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by

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SMALL-SCALE YIELDING AT THE TIP OF A THROUGH-CRACK IN A SHELL

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ABSTRACT

Deformation theory is used to model plastic deformation at the tip of a through-crack in a thin shell. In the vicinity of the crack the shell is subjected to both stretching and bending, but stretching is assumed to dominate. Thus the stresses are tensile, but with a non-uniform distribution through the thickness, which depends on the material properties as well as on the geometry. The non-linear near-tip fields (which are singular) have been analyzed asymptotically. Cracks in shallow shells and spherical shells have been investigated in some detail. It is shown that the angular variations are the same as for generalized plane-stress plate problems. Assuming small-scale yielding a path-independent integral, which is valid in a region close to the crack edge, is used to connect the nonlinear near-tip fields with the corresponding singular parts of the linear fields. It is shown that the nonlinear behavior significantly affects the through-the-thickness variations of the near-tip fields. The singular parts of the membrane stresses tend to become more uniform through the thickness of the shell with stronger strain hardening.

INTRODUCTION

A considerable body of literature exists on the computation of the fields of stress and deformation near a through-crack in a shell. Most of this work has been motivated by applications in pressure vessel technology. The published analytical work is generally based on classical shell theory in which transverse shear deformation is neglected. It is also almost exclusively concerned with linearly elastic behavior, see e.g. Refs. 1 - 3.

In this paper small scale yielding near the edge of a through-crack in a shell is investigated on the basis of deformation theory with power-law strain hardening. Thus, the strains are small, but the stress-strain relation is nonlinear. Effects of transverse shear deformation are implied in the analysis. In recent work on plates, 4 - 5, as well as shells, 6 - 9, it has been shown that elastic near-tip fields are of a different nature if shear deformation is explicitly excluded.

Deformation theory was used by Hutchinson 10 and Rice and Rosengren 11 to analyze the near-tip fields for pure stretching of a cracked sheet. Extensions to bending of a flat plate, in a formulation which includes transverse shear, were investigated by Achenbach and Bubenik 12. Here we further extend the results of 10 - 12 to through-cracks in shells.

Deformation theory is not valid for unloading, and consequently the results apply only when the stresses are tensile throughout the thickness of the shell. Hence we can consider external loads which give rise to bending and stretching, but the stretching must be dominant in order that the results of this paper will be valid. We also restrict the attention to Mode-I near-tip fields. The transverse normal stress is neglected in the stress-strain law, which implies that the solutions are valid only for thin shells. Both spherical and shallow shells are treated in some detail. The non-linear near-tip fields are related to the corresponding linear fields by a path-independent integral.

CONSTITUTIVE RELATIONS

Similarly to the studies of in-plane deformation reported in Refs. 10 and 12, the simplest deformation theory with linearized strain-displacement relations will be employed. It is assumed that the plastic deformation is incompressible, and that the plastic strains ϵ_{ij}^{p}, and the stresses, σ_{ij}, are related by

$$\epsilon_{ij}^{p} = \frac{3}{2} C(\sigma_e)^{n-1} s_{ij} \tag{1}$$

where s_{ij} is the stress deviator

$$s_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij} \tag{2}$$

and the "effective" stress is defined as

$$\sigma_e^2 = \frac{3}{2} s_{ij} s_{ij} \tag{3}$$

In Eq.(1) n is the power hardening coefficient and C is a material constant. The stresses have been nondimensionalized by a yield stress Y and the strains have been normalized with respect to the corresponding yield strain Y/E, where E is the initial slope of the stress-strain curve for one-dimensional stress.

The dominant terms in the stresses and strains in the vicinity of a crack tip are governed by the nonlinear relation Eq.(1). Considering the case of generalized plane stress, i.e.,

$$\sigma_{33} = 0 , \tag{4}$$

and taking into account that the deformation in the nonlinear range is incompressible, i.e.,

$$\epsilon_{pp} = 0 , \tag{5}$$

Eq.(1) can be inverted to yield

$$\sigma_{ij} = \frac{2}{3} C^{-1/n} E^{(1-n)/n} [\epsilon_{ij} + (\epsilon_{11} + \epsilon_{22}) \delta_{ij}] \tag{6}$$

where

$$E^2 = \frac{2}{3} \epsilon_{ij} \epsilon_{ij} . \tag{7}$$

Equation (6) defines the constitutive relation for power-law strain hardening which will be used in this paper. This constitutive relation is, of course, really a nonlinear elastic stress-strain relation; to represent plastic yielding it can therefore only be used when the crack tip region is being monotonically loaded.

ASYMPTOTIC ANALYSIS OF NEAR-TIP FIELDS

For reference purposes the strain-displacement relations and the equilibrium equations for an arbitrary shell element are stated in the Appendix. We assume that these equations are also valid in the immediate vicinity of a crack tip. The stresses and strains are related by the nonlinear constitutive relations for deformation theory which were presented in the previous section.

The asymptotic analysis of the near-tip fields is based on an assumed separation-of-variables form of the near-tip displacements. In the subsequent substitutions of these displacement expressions into the strain-displacement relations, the constitutive relations and the equilibrium equations only the dominant terms are retained, which simplifies these equations considerably. In addition we consider only shallow shells and spherical shells, for which the extension and bending displacements are uncoupled. This provides further simplifications. We do, however, include transverse shear deformation which, as has been shown for plates and shells, see e.g. Refs. 4 - 9, is necessary for consistent results.

