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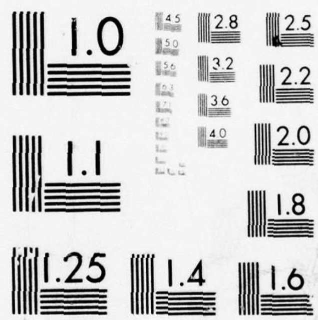

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A NONSTANDARD THEORY OF GAMES, PART III: NONCOOPERATIVE *FINITE GAMES

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Introduction

A nonstandard characterization of Nash-type equilibria in the noncooperative context for games of the form $N^*_{F^*} = \langle F^*, \{X_j\}_{j \epsilon F^*}, \{H_j\}_{j \epsilon F^*} \rangle$, for $F^* = [0,\omega]$, $\omega \, \epsilon \, N^* - N$, is provided in what follows. As an application, the characterization is then used to generate equilibrium points of non-atomic structures, of the variety considered by Schmeidler [4], on the Loeb space generated by $\langle F^*, A(F^*), u_{F^*} \rangle$.

There are two chief features of distinction between the present approach and other versions: First, since the space of players, F*, is internally *Finite, measurability requirements can be replaced with the somewhat weaker assumption that the payoff functions be internal; second, again because of the internality of the space of players, the weak ΓS-topology and the product S-topology coincide. This avoids the requirement of stronger fixed point arguments than that of the usual Kakutani variety.

I. The Noncooperative Formulation

Let F^* be a *Finite set of the form $[0,\omega]$ for $\omega \in N^* - N$. By a *Finite noncooperative game we will mean a triplet, $N^*_{F^*} = \langle F^*, \{X_j\}_{j\in F^*}, \{H_j\}_{j\in F^*}\rangle$, where F^* is the set of participants, $\{X_j\}_{j\in F^*}$ is the collection of feasible strategy spaces for the set of participants, and $\{H_j\}_{j\in F^*}$ is the collection of payoff functions such that $(\forall j \in F^*)(H_j : \hat{X} \to R^*_+)$, where $\hat{X} = \prod_{j\in F^*} X_j$ is the feasible joint strategy space.

We make the assumption that for each $j \in F^*$, $X_j = S(n)$, where $S(n)$ is the normalized simplex of standard n dimension, that is, $S(n) = \{x^* \in (R^*_+)^n : \sum_{i=1}^{n} x^*_i = 1\}$. Then, since F^* is internally *Finite, we are assured that the product space is Q-compact, and of dimension nF^*.

Definition I.1: An arbitrary set $T \subseteq (R^*_+)^n$ for $n \in N^*$ is said to be Q-convex if for $(\forall x^*, y^* \in T)$ and $\lambda \in (0,1)$ (λ standard), $(\lambda x^* + (1-\lambda)y^*) \in T$.

T is said to be S-convex if $\forall x^*, y^* \in T$ and $\lambda \in (0,1) \exists z \in T$ such that $\left[z = (\lambda x^* + (1-\lambda)y^*)\right]_{\text{Mod } M_1}$.

The Q-convex hull of T, Q-con(T), is the set of all combinations $\sum_{j=1}^{n} \alpha_j x^*_j$, for $\sum_{j=1}^{n} \alpha_j = 1$, $\alpha_j \in R^*_+$, $\alpha_j \in [0,1]$ and $x^*_j \in T$, and $n \in N^*$.

Definition I.1.2: A function $f^* : x^* \to Y^*$ for X^* and Y^* defined in R^* is said to be S-concave if for $\lambda \in (0,1)$ and $x^*, y^* \in X^*$

$$f^*(\lambda x^* + (1-\lambda)y^*) \geq \lambda f^*(x^*) + (1-\lambda) f^*(y^*)$$

where the relation $\gtrsim$ is the disjunction of the two relations $\underset{j}{>}$ and $(=)_{\text{Mod } M_1}$.

For each participant $j \in F^*$, we make the assumption that H_j is S-continuous on X and is S-concave in x_j^* for each fixed value of the deleted vector $(x_1^*, \ldots, x_{j-1}^*, x_{j+1}^*, \ldots, x_\omega^*)$, and that H_j is an internal function.

Definition I.1.3: For the game $N^* = \langle F^*, \{X_j\}_{j\varepsilon F^*}, \{H_j\}_{j\varepsilon F^*}\rangle$, an equilibrium point is a vector $\tilde{x}^* \varepsilon \hat{X}$ such that

$$H_j(\tilde{x}^*) = \max_{y_j}\{H_j(\tilde{x}_1^*, \ldots, y_j, \ldots, \tilde{x}_\omega^*)\} \text{ a.e. in } F^*,$$

where a.e. in F^* has the interpretation that the set $T \subseteq F^*$ for which $H_j(\tilde{x}^*) \neq \max_{y_j}\{H_j(\tilde{x}_1^*, \ldots, y_j, \ldots, \tilde{x}_\omega^*)\}$ is such that $\left(\frac{\|T\|}{\|F^*\|} = 0\right)_{\text{Mod } M_1}$.

The definition has the interpretation that only a negligible set of participants can increase their payoffs by a change in their own strategies given the strategic choices of the other participants.

Definition I.1.4: We will require the use of an auxiliary function $\rho : \hat{X} \times \hat{X} \to R_+^*$, where, by definition,

$$\rho(x^*, y^*) \equiv \sum_{j\varepsilon F^*} H_j(x_1^*, \ldots, y_j^*, \ldots, x_\omega^*)$$

▶Lemma I.1.5: The function $\rho(x^*,y^*)$ is S-continuous and S-concave in y^* for a fixed x^*.

