

AD-A072 542

CALIFORNIA UNIV BERKELEY OPERATIONS RESEARCH CENTER
A CURIOUS RENEWAL PROCESS AVERAGE.(U)
JUL 78 W S JEWELL

F/G 12/1

AFOSR-77-3179

UNCLASSIFIED

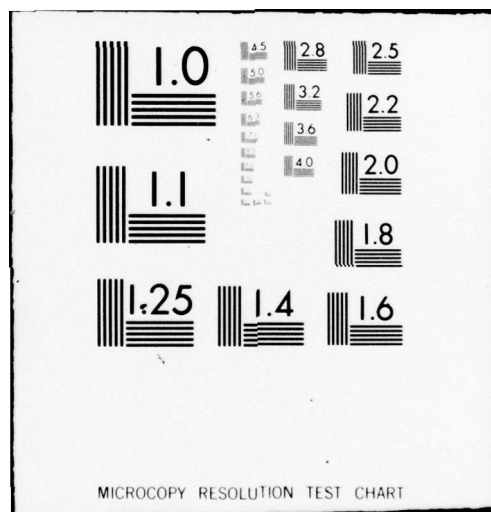
ORC-78-12

NL

| OF |
ADA
072542



END
DATE
FILMED
9-79
DDC



LEVEL

ORC 78-12
JULY 1978

12
B.S.

A CURIOUS RENEWAL PROCESS AVERAGE

by
WILLIAM S. JEWELL

AD A072542

DDC
REFINED
AUG 10 1979
C

This document has been approved
for public release and sale; its
distribution is unlimited.

DDC FILE COPY

OPERATIONS
RESEARCH
CENTER

UNIVERSITY OF CALIFORNIA • BERKELEY

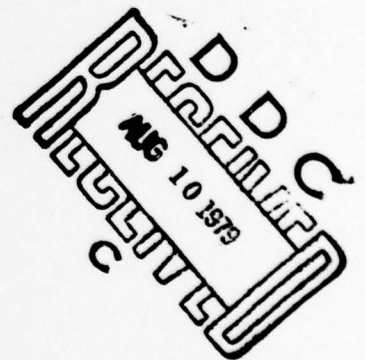
79 08 8 006

(12)

A CURIOUS RENEWAL PROCESS AVERAGE

by

William S. Jewell
Department of Industrial Engineering
and Operations Research
University of California, Berkeley



JULY 1978

ORC 78-12

This research was supported by the Air Force Office of Scientific Research (AFSC), USAF, under Grant AFOSR-77-3179 with the University of California. Reproduction in whole or in part is permitted for any purpose of the United States Government.

79 08 8 006

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 14 ORC-78-12	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) 6 A CURIOUS RENEWAL PROCESS AVERAGE	5. TYPE OF REPORT & PERIOD COVERED 9 Research Report 15	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) 10 William S. Jewell	8. CONTRACT OR GRANT NUMBER(s) 15 AFOSR-77-3179	
9. PERFORMING ORGANIZATION NAME AND ADDRESS Operations Research Center University of California Berkeley, California 94720	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 16 2304/A5 17	
11. CONTROLLING OFFICE NAME AND ADDRESS United States Air Force Air Force Office of Scientific Research Bolling AFB, D.C. 20332	12. REPORT DATE 11 July 1978	13. NUMBER OF PAGES 9
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report) Unclassified	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited. 12 10p.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Renewal Process MTBF Estimation		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) (SEE ABSTRACT)		

DD FORM 1 JAN 73 1473

EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-LF-014-6601

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

270 750

15

ABSTRACT

The lifetimes of a renewal process observed during a fixed interval $(0, t]$ are smaller, on the average, than the process mean lifetime; it is shown that the mean observed lifetime has a particularly simple form.

Accession For	
NTIS GRA&I	☒
DDC TAB	☐
Unannounced	☐
Justification	
By _____	
Distribution/	
Availability Codes	
Dist	Availand/or special
A	

A CURIOUS RENEWAL PROCESS AVERAGE

by

William S. Jewell

1. INTRODUCTION

Consider an ordinary renewal process, in which the nonnegative intervals $\{X_1, X_2, \dots\}$ are independent random variables with common distribution F , and density f . The epoch of the n^{th} renewal,

$Y_n = \sum_{i=1}^n X_i$, has density f^{n*} ; the mean epoch is clearly at $nE\{X\}$.