For linearly elastic behavior the general form of near-tip fields for cracks in shallow shells has been investigated see e.g. Ref. 8. The results of Ref. 8 have been obtained on the basis of the usual assumption for the displacement variation through the thickness of the shell, i.e.,

$$u_i(\alpha_1,\alpha_2,z) = \bar{u}_i(\alpha_1,\alpha_2) + z\, \beta_i(\alpha_1,\alpha_2) \tag{8}$$

$$w(\alpha_1,\alpha_2,z) = \bar{w}(\alpha_1,\alpha_2) \tag{9}$$

where $i = 1,2$. In a polar coordinate system (r,θ) centered at the crack tip, the Mode-I near-tip fields are then of the form

$$\bar{u}_\gamma(r,\theta) = U_\gamma(\theta)\, r^{\frac{1}{2}} \tag{10}$$

$$\beta_\gamma(r,\theta) = B\, U_\gamma(\theta)\, r^{\frac{1}{2}} \tag{11}$$

where $\gamma = r,\theta$. The transverse displacement is of higher order in r, namely.

$$w(r,\theta) = O(r^{3/2}) \tag{12}$$

In Eq.(11), B is a constant, while

$$U_r = \frac{1}{2}(1 + \nu) [(2\kappa-1) \cos(\theta/2) - \cos(3\theta/2)] \tag{13}$$

$$U_\theta = \frac{1}{2}(1 + \nu) [-(2\kappa + 1) \sin(\theta/2) + \sin(3\theta/2)] \tag{14}$$

where ν is Poisson's ratio, and κ is the plane-stress constant

$$\kappa = \frac{3 - \nu}{1 + \nu} \tag{15}$$

For a shell of arbitrary (but smooth) curvature, and for constitutive behavior according to deformation theory, it may be assumed that analogously to the linearly elastic results for shallow shells given by Eqs.(10)-(12), in the vicinity of the crack tip w is again an order or magnitude higher in r than the in-plane displacements. Consequently the third term in both Eq.(73) and (74) may be ignored, as well as the terms containing the stresses σ_{1z} and σ_{2z} in Eqs.(76) and (77). All terms in Eq.(78) are of higher order, and Eq.(78) does, therefore, not enter in the present considerations.

Since the dominant terms in Eqs.(73) - (77) still contain A_1 and A_2 it will be best to consider specific cases. Here we consider shallow shells and spherical shells. For a spherical shell we have

$$R_1 = R_2 = R \tag{16}$$

and polar coordinates (r,θ) in the mid-plane define orthogonal lines of principal curvature. It follows from Eqs.(70) and (71) that

$$A_1 = 1 , \; A_2 = R \sin(r/R) \simeq r \tag{17}$$

For a shallow shell we have

$$1 + \frac{z}{R_1} \simeq 1 + \frac{z}{R_2} \simeq 1 \tag{18}$$

and since curvature effects are neglected in the strain-displacement relations, r and θ can also be used as coordinates, with A_1 and A_2 defined by Eq.(17).

Analogously to the linearly elastic case we now consider the following asymptotically valid expressions for the near-tip displacements for material behavior according to deformation theory

$$u_r = C\,K^n\,U_r(\theta)\,f(z)\,r^p \tag{19}$$

$$u_\theta = C\,K^n\,U_\theta(\theta)\,f(z)\,r^p \tag{20}$$

$$w = 0 \tag{21}$$

The constant C is the same as the one which appears in the constitutive relation, Eq.(1). The function f(z), which need not be linear in z, is positive in accordance with the requirement that no unloading occurs in a cross-sectional area. The purpose of the analysis is to determine $U_i(\theta)$, $f(z)$ and p.

Substitution of (19) - (21) into the strain displacement relations (73) - (75) yields for shallow shells

$$\epsilon_r = C\,K^n\,p\,U_r(\theta)\,f(z)\,r^{p-1} \tag{22}$$

$$\epsilon_\theta = C\,K^n\,[U_r(\theta) + U_\theta'(\theta)]\,f(z)\,r^{p-1} \tag{23}$$

$$\epsilon_{r\theta} = \frac{1}{2}\,C\,K^n\,[U_r'(\theta) + (p-1)\,U_\theta(\theta)]\,f(z)\,r^{p-1} \tag{24}$$

For spherical shells we find

$$\epsilon_r = C\,K^n\,p\,U_r(\theta)\,f(z)\,(1+z/R)^{-1}\,r^{p-1} \tag{25}$$

$$\epsilon_\theta = C\,K^n\,[U_r(\theta) + U_\theta'(\theta)]\,f(z)\,(1+z/R)^{-1}\,r^{p-1} \tag{26}$$

$$\epsilon_{r\theta} = \frac{1}{2}\,C\,K^n[U_r'(\theta)+(p-1)U_\theta(\theta)]\,f(z)(1+z/R)^{-1}r^{p-1} \tag{27}$$