Proof: Straightforward verification of definitions in the light of the internality of F*.

Q.E.D.

Definition I.1.6: Consider the following mapping,

$$\Gamma(\hat{X}) = \bigcup_{x^* \varepsilon \hat{X}} \{y^* : \rho(x^*,y^*) = \max_{z^* \varepsilon \hat{X}} \rho(x^*,z^*)\}.$$

As an easy consequence of Lemma I.1.5, one obtains:

▶Lemma I.1.7: The range of $\Gamma(\hat{X})$ is S-closed and S-convex.

Proof: Successive application of the S-continuity and S-concavity of $\rho(x^*,y^*)$.

Q.E.D.

▶Corollary I.1.8: The range of $\Gamma(\hat{X})$ is Q-compact.

Proof: Since $\gamma_S \subset \gamma_Q$, the Q-topology is finer than the S-topology. The facts that $\Gamma(\hat{X}) \subset \hat{X}$ and $\hat{X}$ is Q-compact yield that $\Gamma(\hat{X})$ is a closed subset of a Q-compact set.

Q.E.D.

We will make use of the following well-known concepts in the context of the Q-topology.

Definition I.1.9: A mapping F, from a topological space X to a topological space Z, is said to be Q-upper semicontinuous at a point x_0^* if for each Q-open set G in Z such that

$F(x_0^*) \subset G$ there is a neighborhood $U(x_0^*)$ in X such that $x^* \in U(x_0^*)$ implies $F(x) \subset G$.

Definition I.1.10: A mapping $F : X \to Z$, where X and Z are topological spaces, is said to be Q-upper semicontinuous if the above definition is satisfied for all $x \in X$ and if, in addition, $F(x)$ is a compact set for each $x \in X$.

▶Lemma I.1.11: A sufficient condition for a mapping $F : X \to Z$, where X and Z are topological spaces, to be Q-upper semicontinuous is that the set $\Gamma(x^*)$ be compact for each $x^* \in X$ and that for each open set $G \subset Z$, the set $F^*G = \{x^* \in X \ldots F(x^*) \subset G\}$ is open.

Proof: Let $x_0^* \in X$ and let G be open in Z, and further let $F(x_0^*) \subset G$. By hypothesis, F^*G is a neighborhood for x_0^*. Obviously, $x^* \in F^*G \Rightarrow F(x^*) \subset G$, and since $F(x_0^*)$ is compact by hypothesis, the conclusion follows from Definition I.1.10.

Q.E.D.

▶Lemma I.1.12: The mapping $\Gamma : \hat{X} \to \hat{X}$, $\Gamma(x^*) = \{y^* : \rho(x^*,y^*) = \max_{z \in \hat{X}} \rho(x^*,z)\}$ is Q-upper semicontinuous on $\hat{X}$.

Proof: Each H_j is S-continuous in x^*, and therefore $\rho(x^*,y^*)$ is S-continuous in x^* and the pre-image of an open set containing $\Gamma(x^*)$ for some x^* is therefore open in $\hat{X}$. By

Corollary I.1.8, $\Gamma(x^*)$ is Q-compact. It is obvious that the hypotheses of Lemma I.1.11 are met.

Q.E.D.

Lemma I.1.13: Let X be a Q-compact, Q-convex set of (internal) *Finite dimension, and let $F : X \to X$ be a Q-upper semicontinuous mapping of X into X. Then there is a point $\tilde{x}^* \in X$ such that $\tilde{x}^* \in F(\tilde{x}^*)$.

Proof: (Machover and Hirschfeld, Lectures on Nonstandard Analysis, Springer Verlag, 1969) This is actually the standard Kakutani Fixed Point Theorem in the Q-context.

Lemma I.1.14: (M. Ali Khan [2]) Let X be internal and S-convex, then for an arbitrary $y^* \in Q\text{-con}(X)$ $\exists z^* \in X$ such that $(y^* = z^*)_{\text{Mod } M_1}$.

Proof: One shows by induction for X S-convex, and $\{x_j^*\}_{j=1}^{n+1}$, $\lambda_j \in [0,1]$ such that $\sum_{j=1}^{n+1} \lambda_j = 1$, there is a $z^* \in X$ such that $\left(z^* = \sum_{j=1}^{n+1} \lambda_j x_j^*\right)_{\text{Mod } M_1}$.

By Definition I.1.1, the statement is true for $n+1 = 2$. Assume it to be true for $n = k-1$. There is no loss in generality to assume $0 < \lambda_k < 1$. Allow $x' = \left(\sum_{j=1}^{k-1} \lambda_j x_j^*\right) \Big/ 1-\lambda_k$. Then since $\sum_{j=1}^{k-1} (\lambda_j/1-\lambda_k) = 1$ for some $c \in X$, $(c = x')_{\text{Mod } M_1}$. However, $\lambda_k x_k^* + (1-\lambda_k)x' = \left(\sum_{j=1}^{k} \lambda_j x_j^* = \lambda_k x_k^* + (1-\lambda_k)c\right)_{\text{Mod } M_1}$. Again, by

Definition I.1.1 for some $z^* \varepsilon X$,

$$\left(z^* = (\lambda_k x_k^* + (1-\lambda_k)c) = \sum_{j=1}^{k} \lambda_j x_j^*\right)_{\text{Mod } M_1}$$

Since X is internal, by Caratheodory's Theorem (transferred), any $y^* \varepsilon$ Q-con(X) can be expressed as $\sum_{j=1}^{n+1} \lambda_j x_j^*$, such that $\sum_{j=1}^{n+1} \lambda_j = 1$ for each $\lambda_j \varepsilon [0,1]$ and each $x_j^* \varepsilon X$.

