)
2	0.423 (-12)	0.209 (-11)	0.135 (-10)	0.733 (-11)
3	0.376 (-10)	0.373 (- 9)	0.677 (- 8)	0.246 (- 6)
4	0.746 (- 9)	0.139 (- 7)	0.763 (- 6)	0.173 (- 6)
5	0.658 (- 8)	0.217 (- 6)	0.384 (- 4)	0.505 (- 5)
6	0.352 (- 7)	0.192 (- 5)	0.118 (- 2)	0.847 (- 4)
7	0.134 (- 6)	0.113 (- 4)	0.258 (- 1)	0.956 (- 3)
8	0.399 (- 6)	0.489 (- 4)	0.445	0.813 (- 2)
9	0.993 (- 6)	0.165 (- 3)	0.620 ( 1)	0.539 (- 1)
10	0.215 (- 5)	0.455 (- 3)	0.627 ( 2)	0.252

TABLE B.II (Continued)

$\frac{0}{r}$	$0$	$\frac{\pi}{2}$	$\pi$	$\pi$ (Relative Error)
<u>n = 7</u>				
1	0.900 (-19)	0.233 (-18)	0.580 (-18)	0.440 (-18)
2	0.140 (-14)	0.722 (-14)	0.472 (-13)	0.256 (-13)
3	0.272 (-12)	0.294 (-11)	0.536 (-10)	0.195 (-10)
4	0.932 (-11)	0.198 (- 9)	0.107 (- 7)	0.242 (- 8)
5	0.125 (- 9)	0.493 (- 8)	0.840 (- 6)	0.110 (- 6)
6	0.925 (- 9)	0.644 (- 7)	0.367 (- 4)	0.263 (- 4)
7	0.462 (- 8)	0.534 (- 6)	0.108 (- 2)	0.400 (- 3)
8	0.173 (- 7)	0.314 (- 5)	0.241 (- 1)	0.440 (- 3)
9	0.524 (- 7)	0.141 (- 4)	0.436	0.379 (- 2)
10	0.134 (- 6)	0.336	0.686 ( 1)	0.276 (- 1)
<u>n = 8</u>				
1	0.370 (-20)	0.100 (-19)	0.100 (-19)	0.759 (-20)
2	0.459 (-17)	0.246 (-16)	0.164 (-15)	0.890 (-16)
3	0.197 (-14)	0.227 (-13)	0.421 (-12)	0.153 (-12)
4	0.118 (-12)	0.275 (-11)	0.150 (- 9)	0.340 (-10)
5	0.239 (-11)	0.108 (- 9)	0.184 (- 7)	0.242 (- 8)
6	0.249 (-10)	0.207 (- 8)	0.115 (- 5)	0.825 (- 7)
7	0.165 (- 9)	0.238 (- 7)	0.461 (- 4)	0.171 (- 5)
8	0.784 (- 9)	0.187 (- 6)	0.133 (- 2)	0.243 (- 4)
9	0.291 (- 8)	0.109 (- 5)	0.301 (- 1)	0.262 (- 3)
10	0.895 (- 8)	0.500 (- 5)	0.565	0.227 (- 2)
<u>n = 9</u>				
1	0.367 (-20)	0.100 (-19)	0.100 (-19)	0.758 (-20)
2	0.500 (-20)	0.500 (-19)	0.377 (-18)	0.205 (-18)
3	0.950 (-17)	0.117 (-15)	0.217 (-14)	0.788 (-15)
4	0.998 (-15)	0.254 (-13)	0.138 (-11)	0.312 (-12)
5	0.311 (-13)	0.106 (-10)	0.263 (- 9)	0.346 (-10)
6	0.456 (-12)	0.442 (-10)	0.236 (- 7)	0.169 (- 8)
7	0.400 (-11)	0.707 (- 9)	0.127 (- 5)	0.470 (- 7)
8	0.242 (-10)	0.743 (- 8)	0.474 (- 4)	0.866 (- 6)
9	0.111 (- 9)	0.564 (- 7)	0.134 (- 2)	0.117 (- 4)
10	0.408 (- 9)	0.329 (- 6)	0.306 (- 1)	0.123 (- 3)

TABLE B.II (Concluded)

$\frac{r}{0}$	$0$	$\frac{\pi}{2}$	$\pi$	$\pi$ (Relative Error)
1	0.367 (-20)	0.100 (-19)	0.100 (-19)	0.759 (-20)
2	0.551 (-20)	0.141 (-19)	0.200 (-20)	0.109 (-20)
3	0.145 (-18)	0.578 (-18)	0.111 (-16)	0.403 (-17)
4	0.850 (-17)	0.231 (-15)	0.126 (-13)	0.285 (-14)
5	0.407 (-15)	0.227 (-13)	0.375 (-11)	0.493 (-12)
6	0.845 (-14)	0.923 (-12)	0.485 (- 9)	0.341 (-10)
7	0.989 (-13)	0.203 (-10)	0.354 (- 7)	0.131 (- 8)
8	0.766 (-12)	0.284 (- 9)	0.172 (- 5)	0.314 (- 7)
9	0.434 (-11)	0.277 (-8)	0.610 (- 4)	0.530 (- 6)
10	0.193 (-10)	0.204 (- 7)	0.170 (- 2)	0.683 (- 5)

TABLE B.III

COEFFICIENTS OF PADÉ APPROXIMATIONS TO  $z^{-1}Si(z)$ 

Note: For the definition of  $Si(z)$  and its Padé approximants, see Section V. The coefficients are given for  $n = 2(2)10$ . The expression attached to the numbers on the right indicates the power of 10 by which the numbers are multiplied.

$$\text{e.g., } z^{-1}Si_2(z) = \frac{1 - 0.02555\dots z^2}{1 + 0.03z^2}$$

	<u>NUMERATOR</u>	<u>DENOMINATOR</u>
N = 02		
	.10000 00000 00000 00000 +01	.10000 00000 00000 00000 +01
	-.25555 55555 55555 55556 -01	.30000 00000 00000 00000 -01
N = 04		
	.10000 00000 00000 00000 +01	.10000 00000 00000 00000 +01
	-.30427 89735 63441 27723 -01	.25127 65819 92114 27833 -01
	.51431 13199 43054 89416 -03	.24362 56643 43689 77376 -03
N = 06		
	.10000 00000 00000 00000 +01	.10000 00000 00000 00000 +01
	-.35595 17688 83526 52167 -01	.19960 37866 72029 03389 -01
	.74030 23630 09289 15834 -03	.18254 56222 98339 34660 -03
	-.44148 31037 35430 42004 -05	.80396 58468 07101 69744 -06
N = 08		
	.10000 00000 00000 00000 +01	.10000 00000 00000 00000 +01
	-.39097 46707 47744 00193 -01	.16458 08848 07811 55363 -01
	.88244 03534 29201 16961 -03	.13011 19356 94820 91197 -03
	-.75392 62090 91062 85582 -05	.60370 25148 69333 23738 -06
	.24448 76200 37211 27450 -07	.14412 35225 57797 97185 -08