In the next step Eqs.(22) - (27) are substituted into the expressions for the stresses, that are given by Eq.(6), to yield for the dominant terms

$$\sigma_r = K\,F(z)\,\Sigma_r(\theta)\,r^{(p-1)/n} \tag{28}$$

$$\sigma_\theta = K\,F(z)\,\Sigma_\theta(\theta)\,r^{(p-1)/n} \tag{29}$$

$$\sigma_{r\theta} = K\,F(z)\,\Sigma_{r\theta}(\theta)\,r^{(p-1)/n} \tag{30}$$

where

$$\Sigma_r(\theta) = \frac{2}{3}\,[(1 + 2p)\,U_r(\theta)+U_\theta'(\theta)]\,(E/C)^{(1-n)/n} \tag{31}$$

$$\Sigma_\theta(\theta) = \frac{2}{3}\,[(2 + p)\,U_r(\theta) + 2\,U_\theta'(\theta)](E/C)^{(1-n)/n} \tag{32}$$

$$\Sigma_{r\theta}(\theta) = \frac{1}{3}\,[U_r'(\theta) + (p-1)\,U_\theta(\theta)]\,(E/C)^{(1-n)/n} \tag{33}$$

and E is defined by

$$\left(\frac{E}{C}\right)^2 = \frac{4}{3}\,[(p^2+p+1)\,U_r^2+(U_\theta')^2+(2+p)\,U_r U_\theta' + \frac{1}{4}(U_r')^2 + \frac{1}{4}\,(p-1)^2\,U_\theta^2 + \frac{1}{2}\,(p-1)\,U_r'U_\theta] \tag{34}$$

For a shallow shell we have

$$F(z) = [f(z)]^{1/n} \tag{35}$$

while for a spherical shell

$$F(z) = [f(z)/(1+z/R)]^{1/n} \tag{36}$$

All other stresses, as well as all other terms in Eqs.(28) - (30) are of higher order near the crack edge, and they can be neglected. The constant K in Eqs.(28) - (30) will henceforth be called the stress-intensity factor.

For both spherical and shallow shells substitution of Eqs.(28) - (30) into Eqs. (76) - (77) yields the following equations for $\Sigma_r(\theta)$, $\Sigma_\theta(\theta)$ and $\Sigma_{r\theta}(\theta)$ when the terms of order $r^{(p-1)/n}$ are collected

$$\left(\frac{p-1}{n} + 1\right)\Sigma_r + \Sigma_{r\theta,\theta} - \Sigma_\theta = 0 \tag{37}$$

$$\Sigma_{\theta,\theta} + \left(\frac{p-1}{n} + 2\right)\Sigma_{r\theta} = 0 \tag{38}$$

The inertia terms do not enter in these equations since they are of order r^{p+1}. The dependence on z cancels because the functional dependence is the same for the leading terms collected in Eqs.(37) and (38).

Governing equations for $U_r(\theta)$ and $U_\theta(\theta)$ are obtained by substituting the equations for the stresses, Eqs.(31) - (33), into Eqs.(37) and (38). We find

$$\{[\tfrac{1}{n}(p-1)+6+2p]\,U_r' + (p-1)[2+\tfrac{1}{n}(p-1)]\,U_\theta+4U_\theta''\}\,E^2 + \frac{1-n}{2n}\,[2(2+p)\,U_r + 4U_\theta']\,(E^2)' = 0 \tag{39}$$

$$\{2(p-1)[1+\tfrac{1}{n}(1+2p)]\,U_r+[p-3+\tfrac{2}{n}(p-1)]\,U_\theta'+U_r''\}\,E^2 + \frac{1-n}{2n}\,[U_r' + (p-1)\,U_\theta]\,(E^2)' = 0 \tag{40}$$

These equations verify that the assumed displacement distributions (19) - (21) lead to consistent results. The governing equations (39) and (40) are identical to those for bending a flat plate, which were derived independently in Ref. 12, and which in turn were shown to be identical to the equations for extension of a flat plate.

Equations (39) and (40) must be supplemented by boundary conditions. For the present applications we consider symmetry relative to $\theta = 0$, see Fig. 1, i.e.,

$$U_r'(0) = 0 \qquad U_\theta(0) = 0 \tag{41}$$

In addition the faces of the crack are free of stresses, i.e.,

$$\sigma_\theta(\pi) = 0 \qquad \sigma_{\theta r}(\pi) = 0 \tag{42}$$

The system of equations (39) - (42) defines a nonlinear eigenvalue problem for $U_r(\theta)$, $U_\theta(\theta)$ and p. This problem can be solved numerically. After solving for $U_r''(\theta)$ and $U_\theta''(\theta)$, a solution is obtained by using a conventional fourth-order Runge-Kutta method with variable stepsize. This procedure is performed from $\theta = 0$ to $\theta = \pi$. Either $U_r(0)$ or $U_\theta'(0)$ can be chosen arbitrarily due to the homogeneous and equidimensional nature of the governing equations. The remaining initial condition and p must be chosen to meet the boundary conditions at $\theta = \pi$. The coefficient p can, however, also be stated a-priori by requiring that certain integrals which will be discussed in the next section have a finite, nonzero value right near the crack edge. This requirement gives

$$p = 1/(n + 1) \tag{43}$$

Hence the problem is considerably simplified as only the remaining initial condition need be varied. Iteration is quick provided a reasonable close first guess is used. The stresses and strains are computed numerically from the displacements and their derivatives. For n = 13, $U_r(\theta)$ and $U_\theta(\theta)$ are shown in Fig. 2. The corresponding strains and stresses are shown in Figs. 3 and 4 respectively. These results completely agree with results of Ref. 10, for plates in pure extension, which were computed on the basis of an Airy stress function formulation.