Since k-1 was chosen at the start for the value n, one obtains $\left(z^* = \sum_{j=1}^{n+1} \lambda_j x_j^* = y^*\right)_{\text{Mod } M_1}$.

Q.E.D.

Theorem I.1.15: An equilibrium point exists for the game $N_{F^*}^* = \langle F^*, \{X_j\}_{j\varepsilon F^*}, \{H_j\}_{j\varepsilon F^*}\rangle$.

Proof: Construct the auxiliary mapping $\psi(x^*) = \{\bar{x}^* \varepsilon \hat{X} : x^* \varepsilon \text{Q-con}(\Gamma(x^*))\}$. Then, by Lemma I.1.13, for some point $x^* \varepsilon \hat{X}$, $\bar{x}^* \varepsilon \psi(\bar{x}^*)$. By Lemma I.1.14, for some $\bar{x}_0^* \varepsilon \Gamma(\bar{x}^*)$, $(\bar{x}_0^* = \bar{x}^*)_{\text{Mod } M_1}$. Suppose that $\bar{x}_0^*$ is not an equilibrium point for N_F^*. Then for a set $T \subset F^*$ $\left(\frac{\|T\|}{\|F^*\|} \neq 0\right)_{\text{Mod } M_1}$, there exists $\bar{z}^* \varepsilon \hat{X}$ such that $H_j(\bar{x}_{01}^*, \ldots, \bar{z}_j^*, \ldots, \bar{x}_{0\omega}^*) \underset{\not\approx}{>} H_j(\bar{x}_{01}^*, \ldots, \bar{x}_{0j}^*, \ldots, \bar{x}_{0\omega}^*)$ for all $j \varepsilon T$. Since $\left(\frac{\|T\|}{\|F^*\|} \neq 0\right)_{\text{Mod } M_1}$ and $\|F^*\| = \omega \varepsilon N^* - N$, $\|T\| \notin N$, then $\rho(\bar{x}_0^*, \bar{z}^*) \underset{\not\approx}{>>} \rho(\bar{x}_0^*, \bar{x}_0^*)$, which contradicts the fact that $\bar{x}^* \varepsilon \psi(\bar{x}^*)$ and $(\bar{x}^* = \bar{x}_0^*)_{\text{Mod } M_1}$ by the S-continuity of ρ in Lemma I.1.5.

Q.E.D.

II. Equilibrium Points for Non-atomic Games

A natural application of the preceding construction is found in characterizing Nash-type equilibria for non-atomic games by considering the non-atomic representation of $N^*_{F^*}$ in the Loeb space of $\langle F^*, A(F^*), u_F{}^* \rangle$.

Definition II.1.1: Let $S(n)$ denote, as before, the n(standard) dimensional simplex. A T-strategy, to use Schmeidler's terminology, is a mapping $x^* : F^* \to S(n)$. Then for $j \in F^*$, $x^*(j) = (x^*_1(j), \ldots, x^*_n(j))$ represents the j^{th} player's mixture of the n pure strategies $\{(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)\}$.

As an immediate consequence of the fact that each component $x^*_i(j)$ is bounded by unity, and Definition I.2 of Chapter 2, one obtains readily:

Lemma II.1.2: The set of T-strategies, $T(F^*)$, is u_{F^*}-measurable on $\langle F^*, A(F^*), u_{F^*} \rangle$.

Proof: For $x^* \in T(F^*)$, $\int_{F^*} x^* d_{uF^*} = \left(\int_{F^*} x^*_1 du_{F^*}, \ldots, \int_{F^*} x^*_\omega du_{F^*}\right)$ and one applies Lemma I.3 of Part II of the series, in succession, to the components of $\int_{F^*} x^* du_{F^*}$.

Q.E.D.

We shall endow $T(F^*)$ with the ΓS-topology, the weak S-topology, as expropriated from Dunford and Schwartz [5], p. 419, to the S-topology. A base for the ΓS-topology can be derived from neighborhoods of the form, $N^0(x_1^*) = \{x_2^* \in T(F^*) : |x_1^*(j) - x_2^*(j)| < e \ \forall j \in F^*\}$, where e is a standard positive real.

An important feature of the present approach is that since F^* is internally *Finite, the ΓS-topology coincides with the product S-topology on $(R^*)^{nF^*}$ (reference Dunford and Schwartz [5], p. 419, once again). This enables the application of the fixed point argument, used above, without appeal to stronger versions of the Fan-Glicksberg variety (compare Schmeidler [4], p. 298). We will refer to $T(F^*)$ endowed with the ΓS-topology as $\tilde{T}(F^*)$. It is easily verified that $\tilde{T}(F^*)$ is Q-convex and closed.

Definition II.1.3: Let the following auxiliary utility functions be defined on $F^* \times \tilde{T}(F^*)$, internally, as $u(\cdot,\cdot) : F^* \times \tilde{T}(F^*) \to (R_+^*)^n$, where $u(j,x^*) = u^1(j,x^*), \ldots, u^n(j,x^*)$ and $u^k(j,x^*)$ is the utility to the j^{th} participant of the k^{th} pure strategy when $F^* - \{j\}$ employs x^*. We will assume that for each $j \in F^*$, $u(j,\cdot)$ is continuous on $\tilde{T}(F^*)$, nonnegative, and bounded in M_0^+. We also require that $(\forall x^* \in \tilde{T}(F^*))(\forall j \in F^*)\{\exists K \in \{1, \ldots, n\} \text{ s.t. } u^K(j,x^*) > 0.\}$.

