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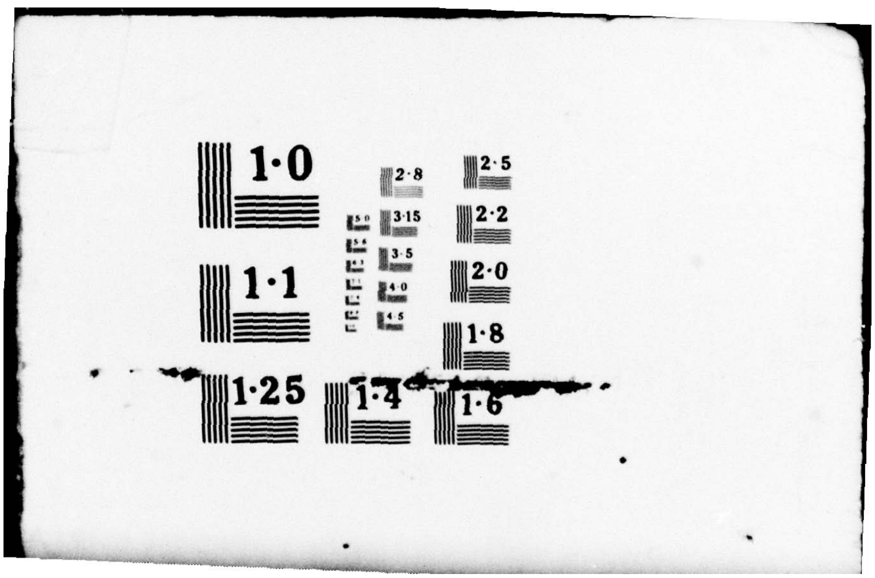
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AN APPROXIMATE REPRESENTATION OF NEUMANN'S SOLIDIFICATION SOLUTION *

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Introduction

The solution of Neumann's change-of-phase problem (i.e., the solidification or melting of a slab, whose temperature is initially uniform and is maintained constant at the surface [1]) by approximate analytical means has received some attention in the recent literature [2]. Most of the solutions available, however, present a single approximation, and are not readily adaptable to obtaining further, and hopefully more accurate, approximations. In the present work a method of so doing is presented, in which the multiple penetration-depth technique of [3] is applied to the formulation of Neumann's problem in the form established in [4]. Some numerical results presented at the end indicate that, at least for the case of constant properties, the proposed approach is workable and satisfactory.

Analysis

The problem at hand refers to the slab $x > 0$, initially liquid at a uniform temperature $T_1 \geqslant T_m$ (for the case of solidification), whose surface temperature $T(0,t)$ is maintained at the constant value $T_0 < T_m$. At any time $t > 0$ the liquid occupies the region $s(t) < x < \infty$, and the solid the region $0 < x < s(t)$. The temperatures T_L and T_S in the liquid and solid respectively must satisfy the Fourier heat-conduction equation (with primes and dots indicating differentiation with respect to x and t respectively):

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

An approximate method of solution of Neumann's change-of-phase problem is presented. The solution makes use of a multiple penetration-depth approach.

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$$\left[k_{L,S}(T_{L,S})\,T'_{L,S}\right]' = \rho c_{L,S}(T_{L,S})\,\dot{T}_{L,S} \tag{1}$$

under the initial and boundary conditions

$$\begin{aligned} T_L(x,0) &= T_1 & x>0 \\ T_L(\infty,t) &= T_1 & t\geq 0 \\ T_S(0,t) &= T_0 & t>0 \end{aligned} \tag{2}$$

and subject to the solid-liquid interface conditions

$$T_L[s(t),t] = T_S[s(t),t] = T_m \quad , x=s(t)\,,\ t>0 \tag{3}$$

$$k_S T_S' - k_L T_L' = \rho l \dot{s} \quad , x=s(t)\,,\ t>0 \tag{4}$$

with $s(0)=0$. We note that the solution is of the form

$$s(t) = 2\lambda\sqrt{\kappa_0 t} \tag{5}$$

where λ is an unknown constant and κ_0 reference diffusivity.

Let now [4] two auxiliary solutions $T_{L,S}(x,t;B,C)$ be defined as satisfying eq. (1) and the following boundary and initial conditions:

$$\begin{aligned} T_S(0,t;B,T_0) &= T_0 & T_L(0,t;T_1,C) &= C \\ T_S(x,0;B,T_0) &= T_0+B & T_L(x,0;T_1,C) &= T_1 \\ T_S(\infty,t;B,T_0) &= T_0+B & T_L(\infty,t;T_1,C) &= T_1 \end{aligned} \tag{6}$$

It is then easily verified that the solution of the desired problem is obtained by choosing the constants B, C and λ so as to satisfy eqs. (3) and (4).

