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CONSISTENT ESTIMATION OF CONTINUOUS-TIME SIGNALS FROM QUANTIZED NOISY SAMPLES

by

Elias Masry and Stamatis Cambanis

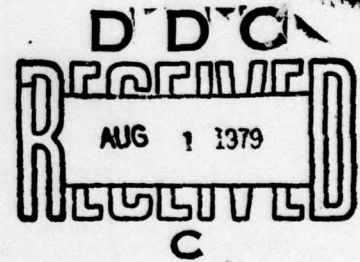
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Abstract

It is well known that a continuous-time signal $f(t)$, $-\infty < t < \infty$, cannot be reconstructed from its 2-level quantized version $\text{sgn}[f(t)]$. It is shown that by deliberately corrupting equally-spaced samples $\{f(k/W)\}$ of f by additive Gaussian noise $\{\xi_k\}$ before hardlimiting, the signal f can be estimated consistently from the binary sequence $\{\text{sgn}[f(k/W) + \xi_k]\}$ as the sampling rate $W \rightarrow \infty$. A class of nonlinear estimates is introduced and bounds on the mean-square error are obtained. The signal f need not be bandlimited.

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CHRISTIAN ESTIMATION OF CONTINUOUS-TIME SIGNALS
FROM QUANTIZED NOISY SAMPLES



It is well known that a continuous-time signal $f(t)$ cannot be reconstructed from its N -level quantized version $g(t)$. It is shown that by deliberately corrupting equally-spaced samples of f by additive Gaussian noise $n(t)$ before quantization, the signal f can be estimated consistently from the binary sequence $g(t) + n(t)$ as the sampling rate $N \rightarrow \infty$. A class of nonlinear estimators is introduced and bounds on the mean-square error are obtained. The signal f need not be bandlimited.

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I. INTRODUCTION

Let $f(t)$, $-\infty < t < \infty$, be a real continuous function. It is well known that, in general, f cannot be determined from $\text{sgn}[f(t)]$. This situation remains true even when f is analytic, such as a bandlimited function [1]. We recall that for a bandlimited function

$$(1) \quad f(t) = \int_{-M}^M e^{it\lambda} f(\lambda) d\lambda, \quad f(\lambda) \in L_1[-M, M],$$

we have by Titchmarsh [1] that f can be represented by the conditionally convergent series

$$(2) \quad f(t) = f(0) \prod_{n=1}^{\infty} \left(1 - \frac{t}{z_n}\right)$$

where $f(0) \neq 0$ and $\{z_n\}$ is the set of all (real and complex) zeros of $f(z)$.

$z = t + iu$, in the complex plane. Thus f is determined, up to a multiplicative constant, by its real and complex zeros; the complex zeros, however, are not observable. Duffin and Schaeffer [2] have shown in 1938 that by subtracting a cosine function $C \cos Wt$ from f , the resulting function has real zeros only. Unaware of Duffin and Schaeffer's result, Bar-David [3] reproduced a weaker version of it. Duffin and Schaeffer's result is given by

Proposition 1 [2]. Let $f(z)$, $z = t + iu$, be an entire function of exponential type with exponent W , $f(z) = O(e^{W|z|})$, such that $|f(t)| \leq 1$. Then the function $g(z) = C \cos Wz - f(z)$, $C > 1$, has real simple zeros only.

Substituting z for t in (1), we have an entire function of exponential type with exponent W and $|f(t)| \leq A$, $A = \int_{-M}^M |f(\lambda)| d\lambda$. It follows by Proposition 1 that for a bandlimited function f given by (1) we have, with $C > A$, that $g(t) = C \cos Wt - f(t)$ has real zeros $\{t_k\}$ only and

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The practical significance of (3) for digital transmission of continuous-time signals f is doubtful, since (i) a synchronous tone $C \cos \omega t$ is needed at the receiver and (ii) no digital reconstruction scheme of f based on $\text{sgn}[g(t)]$ is available.

A new approach is presented in this paper. We do not assume that f is bandlimited, and we do reconstruct f from the sign of deliberately corrupted time-samples of f , as the sampling rate tends to infinity. The approach is motivated by the results of a recent paper [4] by the present authors.

so that

$$g(t) = g(0) \prod_{k=1}^{\infty} \left(1 - \frac{t}{t_k}\right) = [C - f(0)] \prod_{k=1}^{\infty} \left(1 - \frac{t}{t_k}\right)$$

$$(3) \quad f(t) = C \cos \omega t - [C - f(0)] \prod_{k=1}^{\infty} \left(1 - \frac{t}{t_k}\right)$$