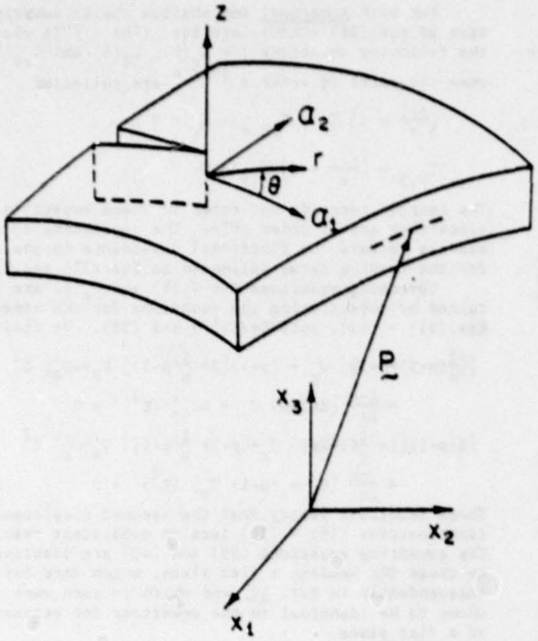

Fig. 1 Through crack in a thin shell

It is noted that the angular variations computed here are valid for static as well as dynamic problems, since in the vicinity of the crack edge the inertia terms can be neglected as compared to the terms that control the asymptotic considerations presented in this Section.

The stress-intensity factor K and the function f(z) can of course not be determined by asymptotic considerations only. They may be determined by connecting the near-tip fields to the far fields.

For the linearly elastic case the equations for $U_r(\theta)$ and $U_\theta(\theta)$ are

$$(p^2-1)(\kappa+1)\, U_r + (\kappa-1)\, U_r'' + 2(p-\kappa)\, U_\theta' = 0 \qquad (44)$$

$$2(p+\kappa)\, U_r' + (p^2-1)(\kappa-1)\, U_\theta + (\kappa+1)\, U_\theta'' = 0 \qquad (45)$$

where κ is defined by Eq.(15). The solution to the eigenvalue problem then yields $p = 1/2$, and the angular distributions defined by Eqs.(13) and (14). The corresponding stresses are

$$\Sigma_r = \frac{1}{2}\,(3 - \cos\theta)\cos(\theta/2) \qquad (46)$$

$$\Sigma_\theta = \frac{1}{2}\,(1 + \cos\theta)\cos(\theta/2) \qquad (47)$$

$$\Sigma_{r\theta} = \frac{1}{2}\,\sin\theta\cos(\theta/2) \qquad (48)$$

For an incompressible material ($\nu = 1/2$), Eqs.(44) and (45) agree with the equations that are obtained from Eqs.(39) and (40) by setting $n = 1$.

AN INTEGRAL WITH PATH-INDEPENDENT PROPERTIES

The asymptotic results presented in the previous section have to be supplemented by a method to compute the stress-intensity factor K and the function f(z). Several known solutions which include the

