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DIGITAL SIMULATION OF A GAUSSIAN SEAWAY BASED ON THE RANDOM PHA--ETC(U)
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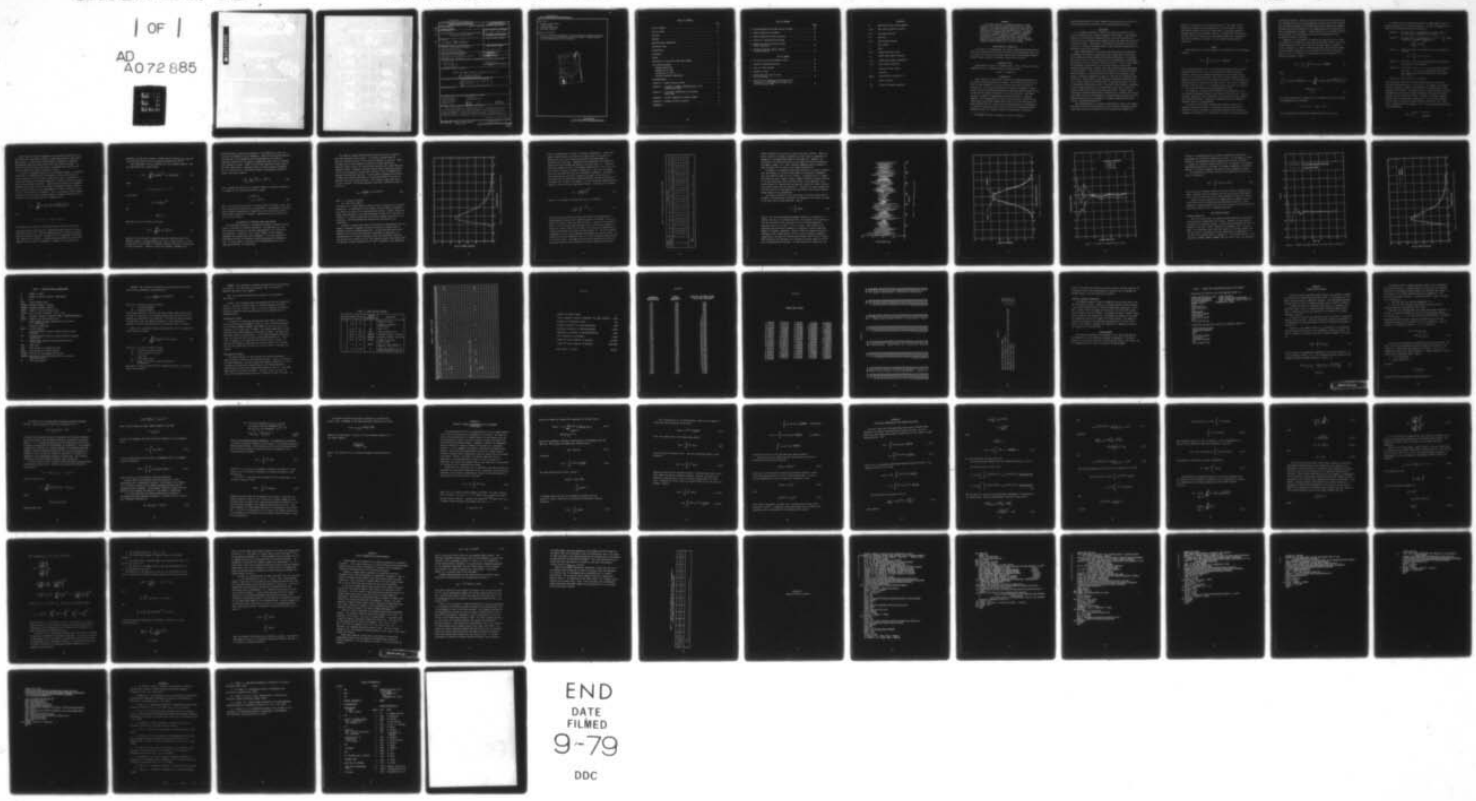

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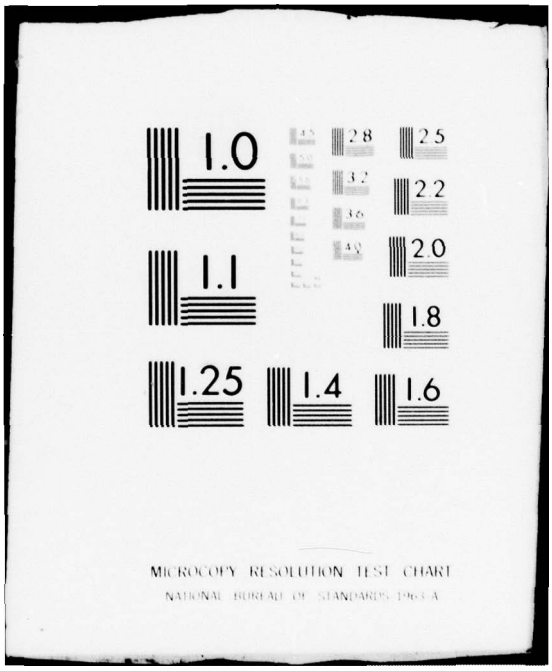
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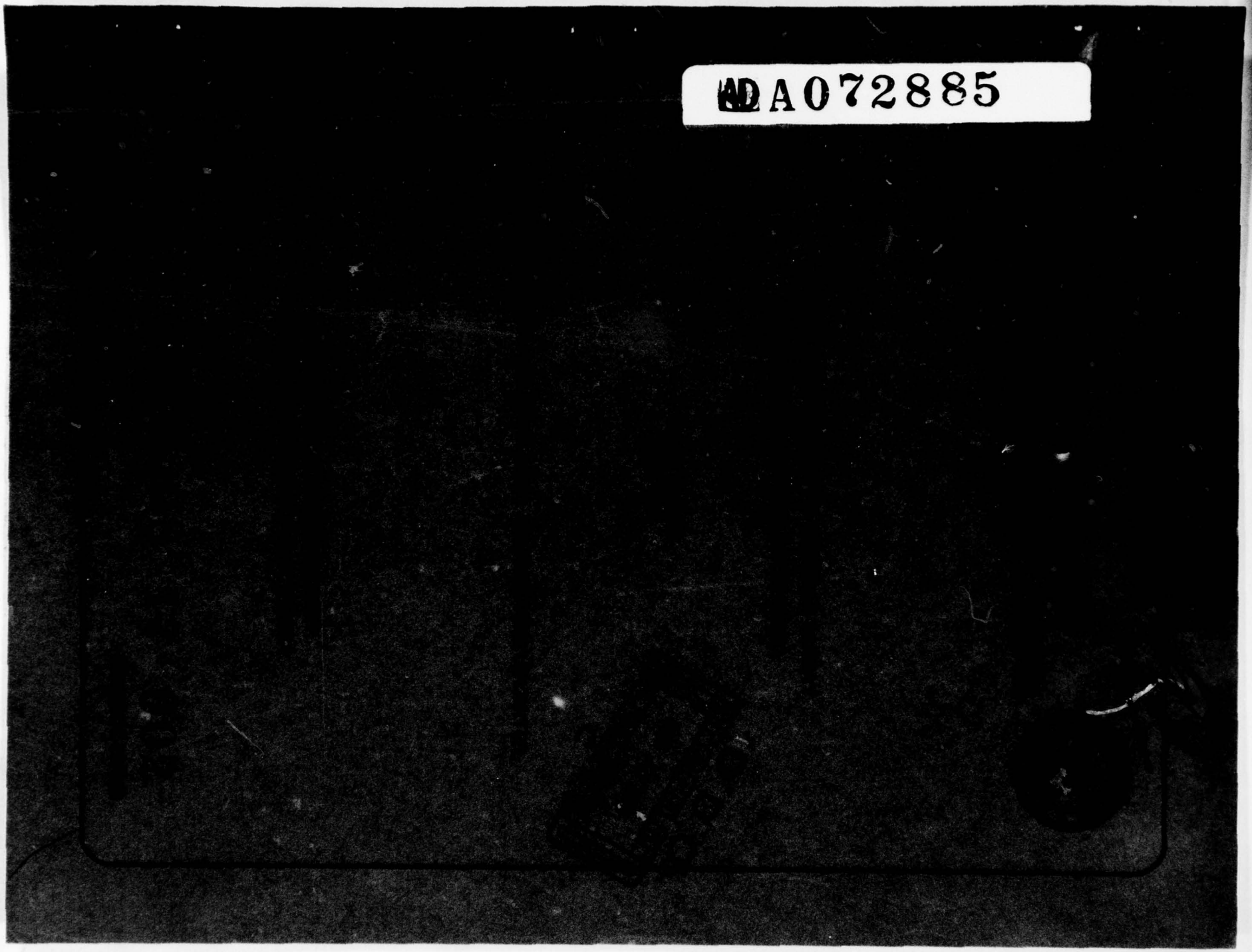
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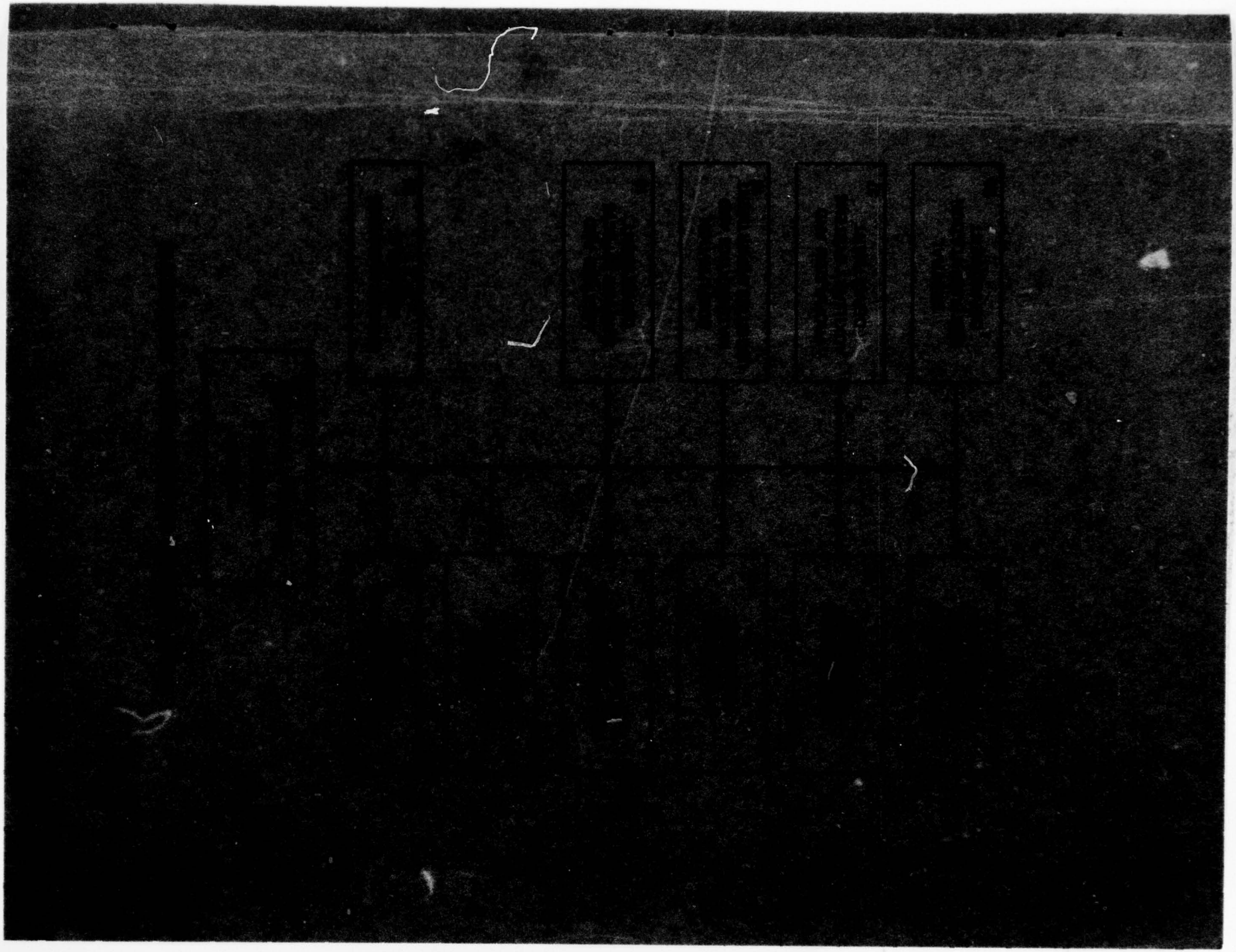