II. NEW RECONSTRUCTION SCHEME

Consider the diagram depicted in Figure 1. A continuous-time signal f is sampled at equally-spaced points $\{f(\frac{n}{W})\}_{n=-\infty}^{\infty}$, where W is the n^{th} sampling rate, $W \rightarrow \infty$ as $n \rightarrow \infty$. Each sample $f(\frac{n}{W})$ is deliberately corrupted by an additive Gaussian variate $\epsilon_{n,k}$, i.e.,

$$(4) \quad y_{n,k} = f(\frac{n}{W}) + \epsilon_{n,k} \quad k = \dots, -1, 0, 1, \dots$$

where for each fixed n , $\{\epsilon_{n,k}\}_{k=-\infty}^{\infty}$ is a sequence of independent identically distributed Gaussian random variables with means zero and variances σ^2 . Only the sign of $\{y_{n,k}\}_{k=-\infty}^{\infty}$ is transmitted, i.e.,

$$(5) \quad z_{n,k} = \text{sgn}[y_{n,k}] = \text{sgn}[f(\frac{n}{W}) + \epsilon_{n,k}], \quad k = \dots, -1, 0, 1, \dots$$

At the receiver, the binary sequence $\{z_{n,k}\}_{k=-\infty}^{\infty}$ is used to obtain a consistent estimate $\hat{f}_n(t)$ of $f(t)$, i.e., $\hat{f}_n(t)$ converges to $f(t)$ in quadratic mean as the sampling rate $W \rightarrow \infty$. Note that without the additive noise $\{\epsilon_{n,k}\}_{k=-\infty}^{\infty}$ it is not possible to reconstruct $f(t)$ from $\{\text{sgn}[f(\frac{n}{W})]\}_{k=-\infty}^{\infty}$ as $n \rightarrow \infty$, i.e., from $\text{sgn}[f(t)]$, $-\infty < t < \infty$.

We shall provide reconstruction procedures for the following class of signals f .

Definition. Let $U(c_0)$ be the class of real bounded uniformly continuous functions f on a finite or infinite interval $I = [a, b]$, $-\infty < a < b < \infty$, such that $|f(t)| \leq c_0$ for all $t \in I$.

The structure of the receiver is as follows: Let

$$(6) \quad u(x) = \sqrt{\frac{\pi}{2}} \int_0^x e^{-u^2/2} du, \quad -\infty < x < \infty,$$

for Bernstein's interpolation kernel. (Here M_n is an integer.)

$$(9b) \quad h_k(n, t) = \binom{M_n}{k} t^k (1-t)^{M_n-k}, \quad k = 0, 1, \dots, M_n \\ t \in I = [0, 1]$$

for Szász's interpolation kernel, and

$$(9a) \quad h_k(n, t) = \frac{e^{-M_n t} (M_n t)^k}{k!}, \quad k = 0, 1, \dots, \quad t \in I = [0, \infty)$$

in [5]. We have

tain bounds on the mean square error. More general kernels are discussed. We now consider two specific sequences of kernels $\{h_k(n, t)\}_{k=-\infty}^{\infty}$ and ob-

each $t \in I$.

Then for every $f \in U(C_0)$ we have $\hat{f}_n(t) \rightarrow f(t)$, in quadratic mean as $n \rightarrow \infty$, for

$$i. \quad \sum_{k=-\infty}^{\infty} h_k^2(n, t) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

$$ii. \quad \sum_{k=-\infty}^{\infty} \left(t - \frac{M_n}{k}\right)^2 h_k(n, t) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

$$i. \quad \sum_{k=-\infty}^{\infty} h_k(n, t) = 1, \quad n = 1, 2, \dots$$

Theorem 1. Assume the kernels $\{h_k(n, t)\}_{k=-\infty}^{\infty}$ satisfy for each $t \in I$

The proofs of the following results can be found in [5].

$$(8) \quad \hat{f}_n(t) = \begin{cases} 0 & |m_n(t)| > \mu(c) \\ \mu^{-1}[m_n(t)] & |m_n(t)| \leq \mu(c) \end{cases}$$

Let $c = c_0 + n$, $n > 0$, and set

where $\{h_k(n, t)\}_{k=-\infty}^{\infty}$ is a sequence of positive kernels to be specified below.

$$(7) \quad \hat{m}_n(t) = \sum_{k=-\infty}^{\infty} Z_{n,k} h_k(n, t), \quad t \in I$$

and note that μ is odd and strictly monotonic on $(-\infty, \infty)$. Let

$$A_1(c) = \frac{\pi c}{8} A_2(c)$$

$$A_2(c) = \left[\frac{\pi c}{2} + \frac{\pi n}{2} \right]$$

We remark that the variance σ^2 of the Gaussian variates $\{\xi_{n,k}\}$ is completely arbitrary. It should be clear on intuitive grounds that as the bound c_0 of f increases, so must σ^2 . It is seen from Theorem 2 that only affects the values of the constants $A_i(c)$, $i = 1, 2$. It turns out that a nearly optimal choice for σ is $\sigma = c = c_0 + n$ for which

probability one as $n \rightarrow \infty$, uniformly on compact subsets of $(0, \infty)$.

c. If $f \in \text{Lip } 1$ and $W_n = O(n^a)$, $a > 2$, then $\hat{f}_n(t)$ converges to $f(t)$ with

$$E[\hat{f}_n(t) - f(t)]^2 = O(W_n^{-\min(1/2, \alpha)})$$

b. If $f \in \text{Lip } \alpha$, $0 < \alpha \leq 1$, then

subsets of $(0, \infty)$.