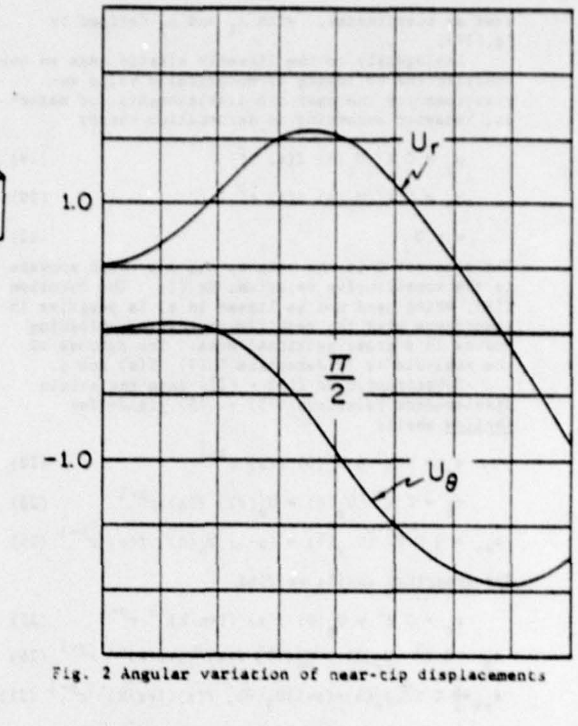

Fig. 2 Angular variation of near-tip displacements

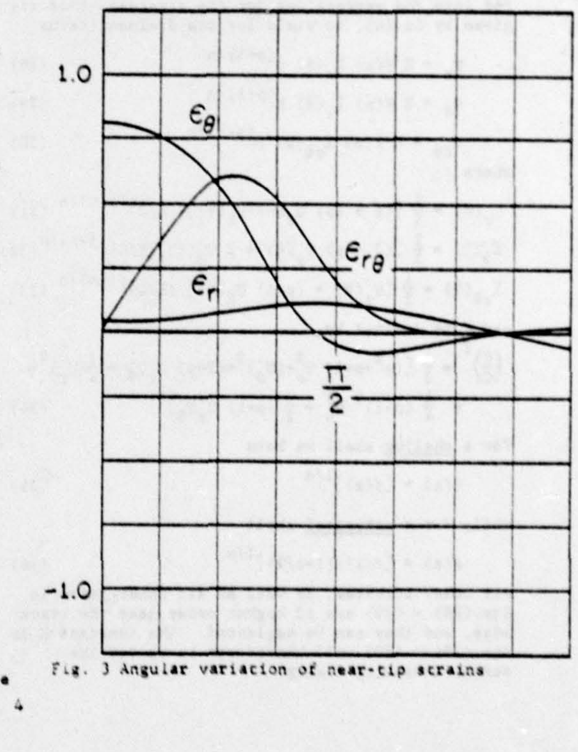

Fig. 3 Angular variation of near-tip strains

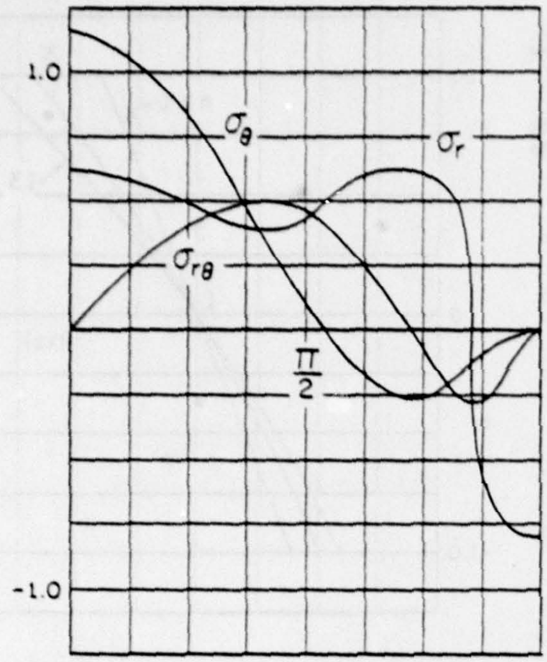

Fig. 4 Angular variation of near-tip stresses

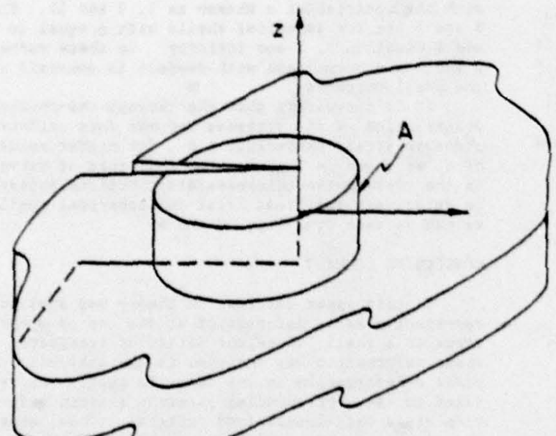

Fig. 5 Surface A enclosing a crack edge in a shell

effect of transverse shear are available for linearly elastic problems, see Refs. 6 - 9. In this section we present a method to compute K and f(z) for deformation theory from the corresponding K and f(z) for linearly elastic behavior.

Figure 5 shows a shell with a through crack, and a surface A which encloses the crack edge. The surface A is normal to the midplane of the shell, and its projection on the shell's mid-plane is Γ. It is now not difficult to show that in the vicinity of the crack edge the following surface integral has the same value for all surfaces of the kind A:

$$J_A = \int_A z^q \left(W \delta_{1j} - \sigma_{ij} \frac{\partial u_i}{\partial \alpha_1}\right) n_j \, dz \, d\Gamma \tag{49}$$

Here i and j are either 1, 2 or 3, where 1 and 2 refer to the curvilinear coordinates shown in Fig. 1, and 3 refers to z, and W denotes the strain energy. To prove this feature of J_A we consider a surface A_T made up of A^+ and A^-, and two cylindrical surfaces A_1 and A_2 of the type A, where A^+ and A^- are the surface areas between the intersections of A_1 and A_2 with the top and bottom faces of the shell. The volume enclosed by A_T is denoted by V. To show that

$$J_{A_1} = J_{A_2} \tag{50}$$

we consider an extension of Eq.(49) to the closed surface A_T:

$$J_{A_T} = \int_{A_T} z^q \left(W \delta_{1j} - \sigma_{ij} \frac{\partial u_i}{\partial \alpha_1}\right) n_j \, dA \tag{51}$$

An application of Gauss' theorem yields

$$J_{A_T} = \int_V \frac{\partial}{\partial \alpha_j} \left[z^q \left(W \delta_{1j} - \sigma_{ij} \frac{\partial u_i}{\partial \alpha_1}\right)\right] dV \tag{52}$$

Integration by parts and the use of the relations

$$\frac{\partial W}{\partial \alpha_1} = \sigma_{ij} u_{i,1j} \tag{53}$$

$$\sigma_{ij,j} = 0 \tag{54}$$

subsequently reduces Eq.(52) to

$$J_{A_T} = \int_V q z^{q-1} \left(-\sigma_{iz} \frac{\partial u_i}{\partial \alpha_1}\right) dV \tag{55}$$

Since $A_T = A_1 + A_2 + A^+ + A^-$, and since for traction-free shell faces we have

$$J_{A^+} = J_{A^-} = 0 ,$$

the result given by Eq.(50) follows, provided that

$$\text{either } q = 0 \tag{57}$$

$$\text{or } \sigma_{iz} = 0 \tag{58}$$

The case q = 0 corresponds to the usual J-integral for three-dimensional deformation. Generally $\sigma_{iz} \neq 0$, except in the immediate vicinity of the crack tip where σ_{iz} can indeed be ignored as it is of higher order as compared to the other stresses. Thus near the crack edge J_A as given by Eq.(49) has the same value for all surfaces of the type A.