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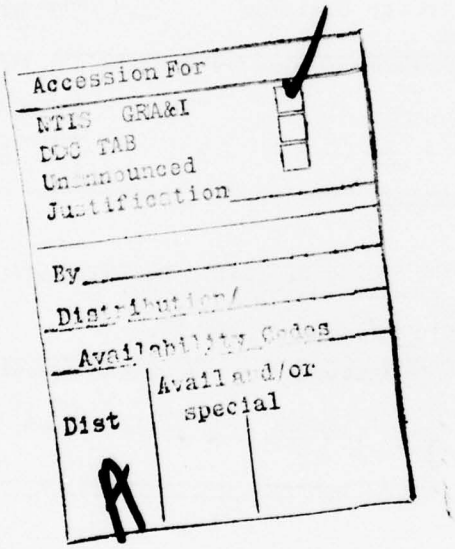

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NOTATION

E	Expectation (mean value) operator
$f_\eta(\omega)$	Wave spectral density function
$R(\tau)$	Covariance function
Re	Real part
$S(\omega)$	Wave energy spectrum
T	Wave period
t	Time
$X(t)$	Random function of time
ε_k	Random phase angle (discrete)
$\varepsilon(\omega)$	Random phase angle (continuous)
$\eta(t)$	Wave surface elevation
σ^2	Variance of wave record
τ	Time lag
$\varphi_x(p)$	Characteristic function of X
ω	Radian frequency
ω_k^*	Discrete frequency component

ABSTRACT

This report presents a convenient method for simulating time histories of ocean wave surfaces on a digital computer. The simulation model employed is basically the spectral energy wave surface representation of St. Denis and Pierson. A random selection of frequency components within each frequency interval is employed to increase the wave period of the digital process. Analytical properties of the resulting wave system are derived and comparison with computed wave records is shown. Detailed computer program documentation is also provided.

ADMINISTRATIVE INFORMATION

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CONVERSION TABLE

Measurements are given in customary units. Their conversion to metric equivalents can be computed using the following values:

$$1 \text{ ft/sec}^2 = 0.3048 \text{ m/s}^2$$

INTRODUCTION

Time histories of wave surface elevation are employed extensively in the study of ship performance in a seaway. Such time series can be used both to study properties of the wave system itself and to provide excitation for the various ship motions and propulsive responses. Current interest in the time domain simulation of ship-wave dynamic systems has led to the need for digital synthesis of wave time histories. This report presents one method for generating such time histories on a digital computer. The representation of wave surface elevation is based on the random phase wave energy integral model used by St. Denis and Pierson,[1*] with two modifications: first, the wave representation is interpreted as a stochastic integral which is closely related to the Riemann-Stieltjes integral; second, the frequency values for determining the spectral

*A complete listing of references is given on page 65.

amplitudes and phases are chosen randomly within each interval in order to increase indefinitely the period of the resulting digital process.

BACKGROUND

It is commonly accepted in linear wave theory that points of wave surface elevation follow a normal (Gaussian) distribution. This Gaussian property of ocean waves was demonstrated in a statistical analysis of field data over two decades ago by Rudnick[2] and it has since been supported by various studies. Various past investigations have dealt with the sea surface as a stationary Gaussian process, which represents approximately a relatively stable sea condition.

Many representations of stationary Gaussian processes are possible. For example, the earliest representation was through the use of Fourier series expansions having normally distributed coefficients. Such models were used as early as 1910 by Einstein in his investigation of black-body radiation and they are discussed in detail by Rice.[3] Fourier series models are appealing for application to ocean waves because of their nice physical interpretation. However, they have the unfortunate disadvantage of being periodic. Their periods can be increased by lowering the fundamental frequency of oscillation which may result in an extremely large number of frequency components in order to cover the full frequency range of interest. More recently, Levy[4] has given a different representation, the so called "canonical representation" of Gaussian processes, in terms of integrated Gaussian white noise. This model, which requires independent Gaussian random variables for its generation, would seem to involve extensive computational time in practical application. Fourier series expansions containing deterministic coefficients but uniformly distributed phase angles have also been employed to obtain periodic processes that are asymptotically normal as the number of frequency components approaches infinity (Grenander-Rosenblatt[5]).

One representation of a stationary Gaussian wave surface that appears well-suited for practical application is the continuous frequency random phase model which was introduced by Levy[6] and which was proposed as a

model for ocean waves by St. Denis and Pierson.[1] This model yields a nonperiodic random phase representation of the wave surface. The St. Denis-Pierson model has been used extensively in frequency domain investigations of ship performance; however, it appears that the model has not been explicitly exploited for time domain prediction or simulation. This report implements the model in a time domain digital-computer simulation.

THEORY

The random phase ocean wave model may be represented by the stochastic integral

$$\eta(t) = \int_0^\infty \cos\ (\omega t - \varepsilon(\omega))\ \sqrt{2S(\omega)d\omega} \qquad (1)$$

In this representation $\eta(t)$ is the unidirectional wave surface elevation. The random phase angles, $\varepsilon(\omega)$, are a function of frequency and are assumed to belong to an uniform distribution over the interval $[0,2\pi)$. $S(\omega)$ is the wave energy spectrum, which gives the mean squared value of amplitude associated with each non-negative frequency. The area under the spectrum equals the variance of the wave record. Thus, $S(\omega)$ is intimately related to the sea state. It is assumed that the sea state is stationary so that the energy spectrum will be independent of time.

For simplicity, the wave surface model is here assumed to be unidirectional and spatially fixed at the origin of the wave field. That is, the model represents surface elevations at a point in space and considers only long-crested waves. Extension to multidirectional and spatially dependent models is straightforward.

It should be noted that the random phase model of Equation (1) represents an entire ensemble of sea-surface time histories. Here, ensemble is used in the sense of statistical mechanics and indicates an assignment of probability to the functions of time which describe possible

sea-surface histories. Further, any specific time history resulting from this representation will be a deterministic function, arbitrarily selected from the infinite number of such time histories that make up the ensemble. Detailed theoretical discussion of the random phase wave surface model is given in Appendices A through C. There, Equation (1) is derived from first principles and its relationship to the more usual Riemann-Stieltjes integration notation is defined.