TABLE B.III (Concluded)

N = 10

.10000	00000	00000	00000	+01	.10000	00000	00000	00000	+01
-.41604	56609	13051	18594	-01	.13950	98946	42504	36962	-01
.98695	05591	58116	83815	-03	.95338	86272	75855	58234	-04
-.99826	03983	42126	42340	-05	.40702	15961	74597	46660	-06
.48154	24911	89950	63472	-07	.11122	28953	69313	70588	-08
-.89943	23724	44804	92329	-10	.16036	02483	25839	25315	-11

TABLE B.IV

ERROR OF PADE APPROXIMATION TO  $z^{-1}Si(z)$ 

Let  $z^{-1}Si_n(z)$  be the  $n^{\text{th}}$  order main diagonal (see B.III) Pade' approximant to  $z^{-1}Si(z)$  and define  $\epsilon_n(z) = |z^{-1}Si(z) - z^{-1}Si_n(z)|$ . The tables give  $\epsilon_n(z)$  for  $n = 2(2)10$  and  $z = 1(1)10$ ,  $z$  real. The expression in parentheses to the right of each number indicates the power of 10 by which the number is multiplied.

$\frac{n}{z}$	<u>2</u>	<u>4</u>	<u>6</u>	<u>8</u>	<u>10</u>
1	0.210 (-4)	0.142 (-8)	0.251 (-13)	0.170 (-18)	0.100 (-20)
2	0.112 (-2)	0.127 (-5)	0.369 (- 9)	0.392 (-13)	0.186 (-17)
3	0.990 (-2)	0.590 (-4)	0.898 (- 7)	0.498 (-10)	0.122 (-13)
4	0.401 (-1)	0.775 (-3)	0.392 (- 5)	0.714 (- 8)	0.570 (-11)
5	0.104	0.496 (-2)	0.647 (- 4)	0.302 (- 6)	0.616 (- 9)
6	0.234	0.198 (-1)	0.565 (- 3)	0.579 (- 5)	0.254 (- 7)
7	0.310	0.563 (-1)	0.313 (- 2)	0.630 (- 4)	0.539 (- 6)
8	0.415	0.125	0.123 (- 1)	0.449 (- 3)	0.692 (- 5)
9	0.497	0.227	0.370 (- 1)	0.229 (- 2)	0.599 (- 4)
10	0.555	0.335	0.896 (- 1)	0.889 (- 2)	0.376 (- 3)

TABLE B.V

COEFFICIENTS OF PADÉ APPROXIMATIONS TO  $4z^{-2}H(z)$

Note: For the definition of  $H(z)$  and its Padé approximants, see Section V. The coefficients are given for  $n = 2(2)10$ . The expression attached to the numbers on the right indicates the power of 10 by which the numbers are multiplied.

$$\text{e.g., } 4z^{-2}H_2(z) = \frac{1 - 0.019444\dots z^2}{1 + 0.0222\dots z^2}$$

NUMERATOR

DENOMINATOR

N = 02

.10000	00000	00000	00000	+01	.10000	00000	00000	00000	+01
-.19444	44444	44444	44444	-01	.22222	22222	22222	22222	-01

N = 04

.10000	00000	00000	00000	+01	.10000	00000	00000	00000	+01
-.21441	94756	55430	71161	-01	.20224	71910	11235	95506	-01
.23504	84513	40586	17205	-03	.15181	91546	28143	39219	-03

N = 06

.10000	00000	00000	00000	+01	.10000	00000	00000	00000	+01
-.24662	60143	38362	60391	-01	.17004	06523	28304	06275	-01
.34682	10542	75433	98340	-03	.12939	78463	84108	31895	-03
-.15875	89095	57136	15182	-05	.46027	66426	80130	08786	-06

N = 08

.10000	00000	00000	00000	+01	.10000	00000	00000	00000	+01
-.27191	18247	38415	96633	-01	.14475	48419	28250	70034	-01
.42197	89824	50745	07333	-03	.99198	23122	58636	82145	-04
-.27389	37015	82204	22581	-05	.39189	01649	14937	85289	-06
.70221	55017	22417	74512	-08	.77891	33652	63414	35304	-09

TABLE B.V (Concluded)

<u>NUMERATOR</u>					<u>DENOMINATOR</u>				
N = 10									
.10000	00000	00000	00000	+01	.10000	00000	00000	00000	+01
-.29128	03421	36361	44413	-01	.12538	63245	30305	22254	-01
.47973	22804	11881	21947	-03	.76249	37336	22270	54117	-04
-.36817	31277	22802	06720	-05	.28627	47330	74607	14153	-06
.14091	69854	52941	91735	-07	.67825	76089	43662	46538	-09
-.21678	54057	76489	52502	-10	.83325	19383	71648	11691	-12

TABLE B.VI

ERROR OF PADE' APPROXIMATION TO  $4z^{-2}H(z)$ 

Let  $4z^{-2}H_n(z)$  be the  $n^{\text{th}}$  order main diagonal (see B.V) Padé' approximant to  $4z^{-2}H(z)$  and define  $\epsilon_n(z) = |4z^{-2}H(z) - H_n(z)|$ . The tables give  $\epsilon_n(z)$  for  $n = 2(2)10$  and  $z = 1(1)10$ . The expression in parentheses to the right of each number indicates the power of 10 by which the number is multiplied.