NEAR-TIP FIELDS ACCORDING TO DEFORMATION THEORY

In this paper it is assumed that the material behaves nonlinearly close to the crack edge, and linearly elsewhere. We now make the additional assumption of small-scale yielding, to connect the nonlinear fields to the singular parts of the linear fields via the integral J_A given by Eq.(49), which is valid for both the linear and nonlinear fields close to the crack tip. Thus we can take two circular cylindrical surfaces A_1 and A_2 centered at the crack edge, and employ the nonlinear results of this paper to compute J_{A_1} and the corresponding linear results to compute J_{A_2}. The coefficient q in Eq. (49) is however an arbitrary integer, which implies that

$$J(z)\big|_{nonlinear} = J(z)\big|_{linear} \tag{59}$$

for a contour near the crack tip, where

$$J(z) = \int_\Gamma \left[W\,\delta_{1j} - \sigma_{ij}\,\frac{\partial u_i}{\partial x_1}\right] n_j\,d\Gamma \tag{60}$$

The integral $J(z)$ is of the same form as the usual J-integral. It depends, however, on z in a manner prescribed by the particular shell theory.

For the nonlinear field $J(z)$ gives for spherical shells,

$$J(z) = C\,K^{(1+n)}\,I\left[f(z)\right]^{(1+n)/n}\left(1 + \frac{z}{R}\right)^{-1/n} \tag{61}$$

and for shallow shells,

$$J(z) = C\,K^{(1+n)}\,I\left[f(z)\right]^{(1+n)/n} \tag{62}$$

where I is just the same as defined in Ref. 10 and 12, i.e.,

$$I = \int_{-\pi}^{\pi}\left\{\frac{n}{n+1}\,\Sigma_e^{n+1}\cos\theta - \sin\theta\left[\Sigma_r(U_\theta - U_r') - \Sigma_{r\theta}(U_r + U_\theta')\right] - p\cos\theta\,(\Sigma_r U_r + \Sigma_{r\theta} U_\theta)\right\}d\theta \tag{63}$$

where Σ_r and $\Sigma_{r\theta}$ are defined by Eqs. (31) and (33). In Eq. (63) we have also used that

$$\sigma_e = K\,F(z)\,\Sigma_e(\theta)\,r^{(p-1)/n} \tag{64}$$

As an example, we consider the case when the elastic near-tip displacements are linearly distributed, i.e.

$$f(z)\big|_{linear} = \left(1 + \frac{z}{R}\right)F(z)\big|_{linear} = 1 + \zeta z \tag{65}$$

Here ζ is a function of both the applied bending and extension loads, see for example Ref. 7. By using Eq.(65) the J-integral can be evaluated as

$$J(z)\big|_{linear} = 2\pi\,K_\ell^2\,\frac{(1 + \zeta z)^2}{(1 + z/R)} \tag{66}$$

where K_ℓ is the linear stress-intensity factor. Equating Eqs.(66) and (61) or (62) yields

$$K = \left[\frac{2\pi K_\ell^2}{C\,I}\right]^{1/(n+1)} \tag{67}$$

and, after an expansion for small z, one finds for spherical shells,

$$f(z) = 1 + \frac{1}{n+1}\left(2n\zeta + \frac{1-n}{R}\right)z + \left\{\frac{n}{n+1}\left(\zeta - \frac{1}{R}\right)^2 + \frac{1}{nR}\left[\frac{2n\zeta - (1-n)/R}{n+1} - \frac{1}{2R}\right] - \frac{1}{2n}\left[\frac{2n\zeta - (1-n)/R}{n+1}\right]^2\right\}z^2\ldots \tag{68}$$

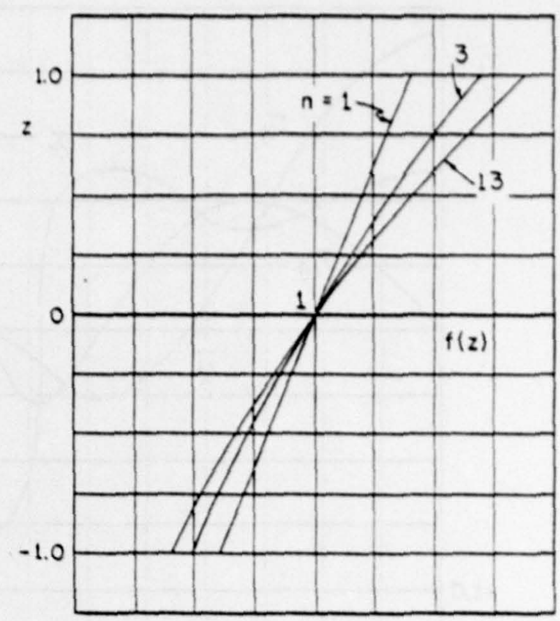

Fig. 6 Through-the-thickness variation of the near-tip displacements in a shallow shell

The corresponding expression for shallow shells is identical to the above when R is set equal to infinity.

For $\zeta = 0.4$ curves for the displacements and stresses corresponding to $f(z)\big|_{linear}$ are shown in Figs. 6 - 9. Figs. 6 and 7 are for shallow shells with the coefficient n chosen as 1, 3 and 13. Figs. 8 and 9 are for spherical shells with n equal to 3 and R equal to 3, 7 and infinity. In these curves, z and R are normalized with respect to one-half of the shell thickness.