Definition II.1.4: In terms of the $u(j,x^*)$, let the payoff functions for each participant be defined as:

$\tilde{H}_j(x^*) = x^*(j) \cdot u(j,x^*)$, where '·' denotes the inner product in $(R^*_+)^n$. Obviously, $\tilde{H}_j(x^*) \in M_0^+$ $\forall x^* \in \tilde{T}(F^*)$ and the internality of $u(j,x^*)$ implies that of $\tilde{H}_j(x^*)$.

A T-strategy, $\tilde{x}^* \in \tilde{T}(F^*)$, is an equilibrium point of the game $N^*_{\tilde{F}^*} = \langle F^*, \tilde{T}(F^*), \{\tilde{H}_j\}_{j \in F^*} \rangle$ if $\forall s \in S(n)$ $\tilde{H}_j(\tilde{x}^*) \geq s \cdot u(j,\tilde{x}^*)$ a.e. in F^*.

▶Theorem II.1.5: There exists an equilibrium T-strategy for the game $\tilde{N}^*_{F^*}$.

Proof: The result is essentially an instance of Theorem I.1.15 with modifications inclusive replacing $\hat{X}$ with $\tilde{T}(F^*)$ which is permissible as remarked on p. 126, and replacing the mapping used, $\Gamma(x^*)$, with the following construction of $\tilde{\Gamma}(x^*)$.

Let $G(j,x^*) = \{s \in S(n) : \forall t \in S(n)(s \cdot u(j,x^*) \geq t \cdot u(j,x^*))\}$. Then it is easily seen that $G(j,x^*) \neq \phi$ for any $j \in F^*$ and is S-convex owing to the properties of $S(n)$.

Construct next the mapping $\tilde{\Gamma} : \tilde{T}(F^*) \to \tilde{T}(F^*)$,

$\tilde{\Gamma}(x^*) = \{y^* \in \tilde{T}(F^*) : \forall j \in F^* \; y^*(j) \in G(j,x^*)\}$

▶Lemma II.1.6: For any $x^* \in \tilde{T}(F^*)$, $\tilde{\Gamma}(x^*)$ is non-empty and S-convex.

Proof: Define the sets T_K, for $K = 1, \ldots, n$, to be as follows: $T_K = \left\{ j \in F^* : u^K(j,x^*) \geq u^i(j,x^*) \ i = 1, \ldots, n \right\}$. By Definition II.1.3, $\left(\bigcup_{K=1}^{n} T_K \right) = F^*$ and, rather obviously, if $j \in T_K$, then $G(j,x^*)$ contains the vector in $S(n)$, $e_K = \begin{cases} 0 & i \neq K. \\ 1 & i = K \end{cases}$ Next, inductively define the sets $S_1 = T_1$, $S_K = T_K - \left(\bigcup_{i=1}^{K-1} T_i \right)$, $K = 2, \ldots, n$. Then the T-strategy $\tilde{x}^*(j) = e_K$ for $j \in T_K$ is such that $\tilde{x}^* \in \tilde{\Gamma}(x^*)$.

The S-convexity of $\tilde{\Gamma}(x^*)$ is implied by the S-convexity of $G(j,x^*)$ for each $j \in F^*$.

Q.E.D.

Lemma II.1.6 is a construction based on an argument found in Schmeidler [4], p. 297.

The proof of the theorem is now an exact analog of Theorem I.1.15 with the above replacements.

Q.E.D.

* * *

The procedure used to obtain a non-atomic representation of $\tilde{N}^*_{F*}$ is similarly performed as was done in Part II of the series.

The set of players for $\tilde{N}^*_{F*}$, $F^* = [0,\omega]$, generates a *Finitely additive measure space, $<F^*, A(F^*), u_{F*}>$ where $A(F^*)$ is the internal algebra of sets in F^* and $u_{F*}(S) = \frac{||F^* \cap S||}{||F^*||}$ for $S \in A(F^*)$. Then by Theorem I.7 of Part II of the series, there exists a non-atomic extension of $stu_{F*}(\cdot)$ on $<X(F^*), X(A(F^*)), m>$, the Loeb space of $<F^*, A(F^*), u_{F*}>$, where $X(A(F^*))$ is the smallest σ-algebra containing $A(F^*)$.

Let $\psi(\tilde{N}^*_{F*})$ be defined as follows: Map each player $j \in F^*$ to himself via the identity mapping into $X(A(F^*))$ and let S_x and j_x indicate the images of sets in $A(F^*)$, and players in F^*, in $X(A(F^*))$, respectively. Define the set $\tilde{T}_x(X(F^*))$ as $\tilde{T}_x(X(F^*)) = \{st(x^*) : x^* \in \tilde{T}(F^*)\}$ and let the payoff functions $\tilde{H}_{jx}$ be defined as $\tilde{H}_{jx} = st(\tilde{H}_j)$. Then a representation in the Loeb space of $\phi^* = <F^*, A(F^*), u_{F*}>$, denoted as $\psi = <X(F^*), X(A(F^*)), m>$, of the game $\tilde{N}^*_{F*}$ is obtained by $\psi(\tilde{N}^*_{F*}) = <X(F^*), \tilde{T}_x(X(F^*)), \{\tilde{H}_{jx}\} j_x \in X(F^*)>$.