An approximate form for the auxiliary temperatures can be taken in terms of m penetration depths $q_i(t)$ as [3]:

$$\begin{aligned} T_S(x,t;B,T_0) &= T_0 + B - B\sum_{i=0}^{m-1} d_{iS}\,x^i\left(1-\frac{x}{q_{iS}}\right)^2 H(x,q_{iS}) \\ T_L(x,t;T_1,C) &= T_1 + (C-T_1)\sum_{i=0}^{m-1} d_{iL}\,x^i\left(1-\frac{x}{q_{iL}}\right)^2 H(x,q_{iL}) \end{aligned} \tag{7}$$

where

$$H(x, x_1) = \begin{cases} 1 & x < x_1 \\ 0 & x > x_1 \end{cases} \tag{8}$$

$$d_{0_{S,L}} = 1;\ d_{i_{S,L}} = d_{i_{S,L}}(t) \text{ for } i \geq 1;\ q_{i_{S,L}} = q_{i_{S,L}}(t) \text{ for } i \geq 0$$

and where it is assumed that

$$q_{i_{S,L}}(t) \geq 0\ ,\ i = 0, 1, \ldots, m-1 \tag{9}$$

The unknowns d_i and q_i are calculated by satisfying (1) approximately by means of the "method of moments",i.e., by setting

$$\int_0^\infty \left[(k\, T')' - \rho c \dot{T} \right] x^n dx = 0,\ n = 0, 1, \ldots, 2(m-1) \tag{10}$$

For the case of constant properties, it is found [3] that

$$q_{i_{S,L}} = A_{i_{S,L}} \sqrt{\kappa_{S,L}\, t} \qquad ,\qquad i \geq 0$$

$$d_{i_{S,L}} = \frac{D_{i_{S,L}}}{\sqrt{\kappa_{S,L}\, t}} \qquad ,\qquad i > 0 \tag{11}$$

where the values of A_i and D_i are to be obtained numerically under the restriction of eqs. (9).

There now remain to satisfy the interface relations (3) and (4). The former gives

$$B = \frac{T_m - T_0}{1 - \sum_{i=0}^{m-1} d_{i_S} s^i \left(1 - s/q_{i_S}\right)^2 H\left(s, q_{i_S}\right)}$$

$$C = \frac{T_m - T_1}{\sum_{i=0}^{m-1} d_{i_L} s^i \left(1 - s/q_{i_L}\right)^2 H\left(s, q_{i_L}\right)} \tag{12}$$

and the latter reduces to the form

$$R = \frac{\lambda}{\mathfrak{z} \sqrt{\pi}} \tag{13}$$

where

$$R = \frac{c\,(T_m - T_0)}{l\,\sqrt{\pi}}\ ;\ P = \frac{T_1 - T_m}{T_m - T_0}\ ;\ \alpha = \sqrt{\frac{\kappa_S}{\kappa_L}}\ ;\ s = 2\lambda\sqrt{\kappa_S t} \tag{13a}$$

and where

$$\zeta = -\frac{\sum_{i=0}^{m-1}\left\{D_i(2\lambda)^i\left[\frac{i}{2\lambda}\left(1-\frac{2\lambda}{A_i}\right)^2-\frac{2}{A_i}\left(1-\frac{2\lambda}{A_i}\right)\right]\right\}H(2\lambda,A_i)}{1-\sum_{i=0}^{m-1}\left\{D_i(2\lambda)^i\left(1-\frac{2\lambda}{A_i}\right)^2\right\}H(2\lambda,A_i)} + \\ + \frac{\frac{k_l}{k_s}p\sum_{i=0}^{m-1}\left\{D_i(2\lambda\alpha)^i\left[\frac{i}{2\lambda}\left(1-\frac{2\lambda\alpha}{A_i}\right)^2-\frac{2\alpha}{A_i}\left(1-\frac{2\lambda\alpha}{A_i}\right)\right]\right\}H(2\lambda,\alpha A_i)}{\sum_{i=0}^{m-1}D_i(2\lambda\alpha)^i\left(1-\frac{2\lambda\alpha}{A_i}\right)^2H(2\lambda,\alpha A_i)} \qquad (13b)$$

The solution of eqs. (13) involves a trial and error procedure, in order to obtain values of $s(t)$ which lie (since $T<T_m$ for $x<s$) between zero and the largest of the $q_i(t)$. In the cases of one and two penetration-depths, $s(t)$ was smaller than the smallest penetration-depth, so that the step function in these equations could be ignored; but for the case of three penetration-depths this was not the case and the complete expressions had to be used. A numerical comparison of the result obtained from one, two and three penetration depth with the exact values is presented in the accompanying figures, for the case in which the solid and the liquid have the same properties ($\alpha=1$), for various values of p. It is clear that improvements result from the use of several penetration depths, although for this particular problem even one penetration depth leads to quite accurate results.