It is noteworthy that the through-the-thickness distribution of the stresses becomes more uniform for stronger strain hardening, i.e., for higher values of n, as shown in Fig. 7. The influence of curvature on the through-the-thickness distributions appears to be relatively small, at least for spherical shells, as can be seen from Figs. 8 and 9.

CONCLUDING COMMENT

In this paper deformation theory was applied to represent plastic deformation at the tip of a through-crack in a shell, where the effect of transverse shear deformation was included in the analysis. The plastic deformation in the near-tip region was related to the corresponding linearly elastic deformation via a path-independent integral. Thus, once the elastic fields, which are represented by the stress-intensity factor, have been computed, the method of this paper can be used to compute, for example, near-tip plastic strains. These strains can then be related to a fracture criterion, yielding a final result which is completely in terms of linearly elastic results, as is consistent with small-scale

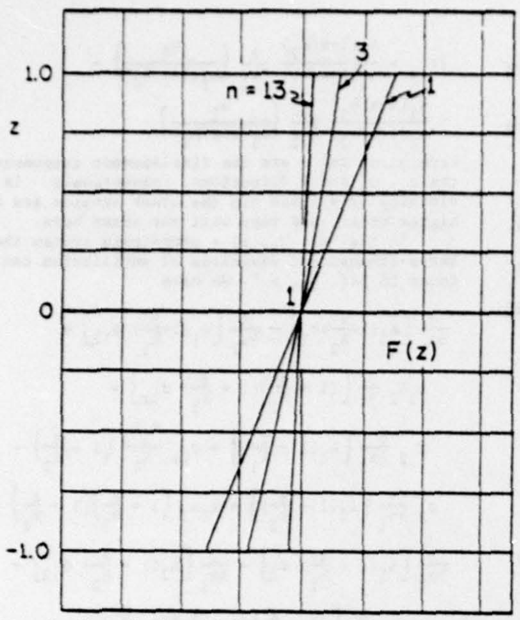

Fig. 7 Through-the-thickness variation of the near-tip stresses in a shallow shell.

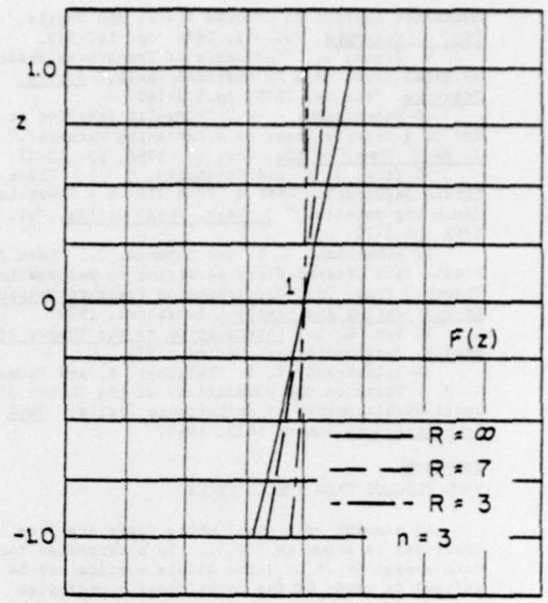

Fig. 9 Through-the-thickness variation of the near-tip stresses in a shpherical shell.

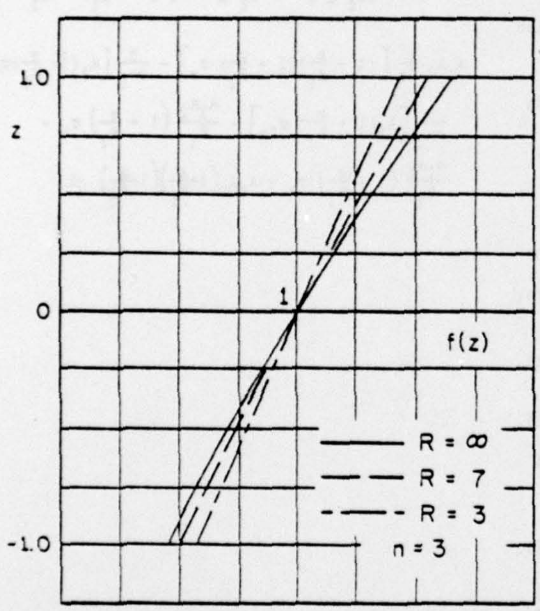

Fig. 8 Through-the-thickness variation of the near-tip displacements in a spherical shell

yielding.