▶Theorem II.2.1: For $x^* \in \tilde{T}(F^*)$, $st(x^*)$ is m-measurable.

Proof: Lemma II.1.2 and an analog of Theorem of I.9 of Part II of the series.

Q.E.D.

Theorem II.2.2: If $\tilde{x}^*$ is an equilibrium T-strategy for $\tilde{N}^*_{F^*}$, then $st(\tilde{x}^*)$ is an equilibrium T-strategy for $\psi(\tilde{N}^*_{F^*})$.

Proof: If $\forall s \in S(n)$ $\tilde{H}_j(\tilde{x}^*) \geq s \cdot u(j,\tilde{x}^*)$ a.e. in F^*, for some $\tilde{x}^* \in \tilde{T}(F^*)$, then obviously, $st\left(\tilde{H}_j(\tilde{x}^*)\right) \geq s \cdot st\left(u(j,\tilde{x}^*)\right)$ a.e. in F^*. However, $st\left(\tilde{H}_j(\tilde{x}^*)\right) = st(\tilde{x}^*) \cdot st\left(u(j,\tilde{x}^*)\right)$ and since $u(j,\cdot)$ is S-continuous on $\tilde{T}(F^*)$, $st(\tilde{x}^*) \cdot st\left(u(j,\tilde{x}^*)\right) = st(\tilde{x}^*) \cdot u\left(j, st(\tilde{x}^*)\right)$, and one then obtains that for some $st(\tilde{x}^*) \in \tilde{T}_x\left(X(F^*)\right)$ $\left(\forall s \in S(n)\right)$ $\tilde{H}_{jx}\left(st(\tilde{x}^*)\right) \geq s \cdot u\left(j, st(\tilde{x}^*)\right)$ a.e. in $X(F^*)$, since negligible sets in $A(F^*)$ are null in $X\left(A(F^*)\right)$.

Q.E.D.

It can be observed that Theorem II.1.5 along with Theorem II.2.2 provides an alternative means for arriving at Theorem 1 of Schmeidler [4]. Professor Salim Rashid of Dartmouth College has obtained results that provide an alternative means for arriving at Theorem 2 of Schmeidler by similar constructions.

III. Ordinal Cores of *Finite Simple Games

As an additional framework of application, we consider a *Finite characterization of a class of simple games that arise in the theory of social choice. The characterization provides, by the means established above, generalization of results that have been obtained for the case of standard games, having a finite number of participants, to the non-atomic setting.

The concept of a simple game originates in Von Neumann and Morgenstern [6], Chapter X, to characterize those cooperative games in which any given coalition is either completely efficacious or impotent with respect to a set of outcomes. Such coalitions are termed, respectively, winning or losing. The first author to note the formal similarity between the structure of simple games and the analysis of social choice, as formulated by Arrow [7], was H. J. Blau [8].

A closely related area to the general theory of social choice is that of n-person voting games, the canonical form of which is noncooperative. For a specified set of outcomes, Q, which is usually finite, a finite set of participants $N = \{1, \ldots, n\}$ to which is ascribed a collection of ordinal preference rankings $\{R_j\}_{j \in N}$, each of which is a weak ordering on $Q \times Q$. Winning sets, in terms of a voting interpretation, are characterized as simple majorities, i.e., $\{S \in 2^N : |S| > \frac{N}{2}\}$ for $\frac{N+1}{2}$ not an integer. An outcome, $\hat{q} \in Q$, is said to carry the ballot, so to speak, if, for some winning set T, it is the case that $(\forall j \in T)(\forall q \in Q)(\hat{q}R_j q)$.

The chief matter of concern, in such games, is not so much that of constructing social welfare functions of a specific character, as it is the case in the general theory of social choice, but rather it is the characterization of what Dummett and Farquharson [9] have termed, stable outcomes. The distinction and similarity between the two theories is made by those authors on page 35 in Reference 9.

On the assumption that each player has an identical strategy set, $S_j = \{s_{1j}, \ldots, s_{\ell j}\}$, where s_{ij} signifies that the j^{th} player chooses the i^{th} item or issue, an outcome, viewed as a round of voting, is a member of the quotient space $\tilde{S} = \left(\prod_{j=1}^{n} S_j\right)\Big/[\sim] = Q$, where $\sim$ is derived from the R_j to mean $s \sim t$ for $s,t \in S = \prod_{j=1}^{n}(S_j)$, if and only if $(\forall j \in N)(sR_jt . \wedge . tR_js)$. An outcome is then viewed as an equivalence class of joint strategy choices by each participant from his respective strategy set. The assumption that $S_j = S_i$ for all $i,j \in N$, simply means that each voter is faced with the same issues. In the terminology of Dummett and Farquharson [9], an element of $\prod_{j=1}^{n} S_j = S$ is a situation, and a situation $\hat{s} \in S$ is said to be stable if it is the case that $(\sim\exists t \in S)(\forall j \in N)\left(tP_j\hat{s} . \vee . t(j) = \hat{s}(j)\right)$, where $tP_j\hat{s}$ is derived from R_j to mean that $(tR_j\hat{s}) . \wedge . \sim(\hat{s}R_jt)$. The interpretation that is given to the above condition is that it implies that no group of participants can obtain a result that all members prefer to $\hat{s}$ by voting differently, that is, by choosing a different element from their strategy sets. A stable outcome is then one which contains a stable situation.

