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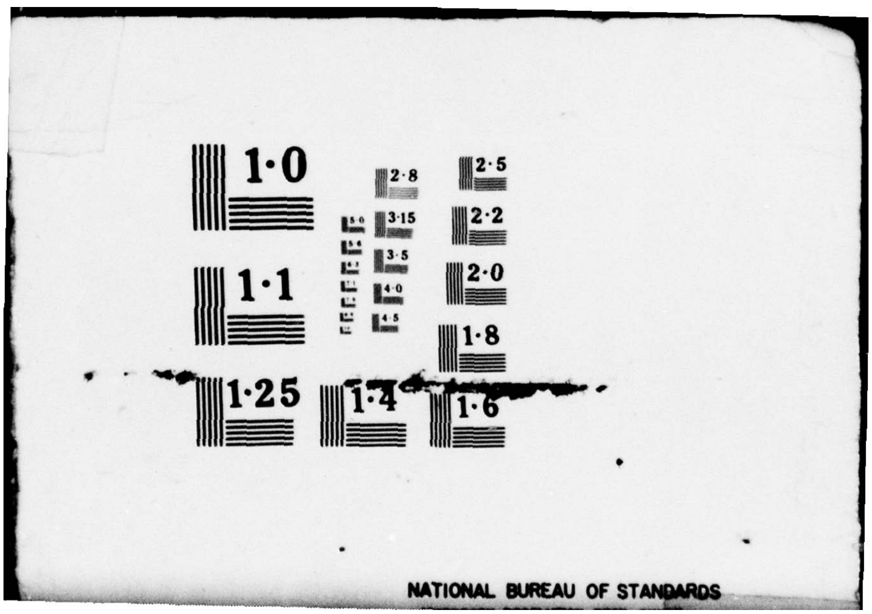
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THE ROLE OF THE ASYMMETRIC TERMS OF THE DIVERGENCE AND VORTICITY EQUATIONS IN THE ZERO LAPLACIAN VORTEX

by

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1.0 Introduction

A model of the axially symmetric vortex has been proposed by Nicholson and Johnson (1978), which describes the vortex in terms of its vorticity distribution as a function of radius. The vortex model is a composite of three lateral regimes in which the vorticity distribution is a piecewise smooth and continuous function of radius. This distribution along with a hypothesized divergence distribution is shown in Figure 1a, with the concurrent tangential and radial velocity distributions in 1b.

Nicholson and Johnson (1978) present data from hurricanes, tornadoes and waterspouts to demonstrate that the model is a good and reasonable approximation to the tangential wind above the boundary layer. This distribution satisfies Laplace's equation, hence the name, the zero Laplacian vortex. Examples of the fit of the velocity to the data are given below in Figures 2a and 2b.

Satisfaction of Laplace's equation by the vorticity distribution suggests that the vorticity may be part of an harmonic function (the imaginary part) and that the real part of the function, the divergence, also satisfies Laplace's equation, while observing matching boundary conditions.

This assumption implies that a distribution of divergence accompanies the vorticity distribution. Assuming, as a first approximation, that the atmosphere is horizontally incompressible, then from continuity the vertical velocity distribution may also be postulated.

Laplace's equation is a classic elliptic equation which has complex characteristics. The axially symmetric solution (linearity in ℓnr) is not the only solution to Laplace's equation. The asymmetric solution is the product of an exponential and periodic function such that the Laplacian in Cartesian coordinates is given by

$$\nabla_2^2 a = \frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2} \qquad (1)$$

then if

$$a = \gamma(n)e^{nx} \cos ny \text{ or}$$
$$a = \gamma(n)e^{ny} \cos nx \text{ or}$$
$$a = \gamma(n)e^{nx} \sin ny \text{ or}$$
$$a = \gamma(n)e^{ny} \sin nx \qquad (2)$$

where n may be an integer in a Fourier series, and $\gamma(n)$ is a Fourier amplitude dependent on n,

$$\nabla_2^2 a = 0 \qquad (3)$$

The trick is to ascertain the relevance of the solution to the phenomena of the asymmetric vortex. This will be addressed in the following section.

2.0 Spiral Cloud Bands

It has long been established that the spiral cloud bands in hurricanes occur in the form of equiangular or logarithmic spirals. That is to say

$$\tan \alpha = \frac{r\partial\theta(r)}{\partial r} = \frac{\partial\theta(r)}{\partial \ell n r} = \text{constant} \qquad (4)$$

where α is the angle a log spiral makes with a ray emanating from the center of a polar coordinate system.

Since a log spiral satisfies (4) then it should plot in the semilog coordinates, θ, $\ell n r$ as a straight line. Interestingly enough when multiple cloud bands are plotted in this coordinate system they not only form straight lines, but are parallel to one another and exhibit a certain amount of periodicity. Since the bands increase in intensity in a direction orthogonal to the periodicity one wonders whether they may exhibit behavior which would satisfy (2) and (3).