If we fix an interval $(0, t]$, the random number of renewals in that interval, $N(t)$, has discrete density

$$P_n(t) = \Pr \{N(t) = n\} = \int_0^t P_{n-1}(t-x) dF(x) = F^{n*}(t) - F^{(n+1)*}(t).$$

The mean number of renewals, $M(t)$, has a special role in renewal theory; it is given by the integral equation

$$M(t) = F(t) + \int_0^t M(t-x) dF(x).$$

One interesting well-known "Waldian" formula concerns the next following renewal after time t ; it occurs at mean value

$$E\{Y_{N(t)+1}\} = E\{X\} \cdot [M(t) + 1].$$

However, simple results about $Y_{N(t)}$ do not seem to be obtainable; for example:

$$E\{Y_{N(t)} \mid N(t) > 0\} = \frac{\int_0^t [1 - F(t-x)] dM(x)}{F(t)} .$$

2. THE AVERAGE COMPLETED INTERVAL

However, the mean value of the completed intervals in $(0, t]$ has an unexpected simple form; this was discovered while investigating biases in renewal testing [1].

Theorem:

$$E \left\{ \frac{Y_{N(t)}}{N(t)} \mid N(t) > 0 \right\} = E(X_1 \mid X_1 \leq t) = \frac{\int_0^t x dF(x)}{F(t)} = t - \int_0^t \left[\frac{F(x)}{F(t)} \right] dx .$$

A direct proof uses Laplace transforms on the numerator of:

$$E \left\{ \frac{Y_{N(t)}}{N(t)} \mid N(t) > 0 \right\} = \frac{\sum_{n=1}^{\infty} \frac{1}{n} \int_0^t [y f^{n*}(y)] [1 - F(t-y)] dy}{F(t)} .$$

First, the transforms of the two terms in square brackets in the numerator are found in terms of the transform of f , and then multiplied, as the integral is a convolution; a term in n is created, fortuitously cancelling $1/n$, and the sum introduces further cancellation. The resulting expression is then recognized as the transform of $\int_0^t x dF(x)$.

A simple proof that perhaps explains the theorem better was suggested by H. Gerber, University of Michigan:

(1) It is easily verified that, given $N(t) = n > 0$,

$X_1, X_1, \dots, X_n$ are *exchangeable random variables*;

(2) Therefore, given $N(t) = n > 0$, $E\{Y_n/n\} = E\{X_1\}$;

(3) But, cases in which $n = 1, 2, \dots$ are exactly those cases
in which $X_1 \leq t$. Q.E.D.

The average completed interval is approximately $t/2$ for small values of t , and increases monotonically to $E\{X_1\}$, usually slowly. For example, if f is exponential with parameter λ ,

$$E\{X_1 \mid X_1 \leq t\} = \frac{1}{\lambda} \left[\frac{1 - (1 + \lambda t)e^{-\lambda t}}{1 - e^{-\lambda t}} \right].$$

Unfortunately, the simplicity of the above result does not generalize to other functions of $W(t) = Y_{N(t)}/N(t)$, ($N(t) > 0$). For example, the density of $W = w$ consists of different sums of convolutions over different intervals. The exponential case,

$$\lambda p(w) = \frac{\sum_{n=1}^{[t/u]} \frac{n^n (\lambda w)^{n-1}}{(n-1)!}}{(e^{\lambda t} - 1)},$$

is a real wonder, being level over $(t/2, t]$, linear over $(t/3, t/2]$, quadratic over $(t/4, t/3]$, etc. Figure 1 attempts to show this behavior for $\lambda t = 2$.