It is noted by Dummett and Farquharson that the concept of stable outcome is in actuality a generalization of a Nash equilibrium point ([9], fn. 5, p. 36). Dummett and Farquharson also provide sufficient conditions for a stable outcome to exist. The conditions are based on Black's concept of single-peakedness, but are a weakened version.

More recently, several authors, and notably Kramer [10] and Nakamura [11], have combined the frameworks of simple games and the characterization of stable outcomes, found in Dummett and Farquharson, to consider a class of n-person majority games possessing what we choose to call ordinal cores. We present in what follows, a typical characterization of such games in a *Finite context and indicate the manner in which the results of the above authors can be generalized by the means presented in earlier sections, to the case of non-atomic games. The characterization follows closely that of Nakamura [11], with the exception that we do not restrict the space of outcomes to finite sets. The principal theorem can then be construed as a slight generalization of Nakamura's Theorem 1.

* * *

We shall be concerned with the games of the form, $N^*_S = \langle F^*, \hat{X}^*, W, \{R_j\}_{j \epsilon F^*} \rangle$, where F^* is an internal *Finite set, $[0,\omega]$, for $\omega \epsilon N^* - N$, an infinite integer; $\hat{X}^*$ a Q-compact set; W a collection of subsets in $A(F^*)$, the internal algebra of sets in F^*, termed the winning coalitions; and $\{R_j\}_{j \epsilon F^*}$, an internal collection of weak orderings on $\hat{X}^*$.

We will assume that W will satisfy:

(i) $F^* \epsilon W$ and $\phi \notin W$

(ii) $S \epsilon W$ and $S \subset T$ implies $T \epsilon W$

Then in accordance with (i) and (ii), respectively, we will say that N^*_S is comprehensive and monotonic. If, in addition, N^*_S satisfies the following,

(iii) $S \epsilon W$ implies $(F^* - S) \notin W$

(iv) $S \notin W$ implies $(F^* - S) \epsilon W$

then N^*_S will be said to be, respectively, proper or strong.

The assumption that will be placed on $\hat{X}^*$ is that it be generated by the product of the participant strategy sets, X^*_j, $j \epsilon F^*$, where each X^*_j is an n-dimensional simplex of standard dimension. An element $\tilde{x}^* \epsilon \hat{X}^*$ will be termed, in the manner of Dummett and Farquharson, a situation. Then if $\tilde{x}^*$ is a situation, $\tilde{x}^*(j) \epsilon X^*_j$ for each $j \epsilon F^*$ such that $\tilde{x}^*(j) \epsilon [0,1]$.

Each weak ordering R_j is seen to generate a pair of induced orderings of $\hat{X}$ as follows: For distinct pairs $\tilde{x}^*_1$ and $\tilde{x}^*_2$ in $\hat{X}^*$,

(v) $\tilde{x}^*_1 P_j \tilde{x}^*_2$ if $\tilde{x}^*_1 R_j \tilde{x}^*_2$ and $\sim(\tilde{x}^*_2 R_j \tilde{x}^*_1)$

(vi) $\tilde{x}^*_1 I_j \tilde{x}^*_2$ if $\tilde{x}^*_1 R_j \tilde{x}^*_2$ and $\tilde{x}^*_2 R_j \tilde{x}^*_1$

The quotient space, $\hat{X}^*/[I]$, the elements of which are equivalence classes, $[x^*] = y^* : x^* I y^*$, for $x^* I y^*$ defined as $x^* I y^*$ if $(\forall j \in F^*)(x^* I_j y^*)$, will be called the space of outcomes, which we denote as Ω. Then it is readily seen that the ordering, R_j on $\hat{X}^*$, induces an ordering, $\tilde{R}_j$ on Ω, for each $j \in F^*$ as:

(vii) $(\forall x^* \in [x^*])(\forall y^* \in [y^*])\left([x^*]\tilde{R}_j[y^*] \Leftrightarrow x^* R_j y^*\right)$

Then $\tilde{R}_j$ generates the induced orderings $\tilde{P}_j$ and $\tilde{I}_j$ in the manner of (v) and (vi) on Ω.

For a given $x^* \in \hat{X}^*$, and $T \in A(F^*)$, we will denote by x^*_T, the truncation of x^* to the entries of $x^*(j)$ for $j \in T$, which is to say that $x^*_T = \left(x^*(j)\right)_{j \in T}$. Let $\hat{X}^*_T$ denote the set $\{x^*_T : x^* \in \hat{X}^*\}$.

A given situation, $\tilde{x}^* \in \hat{X}^*$, will be called unstable if there is a significant coalition, i.e., $T \in A(F^*)$ for which $\left(\frac{||T||}{||F^*||} \neq 0\right)_{\text{Mod } M_1}$, and such that $(y^*_T, \tilde{x}^*_{F^*-T}) P_j \tilde{x}^*$ for all $j \in T$, for some $y^*_T \in \hat{X}^*_T$. A situation $\tilde{x}^* \in \hat{X}^*$ is said to be stable if it is not unstable. Intuitively, a stable situation is such that no nonnegligible coalition can obtain a result, by choosing different strategies, that all members of the group prefer, on the assumption that all participants not in that coalition leave their strategic choices unaltered. The notion of stable situation is close in spirit to Aumann's concept of the α-core and derives from Farquharson's generalization of Nash-type equilibria [9]. An outcome is said to be stable if it contains a stable situation. The game, N_S^*, is stable if there exists a stable outcome.