In Cartesian coordinates it is axiomatic that rotation of coordinates implies a direct transformation of the form of the Laplacian from the old coordinate axes to the new ones (Figure 3). The Laplacian before rotation is expressed as

$$\nabla_2^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{5}$$

and after rotation as

$$\nabla_2^2 = \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} \tag{6}$$

In polar coordinates the Laplacian is written

$$\nabla_2^2 = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \tag{7}$$

By multiplying the radial term by r/r, recombining and factoring out $1/r^2$ the Laplacian may be rewritten as

$$\nabla_2^2 = \frac{1}{r^2} \left(\frac{\partial^2}{\partial \ell n r^2} + \frac{\partial^2}{\partial \theta^2} \right) \tag{8}$$

Laplace's equation may dispense with the $1/r^2$ to give

$$r^2 \nabla_2^2 \, a = \left(\frac{\partial^2}{\partial \ell n r^2} + \frac{\partial^2}{\partial \theta^2} \right) a = 0 \tag{9}$$

Thus, any linear solution in θ and ℓnr satisfies (9) as well as the product of an orthogonal set of exponential and periodic functions. As can be seen in (6) a rotation of the coordinate system θ, ℓnr by angle α in Figure 4 would give a new expression for Laplace's equation

$$r^2 \nabla_2^2 a = \left(\frac{\partial^2}{\partial S_r^2} + \frac{\partial^2}{\partial S_\theta^2} \right) a$$

where S_r and S_θ are the new coordinate axes illustrated in Figure 4. This rotation brings the cloud bands parallel to S_θ and periodic in S_r. Since both S_θ and S_r satisfy (4) they are log spirals, and the new coordinate system e^{S_θ} and e^{S_r} are log spiral coordinates. S_r and S_θ are in units of radians. They vary in the model from 0 to 2π, increasing to π and decreasing again to zero.

The scalar transformations between θ, ℓnr and S_θ, S_r are

$$S_r = \ell nr \cos\alpha + \theta \sin\alpha \tag{11a}$$

$$S_\theta = \theta \cos\alpha - \ell nr \sin \alpha \tag{11b}$$

Thus (2) may be rewritten as

$a = \gamma(n) e^{n(S_\theta)} \cos(nS_r)$, etc., being exponential in S_θ and periodic and linear in S_r.

3. Spiral Space

3.1 Polar Coordinates

Spiral space is represented pictorially in Figure 5 below. The spirals are defined by S_r and S_θ being constants.

For

$$S_r = K = \ln r \cos\alpha + \sin\alpha$$

and

$$S_\theta = K' = \cos\alpha - \ln r \sin\alpha$$

the orthogonal log spirals which define spiral space are given by

$$r = \exp\left[\frac{K - \sin\alpha}{\cos\alpha}\right]$$

for constant S_r, i.e., the coordinate along which S_θ only changes and

$$r = \exp\left[\frac{\cos\alpha - K'}{\sin\alpha}\right]$$

for constant S_θ, i.e., the coordinate along which only S_r changes. Thus any point in spiral space may be defined in terms of values of the log spirals e^{S_r} and e^{S_θ}, or simply S_r and S_θ.

3.2 Spherical Coordinates

Spiral space may also be defined in geographic or spherical coordinates, so that all algorithms applicable to polar coordinates may be applied to spherical ones as well. The two dimensional Laplacian in spherical coordinates ϕ, λ is given by

$$\nabla_2^2 = \frac{1}{r^2 \sin\phi} \frac{\partial}{\partial\phi} \sin\phi \frac{\partial}{\partial\phi} + \frac{1}{r^2 \sin^2\phi} \frac{\partial^2}{\partial\lambda^2}$$

rearranging terms gives

$$\nabla_2^2 = \frac{1}{r^2} \sin^2\phi \left[\frac{\partial}{\csc\phi\partial\phi} \frac{\partial}{\csc\phi\partial\phi} + \frac{\partial^2}{\partial\lambda^2} \right]$$

or

$$\nabla_2^2 = \frac{1}{r^2} \sin^2\phi \left[\frac{\partial^2}{\partial(\ln \tan(\frac{\phi}{2}))^2} + \frac{\partial^2}{\partial\lambda^2} \right]$$

Rotating the coordinate system $\ln \tan \frac{\phi}{2}$, λ through the angle α produces

$$\nabla_2^2 = \frac{1}{r^2} \sin^2\phi \left[\frac{\partial^2}{\partial S_\phi^{\ 2}} + \frac{\partial^2}{\partial S_\lambda^{\ 2}} \right]$$

where S_ϕ and S_λ are log spirals on a sphere conforming to the equation

$$\tan \alpha = \frac{r \sin \phi \, \partial\lambda \, (\phi)}{\partial \, \phi} = \frac{r \, \partial \, \lambda(\phi)}{\partial \ell n \, \tan(\frac{\phi}{2})} = \text{constant.}$$

The elliptic solution

$$e^{(S_\theta + i \, S_r)}$$

may be equally applied to

$$e^{(S_\lambda + i \, S_\theta)}$$

where

$$S_\phi = \ell n(\tan \frac{\pi}{2}) \cos \alpha + \lambda \sin \alpha$$

and

$$S_\lambda = \lambda \cos \alpha - \ell n(\tan \frac{\phi}{2}) \sin \alpha$$

The spherical space is defined in terms of spherical log spirals so that

$$\phi = 2 \tan^{-1} (\exp \left[\frac{K - \lambda \sin \alpha}{\cos \alpha} \right])$$

for constant S_ϕ and

$$\phi = 2 \tan^{-1} \left(\exp \left[\frac{\lambda \cos \alpha - K'}{\sin \alpha}\right]\right)$$

for constant S_λ .

For local spherical coordinates, where μ is the angle subtending the radial arc, and ν is the azimuthal angle the transformations between geographic ϕ , λ and local spherical μ , ν coordinates are given by

$$\lambda = \lambda_c + \tan^{-1} (\cos \nu / \cot \mu)$$

$$\phi = \phi_c + \cos^{-1} (\cos \mu / \cos (\lambda - \lambda_c))$$

and

$$\mu = \cos^{-1} (\cos \Delta\phi \cos \Delta\lambda)$$

$$\nu = \cos^{-1} (\cot \mu \tan \Delta\lambda)$$

where subscript "c" is the location of the center of the storm and the spiral coordinates are given by

$$S_\mu = \ln \tan \left(\frac{\mu}{2}\right) \cos \alpha + \nu \sin \alpha$$

$$S_\nu = \nu \cos \alpha - \ln \tan \left(\frac{\mu}{2}\right) \sin \alpha$$

4. Transformations in Complex Space

The transformation from polar to spiral coordinates involves the successive transformations of shortening, rotation, scaling and stretching.