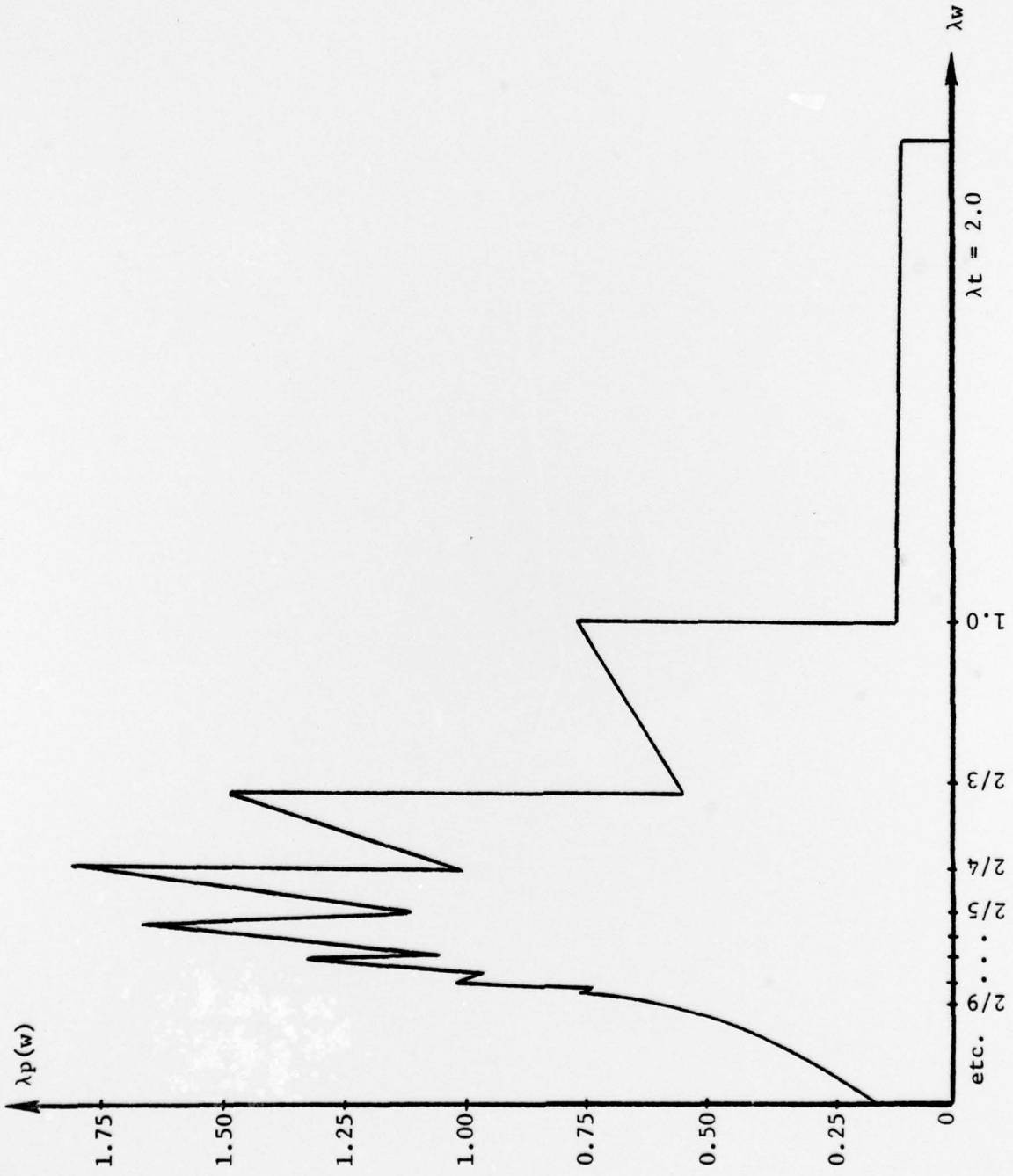


FIGURE 1
DENSITY OF AVERAGE COMPLETED INTERVALS IN $(0, t]$ FOR EXPONENTIAL
INTERVALS WITH PARAMETER λ ($\lambda t = 2$)

REFERENCE

- [1] Jewell, W. S., "'Reliability Growth' as an Artifact of Renewal Testing," ORC 78-9, Operations Research Center, University of California, Berkeley, (June 1978).