Equation (1) is essentially the same as the form given by St. Denis and Pierson[1] as a theoretical ocean wave surface model, except that $2\,S(\omega)$ is used instead of $\eta^2(\omega)$ and the sign of the phase angles is negative rather than positive. The St. Denis-Pierson paper also described roughly how the integral in Equation (1) is obtained as the limit in quadratic mean (ℓ.i.m.) of a sequence of random partial sums (see Appendix A for a detailed treatment). That is

$$\eta(t) = \underset{\Lambda \to \infty}{\text{ℓ.i.m.}} \int_0^{\Lambda} \cos\,(\omega t - \varepsilon(\omega))\,\sqrt{2S(\omega)\,d\omega} \tag{2}$$

where

$$\int_0^{\Lambda} \cos\,(\omega t - \varepsilon(\omega))\,\sqrt{2S(\omega)d\omega} = \text{ℓ.i.m.} \sum_{k=1}^{N} \cos\,(\omega_k^* t - \varepsilon(\omega_k^*))\,\sqrt{2S(\omega_k^*)(\omega_k - \omega_{k-1})}$$

$$\max|\omega_k - \omega_{k-1}| \to 0 \qquad N \to \infty \tag{3}$$

In the right hand side of Equation (3) the summation is over all subintervals appearing in the partition

$$0 = \omega_0 < \omega_1 \ldots < \omega_{N-1} < \omega_N = \Lambda$$

of the interval $[0,\Lambda]$, and ω_k^* is an arbitrary point in (ω_{k-1},ω_k).

It follows from the uniform distribution of phase angles that the random phase model has a Gaussian character. As shown in Appendix C, the model has the following statistical properties:

Property 1 - For each time t, the expected value (mean value) of the wave surface elevation $\eta(t)$, is zero.

Property 2 - For each time t, the variance of $\eta(t)$ is $\sigma^2 = \int_0^\infty S(\omega)d\omega$. The RMS value (standard deviation) is

$$\sigma = \left[\int_0^\infty S(\omega)d\omega\right]^{1/2}$$

Property 3 - The wave process is stationary and has the covariance function

$$R(\tau) = \int_0^\infty S(\omega)\cos(\omega\tau)d\omega$$

Property 4 - For each t, $\eta(t)$ has a normal (Gaussian) distribution with mean zero and variance σ^2.

Property 5 - For distinct times, $t_1, \ldots, t_n$, the joint distribution of $(\eta(t_1), \ldots, \eta(t_n))$ is multivariate normal with mean vector 0 and covariance matrix M, with elements $M_{ij} = R(t_i - t_j)$.

Property 6 - The process $\eta(t)$ is ergodic.

These properties imply that the random phase model is a zero mean, stationary Gaussian process. The process is also ergodic, which means that its statistical moments can be obtained as averages over a single (infinite) time history. A possible point of confusion between the energy spectrum, $S(\omega)$, and the more usual statistical term, spectral density function, may be resolved as follows. Consider a stationary Gaussian process, $\eta(t)$, with covariance function $R(\tau)$. The Fourier transform $f_\eta(\omega)$ of $R(\tau)$ is called the spectral density function of $\eta(t)$. Thus, $f_\eta(\omega)$ is defined over the interval $(-\infty,\infty)$. The wave energy spectrum $S(\omega)$ is then defined by

$$S(\omega) = \begin{cases} 2f_\eta(\omega) & \text{if } \omega\varepsilon[0,\infty) \\ 0 & \text{elsewhere.} \end{cases} \qquad (4)$$

The preceding discussion summarizes the theoretical features of the continuous time, continuous frequency, random phase model of ocean waves. What follows is a description of the authors' procedure for digitally synthesizing ocean wave time histories from the theoretical model. It turns out that the abstract definition of the model given above can be directly exploited to obtain the digital results.

In order to implement Equation (1) on a digital computer, the infinite upper limit of integration must be replaced by a finite value and the limiting sequence of partial sums in Equation (3) must be replaced by a single sum involving N frequency components. Neither of these restrictions poses a real problem in practice. Ocean waves are basically low frequency phenomena and contain very little energy at frequencies beyond π rad/sec. Also, the wave spectrum can be approximated as closely as desired by the use of a finite number of frequencies. Thus, by choosing a finite maximum frequency ω_c, and a large but finite number of frequency subintervals, we obtain the following approximation (to Equation (1))

$$\eta(t) = \sum_{k=1}^{N} \cos\left(\omega_k^* t - \varepsilon(\omega_k^*)\right) \sqrt{2S(\omega_k^*)(\omega_k - \omega_{k-1})} \qquad (5)$$

with

$$0 = \omega_0 < \omega_1 \ldots < \omega_{N-1} < \omega_N = \omega_c$$

By definition, each ω_k^* can be chosen arbitrarily from the subinterval (ω_{k-1}, ω_k) and each $\varepsilon(\omega_k^*)$ must be a random phase angle having an uniform distribution on $[0, 2\pi)$. Standard algorithms exist for generating random numbers on digital computers. Discussion of such algorithms with particular emphasis on the quality of random numbers generated by the CDC 6700 computer system is given in Appendix D. However, by using some such

generator, the ω_k^* can be chosen at random from the interval (ω_{k-1},ω_k) and the $\varepsilon(\omega_k^*)$ and be chosen from an uniform distribution on $[0,2\pi)$.

For convenience, the ω_k in Equation (5) can be equally spaced. Then the approximate wave system becomes

$$\eta(t) = \sum_{k=1}^{N} [2S(\omega_k^*)\Delta\omega]^{1/2} \cos(\omega_k^* t - \varepsilon(\omega_k^*)) \tag{6}$$

where

$$\Delta\omega = \omega_k - \omega_{k-1},\ k = 1, \ldots N \tag{7}$$

If we denote

$$C_k = [2S(\omega_k^*)\Delta\omega]^{1/2} \tag{8}$$

and

$$\varepsilon(\omega_k^*) = \varepsilon_k$$

Equation (6) can be rewritten in the form

$$\eta(t) = \sum_{k=1}^{N} C_k \cos(\omega_k^* t - \varepsilon_k) \tag{9}$$

Equation (9) is then used to digitally generate time histories in the computer program. Although Equation (9) is not a Fourier series, it does contain only a finite number of periodic components and thus, in prinicple, it will be periodic. However, because the frequency components are

selected randomly within each subinterval, the probability is zero that two distinct components will be harmonics. Thus, the wave period resulting from the use of such components will be determined by the number of components employed and the number of significant digits used to represent each component's period. Based on the fifteen significant digits used in CDC 6700 computations, a simulation based on N wave frequency components over the interval $[0,\pi)$, each having a period containing N_k significant digits, would have a period

$$T = \prod_{k=1}^{N} \left(\frac{2\pi}{\omega_k} 10^{N_k} \right) \approx 10^{15N} \text{ sec} \tag{10}$$

Thus, assuming the selection of a moderate number of frequency components, for example 50, the wave period would be roughly

$$\begin{aligned} T &= 10^{750} \text{ sec} \\ &= 3.17 \times 10^{742} \text{ yr} \end{aligned} \tag{11}$$

Hence, for all practical purposes, the wave period is infinite so that the wave system may be considered to be nonperiodic. The advantage of the wave energy integral representation becomes clear upon noting that a Fourier series model containing 100 frequency components over $[0,\pi)$ would have a period of only 200 sec.

DESCRIPTION OF SIMULATED OCEAN WAVE RECORDS

The idealized character of a continuous parameter stochastic process cannot be realized in a digital simulation. The simulated wave time histories must be based on a discrete approximation to the theoretical wave spectrum. In addition, only a finite number of wave frequency components and random phases can be employed. It is thus essential to investigate how well the simulated records approximate the theoretical ones. In this respect, the effects of both approximation error and sampling variability are significant.

The computer program developed herein allows the user to input the wave spectrum model of his choice. It is merely required that the spectral ordinate be input at each discrete frequency of interest. Hence, the model can be easily adopted to model wave systems exhibiting empirical spectra such as the station India spectra used in many ship performance studies. Likewise, the model readily admits theoretical spectra such as the International Towing Tank Conference (ITTC) recommended two-parameter Bretschneider spectrum. In the numerical examples that follow, a 30-knot wind spectrum developed by Pierson and Moskowitz[7] was used to generate the required wave processes. The frequency range selected was [0.2,2.2] rad/sec. The Pierson-Moskowitz spectrum is defined in units of ft^2-sec by

$$S(\omega) = \frac{0.00810}{\omega^5} e^{-0.749/(VK \cdot \omega)^4} \tag{12}$$

where ω = frequency (rad/sec)

VK = wind speed (knots)

The spectrum is a single amplitude spectrum and is continuous over positive frequencies. As implemented in this report, the area under the spectrum will equal the variance of the generated wave process. Figure 1 shows the energy spectrum based on the Pierson-Moskowitz formula for a wind speed of 30 knots. It should be noted that any desired spectrum can be used in the simulation. The Pierson-Moskowitz spectrum was selected only for convenience.