By considering complex space

$$Z = re^{i\theta} \tag{12a}$$

and its conjugate

$$\bar{Z} = re^{-i\theta} \tag{12b}$$

then by the transformation of shortening (operating on the conjugate in the northern hemisphere) we obtain

$$\ln \bar{z} = \ln r - i\theta \tag{13}$$

Scaling by i and n where $i = \sqrt{-1}$ and n is an interger produces

$$ni \ln \bar{z} = n(\theta + i \ln r) \text{ (semilog coordinates)} \tag{14}$$

Rotation by the versor, $e^{i\alpha}$ (equivalent to rotating the semilog coordinates), produces

$$nie^{i\alpha} \ln\bar{z} = n(\theta \pm \ln r)e^{i\alpha} = n(S_\theta + iS_r) \qquad (15)$$

Restretching to obtain real space curvilinear coordinates produces

$$\bar{z}^{\,nie^{i\alpha}} = \exp\left[n(\theta + i \ln r)\, e^{i\alpha}\right] \qquad (16a)$$

$$= \exp\left[n(S_\theta + iS_r)\right] \qquad (16b)$$

The Laplacian in complex coordinates is given by

$$\nabla_2^2 = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \qquad (17)$$

so that

$$4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} (\bar{z}^{nie^{i\alpha}}) = 0 \qquad (18)$$

and 16a and b satisfy Laplace's equation.

Assuming the divergence and vorticity are the real and imaginary parts of an harmonic function, i.e., that they satisfy the Cauchy Riemann equations which in spiral coordinates are

$$\frac{\partial\delta}{\partial S_r} = \frac{\partial\zeta}{\partial S_\theta} , \frac{\partial\zeta}{\partial S_r} = - \frac{\partial\delta}{\partial S_\theta} \quad (19)$$

then we may write the asymmetric solution for divergence and vorticity as

$$\delta_n = e^{nS_\theta}\cos n\, S_r, \ \zeta_n = - e^{nS_\theta}\sin n\, S_r \quad (20)$$

where this solution is $\bar{F}$ where $F = \delta - i\zeta$ so that

$$\mathrm{Re}(\bar{F}) = e^{n(S_\theta)} \cos S_r \quad (21)$$

and

$$\mathrm{Im}(\bar{F}) = -e^{n(S_\theta)} \sin S_r$$

In this case the divergence field leads the vorticity field by $\pi/2$. Thus the full solution for symmetric and asymmetric distributions is

$$\delta = \delta_o + \iota_o \ell nr + \sum_{n=1}^{\infty} \gamma_o(n) e^{n(S_\theta)} \cos n\, S_r \quad (21a)$$

$$\zeta = \zeta_o + \iota_o \ell n r - \sum_{n=1}^{\infty} \gamma_o(n) e^{n(S_\theta)} \sin(n\ S_r) \tag{21b}$$

where r is dimensionless, being understood as r/r_o where r_o = 1L. δ_o, ζ_o, ι_o and γ_o are constants of integration derived from boundary conditions.

5. Covariances in the Divergence and Vorticity Equations

In spiral coordinates the divergence and vorticity may be written:

$$\delta = \exp\left(-2(S_r \cos\alpha - S_\theta \sin\alpha)\right) \times$$

$$\left[\frac{\partial}{\partial S_r} \exp\left(S_r \cos\alpha - S_\theta \sin\alpha\right) U_{S_r} + \frac{\partial}{\partial S_\theta} \exp\left(S_r \cos\alpha - S_\theta \sin\alpha\right) U_{S_\theta}\right] \tag{22a}$$

and

$$\zeta = \exp(-2(S_r \cos\alpha - S_\theta \sin \alpha)) \times$$

$$\left[\frac{\partial}{\partial S_r} \exp (S_r \cos\alpha - S_\theta \sin\alpha) U_{S_\theta} \right. \qquad (22b)$$

$$\left. - \frac{\partial}{\partial S_\theta} \exp (S_r \cos\alpha - S_\theta \sin \alpha) U_{S_r} \right]$$

Since

$$\delta_n = \gamma_o(n)e^{n(S_\theta)} \cos nS_r$$

and

$$\zeta_n = \gamma_o(n)e^{n(S_\theta)} \sin n S_r$$

then by substitution and partial integration both U_{S_r} and U_{S_θ} have as part of their form

$$U_n \propto \int e^{ax} \cos px dx \qquad \text{or}$$

$$U_n \propto \int e^{ax} \sin px dx \qquad (23)$$

which integrate to

$$\frac{e^{ax}(a \cos px \quad p \sin px)}{a^2 + p^2} \qquad \text{or}$$

$$\frac{e^{ax}(a \sin px - p \cos px)}{a^2 + p^2} \tag{24}$$

respectively. Hence both U_{S_r} and U_{S_θ} are out of phase by a non multiple of $\pi/2$ with the divergence and vorticity fields since in the velocity fields there exists the sum of sine and cosine terms while the divergence and vorticity fields are represented by sine and cosines respectively. (In Table I, found in the Appendix, the vorticity and divergence equations are listed term by term.)