a. $\hat{f}_n(t) \rightarrow f(t)$ in quadratic mean as $n \rightarrow \infty$ uniformly on compact

Corollary. Under the assumptions of Theorem 2,

$i = 1, 2$, are constants independent of n (depending on c and σ^2 only).
modified Bessel function of the first kind of order zero and $A_i(c)$,
where $w(f, \delta)$ is the modulus of continuity of f over $[0, \infty)$, $I_0(x)$ is the

$$A_2(c) e^{-2W_n t} I_0(2W_n t)$$

$$E[\hat{f}_n(t) - f(t)]^2 \leq A_1(c) \omega(f, \sqrt{t/W_n}) +$$

polation kernel (9a). Then for every $t \in [0, \infty)$

Theorem 2. Let $f \in U(c_0)$ and $\hat{f}_n(t)$ be given by (8) with Szász's inter-

Similar results can be obtained for the estimate (8) with Bernstein's

kernel (9b). We have

Theorem 3. Let f be continuous on $[0, 1]$ such that $|f(t)| \leq c_0$ for all $t \in [0, 1]$. Then for every $t \in [0, 1]$ the estimate (8) with Bernstein's

kernel (9b) satisfies

$$E[\hat{f}_n(t) - f(t)]^2 \leq A_1(c) \omega_2(f, \sqrt{t(1-t)/W_n}) + A_2(c) [W_n t(1-t)]^{-1/2(1+\alpha(1))}$$

where $\omega(f, \delta)$ is the modulus of continuity of f over $[0, 1]$ and $A_i(c)$, $i = 1, 2$, are constants independent of n .

Corollary. Under the assumptions of Theorem 3,

- a. $\hat{f}_n(t) \rightarrow f(t)$ in quadratic mean as $n \rightarrow \infty$ uniformly on $[a, b] \subset (0, 1)$.
- b. If $f \in \text{Lip } \alpha$, $0 < \alpha \leq 1$, then

$$E[\hat{f}_n(t) - f(t)]^2 = O(W_n^{-\min(1/2, \alpha)}) \text{ uniformly on } [a, b] \subset (0, 1).$$

- c. If $f \in \text{Lip } 1$ and $W_n = O(n^a)$, $a > 2$, then $\hat{f}_n(t) \rightarrow f(t)$ with probability one as $n \rightarrow \infty$, uniformly on $[a, b] \subset (0, 1)$.

We remark that the interval $[0, 1]$ in Theorem 3 can be replaced by any

finite interval $[a, b]$ by proper scaling.

We also note that the Gaussian density of the variates $\{\varepsilon_{n,k}\}$ in (4)

can be replaced by other symmetric densities which are positive over $(-\infty, \infty)$. For example, one could use Laplacian density $\phi(x) = \frac{\alpha}{2} e^{-\alpha|x|}$ for which $u(x) = (1 - e^{-\alpha|x|}) \text{sgn } x$.

We finally remark that the more general case where the nonlinearity

$\text{sgn } x$ in Figure 1 is replaced by a, possibly nonmonotonic, general non-

linearity is discussed in [5].

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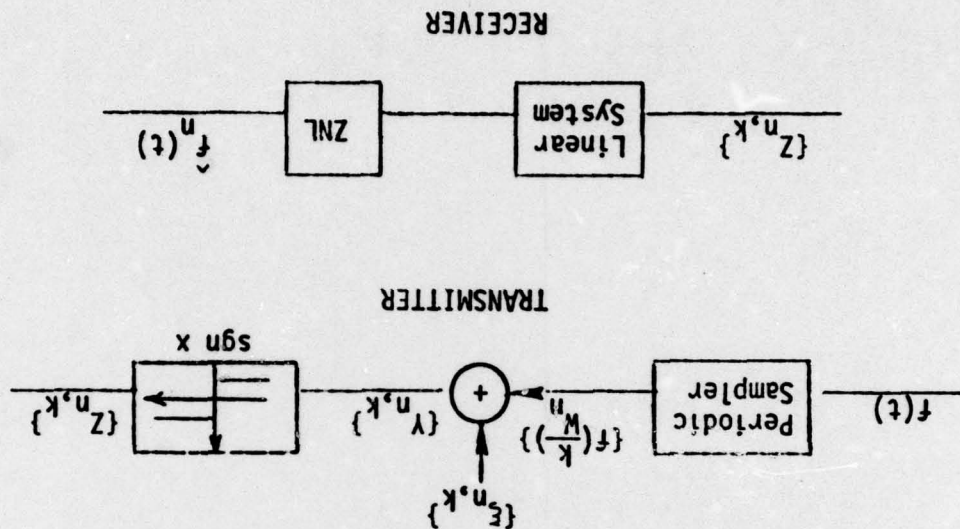


Figure 1. Transmitter/Receiver Structure

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