$\frac{n}{z}$	<u>2</u>	<u>4</u>	<u>6</u>	<u>8</u>	<u>10</u>
1	0.584 (-5)	0.336 (-9)	0.433 (-14)	1.000 (-20)	1.000 (-20)
2	0.443 (-3)	0.308 (-6)	0.646 (-10)	0.535 (-14)	0.210 (-18)
3	0.414 (-2)	0.148 (-4)	0.161 (- 7)	0.691 (-11)	0.138 (-14)
4	0.179 (-1)	0.205 (-3)	0.728 (- 6)	0.102 (- 8)	0.657 (-12)
5	0.599 (-1)	0.140 (-2)	0.126 (- 4)	0.443 (- 7)	0.721 (-10)
6	0.104	0.597 (-2)	0.116 (- 3)	0.881 (- 6)	0.308 (- 8)
7	0.177	0.184 (-1)	0.680 (- 3)	0.100 (- 4)	0.675 (- 7)
8	0.341	0.441 (-1)	0.286 (- 2)	0.751 (- 4)	0.900 (- 6)
9	0.339	0.873 (-1)	0.920 (- 2)	0.404 (- 3)	0.812 (- 5)
10	0.410	0.149	0.240 (- 1)	0.167 (- 2)	0.535 (- 4)

TABLE B.VII

COEFFICIENTS OF RATIONAL APPROXIMATIONS TO E(z)

Here we present coefficients of the numerator and denominator polynomials of the rational approximation to  $E(z)$  defined in (5.11). We give coefficients for  $n = 0(1)10$ . The sequence of numbers given is for the lowest power to the highest power, respectively. The expression to the right of each number is the power of 10 by which it is multiplied.

$$\text{e.g., } E_2(z) = \frac{3z + 36}{z^2 + 12z + 36}$$

	<u>NUMERATOR</u>	<u>DENOMINATOR</u>
N = 00	. 10000 00000 00000 00000 01	. 10000 00000 00000 00000 01
N = 01	. 40000 00000 00000 00000 01 . 00000 00000 00000 00000 00	. 40000 00000 00000 00000 01 . 10000 00000 00000 00000 01
N = 02	. 36000 00000 00000 00000 02 . 30000 00000 00000 00000 01 . 00000 00000 00000 00000 00	. 36000 00000 00000 00000 02 . 12000 00000 00000 00000 02 . 10000 00000 00000 00000 01
N = 03	. 48000 00000 00000 00000 03 . 60000 00000 00000 00000 02 . 56666 66666 66666 66667 01 . 00000 00000 00000 00000 00	. 48000 00000 00000 00000 03 . 18000 00000 00000 00000 03 . 24000 00000 00000 00000 02 . 10000 00000 00000 00000 01

TABLE B.VII (Continued)

<u>NUMERATOR</u>					<u>DENOMINATOR</u>				
N = 04									
.84000	00000	00000	00000	+04	.84000	00000	00000	00000	+04
.12600	00000	00000	00000	+04	.33600	00000	00000	00000	+04
.16666	66666	66666	66667	+03	.54000	00000	00000	00000	+03
.41666	66666	66666	66667	+01	.40000	00000	00000	00000	+02
.00000	00000	00000	00000	-99	.10000	00000	00000	00000	+01
N = 05									
.18144	00000	00000	00000	+06	.18144	00000	00000	00000	+06
.30240	00000	00000	00000	+05	.75600	00000	00000	00000	+05
.46200	00000	00000	00000	+04	.13440	00000	00000	00000	+05
.21000	00000	00000	00000	+03	.12600	00000	00000	00000	+04
.65666	66666	66666	66667	+01	.60000	00000	00000	00000	+02
.00000	00000	00000	00000	-99	.10000	00000	00000	00000	+01
N = 06									
.46569	60000	00000	00000	+07	.46569	60000	00000	00000	+07
.83160	00000	00000	00000	+06	.19958	40000	00000	00000	+07
.13776	00000	00000	00000	+06	.37800	00000	00000	00000	+06
.81900	00000	00000	00000	+04	.40320	00000	00000	00000	+05
.41160	00000	00000	00000	+03	.25200	00000	00000	00000	+04
.49000	00000	00000	00000	+01	.84000	00000	00000	00000	+02
.00000	00000	00000	00000	-99	.10000	00000	00000	00000	+01
N = 07									
.13837	82400	00000	00000	+09	.13837	82400	00000	00000	+09
.25945	92000	00000	00000	+08	.60540	48000	00000	00000	+08
.45276	00000	00000	00000	+07	.11975	04000	00000	00000	+08
.31416	00000	00000	00000	+06	.13860	00000	00000	00000	+07
.19580	40000	00000	00000	+05	.10080	00000	00000	00000	+06
.46480	00000	00000	00000	+03	.45360	00000	00000	00000	+04
.71857	14285	71428	57143	+01	.11200	00000	00000	00000	+03
.00000	00000	00000	00000	-99	.10000	00000	00000	00000	+01

TABLE B.VII (Concluded)

<u>NUMERATOR</u>					<u>DENOMINATOR</u>				
N = 08									
.46702	65600	00000	00000	+10	.46702	65600	00000	00000	+10
.90810	72000	00000	00000	+09	.20756	73600	00000	00000	+10
.16432	41600	00000	00000	+09	.42378	33600	00000	00000	+09
.12612	60000	00000	00000	+08	.51891	84000	00000	00000	+08
.89073	60000	00000	00000	+06	.41580	00000	00000	00000	+07
.29106	00000	00000	00000	+05	.22176	00000	00000	00000	+06
.78274	28571	42857	14286	+03	.75600	00000	00000	00000	+04
.54357	14285	71428	57143	+01	.14400	00000	00000	00000	+03
.00000	00000	00000	00000	-99	.10000	00000	00000	00000	+01
N = 09									
.17643	22560	00000	00000	+12	.17643	22560	00000	00000	+12
.35286	45120	00000	00000	+11	.79394	51520	00000	00000	+11
.65585	52000	00000	00000	+10	.16605	38880	00000	00000	+11
.54054	00000	00000	00000	+09	.21189	16800	00000	00000	+10
.41441	40000	00000	00000	+08	.18162	14400	00000	00000	+09
.16336	32000	00000	00000	+07	.10810	80000	00000	00000	+08
.57052	28571	42857	14286	+05	.44352	00000	00000	00000	+06
.83442	85714	28571	42857	+03	.11880	00000	00000	00000	+05
.76579	36507	93650	79365	+01	.18000	00000	00000	00000	+03
.00000	00000	00000	00000	-99	.10000	00000	00000	00000	+01
N = 10									
.73748	68300	80000	00000	+13	.73748	68300	80000	00000	+13
.15084	95788	80000	00000	+13	.33522	12864	00000	00000	+13
.28621	23264	00000	00000	+12	.71455	06368	00000	00000	+12
.24872	04720	00000	00000	+11	.94097	20320	00000	00000	+11
.20211	87168	00000	00000	+10	.84756	67200	00000	00000	+10
.90210	12000	00000	00000	+08	.54486	43200	00000	00000	+09
.36810	65142	85714	28571	+07	.25225	20000	00000	00000	+08
.76668	42857	14285	71429	+05	.82368	00000	00000	00000	+06
.12887	46031	74603	17460	+04	.17820	00000	00000	00000	+05
.58579	36507	93650	79365	+01	.22000	00000	00000	00000	+03
.00000	00000	00000	00000	-99	.10000	00000	00000	00000	+01