In order to compute wave surface time histories from Equation (9), the number of frequency components N must be selected in a manner that is dependent upon the intended application. In many instances where strict adherence to the normal distribution of amplitudes is important, the minimum number of frequency components should be sufficiently large for the central limit theorem to hold. The normality property was investigated by applying chi-square goodness-of-fit tests to samples of wave amplitude

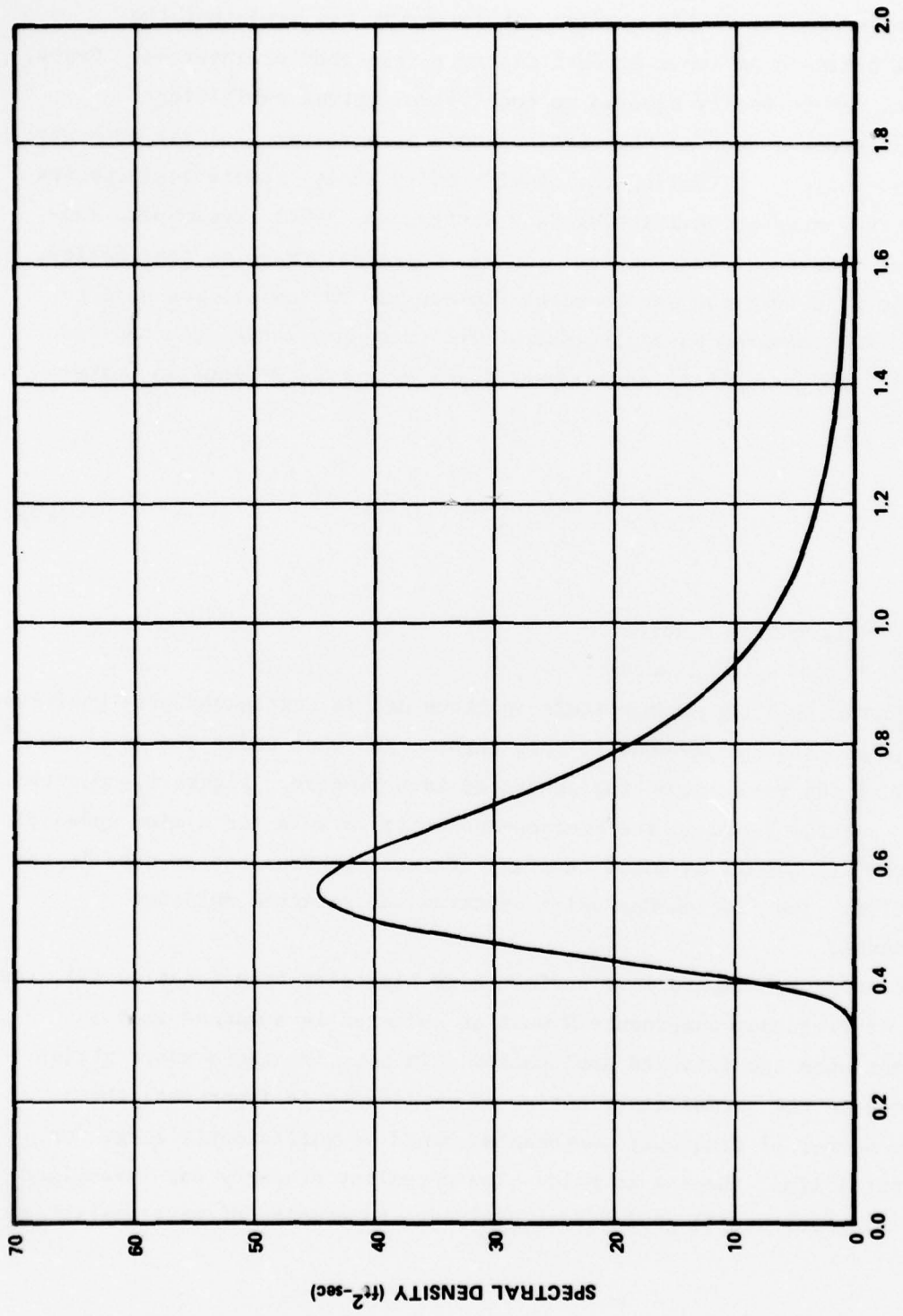

Figure 1 - Pierson-Moskowitz Wave Spectrum at 30 Knots

points corresponding to a number of frequency components N. Since independent sampled points are required for application of the usual chi-square tests, such tests cannot be performed straightforwardly on correlated wave histories. Independent wave amplitudes were obtained by sampling the wave ensemble 600 times at the fixed time t = 0. This procedure was applied to each of ten samples consisting of 5, 10, 15, 25, 50, and 100 wave frequency components, respectively. Thus, ten chi-square tests were performed for each set of frequency components. The number of class intervals was selected to maximize the power of the tests in accordance with the procedure of Mann and Wald[8] as detailed by Williams.[9] Following this procedure, the number K, of class intervals depends upon the number of points N and the significance level α, according to the formula

$$K = 4 \left(\frac{2(N-1)^2}{\alpha^2} \right)^{1/5} \tag{13}$$

where N is the number of sample points and α is defined by

$$\frac{1}{(2\pi)^{1/2}} \int_0^\infty e^{-\frac{y^2}{2}} \, dy = \alpha \tag{14}$$

Fifty-two class intervals were used and the data were grouped into 52 equiprobability classes which yielded 51 degrees of freedom. The results from the chi-square tests are shown in Table 1. Based on the significance level $\alpha = 0.10$, no more than one sample out of ten would be expected to exceed the critical value, $\chi^2 = 64.295$, if the data were representative of a normal distribution with mean zero and variance σ^2. For five frequency components, all ten of the computed values exceed the critical value. For twenty-five or more components, none of the ten chi-square values exceeds the critical value. Chi-square tests were also computed on the data by using the fifty-two class intervals but without grouping the data into

TABLE 1 - CHI-SQUARE VALUES FOR NORMALITY TESTS

Number Frequency Components	Sequence Number									
	1	2	3	4	5	6	7	8	9	10
5	120.165	116.704	97.842	123.626	130.028	119.819	128.298	143.353	107.705	113.935
10	53.542	71.020	59.253	45.409	64.271	58.215	56.657	36.065	53.889	52.677
15	59.080	53.715	69.636	52.677	52.331	63.925	76.038	61.676	71.885	48.524
25	48.351	42.468	47.313	57.522	40.737	44.890	43.679	62.887	52.331	48.697
50	52.158	57.522	41.429	41.948	36.065	47.140	43.506	44.025	59.080	46.794
100	37.795	58.907	45.063	51.985	50.428	49.216	56.138	48.524	55.100	37.622

NOTE:
Critical value for the 0.10 probability level with 51 degrees of freedom is 64.295.

equally probable intervals and the same trends were observed. Hence, as few as twenty-five frequency components are sufficient to guarantee convergence of the random phase model to the normal law. However, this conclusion is strictly valid only for the range of variation observed in the samples, namely $\pm 3\sigma$. More frequency components may be required to imply normality for the tails of the sample distributions.

An example of a generated ocean wave time history is shown in Figure 2. This time history is a 30 min record based on 100 frequency components over the frequency range [0.2,2.2] rad/sec. A sampling rate of 1 sample per second was used. Figure 3 shows a sample probability density function of the wave amplitudes from the 30 min record. Close agreement is observed between the computed wave data and the theoretical normal distribution based on mean zero and variance σ^2. The usual chi-square goodness-of-fit test could not be performed on the time history data because adjacent sample points are not independent.

An important property of the wave record is its mean square value or variance. The theoretical variance of the generated wave process is given by the area under the wave spectrum. That is

$$\sigma^2 = \int_a^b S(\omega)d\omega \tag{15}$$

where a and b are the minimum and maximum frequencies, respectively, in rad/sec, and $S(\omega)$ is the energy spectrum. The variance of the theoretical wave process based on the continuous form of the Pierson-Moskowitz spectrum, where b = 2.2 and a = 0.2, is 17.287. The variance of the generated wave process was computed as a function of record length and number of frequency components. Figure 4 shows the computed variance of additional wave records based on 25, 50, and 100 frequency components, respectively, and based on time histories between 0 and 30 min in length. The horizontal line represents the theoretical variance based on the 30-knot Pierson-Moskowitz spectrum. In time histories of length less than