ACKNOWLEDGEMENT

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APPENDIX
SOME RESULTS FROM SHELL THEORY

An element of a shell whose faces are free of tractions is shown in Fig. 1. In a cartesian coordinate system (x_1,x_2,x_3) the middle surface may be defined in terms of the curvilinear coordinates α_1 and α_2, which are taken to coincide with the orthogonal lines of principal curvature, i. e.,

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$$\underset{\sim}{\rho} = f_i(\alpha_1,\alpha_2)\, \hat{\underset{\sim}{e}}_i \tag{69}$$

where $\underset{\sim}{\rho}$ is the position vector of points in the middle-surface of the shell. The position of a point in the shell is specified by α_1, α_2 and the normal coordinate z. The magnitude of a differential element of length is then given by

$$ds^2 = A_1^2\left(1+\frac{z}{R_1}\right)^2 d\alpha_1^2 + A_2^2\left(1+\frac{z}{R_2}\right)^2 d\alpha_2^2 + dz^2 \tag{70}$$

where R_1 and R_2 are the principal radii of curvature, and

$$A_1^2 = \frac{\partial\underset{\sim}{\rho}}{\partial\alpha_1}\cdot\frac{\partial\underset{\sim}{\rho}}{\partial\alpha_1} \tag{71}$$

$$A_2^2 = \frac{\partial\underset{\sim}{\rho}}{\partial\alpha_2}\cdot\frac{\partial\underset{\sim}{\rho}}{\partial\alpha_2} \tag{72}$$

The strain-displacement relations can be found in books on shell theory, see e.g. Ref. 13, p. 25. The strains relevant to the present paper are

$$\epsilon_1 = \frac{1}{A_1(1+z/R_1)}\left\{\frac{\partial u_1}{\partial\alpha_1} + \frac{u_2}{A_2}\frac{\partial A_1}{\partial\alpha_2} + A_1\frac{w}{R_1}\right\} \tag{73}$$

$$\epsilon_2 = \frac{1}{A_2(1+z/R_2)}\left\{\frac{\partial u_2}{\partial\alpha_2} + \frac{u_1}{A_1}\frac{\partial A_2}{\partial\alpha_1} + A_2\frac{w}{R_2}\right\} \tag{74}$$

$$2\epsilon_{12} = \frac{A_2(1+z/R_2)}{A_1(1+z/R_1)}\frac{\partial}{\partial\alpha_1}\left\{\frac{u_2}{A_2(1+z/R_2)}\right\} + \frac{A_1(1+z/R_1)}{A_2(1+z/R_2)}\frac{\partial}{\partial\alpha_2}\left\{\frac{u_1}{A_1(1+z/R_1)}\right\} \tag{75}$$

Here u_1, u_2 and w are the displacement components in the α_1, α_2 and z directions, respectively. In the vicinity of a crack tip the other strains are of higher order, and they will not enter here.

In the (α_1, α_2, z) - coordinate system the three-dimensional equations of equilibrium can be found in Ref. 14, p.7. We have

$$\begin{aligned}&\frac{\partial}{\partial\alpha_1}\left[A_2\left(1+\frac{z}{R_2}\right)\sigma_1\right] + \frac{\partial}{\partial\alpha_2}\left[A_1\left(1+\frac{z}{R_1}\right)\sigma_{12}\right] + \\ &A_1A_2\frac{\partial}{\partial z}\left[\left(1+\frac{z}{R_1}\right)\left(1+\frac{z}{R_2}\right)\sigma_{1z}\right] + \\ &\sigma_{12}\frac{\partial}{\partial\alpha_2}\left[A_1\left(1+\frac{z}{R_1}\right)\right] + \sigma_{1z}\frac{A_1A_2}{R_1}\left(1+\frac{z}{R_2}\right) - \\ &\sigma_2\frac{\partial}{\partial\alpha_1}\left[A_2\left(1+\frac{z}{R_2}\right)\right] = A_1A_2\left(1+\frac{z}{R_1}\right)\left(1+\frac{z}{R_2}\right)\rho\ddot{u}_1\end{aligned} \tag{76}$$

$$\begin{aligned}&\frac{\partial}{\partial\alpha_2}\left[A_1\left(1+\frac{z}{R_1}\right)\sigma_2\right] + \frac{\partial}{\partial\alpha_1}\left[A_2\left(1+\frac{z}{R_2}\right)\sigma_{12}\right] + \\ &A_1A_2\frac{\partial}{\partial z}\left[\left(1+\frac{z}{R_1}\right)\left(1+\frac{z}{R_2}\right)\sigma_{2z}\right] + \\ &\sigma_{12}\frac{\partial}{\partial\alpha_1}\left[A_2\left(1+\frac{z}{R_2}\right)\right] + \sigma_{2z}\frac{A_1A_2}{R_2}\left(1+\frac{z}{R_1}\right) \\ &-\sigma_1\frac{\partial}{\partial\alpha_2}\left[A_1\left(1+\frac{z}{R_1}\right)\right] = A_1A_2\left(1+\frac{z}{R_1}\right)\left(1+\frac{z}{R_2}\right)\rho\ddot{u}_2\end{aligned} \tag{77}$$

$$\begin{aligned}&A_1A_2\frac{\partial}{\partial z}\left[\left(1+\frac{z}{R_1}\right)\left(1+\frac{z}{R_2}\right)\sigma_z\right] + \frac{\partial}{\partial\alpha_1}\left[A_2\left(1+\frac{z}{R_2}\right)\sigma_{1z}\right] + \\ &\frac{\partial}{\partial\alpha_2}\left[A_1\left(1+\frac{z}{R_1}\right)\sigma_{2z}\right] - \frac{A_1A_2}{R_1}\left(1+\frac{z}{R_2}\right)\sigma_1 - \\ &\frac{A_1A_2}{R_2}\left(1+\frac{z}{R_1}\right)\sigma_2 = A_1A_2\left(1+\frac{z}{R_1}\right)\left(1+\frac{z}{R_2}\right)\rho\ddot{w}\end{aligned} \tag{78}$$