TABLE B.VIII

ERROR OF RATIONAL APPROXIMATION TO  $E(z)$ 

Let  $E_n(z)$  be the  $n^{\text{th}}$  order rational approximation to  $E(z)$  as defined in (5.11) and Table B.VII. Set  $\epsilon_n(z) = |E(z) - E_n(z)|$ . The tables give  $\epsilon_n(z)$  for  $n = 3(1)10$ ,  $r = 1(1)10$  and  $\theta = 0, \pi/2, \pi$  where  $z = re^{i\theta}$ . The expression in parentheses to the right of each number indicates the power of 10 by which the number is multiplied.

$r \backslash \theta$	0	$\pi/2$	$\pi$	$\pi$ (Relative Error)
<u><math>n = 3</math></u>				
1	0.593 (-5)	0.177 (-4)	0.487 (-4)	0.369 (-4)
2	0.272 (-4)	0.291 (-3)	0.219 (-2)	0.119 (-2)
3	0.134 (-4)	0.148 (-2)	0.317 (-1)	0.115 (-1)
4	0.870 (-4)	0.444 (-2)	0.284	0.643 (-1)
5	0.286 (-3)	0.975 (-2)	0.175 ( 1)	0.230
6	0.566 (-3)	0.172 (-1)	0.719 ( 1)	0.517
7	0.895 (-3)	0.259 (-1)	0.206 ( 2)	0.762
8	0.125 (-2)	0.345 (-1)	0.490 ( 2)	0.896
9	0.160 (-2)	0.413 (-1)	0.110 ( 3)	0.957
10	0.193 (-2)	0.454 (-1)	0.243 ( 3)	0.976
<u><math>n = 4</math></u>				
1	0.188 (-6)	0.287 (-6)	0.754 (-6)	0.571 (-6)
2	0.407 (-5)	0.543 (-5)	0.703 (-4)	0.382 (-4)
3	0.226 (-4)	0.155 (-4)	0.180 (-2)	0.655 (-3)
4	0.716 (-4)	0.198 (-3)	0.276 (-1)	0.624 (-2)
5	0.166 (-3)	0.102 (-2)	0.326	0.429 (-1)
6	0.315 (-3)	0.317 (-2)	0.374 ( 1)	0.269
7	0.524 (-3)	0.724 (-2)	0.274 ( 3)	0.101 ( 2)
8	0.787 (-3)	0.133 (-1)	0.773 ( 2)	0.141 ( 1)
9	0.110 (-2)	0.209 (-1)	0.126 ( 3)	0.110 ( 1)
10	0.145 (-2)	0.288 (-1)	0.256 ( 3)	0.103 ( 1)

TABLE B.VIII (Continued)

$r \setminus \theta$	$0$	$\pi/2$	$\pi$	$\pi$ (Relative Error)
<u>n = 5</u>				
1	0.376 (-8)	0.632 (-8)	0.125 (-7)	0.947 (-8)
2	0.140 (-6)	0.379 (-6)	0.188 (-5)	0.102 (-5)
3	0.918 (-6)	0.449 (-5)	0.674 (-4)	0.245 (-4)
4	0.290 (-5)	0.306 (-4)	0.148 (-2)	0.335 (-3)
5	0.598 (-5)	0.145 (-3)	0.242 (-1)	0.318 (-2)
6	0.916 (-5)	0.504 (-3)	0.314	0.226 (-1)
7	0.107 (-4)	0.136 (-2)	0.318 ( 1)	0.118
8	0.860 (-5)	0.298 (-2)	0.216 ( 2)	0.395
9	0.117 (-5)	0.553 (-2)	0.848 ( 2)	0.737
10	0.129 (-4)	0.895 (-2)	0.228 ( 3)	0.916
<u>n = 6</u>				
1	0.760 (-10)	0.125 (-9)	0.225 (-9)	0.170 (-9)
2	0.602 (-8)	0.154 (-7)	0.559 (-7)	0.304 (-7)
3	0.665 (-7)	0.253 (-6)	0.233 (-5)	0.847 (-6)
4	0.337 (-6)	0.184 (-5)	0.603 (-4)	0.136 (-4)
5	0.113 (-5)	0.883 (-5)	0.129 (-2)	0.170 (-3)
6	0.295 (-5)	0.290 (-4)	0.229 (-1)	0.165 (-2)
7	0.649 (-5)	0.936 (-4)	0.343	0.127 (-1)
8	0.126 (-4)	0.288 (-3)	0.465 ( 1)	0.850
9	0.221 (-4)	0.778 (-3)	0.825 ( 2)	0.717
10	0.360 (-4)	0.178 (-2)	0.508 ( 3)	0.204
<u>n = 7</u>				
1	0.144 (-11)	0.236 (-11)	0.407 (-11)	0.308 (-11)
2	0.229 (- 9)	0.579 (- 9)	0.185 (- 8)	0.101 (- 8)
3	0.370 (- 8)	0.137 (- 7)	0.936 (- 7)	0.340 (- 7)
4	0.236 (- 7)	0.113 (- 6)	0.240 (- 5)	0.543 (- 6)
5	0.905 (- 7)	0.526 (- 6)	0.551 (- 4)	0.725 (- 5)
6	0.251 (- 6)	0.142 (- 5)	0.120 (- 2)	0.863 (- 4)
7	0.558 (- 6)	0.627 (- 5)	0.229 (- 1)	0.848 (- 3)
8	0.105 (- 5)	0.341 (- 4)	0.367	0.671 (- 2)
9	0.172 (- 5)	0.130 (- 3)	0.493 ( 1)	0.429 (- 1)
10	0.255 (- 5)	0.380 (- 3)	0.506 ( 2)	0.203