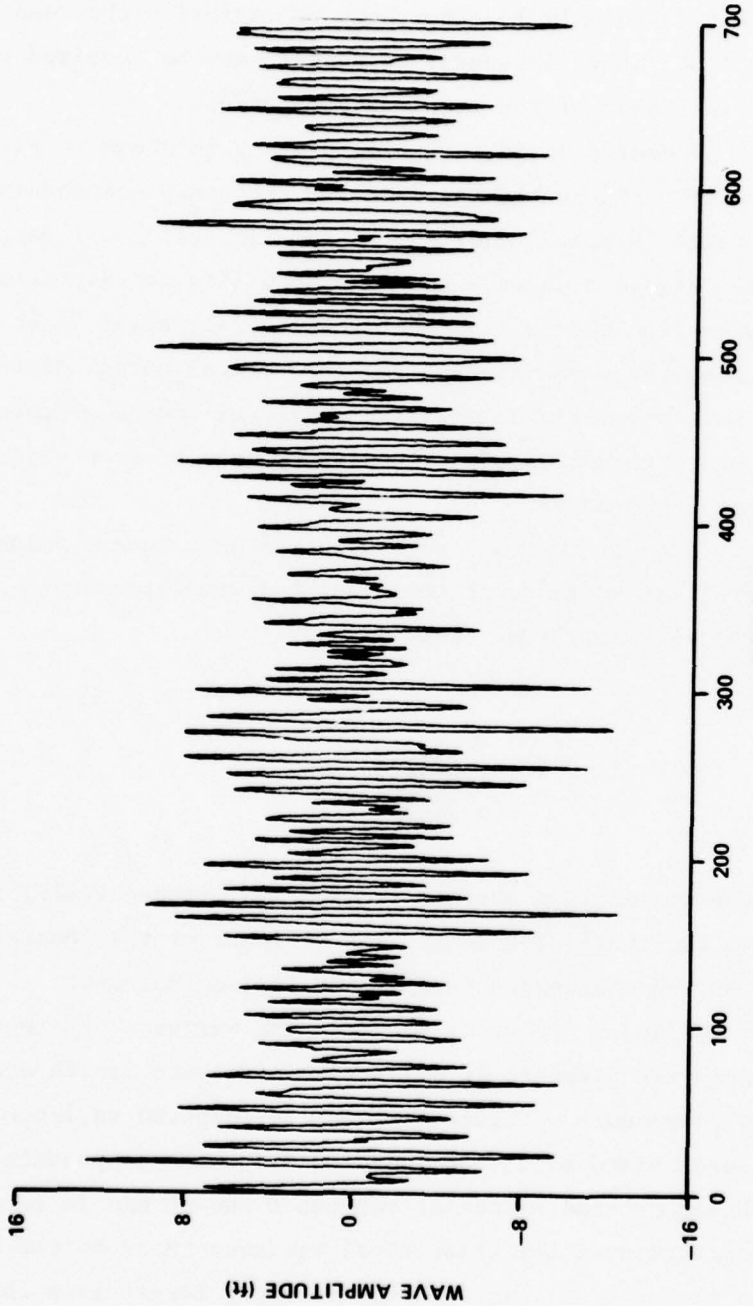

Figure 2 - Sample Simulated Wave System

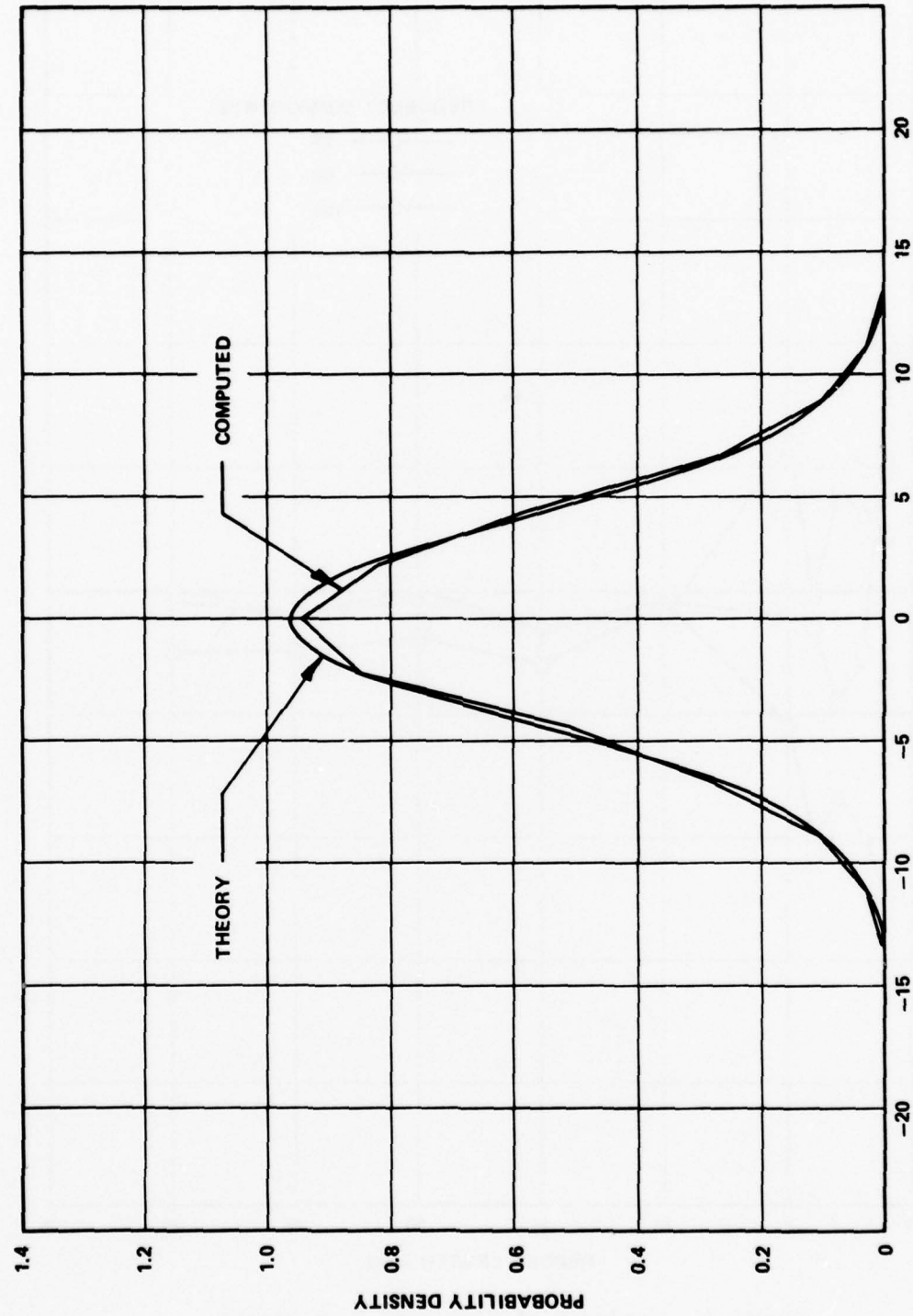

Figure 3 - Sample Probability Density Function

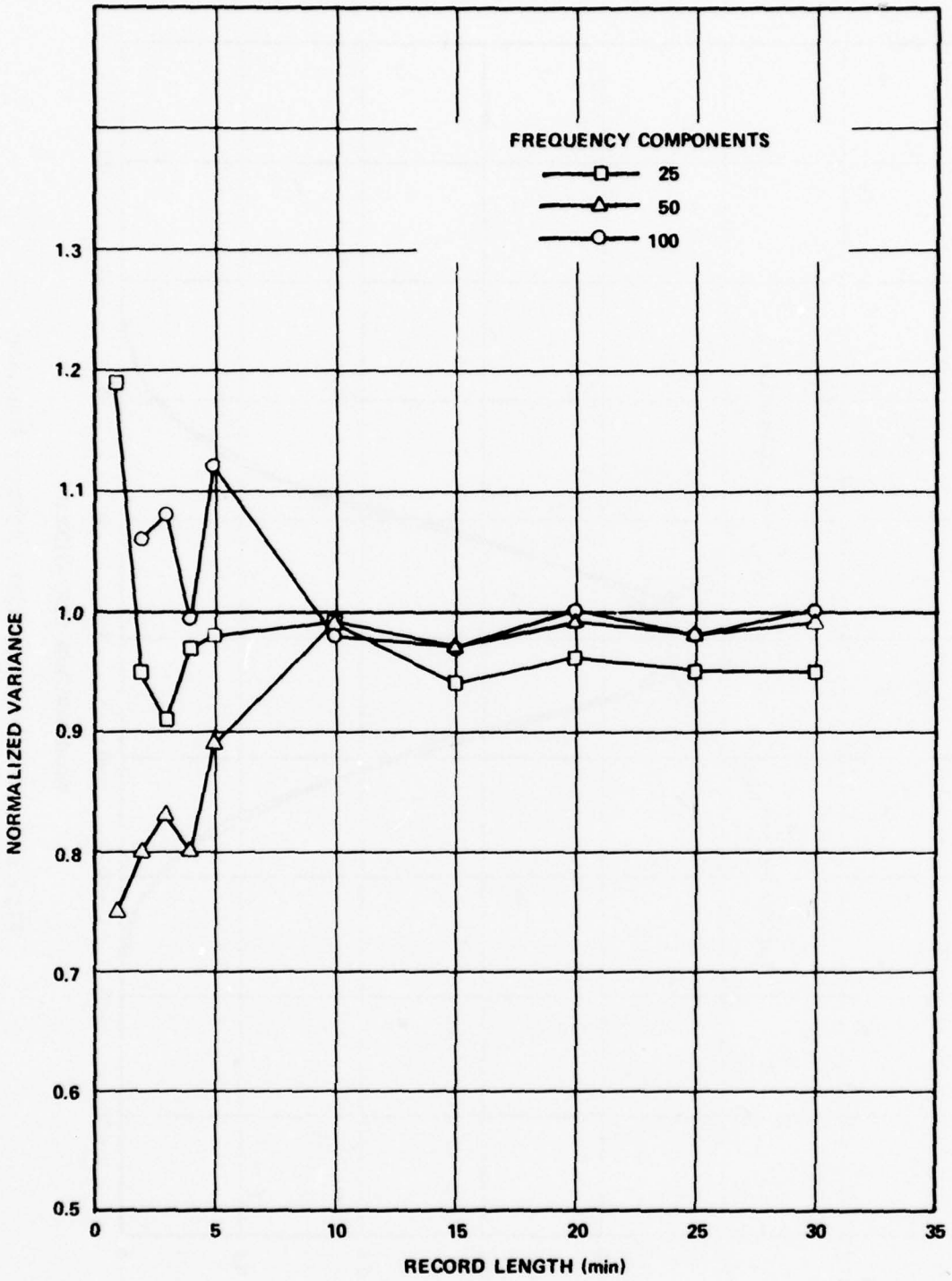

Figure 4 - Variance of Simulated Wave Records

five min, the computed variance fluctuates considerably and does not approximate the theoretical variance too closely. As the length of record increases, the approximated quality improves and the error is within ± 5 percent for records no shorter than 10 min. As expected, the variance approximation improves as the number of frequency components is increased. However, 25 frequency components are seen to yield reasonable accuracy.