As a first order approximation assume the vertical velocity, w, has the same periodic distribution as the negative divergence. The terms then may be viewed as products term by term of a Fourier series. If the terms are in phase then

$$\int_0^{2\pi} \sin^2\theta d\theta = \int_0^{2\pi} \cos^2\theta d\theta = \pi \tag{25}$$

and their azimuthal averages are non zero. If they are out of phase then

$$\int_{o}^{2\pi} \sin\theta \cos \theta d\theta = 0 \tag{26}$$

and their azimuthal averages are zero, i.e., there exist no positive and negative covariances from the asymmetric terms which may contribute substantially to the hurricane budget.

In spiral coordinates the integral over 2π radians around the closed curve

of the real or imaginary part of the function

$$e^{n}(S_\theta + i S_r)$$

can be shown to be zero.

$$\mathrm{Re} \quad \left[e^{n(S_\theta + i S_r)} \right]$$

may be expressed by

$$e^{nS_\theta} \cos(n S_r).$$

The integral over the closed curve above is given by

$$\int_0^{2\pi} \int_0^{2\pi} e^{nS_\theta} \cos(n\ S_r) r\ d\ S_r\ r\ d\ S_\theta \tag{27}$$

In polar coordinates, where $\alpha = 0$, $r\ d\ S_r\ r\ d\ S_\theta$ reverts to $r\ d\ \ell nr\ r\ d\ \theta$, or $r\ d\ r\ d\ \theta$.

By substituting

$$r^2 = \exp 2(S_r \cos \alpha - S_\theta \sin \alpha)$$

into (27) we obtain

$$\int_0^{2\pi} \int_0^{2\pi} e^{nS_\theta} \cos(n\ S_r) e^{2\ S_r \cos\alpha} e^{-2\ S \sin\alpha} d\ S_r\ d\ S_\theta \tag{28a}$$

$$\int_0^{2\pi} \int_0^{2\pi} e^{S_\theta(n-2\ \sin\ \alpha)} e^{2S_r\ \cos\alpha} \cos(n\ S_r) dS_r dS_\theta \tag{28b}$$

Integrating S_θ between 0 and 2π we obtain

$$\frac{e^{2\pi(n-2\sin\alpha)} - e^{(n-2\sin\alpha)}}{n-2\sin\alpha} \int_{o}^{2\pi} e^{2S_r\cos\alpha} \cos(n\, S_r)dSr \qquad (29)$$

Letting $p = n$ and $a = 2\cos\alpha$ and $x = S_r$ and remembering that

$$\int e^{ax}\cos px\, dx - e^{ax}\frac{(a\cos px + p\sin px)}{a^2 + p^2} \qquad (30)$$

we obtain

$$\frac{e^{2\pi(n-2\sin\alpha)} - e^{(n-2\sin\alpha)}}{n-2\sin\alpha} \quad e^{2S_r\cos\alpha} \quad x$$

$$\left.\frac{(2\cos\alpha\cos(nS_r) + n\sin(n\,S_r))^2}{4\cos^2\alpha + n^2}\right|_0^{2\pi} = 0$$

The same procedure obtains for the imaginary part of the function. The importance of this fact is that azimuthally integrated terms of a simple Fourier series make no contribution to the overall storm budget. Unless the term is the product of a Fourier series with the terms in phase as either the products of sines or cosines as indicated above it makes no contribution.

Hence, if we look at the product of two terms, such as occurs in $-\zeta\delta$ in the vorticity equation or ζ^2 in the divergence equation then we must consider the products of the whole term, mean and Fourier components as in (21a) and (21b). The investigation of the products of terms A and B in the relevant equations gives us

$$(\bar{A} + A')(\bar{B} + B') = \bar{A}\,\bar{B} + \bar{A}\,B' + A'\bar{B} + A'B'$$

where $\bar{A}$ is the azimuthal average of A and A' is the Fourier component. Both $\bar{A}B'$ and $A'\bar{B}$ vary in a simple sinusoidal fashion. $\bar{A}\,\bar{B}$ does not vary at all, and A'B' varies as the product of sinusoidal functions. Thus $\bar{A}\,B'$ and $A'\bar{B}$ contribute nothing to the azimuthally integrated storm. $\bar{A}\,\bar{B}$, the product of the means, contributes to the symmetric part and A'B' the product of the variances contributes to the asymmetric part.

This would at first appear contrary to customary usage of perturbation theory. The underlying assumptions here, of course, are that the asymmetries are not negligibly small in comparison to the symmetric part of the storm.

The asymmetric forms of the terms for the divergence and vorticity equations given in Table A1 in the appendix are indicated in *Tables I and II below*, assuming that the specific volume varies as the divergence and the pressure varies as the vorticity.

It appears that as in the symmetric case, advective and tilting terms may balance one another, while the convective and enstrophy terms sum to be equal and opposite to the energy flux divergence. As opposed to the symmetric case, where the brake on storm development appears to be the frictional terms, in the asymmetric budget, the brake on the concentration of kinetic energy, the energy flux convergence arises out of the creation of divergence by enstrophy and the export of convergence by the convective terms. The advective and tilting terms appear uncoupled from the other terms. Possible subsets of equations in the asymmetric case of the divergence equation are listed below.

$$-U_{S_\theta} \; \partial\zeta/\partial e^{S_r} = -\frac{\partial w}{\partial e^{S_\theta}} \; \frac{\partial U_{S_\theta}}{\partial Z}$$

and

$$-U_{S_r} \; \partial\zeta/\partial e^{S_\theta} = -\frac{\partial w}{\partial e^{S_r}} \; \frac{\partial U_{S_r}}{\partial Z}$$

Only the advective, twisting and solenoidal terms contribute. They must sum to zero and be in phase for a steady state storm.