We define next a concept of domination on the space of outcomes. An outcome $[x^*]$ dominates an outcome $[y^*]$, symbolized as $[x^*]\mathrm{dom}[y^*]$, if there is an $S \in W$ such that $[x^*]\tilde{P}_j[y^*]$ for all $j \in S$. For any non-empty subset $\Theta \subseteq \Omega$, following Nakamura [11], we term the Θ-core of the game N_S^* as:

(viii) $C(\Theta) = \{[x^*] \in \Theta : \forall [y^*] \in \Theta \sim([y^*]\mathrm{dom}[x^*])\}$

The Ω-core of the game, N_S^*, we will term the ordinal core of the game.

To more precisely characterize the simple nature of the class of games in consideration, we give the following definition in terms of outcomes and situations. The game N_S^* is simple if for any $[x^*] \in \Omega$, there is an $s^* \in \hat{X}^*$, such that $t^* \in [x^*]$ for any $t^* \in \hat{X}^*$ for which $t_T^* = s_T^*$ for some $T \in W$. The definition has the interpretation that outcomes are determined by situations that are brought about by the strategic choices of the winning coalitions. We will implicitly assume that members of W are nonnegligible and such that N_S^* is proper.

Following Kramer [10], we will say that the collection of orderings, $\{R_j\}_{j \in F^*}$, is weakly single-peaked if for any distinct triple of situations, $x_1^*, x_2^*, x_3^* \in \hat{X}^*$, one has

$$(x_1^* R_j x_2^* \ldots x_1^* R_j x_3^*) \quad (\forall j \in F^*)$$

or

$$(x_2^* R_j x_3^* \ldots x_2^* R_j x_1^*) \quad (\forall j \in F^*)$$

or

$$(x_3^* R_j x_1^* \ldots x_3^* R_j x_2^*) \quad (\forall j \in F^*)$$

▶Theorem III.2.1: If N_S^* is both simple and strong and if the preferences $\{R_j\}_{j \in F^*}$ are weakly single-peaked, then N_S^* is stable if and only if $C(\Omega) \neq \phi$.

Proof: The argument is essentially the same as that given by Dummett and Farquharson [9] for the class of simple majority games which can be seen as a special case of strong simple games.

Suppose $\tilde{x}^*$ were stable and $\tilde{x}^* \in [\tilde{x}^*]$. Then $[\tilde{x}^*] \in C(\Omega)$ for, if not and some $[y^*] \in \Omega$ is such that $[y^*] \text{dom} [\tilde{x}^*]$, and therefore for some $S \in W$ $y^* P_j x^*$ for all $j \in S$ and some $y^* \in [y^*]$. Then, $\tilde{x}^*$ is not stable since, by the simple character of N_S^*, $(y_T^*, \tilde{x}_{F^*-T}^*) P_j \tilde{x}^*$ for all $j \in S$.

Suppose next that $[\tilde{x}^*] \in C(\Omega)$, then for some $\tilde{x}^* \in [\tilde{x}^*]$, $\tilde{x}^*$ is stable, for if $\tilde{x}^*$ were not stable then for some $T \in W$, $(y_T^*, \tilde{x}_{F^*-T}^*) P_j \tilde{x}^*$ for all $j \in T$. But then since $y^* \notin [\tilde{x}^*]$, $y^* \in [y^*]$ and by definition $[y^*] \text{dom} [\tilde{x}^*]$, which contradicts

$[\tilde{x}] \in C(\Omega)$. Since N_S^* is strong, any negligible coalition in W cannot alter the outcome $[\tilde{x}^*]$ by choosing differently.

Q.E.D.

Lemma III.2.2: The relation $[x^*]\text{dom}[y^*]$ is acyclic on Ω if the $\{R_j\}_{j \in F^*}$ are weakly single-peaked and if N_S^* is both simple and strong.

Proof: Nakamura [11], Theorem 1.

Q.E.D.

Theorem III.2.3: If N_S^* is simple and strong, and if the $\{R_j\}_{j \in F^*}$ are weakly single-peaked, and if for each $j \in F^*$, the set $\{y^* \in \hat{X}^* : x^* P_j y^*\} = P_j(x^*)^{-1}$ for any $x^* \in \hat{X}^*$, is Q-open, then there is at least one stable outcome for the game.

Proof: We employ the result of Bergstrom [12], and thank Jonathan Cave of Stanford's IMSSS for suggesting the reference, to establish that $C(\Omega) \neq \phi$.

A simple argument suffices to show that Ω is compact in the quotient topology on $\hat{X}^*/[I]$, the open sets of which are $\{0 \subset \Omega : \Pi^{-1}(0) \in \gamma_{\hat{X}^*}\}$, where $\Pi(x^*)$ is the mapping $\Pi : \hat{X}^* \rightarrow [\hat{X}^*]_I$, i.e., for $x^* \in \hat{X}^*$, $\Pi(x^*) = \{y^* \in \hat{X}^* : x^* I y^*\}$ for $I = \prod_{j \in F^*} [(I_j)]$ and $\gamma_{\hat{X}^*}$ is the product Q-topology on $\hat{X}^*$. The assumption that $P_j(x^*)^{-1}$ is Q-open then yields that the relation $[x^*]\text{dom}[y^*]$

is such that $\mathrm{dom}([x^*])^{-1}$ is Q-open on Ω. The following sub-lemma is due to T. Rader and is proved in Reference 12.