The theoretical covariance function of the generated wave process is given by (Appendix C)

$$R(\tau) = \int_0^\infty (S(\omega) \cos (\omega\tau))d\tau \tag{16}$$

This theoretical covariance function is shown in Figure 5 along with the sample covariance function computed from the 30 min record (100 frequencies). The two curves are in very close agreement for lags up to 60 sec.

Figure 6 shows a comparison of a sample spectrum computed from the 30 min record and the theoretical Pierson-Moskowitz spectrum. Close agreement is observed between the two curves, although the sample spectrum shows some oscillation due to sampling variability.

THE COMPUTER PROGRAM

PROGRAM DESCRIPTION

This section describes the computer program developed for generating and plotting the random phase wave records. WAVESIM, the main program, functions as an executive program. It reads input parameters that must be supplied by the user. Table 2 explains the parameters used in the program. The subroutines that follow are called from the main program, and the input data is transferred to these subroutines for computation. A listing of the program appears in Appendix E. The program consists of the subroutines SPECTRA, WAVES, ARAMEAN, VAR, and PLOT W, which are discussed.

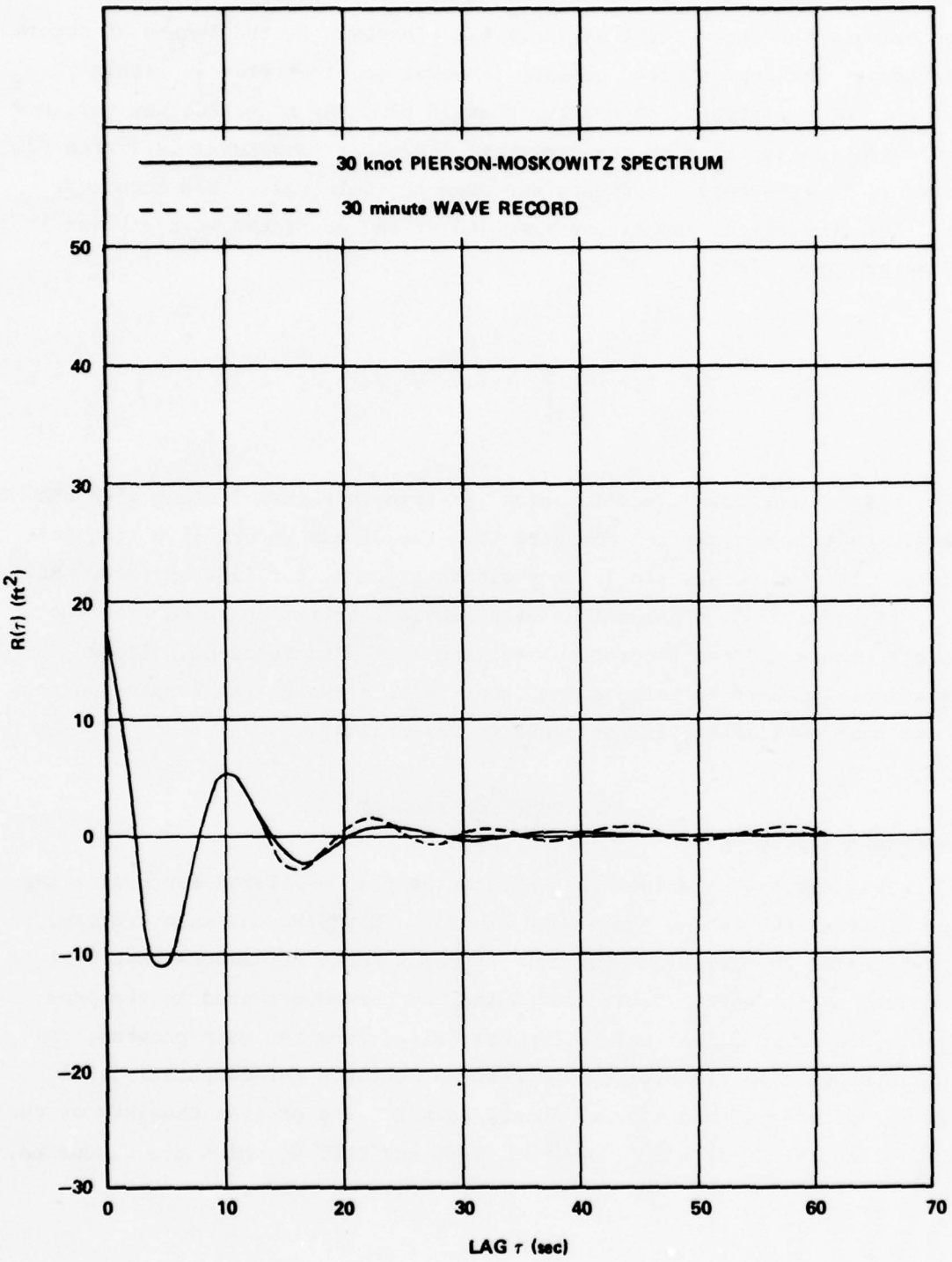

Figure 5 - Sample Covariance Function for Wave Surface Elevation

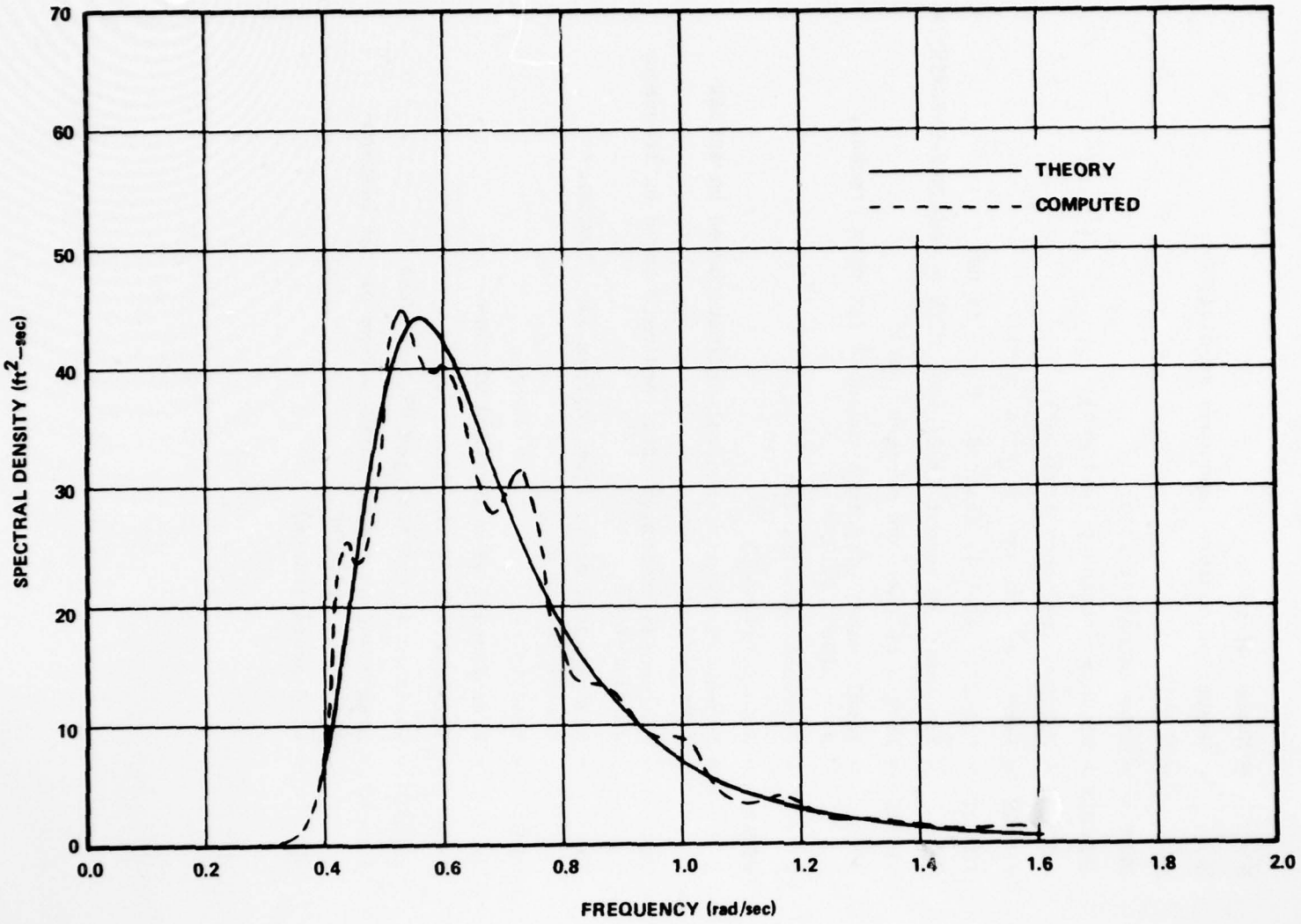

Figure 6 - Estimated Spectral Density Function (30 Minute Record)