The advective term creates convergence in most of the storm. The wind, U_{S_θ} ,must be sufficiently high in order for the tilting term to balance it. In the tropical storm where U_{S_r} is trivial, even near the core, and enstrophy and tilting are still at a low value compared to the mature storm, the convective term may provide the braking influence on the advective term. Since the convective term is related to the magnitude of the low level convergence, a measure of storm intensity, the asymmetries would need to be well developed for this balance to occur. Otherwise, the advective term would lead the convective in storm intensification.

Again, let it be stressed that this is a highly hypothetical, partial model of the two dimensional divergence and vorticity fields in the tropical cyclone. Physical scenarios are suggested by a combination of phenomenology and the mathematical possibility that both the divergence and vorticity fields satisfy Laplace's equation in both symmetric and asymmetric cases. Thus far it has only been shown that the vorticity field satisfies the symmetric version in a large majority of storms studied. Supporting evidence for asymmetric satisfaction by divergence and vorticity of Laplace's equation and symmetric satisfaction by divergence, while indicative, is not yet firm enough to warrant definite conclusions. Thus far, the model is phenomenologically realistic and seems to be internally consistent despite its elaborate ramifications.

The model may be tested numerically, using the model as an analytic parameterization of a fully developed numerical model including energy and thermodynamic considerations. The partial analytical model here deals only with the divergence and vorticity equations, forms of the equations of motion, and a simple form of the continuity equation. The model does not address the first law of thermodynamics, the equation of state or the conservation equation for the water substance. Further corroborating evidence may be obtained by a Fourier analysis of available spiral data of hurricanes from satellites such as the microwave detection of water droplet concentration in spiral rainbands studied by Allison, et al., (1974). A study of phenomena, such as water droplet content which may be correlated to the divergence field may prove to be indicative of the relevance of the mathematical solution to the actual processes of the storm.

6. Conclusion

The generation of divergence and to a lesser extent vorticity are highly dependent upon the development of asymmetries in this model. Because of the laborious mathematical analysis, a numerical model should be parameterized using the asymmetries of this model to assess their contributions to the heat, momentum and water vapor budgets of the hurricane.

Table A.1 Comparison of Vorticity and Divergence Equations by Terms

	Vorticity $\frac{\partial \zeta}{\partial t}$ =	Divergence $\frac{\partial \delta}{\partial t}$ =
1)	Advective $-\underset{\sim}{U} \cdot \nabla \zeta$	Advective $-\underset{\sim}{U} \cdot \nabla \zeta \; x \underset{\sim}{k}$
2)	Convective $-w \frac{\partial \zeta}{\partial z}$	Convective $-w \frac{\partial \delta}{\partial z}$
3)	Twisting $-\nabla w \cdot \frac{\partial \underset{\sim}{U}}{\partial z} x \underset{\sim}{k}$	Tilting $-\nabla w \cdot \frac{\partial \underset{\sim}{U}}{\partial z}$
4)	Divergence $-\delta \zeta$	Enstrophy $+\zeta^2$
5)	Solenoidal $-\nabla \alpha x \nabla \rho \cdot \underset{\sim}{k}$	Toroidal $-\nabla \alpha \cdot \nabla \rho$
6)	Vertical mixing $\frac{\partial}{\partial z} \nu_z \frac{\partial \zeta}{\partial z}$	Vertical mixing $\frac{\partial}{\partial z} \nu_z \frac{\partial \delta}{\partial z}$
7)	Lateral mixing $\nu \nabla_2^2 \zeta$	Lateral mixing $\nu \nabla_2^2 \delta$
8)	-	Turbulent P $-\alpha \nabla^2 p$
9)	-	Energy dissipation $-\nabla^2 (\underset{\sim}{U} \cdot \underset{\sim}{U}/2)$

APPENDIX A

A.1 The Role of Terms in the Axially Symmetric Vortex

A.1.1 Vorticity and Divergence Equations

In the axially symmetric vortex, the interactive terms of the vorticity and divergence equations come into play to reinforce the regime structure. These terms are listed and named in Table A.1 where both equations are presented vertically (read down) each term being named (∇ is the two dimensional gradient operator).

The first and third terms, which are difficult to visualize because they involve the cross product with the unit vertical vector, $\underset{\sim}{k}$, are illustrated in Figure A.1 for observed conditions under which they are positive and negative.

A.1.2 The Characteristic Equations for Given Regimes

In the vortex model ordinarily there are more unknowns than there are equations. This entails guessing at the values of some of the unknowns in order to obtain a consistent and coherent model. In this section we shall see how we may increase the number of equations through grouping analytically similar terms which should therefore sum up in a self consistent way to satisfy the model.

Since the vortex is modeled in terms of vorticity and divergence distribution the model gives an indication of how the individual terms in the vorticity and divergence equations behave. For the steady state vortex the terms in scalar form appear to fall into discrete analytical categories. The categorization is based on the terms having similar analytical functions of radius and are of three types. These are given in Table A.2. The categorization is a function of regime and is given in Table A.3 for the three regimes.

In Table A.3 the terms are listed by class for the vorticity and divergence equations. The sign of the term is given in parenthesis. In the core regime, the Class II terms in the vorticity equation change sign from the "eye" portion of the core to the "eye wall" and hence have a sign for each.

Assuming the coefficients of Class I sum to Af(r), Class II sum to Bg(r) and Class III sum to Ch(r) then the steady state equation may be written:

$$Af(r) + Bg(r) + Ch(r) = 0 \tag{A.1}$$

In the core C is zero since $\nabla^2_2\zeta = -\nabla\alpha x\nabla p = 0$ so that

$$Af(r) + Bg(r) = 0 \tag{A.2}$$

If A and B are non zero then Af(r) = Bg(r). But this means that $f(r) - -C_o g(r)$ where C_o is some other constant. This may be true at one point but is not true elsewhere. Thus, if A and B are zero $f(r) \neq C_o g(r)$ since $C_o = -B/A$ is undefined.