TABLE B.VIII (Concluded)

$\frac{e}{r}$	$\underline{0}$	$\frac{\pi}{2}$	$\underline{\pi}$	$\underline{\pi}$ (Relative Error)
<u>n = 8</u>				
1	0.259 (-13)	0.420 (-13)	0.712 (-13)	0.539 (-13)
2	0.825 (-11)	0.208 (-10)	0.626 (-10)	0.340 (-10)
3	0.201 (- 9)	0.759 (- 9)	0.431 (- 8)	0.157 (- 8)
4	0.173 (- 8)	0.939 (- 8)	0.116 (- 6)	0.262 (- 7)
5	0.853 (- 8)	0.648 (- 7)	0.237 (- 5)	0.312 (- 6)
6	0.297 (- 7)	0.327 (- 6)	0.528 (- 4)	0.380 (- 5)
7	0.820 (- 7)	0.142 (- 5)	0.120 (- 2)	0.444 (- 4)
8	0.192 (- 6)	0.555 (- 5)	0.242 (- 1)	0.442 (- 2)
9	0.399 (- 6)	0.189 (- 4)	0.418	0.363 (- 2)
10	0.754 (- 6)	0.558 (- 4)	0.638 ( 1)	0.256 (- 1)
<u>n = 9</u>				
1	0.439 (-15)	0.709 (-15)	0.119 (-14)	0.902 (-15)
2	0.280 (-12)	0.708 (-12)	0.206 (-11)	0.112 (-11)
3	0.102 (-10)	0.391 (-10)	0.205 (- 9)	0.745 (-10)
4	0.117 (- 9)	0.655 (- 9)	0.657 (- 8)	0.149 (- 8)
5	0.710 (- 9)	0.571 (- 8)	0.127 (- 6)	0.167 (- 7)
6	0.290 (- 8)	0.330 (- 7)	0.235 (- 5)	0.169 (- 6)
7	0.904 (- 8)	0.142 (- 6)	0.535 (- 4)	0.198 (- 5)
8	0.231 (- 7)	0.474 (- 6)	0.126 (- 2)	0.230 (- 4)
9	0.509 (- 7)	0.124 (- 5)	0.266 (- 1)	0.231 (- 3)
10	0.995 (- 7)	0.283 (- 5)	0.484	0.194 (- 2)
<u>n = 10</u>				
1	0.658 (-17)	0.114 (-16)	0.188 (-16)	0.142 (-16)
2	0.898 (-14)	0.227 (-13)	0.646 (-13)	0.351 (-13)
3	0.492 (-12)	0.189 (-11)	0.952 (-11)	0.346 (-11)
4	0.747 (-11)	0.425 (-10)	0.392 (- 9)	0.887 (-10)
5	0.566 (-10)	0.463 (- 9)	0.831 (- 8)	0.109 (- 8)
6	0.267 (- 9)	0.315 (- 8)	0.133 (- 6)	0.956 (- 8)
7	0.101 (- 8)	0.151 (- 7)	0.241 (- 5)	0.893 (- 7)
8	0.296 (- 8)	0.546 (- 7)	0.569 (- 4)	0.104 (- 5)
9	0.742 (- 8)	0.154 (- 6)	0.139 (- 2)	0.121 (- 4)
10	0.164 (- 7)	0.426 (- 6)	0.256 (- 1)	0.103 (- 3)

APPENDIX C

FORTRAN PROGRAM FOR COMPUTING RATIONAL APPROXIMATIONS

Here we describe a FORTRAN program which computes the rational approximations (2.1) and (2.18) for the incomplete gamma function and its special cases. The selection of input data determines the function to be approximated. We also include a description of input data, operating procedures, output, and a listing of the FORTRAN program.

The input data are read in the order  $r, \theta, A, B, NCODE, IOPT, LOPT,$   
and  $\left\{ \begin{array}{l} n \\ \text{or} \\ \text{Error} \end{array} \right\}.$

The value IOPT determines which value in the braces  $\left\{ \begin{array}{l} n \\ \text{or} \\ \text{Error} \end{array} \right\}$  should be read. If the  $n^{\text{th}}$  approximant is desired,  $IOPT = 7$  is entered and the corresponding value in braces,  $n$ , is entered.  $IOPT \neq 7$  instructs the computer to iterate until an integer  $m$  is found such that  $|v^{-1} z^v e^z ||V_m(z, v) - V_{m-1}(z, v)| < \text{Error}$ , and therefore Error should accompany this choice of IOPT. Selection of LOPT offers the choice of computing either  $V_n(z, v)$  or  $S_n(z, v)$  (see (2.1) and (2.18)). If  $LOPT = 9$ , the rational approximate is  $V_n(z, v)$  and if  $LOPT \neq 9$ , the approximate is  $S_n(z, v)$ .

For  $z = re^{i\theta}$  ( $\theta$  in degrees) and  $v = A+iB$ , the following table indicates the values of  $v$  and NCODE to select to compute the approximations to the designated functions. A listing of the FORTRAN program concludes this Appendix.