TABLE 2 - COMPUTER PROGRAM NOMENCLATURE

N	= Number of cases
M	= Number of nonzero frequency subdivisions
MM	= M + 1
DELT	= Time increment (sec)
OMEGMIN	= Minimum frequency (rad/sec)
OMEGMAX	= Maximum frequency (rad/sec)
TINMIN	= Length of the wave process (min)
IFPLOT	= Use 1 if plot is desired. Use 2 if not
DELW	= Frequency increment (rad/sec), DELW = (OMEGMAX-OMEGMIN)/M
TINSEC	= Length of the wave process (sec)
NO	= Total number of points computed for wave process, NO = TINSEC/DELT + 1
W	= Frequency subdivision
WSTAR	= Actual frequency
S	= Pierson-Moskowitz spectral ordinate based on actual frequency
PMS	= Pierson-Moskowitz spectral ordinate based on frequency subdivision
XX	= Random phase angle from uniform distribution on $[0,2\pi]$ (rad)
TIME	= Time (sec)
WAVE	= Wave Process as a function of time
RMEAN	= Time average of the wave process
VARANCE	= Variance of the computed wave process
AREAPMS	= Area based on the continuous form of the Pierson-Moskowitz spectrum
VK	= Wind speed (knots)

SPECTRA: This subroutine calculates the spectral density function using the Pierson-Moskowitz formula given by:

$$S(\omega) = \frac{0.00810}{\omega^5} e^{-0.749/(VK \cdot \omega)^4} \tag{17}$$

where $S(\omega)$ = ordinate of spectral density

ω = circular frequency

VK = wind speed (knots)

This formula requires a value for wind speed in knots (VK) which is read as input in the program. The frequency subdivisions and the actual frequencies are generated in this subroutine. The spectral density function is defined only over non-negative frequencies. This subroutine is optional as the user may substitute any spectrum $S(\omega)$ that is desired.

WAVES: This subroutine computes (using Equation (9)) the random wave process using the formula:

$$\eta(t) = \sum_{k=1}^{N} [2S(\omega_k^*)\Delta\omega]^{1/2} \cos(\omega_k^* t - \varepsilon_k) \tag{18}$$

where $S(\omega)$ = the spectral ordinate values

$\Delta\omega$ = frequency increment (rad/sec)

ω_k^* = actual frequency (rad/sec)

t = time (sec)

ε_k = random phase angle

N = number of discrete frequency components to be employed

The values for time in seconds and the random phase angles are generated within this subroutine.

ARAMEAN: This subroutine calculates the mean value of the generated wave process. The area under the continuous form of the Pierson-Moskowitz spectrum is also computed.

VAR: This subroutine computes the variance of the simulated time history.

PLOT W: This subroutine plots the generated wave on the CALCOMP 936 plotter. The value of IFPLOT (input to the main program) determines whether or not this routine is used. If a plot is desired, the program generates a magnetic tape from which plots can be obtained from the CALCOMP plotter.

PREPARATION OF INPUT

The input for calculating and plotting a wave history consists of three cards which are described in Table 3. There are seven parameters, N, M, MM, DELT, OMEGMIN, OMEGMAX, and TINMIN, on the first card. The parameter on the second card is IFPLOT. All integer names must be right justified since blank columns are interpreted as zeros. Floating point names must include a decimal point. If the Pierson-Moskowitz spectrum is used, the third card must contain the wind speed parameter VK. Table 4 shows an example of the input with and without the plot routine. If another case follows, an additional set of data (Cards 1-3) is needed. The last case should have a value of 1 for N, in order to terminate the program.

EXPLANATION OF OUTPUT

The output consists of eight sections of computed information. A sample printout is shown in the following sections. Section 1 consists of the input parameters. Section 2 shows the frequency subdivisions, actual frequencies, and spectral ordinate values of the Pierson-Moskowitz spectrum based on the actual frequencies between 0.2 and 1.0. The random phase angles are listed in Section 3. Section 4 shows the first 136 points of the generated wave process as a function of time in seconds. In

TABLE 3 - INPUT FOR MAIN PROGRAM

Card	Columns	Format	FORTRAN Designation	Definition
1	1- 5	I5	N	Number of cases
	6-10	I5	M	Number of frequency subdivisions
	11-15	I5	MM	MM = M + 1
	16-25	F10.2	DELT	Time increment (sec)
	26-35	F10.2	OMEGMIN	Minimum frequency (rad/sec)
	36-45	F10.2	OMEGMAX	Maximum frequency (rad/sec)
	46-55	F10.2	TINMIN	Length of the wave process (min)
2	1- 5	I5	IFPLOT	Use 1 if plot routine is desired. Use 2 if not.
3	1-10	F10.2	VK	Wind speed (knots) for the Pierson-Moskowitz formula

TABLE 4 - EXAMPLE OF INPUT

```
WITH PLOT ROUTINE
    1   100  101      .5        .2          2.2          30.
    1
    30.

WITHOUT PLOT ROUTINE
    1   100  101      .5        .2          2.2          30.
    2
    30.
```

SECTION 1

NUMBER OF INPUT CASES	1
TOTAL NUMBER OF POINTS COMPUTED FOR WAVE PROCESS	1801
NUMBER OF FREQUENCIES USED	101
MINIMUM FREQUENCY IN RADIANS/SECOND	.200
MAXIMUM FREQUENCY IN RADIANS/SECOND	2.200
FREQUENCY INCREMENT IN RADIANS/SECOND	.020
TIME INCREMENT IN SECONDS	1.000
LENGTH OF WAVE PROCESS IN MINUTES	30.000
LENGTH OF WAVE PROCESS IN SECONDS	1800.000
WIND SPEED IN KNOTS	30.000

SECTION 2

FREQUENCY SUBDIVISION	ACTUAL FREQUENCY	SPECTRAL ORDINATE BASED ON ACTUAL FREQUENCY
.200	.200	.000
.220	.221	.000
.240	.252	.000
.260	.261	.000
.280	.294	.000
.300	.302	.002
.320	.337	.161
.340	.355	.737
.360	.376	2.729
.380	.382	3.619
.400	.405	8.729
.420	.429	16.415
.440	.449	23.645
.460	.461	27.778
.480	.495	37.897
.500	.515	41.606
.520	.538	44.061
.540	.555	44.648
.560	.569	44.478
.580	.595	42.975
.600	.602	42.340
.620	.624	40.052
.640	.652	36.461
.660	.664	34.920
.680	.688	31.710
.700	.711	28.772
.720	.735	25.884
.740	.743	24.945
.760	.779	21.077
.780	.784	20.541
.800	.807	18.463
.820	.836	16.048
.840	.858	14.461
.860	.862	14.186
.880	.893	12.221
.900	.917	10.883
.920	.931	10.222
.940	.952	9.262
.960	.974	8.384
.980	.983	8.021
1.000	1.001	7.396

SECTION 3

RANDOM PHASE ANGLES

4.296128	5.648159	2.762822	.761861	5.094957
4.373786	1.421783	2.834857	4.681262	5.759672
2.743953	4.526702	4.427508	5.621681	2.713662
4.247852	4.246169	4.790832	2.210933	1.973443
2.920811	.006001	5.817229	5.293417	2.636457
4.952821	5.108381	4.354680	4.912478	4.862324
5.426944	.579937	2.156256	2.009325	1.669190
2.060736	2.832687	2.521395	3.426094	.471848
5.234771	4.813573	3.441344	1.295194	3.138264
5.170637	6.179425	5.253782	2.337308	6.022332
3.159101	3.479571	3.249147	.953576	3.601916
4.185023	6.236676	.838576	1.062601	5.199075
3.002502	5.552497	2.112288	5.060934	.089845
3.887805	2.693324	4.536166	1.824741	5.764042
.017589	2.827710	5.295909	1.889345	1.405937
4.277415	.783155	.708669	5.368163	2.675637
2.326224	2.174309	3.432989	3.190898	1.607190
1.256245	4.906084	1.297543	.872652	2.645112
2.370641	1.055278	4.611419	1.917550	2.652305
3.891230	5.905635	4.989108	1.775157	4.563567
3.942509				