Assuming the parallel terms in the divergence equations behave in the same fashion, then the terms $-\nabla\alpha\cdot\nabla_p - \nabla^2 K$ must be of the form Ch(r) = 0.

Therefore there exists a whole series of term couplings which shed light upon the vertical distribution of the terms. Thus, there exist the equations:

$$\overset{(-)}{-\underset{\sim}{U}\cdot\nabla\zeta} - \overset{(-)}{\nabla w\cdot \frac{\partial U}{\partial z} x\underset{\sim}{k}} + \overset{(+)}{\frac{\partial}{\partial z} \nu_z \frac{\partial \zeta}{\partial z}} = 0 \tag{A.3}$$

and

$$-w \frac{\partial \zeta}{\partial z} - \delta\zeta = 0 \tag{A.4}$$

Since we may solve (A.4) to obtain $\frac{\partial \zeta}{\partial z} = \zeta\delta/w$

and we know that $U_\theta = \frac{1}{2\pi} \int_o^r \zeta dr$ so that

$\frac{\partial U_\theta}{\partial z} = \frac{1}{2\pi} \int_o^r \frac{\partial \zeta}{\partial z} dr$ then we obtain by substituting (A.4)

into (A.3)

$$-U_r \frac{\partial \zeta}{\partial r} - \frac{\partial w}{\partial r} \int_o^r \frac{\zeta\delta}{w}\, dr + \frac{\partial}{\partial z} \nu_z \frac{\partial \delta}{w} = 0 \qquad (A.5)$$

or

$$\nu_z = \frac{w}{\zeta\delta} \Big| z \int_o^r U_r \frac{\partial \zeta}{\partial r} + \frac{\partial w}{\partial r} \int_o^r \frac{\zeta\delta}{w}\, dr \quad dz \qquad (A.6)$$

We therefore may obtain the value of the vertical coefficient of eddy viscosity in the core.

By specifying the analytical form of the scalar terms for a given regime we may solve for variables otherwise unattainable such as the eddy viscosity, ν_z, as a function of height and the specific volume, α, as a function of radius. In this way we are able to diminish the number of unknowns in the model because we have increased the number of equations.

Appendix B

Creation of Regimes in the Vortex

B.1 Physical Parameters Governing the Vortex Structure

Perhaps the easiest way to understand the rationale for the lateral distribution of the regimes of the vortex model is to follow the suggested sequence of events which lead to the formation of the vortex.

In order to do this it is necessary to have an understanding of the role of the vorticity and divergence equations in the makeup of the model. Moreover, the role of the Richardson number is essential to understanding how order may be imparted to an otherwise amorphous structure.

B.2 Richardson Number

The Richardson number expresses the characteristic ratio of the energy of a work done against gravitational stability to energy transferred from mean to turbulent motion. Theoretical studies have placed the critical Richardson number from .25 to 2 with stability for higher values and instability for lower. The Richardson number may be expressed by

$$R_i = g \frac{\partial \ln \theta}{\partial Z} \bigg/ \left(\frac{\partial U}{\partial Z}\right)^2 \quad \text{or} \quad R_i = g \frac{\partial p(\sigma\ ,\ T)}{\partial Z} \bigg/ \left(\frac{\partial U}{\partial Z}\right)^2$$

atmospheric oceanic

Since air spiralling into the hurricane must cross isobars to lower pressure the air experiences cooling. Air within the cloud layer, however, cools moist adiabatically, i.e., more slowly due to the release of the latent heat of condensation of water vapor while air below the cloud cools dry adiabatically. Since water vapor liberates 597.3 calories per gram upon condensation, the lower air cools much more rapidly than the upper. The brake in the process, if the vortex is a hurricane, occurs due to transfer of sensible heat from the sea surface into the vortex boundary layer. The resulting stability is only counteracted by increased turbulence due to a higher mean velocity as one approaches the center of the vortex. The more vigorous mixing tends to destroy the inversion caused by the differential cooling. It is thus, however, that the vertical velocity field tends to be concentrated into a narrow throat.

The moist expansion in the cloud layer leads to a higher potential temperature, θ of the air than its lower neighbor. This is expressed by

$$\theta_e = \theta \exp \left[\frac{Lw_s}{C_p T} \right]$$

where L is the latent heat of evaporation, 597.3 cals/gm, w_s the saturation mixing ratio of the water vapor, C_p the specific heat of dry air, and T the temperature of dry air.

B.3 Sequence of Events in the Formation of a Severe Atmospheric Vortex

Given an arbitrary distribution of vorticity, divergence and pressure, the following sequence of events is hypothesized as leading to the formation of a severe vortex.

Assume an initial parabolic distribution of pressure, divergence and vorticity, with a subsequent radial and tangential velocity distribution as shown in Figures Bla and Blb. Assume also that the total amount of vorticity and divergence are constant but may be and indeed are rearranged in response to the influence of the various terms of the relevant equations.

The first step in this process is the rearrangement of the distribution by the two dimensional Laplacians in the vorticity, divergence and pressure tendency equations. The distributions then become as illustrated in Figures B2a and B2b.

The second modification occurs when, in response to the concentration of the wind peaks inward, the inspiralling air experiences differential cooling. The extent of the cooling depends significantly upon whether the trajectory of the inspiralling air is within a cloud or below it. This differential cooling causes an inversion to form faster than turbulence can break it up in the outer regions of the vortex. This in turn suppresses vertical velocity making the formatiom of the inversion even more pronounced. Thus the critical Richardson number Ri defines the boundary between the inner and outer regimes (Figures B3a, B3b).