Lemma III.2.3.1: Let X^* be a Q-compact set and let J be an irreflexive, transitive relation on X^* such that for any $x^* \in X^*$, $\overset{-1}{J}(x^*)$ is open in the quotient topology $X^*/[I]$. Then for some $\tilde{x}^* \in X^*$, $J(\tilde{x}^*) = \phi$, for $J(\tilde{x}^*) = \{y^* \in X^* : y^* J \tilde{x}^*\}$.

Let J be the S-transitive closure of $[x^*]\mathrm{dom}[y^*]$ on $\hat{X}^*$. Then $[x^*]J[y^*]$ if and only if $[x^*]\mathrm{dom}[y^*]$ or there is a finite set $\{x_j^*\}_{j=1}^{K} \subset X^*$ such that $[x^*]\mathrm{dom}[x_1^*]\mathrm{dom}[x_2^*] \ldots [x_{K-1}^*]\mathrm{dom}[x_K^*]\mathrm{dom}[y^*]$. J is manifestly transitive on Ω by Lemma III.2.2 and it is also true that $\overset{-1}{J}([x^*])$ is Q-open in Ω, since if $[y^*] \in \overset{-1}{J}([x^*])$, then for some $[x_K^*] \in \Omega$ $[x^*]J[x_K^*]$ and also $[y^*] \in \mathrm{dom}([x_K^*])^{-1} \subset \overset{-1}{J}([x^*])$. However, by the hypothesis of the theorem, $\mathrm{dom}([x_K^*])^{-1}$ is Q-open and since $[x_K^*] \in \overset{-1}{J}([x^*])$, $\overset{-1}{J}([x^*])$ contains a Q-open set containing $[x_K^*]$. Therefore $\overset{-1}{J}([x^*])$ is Q-open. Then, by Lemma III.2.3.1, for some $[\tilde{x}^*] \in \Omega$, $J([\tilde{x}^*]) = \phi$. However, since $\mathrm{dom}([\tilde{x}^*]) \subset J([\tilde{x}^*])$ $\mathrm{dom}([\tilde{x}^*]) = \phi$. This proves the theorem in the light of Theorem III.2.1 and that $\mathrm{dom}([\tilde{x}^*]) = \phi$ implies $[\tilde{x}^*] \in C(\Omega)$.

Q.E.D.

A routine extension of Theorem III.2.3, using the results of the section on equilibrium points for non-atomic games, with the appropriate modifications, provides the existence of an m-measurable outcome, $\mathrm{st}([\tilde{x}^*])$, that is stable for the game $\psi(N^*_S)$, the non-atomic representation of N^*_S on the Loeb space of $\Phi^* = \langle F^*, A(F^*), u_{F^*} \rangle$.

References

[1] Claude Berge, Espaces Topologiques, Fonctions Multi-voques, Dunod, Paris, 1959.

[2] M. Ali Khan, "Some Approximate Equilibria," Journal of Mathematical Economics, II, 1975.

[3] J. B. Rosen, "Existence and Uniqueness of Equilibrium Points for Concave N-Person Games," Econometrica, XXX:3, 1965.

[4] D. Schmeidler, "Equilibrium Points of Non-Atomic Games," Journal of Statistical Physics, April 1973.

[5] N. Dunford and J. Schwartz, Linear Operations, Part I, Wiley-Interscience, 1969.

[6] J. Von Neumann and O. Morgenstern, The Theory of Games and Economic Behavior, Princeton University Press, 1944.

[7] K. J. Arrow, Social Choice and Individual Values, John Wiley & Sons, 1951.

[8] H. J. Blau, "Social Choice Functions and Simple Games," Bull. AMS, 63, 1957.

[9] M. Dummett and R. Farquharson, "Stability in Voting," Econometrica, Vol. 29, 1961.

[10] G. Kramer, "On a Class of Equilibrium Conditions for Majority Rule," Econometrica, Vol. 41, 1973.

[11] K. Nakamura, "The Core of a Simple Game with Ordinal Preferences," International Journal of Game Theory, Vol. IV, 1975.

[12] T. Bergstrom, "Maximal Elements of Acyclic Relations on Compact Sets," Journal of Economic Theory, 10, 1975.

[13] Alain A. Lewis, "A Nonstandard Theory of Games, Part I: On the Existence of the Quasi-Kernel and Related Solution Concepts for *Finite Cooperative Games," Discussion Paper, Center on Decision and Conflict in Complex Organizations, Harvard University, 1979.

[14] Alain A. Lewis, "A Nonstandard Theory of Games, Part II: On Non-Atomic Representations of *Finite Games," Discussion Paper, Center on Decision and Conflict in Complex Organizations, Harvard University, 1979.

[15] Alain A. Lewis, "A Nonstandard Theory of Games, Part III: Noncooperative *Finite Games," Discussion Paper, Center on Decision and Conflict in Complex Organizations, Harvard University, 1979.

[16] Alain A. Lewis, "A Nonstandard Theory of Games, Part IV: Equilibrium Points for *Finite Games," Discussion Paper, Center on Decision and Conflict in Complex Organizations, Harvard University, 1979.

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13. ABSTRACT

A noncooperative framework for *finite games is provided, where the number of participants is assumed to be indexed by an infinite *Finite set. Making use of results obtained in Part II of the series, it is shown that such games have standard non-atomic representations, thus enabling an alternative approach to Theorem 1 of Schmeidler[4].

As an additional framework of application, a slight generalization of existing results, on the Ordinal Cores of Simple Games, is provided to the *Finite context.

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