<u>NCODE</u>	<u>A</u>	<u>B</u>	<u>Function</u>	<u>Expressions Computed</u>	<u>Output</u>
1	A	B	$\int_0^z t v^{-1} e^{t^2} dt = v^{-1} z v e^z \left\{ V_n(z, v) + R_n(z, v) \right\},$ $\operatorname{Re}(v) > 0$	$V_n^*(z, v) = v^{-1} z v e^{z^2} V_n(z, v)$	$\operatorname{Re} \left[ V_n^*(z, v) \right], \operatorname{Im} \left[ V_n^*(z, v) \right]$
2	0.5	0	$\int_0^z e^{-t^2} dt = z e^{-z^2} \left\{ V_n(z^2 e^{-i\pi}, \frac{1}{2}) + R_n(z^2 e^{-i\pi}, \frac{1}{2}) \right\}$	$Erf(z) = z e^{-z^2} V_n(z^2 e^{-i\pi}, \frac{1}{2})$ and $Erfc(z) = \frac{1}{2} \pi^{\frac{1}{2}} - Erf(z)$	$\operatorname{Re} \left[ Erf^*(z) \right], \operatorname{Im} \left[ Erf^*(z) \right]$ $\operatorname{Re} \left[ Erfc^*(z) \right], \operatorname{Im} \left[ Erfc^*(z) \right]$
3	0.5	0	$\int_0^z e^{-t^2/2} dt = z e^{-z^2/2} \left\{ V_n(z^2 e^{-i\pi/2}, \frac{1}{2}) + R_n(z^2 e^{-i\pi/2}, \frac{1}{2}) \right\}$	$a(z) = z e^{-z^2/2} V_n(z^2 e^{-i\pi/2}, \frac{1}{2})$ and $a^*(z) = (\pi/2)^{\frac{1}{2}} - a(z)$	$\operatorname{Re} \left[ a(z) \right], \operatorname{Im} \left[ a(z) \right]$ $\operatorname{Re} \left[ a^*(z) \right], \operatorname{Im} \left[ a^*(z) \right]$
4	0.5	0	$C(z) = \int_0^z t^{-\frac{1}{2}} \cos t dt = \frac{1}{2} (J_1 + J_2),$ $J(z) = \int_0^z t^{-\frac{1}{2}} \sin t dt = \frac{1}{2\sqrt{1}} (J_1 - J_2),$ $C(z) = \int_z^\infty t^{-\frac{1}{2}} \cos t dt = (\pi/2)^{\frac{1}{2}} - C(z),$	$J_1^* = 2z^{\frac{1}{2}} e^{iz} V_n(\frac{1}{2}, iz),$ $J_2^* = 2z^{\frac{1}{2}} e^{-iz} V_n(\frac{1}{2}, -iz),$ $C^*(z) = \frac{1}{2} (J_1^* + J_2^*),$	$\operatorname{Re} \left[ C^*(z) \right], \operatorname{Im} \left[ C^*(z) \right]$ $\operatorname{Re} \left[ J^*(z) \right], \operatorname{Im} \left[ J^*(z) \right]$ $\operatorname{Re} \left[ C^*(z) \right], \operatorname{Im} \left[ C^*(z) \right]$
			and		
			$S(z) = \int_z^\infty t^{-\frac{1}{2}} \sin t dt = (\pi/2)^{\frac{1}{2}} - J(z)$	$J^*(z) = \frac{1}{2\sqrt{1}} (J_1^* - J_2^*),$	$\operatorname{Re} \left[ S^*(z) \right], \operatorname{Im} \left[ S^*(z) \right]$

<u>NCODE</u>	<u>A</u>	<u>B</u>	<u>Function</u>	<u>Expressions Computed</u>	<u>Output</u>
4	0.5	0	where		
(concluded)			$J_1 = 2z^{\frac{1}{2}}e^{iz} \left\{ V_n(z e^{i\pi/2}, \frac{1}{2}) + R_n(z e^{i\pi/2}, \frac{1}{2}) \right\}$	$C^*(z) = (\pi/2)^{\frac{1}{2}} - C^*(z)$	
			and		
			$J_2 = 2z^{\frac{1}{2}}e^{-iz} \left\{ V_n(z e^{-i\pi/2}, \frac{1}{2}) + R_n(z e^{-i\pi/2}, \frac{1}{2}) \right\}$	$S^*(z) = (\pi/2)^{\frac{1}{2}} - S^*(z)$	
-----					
5	0.5	0	$U(z) = \int_0^z \cos t^2 dt = \frac{1}{2}(K_1 + K_2)$ ,	$K_1^* = ze^{iz^2} V_n(ze^{i\pi/2}, \frac{1}{2})$ ,	$\text{Re}[U^*(z)]$ , $\text{Im}[U^*(z)]$
			$I(z) = \int_0^z \sin t^2 dt = \frac{1}{2i}(K_1 - K_2)$ ,	$K_2^* = ze^{-iz^2} V_n(ze^{-i\pi/2}, \frac{1}{2})$ ,	$\text{Re}[I^*(z)]$ , $\text{Im}[I^*(z)]$
			$F(z) = \int_z^\infty \cos t^2 dt = \frac{1}{2}(\pi/2)^{\frac{1}{2}} - U(z)$ ,	$U^*(z) = \frac{1}{2}(K_1^* + K_2^*)$ ,	$\text{Re}[F^*(z)]$ , $\text{Im}[F^*(z)]$
			and		
			$L(z) = \int_z^\infty \sin t^2 dt = \frac{1}{2}(\pi/2)^{\frac{1}{2}} - I(z)$	$I^*(z) = \frac{1}{2i}(K_1^* - K_2^*)$ ,	$\text{Re}[L^*(z)]$ , $\text{Im}[L^*(z)]$
			where	$F^*(z) = \frac{1}{2}(\pi/2)^{\frac{1}{2}} - U^*(z)$ , and	
			$K_1 = ze^{iz^2} \left\{ V_n(z^2 e^{i\pi/2}, \frac{1}{2}) + R_n(z^2 e^{i\pi/2}, \frac{1}{2}) \right\}$	$L^*(z) = \frac{1}{2}(\pi/2)^{\frac{1}{2}} - I^*(z)$	
			and		
			$K_2 = ze^{-iz^2} \left\{ V_n(z^2 e^{-i\pi/2}, \frac{1}{2}) + R_n(z^2 e^{-i\pi/2}, \frac{1}{2}) \right\}$		

```

C      RATIONAL APPROX., ASCENDING SERIES
      DIMENSION VR(40), VI(40)
1      READ 400, R, TH, A, B, NCODE, IOPT, KOPT, LOPT,
      PRINT 403, R, TH, A, B
      PI=3.1415926
      THA=TH*PI/180.
      HPI=.5*PI
      RTPI=1.7724538
      NN=1
      X=R*COSF(THA)
      Y=R*SINF(THA)
      RO=LOGF(R)
203   IF (IOPT-7) 101, 102, 101
101   READ 401, ERROR
      ERROR=ERROR*ERROR
      GO TO 103
102   READ 402, NT
      MT=NT+1
103   GO TO (70, 80, 90, 100, 110), NCODE
90    CM=.5
      GO TO 81
80    CM=1.0
81    RZ=(Y*Y-X*X)*CM
      RZI=-2.*X*Y*CM
      TH=THA+RZI
      C=COSF(TH)
      D=SINF(TH)
      DEN=R*EXPF(RZ)
      CR=C*DEN
      CI=D*DEN
      X=RZ
      Y=RZI
      GO TO 300
70    C=A*RO-B*THA+X
      DN=1./(A*A+B*B)
      TH=B*RO+A*THA+Y
      DN=EXPF(C)*DN
      CR=DN*(A*COSF(TH)+B*SINF(TH))
      CI=DN*(A*SINF(TH)-B*COSF(TH))
      GO TO 300
100   THA=.5*THA
      C=X
      X=-Y
      Y=C
105   DN=SQRTF(R)*EXPF(X)
      CR=DN*COSF(THA+Y)
      CI=DN*SINF(THA+Y)
      GO TO 300
110   C=X*X-Y*Y
      D=-2.*X*Y