The final stage occurs when the enstrophy term, ζ^2, becomes so large at small values of radius (due to crowding of the vorticity and divergence into a smaller and smaller area) that divergence is created so fast by this term that the vorticity peak collapses like a volcanic caldera bringing about the final stage in the cycle illustrated in Figures B4a and B4b with the creation of a core interior to the inner regime.

The fit of the model to the hurricane's pressure profile is also very good and substantiates the presence of the three regimes mentioned above. An extension of the concept which accounts for the pressure and tangential velocity fields presents a logical explanation for the radial and vertical velocity fields.

The organization of the storm from a preexistent but amorphous field of vorticity and convergence into the highly structured vortex is seen as the effect of turbulent lateral diffusion constrained by critical values of the Richardson number and of enstrophy (the square of the vorticity). The interaction of these mechanisms provides a plausible scenario for the structuring of the mature storm. The application of these concepts to the ocean, changing what must be changed, will provide valuable insights into the behavior of the upper ocean.

References

Allison, Lewis J., Edward B. Rodgers, Thomas T. Wilheit and Robert W. Fett, 1974: Tropical cyclone rainfall as measured by the Nimbus 5 electrically scanning microwave radiometer, BAMS, vol 55, No. 9, pp 1074-1089.

Colon, J. A., 1964: On the Structure of Hurricane Helene (1958), National Hurricane Research Project, Report No. 72, pp. 56.

Nicholson, F. H. and D. R. Johnson, 1978: A model for the tangential velocity profile of some azimuthally averaged vortices. Unpublished manuscript.

Riehl, H., 1963: Some Relations Between Wind and Thermal Structure of Steady State Hurricanes, J. of At. Sciences, 20, 276-287.

Rosmond, Thomas E., and Frank D. Faulkner, 1976: Direct Solution of Elliptic Equations by Block Cyclic Reduction and Factorization, MWR, 104, pp. 641-649.

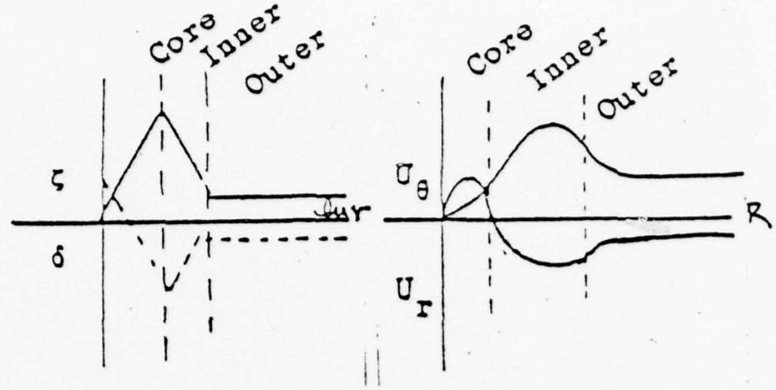


Figure 1a and 1b. Vorticity, divergence and velocity regimes in the zero Laplacian vortex.

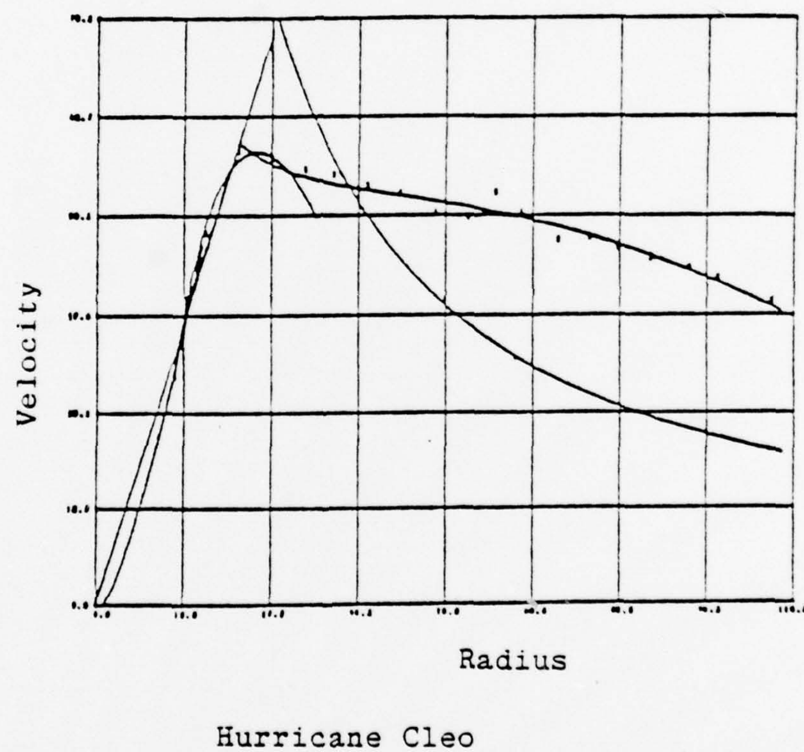
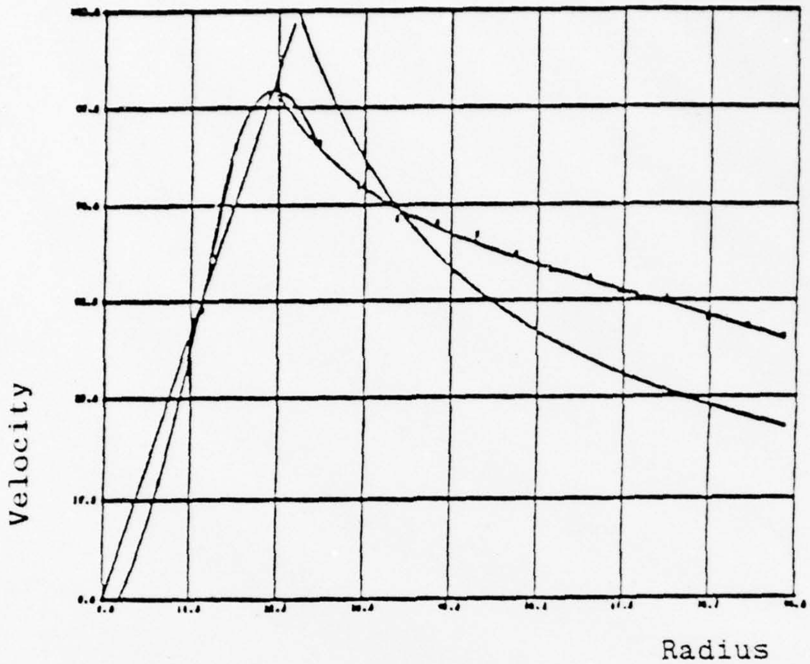