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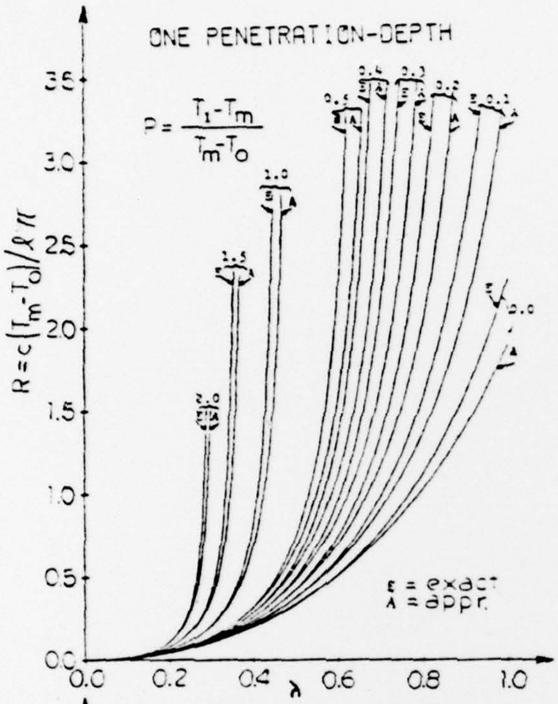
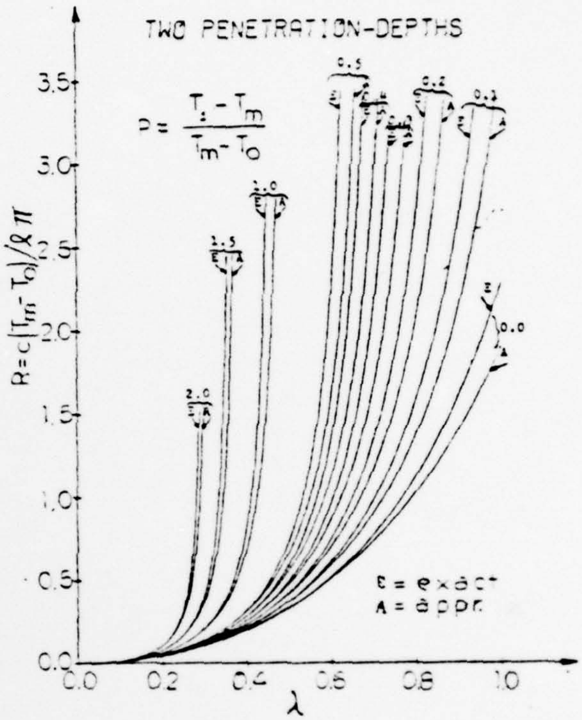

ONE PENETRATION-DEPTH
$P = \frac{T_1 - T_m}{T_m - T_o}$
$R = c(T_m - T_o)/\ell\pi$
E = exact
A = appr.
λ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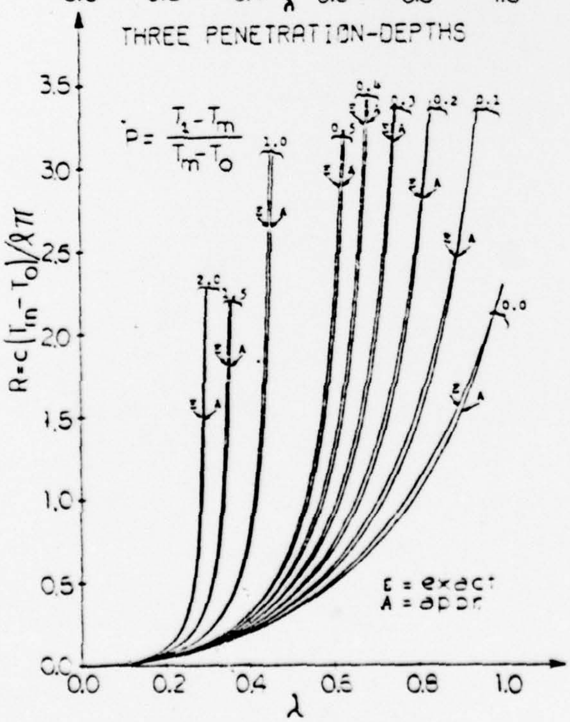
$R = c(T_m - T_o)/\ell\pi$
E = exact
A = appr.
λ


THREE PENETRATION-DEPTHS
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$R = c(T_m - T_o)/\ell\pi$
E = exact
A = appr.
λ