```

```

Y=C
X=D
106 DN=R*EXPF(X)
CR=DN*COSF(THA+Y)
CI=DN*SINF(THA+Y)
GO TO 300
60 IF (NN-2) 61,62,62
61 NN=2
X=-X
Y=-Y
CC=C
DD=D
IF(NCODE-4)106,105,106
300 IF (LOPT-9) 202,205,202
202 A=A-1.
205 AI=(A+1.)*(A+1.)+B*B
AI=1./AI
IF(LOPT-9)11,10,11
10 P1=1.
S2=A+2.+X
S1=1.
P2=(A+2.)-((A+1.)*X+B*Y)*AI
Q1=0.
Q2=B+(B*X-Y*(A+1.))*AI
T1=0.
T2=B+Y
GO TO 301
11 P1=0.0
P2=A+2.
Q1=0.0
Q2=B
S1=1.0
S2=X+A+2.
T1=0.0
T2=Y+B
301 J=1
VR(1)=(P2*S2+Q2*T2)/(S2*S2+T2*T2)
VI(1)=(Q2*S2-P2*T2)/(S2*S2+T2*T2)
302 K=J+1
F=J
G=K
H=2.*F+A
AL=H*(G+A)-B*B
BL=B*(H+G+A)
C=(H+1.)*(H*(H+2.)-B*B+A*X-B*Y)
C=C-B*(B*X+A*Y+B*(2.*H+2.))
D=B*(H*(H+2.)-B*B+A*X-B*Y)
D=D+(H+1.)*(B*X+A*Y+B*(2.*H+2.))
E=F*((X*X-Y*Y)*(H+2.)-2.*B*X*Y)
FE=F*(2.*X*Y*(H+2.))+B*(X*X-Y*Y)
DI=1./(AL*AL+BL*BL)

```

```

ACPBD=AL*C+BL*D
BCMAD=BL*C-AL*D
AEPBF=AL*E+BL*FE
BEMAF=BL*E-AL*FE
P3=DN*(P2*ACPBD+Q2*BCMAD+P1*AEPBF+Q1*BEMAF)
Q3=DN*(-P2*BCMAD+Q2*ACPBD-P1*BEMAF+Q1*AEPBF)
S3=DN*(S2*ACPBD+T2*BCMAD+S1*AEPBF+T1*BEMAF)
T3=DN*(-S2*BCMAD+T2*ACPBD-S1*BEMAF+T1*AEPBF)
Q1=Q2
Q2=Q3
P1=P2
P2=P3
S1=S2
S2=S3
T1=T2
T2=T3
L=K+1
IF (IOPT-7) 304, 303, 304
303 IF (NT-L) 304, 304, 305
305 J=J+1
GO TO 302
304 DEN=1. / (S2*S2+T2*T2)
VR(K)=(P2*S2+Q2*T2)*DEN
VI(K)=(Q2*S2-P2*T2)*DEN
306 IF (IOPT-7) 308, 307, 308
307 IF (NT-K) 308, 308, 305
308 RER=VR(K)-VR(K-1)
REI=VI(K)-VI(K-1)
IF (IOPT-7) 309, 310, 309
309 C=CR*RER-CI*REI
D=CI*RER+CR*REI
ERT=C*C+D*D
IF (ERT-ERROR) 310, 310, 305
310 C=CR*VR(K)-CI*VI(K)
D=CI*VR(K)+CR*VI(K)
GO TO (30, 40, 50, 60, 60), NCODE
30 PRINT 500,
PRINT 501, K, C, D
GO TO 68
40 PRINT 502,
PRINT 501, K, C, D
C=.88622692-C
D=-D
PRINT 501, K, C, D
GO TO 68
50 PRINT 502,
PRINT 501, K, C, D
C=1.2533141-C
D=-D
PRINT 501, K, C, D
GO TO 68

```

```

62  CM=1.
    IF (ICODE-4) 64,65,64
64  CM=.5
65  SIZR=CM*(DD-D)
    SIZI=CM*(C-CC)
    CIZR=CM*(C+CC)
    CIZI=CM*(D+DD)
    PRINT 505,
    PRINT 506,K,SIZR,SIZI,CIZR,CIZI
    DN=1.2533141
    IF (ICODE-5) 66,67,66
67  DN=.5*DN
66  CIZR=DN-CIZR
    SIZR=DN-SIZR
    CIZI=-CIZI
    SIZI=-SIZI
    PRINT 506,K,SIZR,SIZI,CIZR,CIZI
68  IF (SENSE SWITCH 1) 69,1
69  PRINT 501,K,VR(K),VI(K)
    GO TO 1
400  FORMAT(4F12.0,4I3)
401  FORMAT(E14.7)
402  FORMAT(I3)
403  FORMAT(4(E14.7,2X)/)
500  FORMAT(15H INC. GAMMA FN./)
501  FORMAT(13,1X,2(E14.7,2X)/)
502  FORMAT(15H ERROR FUNCTION/)
505  FORMAT(18H FRESNEL INTEGRALS/)
506  FORMAT(13,2X,4(E14.7,2X)/)
    END

```

APPENDIX D

FORTRAN PROGRAM FOR COMPUTATION OF THE EXPONENTIAL  
AND CIRCULAR FUNCTIONS

The following FORTRAN program is based on the results of Section IV.

Input: (Cards) Let  $n$  be the order of the rational approximations to the exponential and circular functions, and suppose that these approximations are desired for  $n = n_1(1)n_k$  and  $x = x_1(\Delta x)x_m$ . The data are entered in the order  $n_1, n_k, x_1, x_m, \Delta x$ .

Switch settings: None.