Figure 2a. Piecewise continuous fit of Zero Laplacian Vortex to Riehl's (1963) hurricane Cleo data. Rankine's combined vortex is included for comparison.



Hurricane Hannah

Figure 2b. Piecewise continuous fit of Zero Laplacian Vortex to Riehl's (1963) hurricane Hannah data. Rankine's combined vortex is included for comparison.

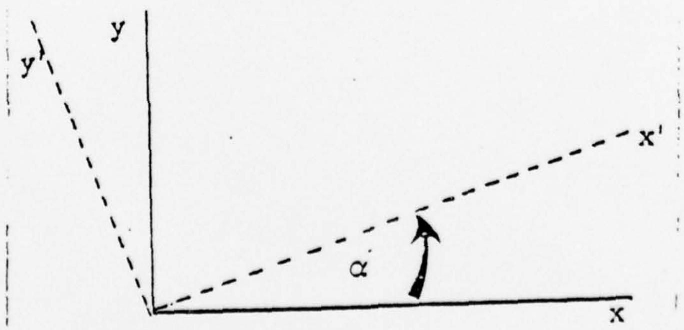


Figure 3. Rotation of Cartesian Coordinate System through angle α.

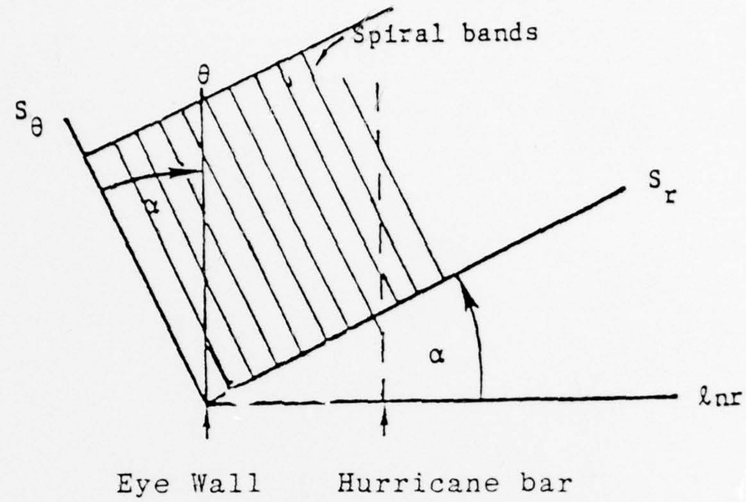


Figure 4. Rotation of semilog coordinate system $\ln r$, θ to produce log spiral system S_r, S_θ.

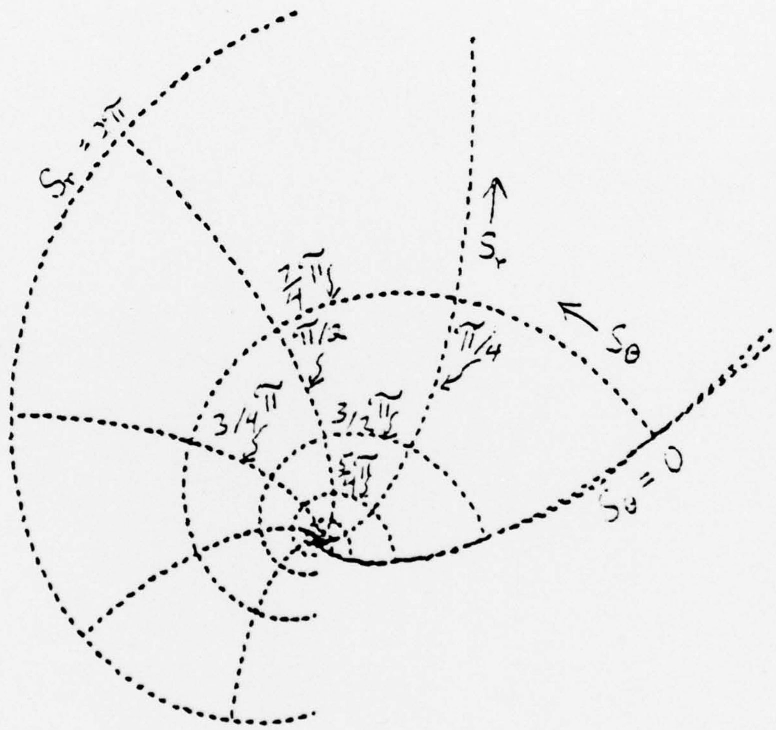


Figure 5. Spiral Space with angle of rotation 20°. The spirals occur at every $\pi/4^{\circ}$. The inner values of S_r would be completely eclipsed by a strong axially symmetric phenomenon at the storm center.