Output: (Printed) The output is of the following form for  $x_1, x_2, \dots, x_m$ .

$$e_{n_1}^{(x_1)}, \sin_{n_1}(x_1), \cos_{n_1}(x_1), \tan_{n_1}(x_1)$$

$$e_{n_2}^{(x_1)}, \sin_{n_2}(x_1), \cos_{n_2}(x_1), \tan_{n_2}(x_1)$$

-----

$$e_{n_k}^{(x_1)}, \sin_{n_k}(x_1), \cos_{n_k}(x_1), \tan_{n_k}(x_1)$$

```

C   RATIONAL APPROX. TO EXP(-Z), COS(Z), SIN(Z), TAN(Z)
1   READ 100,NI,NF,ZI,ZF,ZD
    PRINT 200
    Z=Z1
2   ZP=Z*Z
    Z1=-ZP
    N=2
    FM=2.0
    CM=2.*(2.*FM+1.)
    A1P=2.0
    A2P=12.0+ZP
    B1P=1.0
    B2P=6.0
    A1M=2.0
    A2M=12.0+ZM
    B1M=1.0
    B2M=6.0
3   N=N+1
    A3P=CM*A2P+ZP*A1P
    A3M=CM*A2M+ZM*A1M
    B3P=CM*B2P+ZP*B1P
    B3M=CM*B2M+ZM*B1M
    A1P=A2P
    A2P=A3P
    B1P=B2P
    B2P=B3P
    A1M=A2M
    A2M=A3M
    B1M=B2M
    B2M=B3M
    FM=FM+1.0
    CM=2.*(2.*FM+1.0)
    IF (NI-N) 4,4,3
4   EZ=(A3P-Z*B3P)/(A3P+Z*B3P)
    D=A3M*A3M+ZP*B3M*B3M
    CZ=(A3M*A3M+ZM*B3M*B3M)/D
    SZ=2.*Z*A3M*B3M/D
    D=SZ/CZ
    PRINT 300,N,Z,EZ,SZ,CZ,D
    IF (NF-N) 1,5,3
5   IF (ZF-Z) 1,1,6
6   Z=Z+ZD
    GO TO 2
100  FORMAT(2(I3),3(E15.7))
200  FORMAT(//2H N,7X1HZ,13X7HEXP(-Z),9X6HSIN(Z),10X6HCOS(Z)
1,10X6HTAN(Z)/)
300  FORMAT(12,5(2XE14.7))
    END

```

## APPENDIX E

### FORTRAN PROGRAM FOR COMPUTING RATIONAL APPROXIMATIONS TO $E(z)$

Here we give a brief description of a FORTRAN program which computes the rational approximations to  $E(z)$  defined by (5.11). A listing of the program follows the description of input and output data.

Let  $z = re^{i\theta}$  and suppose we wish to compute the  $n^{\text{th}}$  order rational approximation to  $E(z)$  as defined by (5.11) for  $r_I(\Delta r)r_F$  and  $\theta_I(\Delta\theta)\theta_F$  where I and F subscripts denote initial and final values, respectively.

Input: (Cards) The values  $r_I$ ,  $r_F$ ,  $\Delta r$ ,  $\theta_I$ ,  $\theta_F$  and  $\Delta\theta$  are entered in this order ( $\theta$  in degrees), three numbers per card.

Output: (Typed) For each pair of values of  $r$  and  $\theta$ , the  $n^{\text{th}}$  order rational approximation is printed, real part and then the imaginary part.

Switch settings: None.

```

C      RATIONAL APPROXIMATIONS TO E(Z)
      DIMENSION F1(30),F2(30),F3(30),F4(30),P1(30),
1      P2(30),P3(30),P4(30)
      READ 100,RI,RF,RD,THI,THF,THD
      CF=3.141592653589793238462643/180.
      DO 2 J=1,30
          F1(J)=0.0
          F2(J)=0.0
          F3(J)=0.0
          F4(J)=0.0
          P1(J)=0.0
          P2(J)=0.0
          P3(J)=0.0
2      P4(J)=0.0
          F3(1)=1.0
          F3(2)=24.0
          F3(3)=180.0
          F3(4)=480.0
          F2(1)=1.0
          F2(2)=12.0
          F2(3)=36.0
          F1(1)=1.0
          F1(2)=4.0
          P3(1)=0.0
          P3(2)=17./3.
          P3(3)=60.0
          P3(4)=480.0
          P2(1)=0.0
          P2(2)=3.0
          P2(3)=36.0
          P1(1)=0.0
          P1(2)=4.0
          N=4
3      M=N+1
          H=N
          TWH=2.*H
          A1=(H-2.)*(TWH-1.)/(H*(TWH-3.))
          A2=1.0
          A3=-A1
          B1=2.*(TWH-1.)*(H+1.)/H
          B2=-2.*(TWH-1.)*(H-3.)/H
          F4(1)=A1*F3(1)+A2*F2(1)+A3*F1(1)
          P4(1)=A1*P3(1)+A2*P2(1)+A3*P1(1)
          DO 4 J=2,M
              F4(J)=A1*F3(J)+B1*F3(J-1)+A2*F2(J)+B2*F2(J-1)+A3*F1(J)
4          P4(J)=A1*P3(J)+B1*P3(J-1)+A2*P2(J)+B2*P2(J-1)+A3*P1(J)
          DO 5 J=1,M
              F1(J)=F2(J)
              F2(J)=F3(J)
              F3(J)=F4(J)
              P1(J)=P2(J)

```

```

5     P2(J)=P3(J)
6     P3(J)=P4(J)
7     R0=R1
7     R=1./R0
      TH=TH1
8     PRINT 101,R0,TH
      T=TH*CF
      RE=R*COSF(T)
      RIM=-R*SINF(T)
      SRN=P4(M)*RE+P4(M-1)
      SIN=P4(M)*RIM
      SRD=F4(M)*RE+F4(M-1)
      SID=F4(M)*RIM
      DO 9 J=3,M
      L=M-J+1
      S=RE*SRN-RIM*SIN+P4(L)
      SIN=RE*SIN+RIM*SRN
      SRN=S
      S=RE*SRD-RIM*SID+F4(L)
      SID=RE*SID+RIM*SRD
9     SRD=S
      DEN=SRD*SRD+SID*SID
      QR=(SRN*SRD+SIN*SID)/DEN
      QI=(SIN*SRD-SRN*SID)/DEN
      PRINT 104,QR,QI
      TH=TH+THD
10    IF (THF-TH)10,8,8
      R0=R0+RD
12    IF (RF-R0)12,7,7
13    CONTINUE
      N=N+1
      GO TO 3
100   FORMAT (3E15.7)
101   FORMAT(E14.7,2X,E14.7)
104   FORMAT(2(E32.24))
      END

```

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