SECTION 4

TIME	WAVE	TIME	WAVE	TIME	WAVE	TIME	WAVE
0.00	-3.81	1.00	1.12	2.00	5.83	3.00	7.08
4.00	5.56	5.00	3.19	6.00	.52	7.00	-2.25
8.00	-2.73	9.00	-1.01	10.00	-.66	11.00	-.91
12.00	-.86	13.00	-3.52	14.00	-6.25	15.00	-4.82
16.00	-.54	17.00	6.34	18.00	12.72	19.00	11.02
20.00	3.09	21.00	-3.34	22.00	-7.23	23.00	-9.71
24.00	-9.30	25.00	-4.67	26.00	2.48	27.00	7.06
28.00	7.13	29.00	4.85	30.00	1.27	31.00	-.78
32.00	-.02	33.00	-1.46	34.00	-5.16	35.00	-5.56
36.00	-3.07	37.00	.45	38.00	4.89	39.00	6.48
40.00	3.75	41.00	.53	42.00	-1.21	43.00	-2.13
44.00	-3.35	45.00	-4.73	46.00	-4.11	47.00	-1.43
48.00	1.77	49.00	5.68	50.00	8.41	51.00	6.44
52.00	.04	53.00	-6.74	54.00	-8.73	55.00	-5.22
56.00	-1.37	57.00	1.07	58.00	3.83	59.00	4.68
60.00	3.25	61.00	3.03	62.00	2.71	63.00	.21
64.00	-2.76	65.00	-6.73	66.00	-9.81	67.00	-5.37
68.00	3.52	69.00	7.06	70.00	5.94	71.00	4.87
72.00	2.52	73.00	-.30	74.00	-1.65	75.00	-3.86
76.00	-5.43	77.00	-2.59	78.00	1.24	79.00	2.05
80.00	1.35	81.00	.61	82.00	-.23	83.00	-1.40
84.00	-2.08	85.00	-.26	86.00	2.03	87.00	2.00
88.00	2.89	89.00	4.58	90.00	1.25	91.00	-4.20
92.00	-5.09	93.00	-3.76	94.00	-2.79	95.00	-.02
96.00	2.86	97.00	3.26	98.00	3.21	99.00	2.82
100.00	-.22	101.00	-3.61	102.00	-4.19	103.00	-2.81
104.00	.14	105.00	4.11	106.00	5.29	107.00	2.75
108.00	-.47	109.00	-3.04	110.00	-3.98	111.00	-3.00
112.00	-1.77	113.00	.84	114.00	4.16	115.00	2.71
116.00	-1.24	117.00	-1.03	118.00	-.20	119.00	-2.29
120.00	-1.83	121.00	2.54	122.00	5.52	123.00	4.11
124.00	-.69	125.00	-4.37	126.00	-3.24	127.00	-.65
128.00	-1.12	129.00	-2.43	130.00	-1.18	131.00	.79
132.00	1.65	133.00	2.92	134.00	3.51	135.00	.78

SECTION 5

MEAN OF THE GENERATED WAVE PROCESS	.00000
VARIANCE OF THE COMPUTED WAVE PROCESS	17.27881
AREA BASED ON THE CONTINUOUS FORM OF THE PIERSON-MOSKOWITZ SPECTRUM	17.28684
MINIMUM VALUE OF WAVE PROCESS	-12.56401
MAXIMUM VALUE OF WAVE PROCESS	14.48437

Section 5 the mean and variance of the wave process, the area based on the continuous form of the Pierson-Moskowitz spectrum, and the minimum and maximum values of the wave process are printed.

OPERATING COMPUTER INFORMATION

The WAVESIM program is coded in FORTRAN IV and has been used on the CDC 6700 computer system. However, the program should be easily adapted to any other system provided an uniform random number generator is available. Table 5 shows the sample deck structure for the CDC 6700 system with and without the plot routine. The number of words of computer memory and the computer time required for the program will depend on the number of frequencies used and the length of the wave process. For example, a typical run involving the computation and plotting of a 30 min record based on 100 frequency components and sampled at 1 sample/sec, required 66.5 system sec on the CDC 6700.

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TABLE 5 - SAMPLE DECK STRUCTURE FOR THE CDC 6700 SYSTEM

1. Compilation and Execution with PLOT ROUTINE, IFPLOT = 1.

```
JOBNAME,CM60000,MT1,P3,T100.   *USERS JOB CARD*
CHARGE,CXXX,XXXXXXXXXX.        *USERS CHARGE CARD*,*XXXXXXXXXX  IS
VSN,TAPEN,TAPENAME=SLOT    .   USERS JOB NUMBER*, *CXXX IS USERS ID*
REQUEST,TAPEN,HI.          (TAPENAME,SLOT__/RING)
FTN(T)
ATTACH,CALC936.
LDSET(LIB=CALC936)
LGO.
RETURN(TAPEN)
7/8/9 END OF RECORD
FORTRAN DECK
7/8/9 END OF RECORD
DATA
6/7/8/9 END OF FILE
```

2. Compilation and Execution without PLOT ROUTINE, IFPLOT = 2.

```
JOBNAME,CM60000,P3,T100.
CHARGE,CXXX,XXXXXXXXXX
FTN(T)
LGO.
7/8/9 END OF RECORD
FORTRAN DECK
7/8/9 END OF RECORD
DATA
6/7/8/9 END OF FILE
```

APPENDIX A
RANDOM FUNCTION THEORY

A few basic facts concerning random functions are required to develop and interpret the random phase ocean wave model. A brief summary of this theory is given below for immediate reference. More rigorous discussion and proofs may be found in standard textbooks such as Doob[10] or Lukacs.[11]

A random function X is a function whose domain is a parameter set T, and whose range is a set of random variables, possibly complex valued. Thus, a random function may be considered as an indexed set of random variables, $\{X_t | t\varepsilon T\}$.

In practice, the set T will usually consist of discrete or continuous subsets of the real line. When only continuous subsets are considered, it is conventional to write each member as X(t) and to denote the process by $\{X(t) | t\varepsilon T\}$.

For each t, the random variable X_t is determined by its probability distribution function, $F_{xt}(x)$, as in elementary probability theory. For each X_t, moments of arbitrary order can be computed, whenever they exist, by

$$E\, X_t^n = \int_{-\infty}^{\infty} X^n \, dF_{xt}(x) \tag{A.1}$$

where E denotes the mathematical expectation or mean value operator. At any two different times t_1 or t_2, the covariance function between the two random variables is defined, whenever it exists, by

$$\text{COV}\left(X_{t_1}, X_{t_2}\right) = E\left[\left(X_{t_1} - EX_{t_1}\right)\overline{\left(X_{t_2} - EX_{t_2}\right)}\right] \tag{A.2}$$

$$\triangleq R(t_1, t_2)$$

An important class of random functions, useful in both theoretical work and applications, is characterized by the fact that the principal properties of the indexed random variables are invariant with respect to the location of the random variables within the indexed set. Such processes are called stationary. A process $\{X_t | t\varepsilon T\}$ is said to be wide sense stationary if

1. $\forall t\varepsilon T$ EX_t^2, EX_t^2 exist and are independent of t.
2. $\forall t_1, t_2 \varepsilon T$, the covariance function $R(t_1, t_2)$ exist and are invariant under any translation of the t axis.

Processes for which EX_t^2 exists are called second-order processes. It follows, by Schwartz's inequality, that for any second-order process, X_t, $\text{COV}\left(X_{t_1}, X_{t_2}\right) < \infty$. It is clear from the second property of stationary processes that the covariance function can depend only on the time difference between t_1 and t_2 and not on t_1 or t_2 separately. For this reason, it is customary to denote $\tau = t_2 - t_1$, and write

$$\begin{aligned} R(\tau) &= \text{COV}\left(X_{t_1}, X_{t_2}\right) \\ &= \text{COV}\left(X_{t_1}, X_{t_1+\tau}\right) \end{aligned} \tag{A.3}$$

The notion of integration for random functions can be defined. However, as in the usual Riemann-Stieltjes integral, the concepts of convergence to a limit point and continuity are needed. Let $\{X_n\}$ be a sequence of second order random variables. A random variable X is said to be the limit in quadratic mean of $\{X_n\}$ if

1. X is second order
2. $\underset{n \to \infty}{\ell.i.m.}\ E\left|(X_n - X)\right|^2 = 0$

We write

$$X = \underset{n \to \infty}{\ell.i.m.}\ X_n \tag{A.4}$$

and we say that X_n converges in quadratic mean to X.

Let $\{X(t) | t\epsilon T\}$ be a second order continuous parameter stationary process. Then the process is said to be continuous on T if

$$\forall t\epsilon T \quad \underset{\tau \to 0}{\ell.i.m.} \; X(t+\tau) = X(t) \tag{A.5}$$

A second order stationary process is continuous in the sense of Equation (A.5) if and only if its covariance function is continuous in the usual sense at $\tau = 0$. Continuous processes have the property that they do not fluctuate too wildly and are thus amenable to integration. The stochastic integral may be defined in a fashion analogous to the Riemann-Stieltjes integral, the major difference being that the real line convergence of a sequence of real sums must be replaced by some mode of stochastic convergence for the corresponding sequence of random sums. The following definition can be given based on quadratic mean convergence.

Let X(t) and Z(t) be any two second order continuous processes defined on the finite interval [A,B]. Consider the sequence of partitions of [A,B] given by

$$\Lambda_n : A = t_{n,0} < t_{n,1} \cdots < t_{n,n} = B$$

Then, for every n, let

$$S_n = \sum_{k=1}^{n} X(t^*_{n,k})[Z(t_{n,k}) - Z(t_{n,k-1})]$$

where

$$t^*_{n,k} \; \epsilon [t_{n,k-1}, t_{n,k}]$$

Assume further that

$$\underset{n \to \infty}{\ell.i.m.} \max_{k} \left| t_{n,k} - t_{n,k-1} \right| = 0$$

Then, if there exists an unique random variable S, such that

$$S = \underset{n \to \infty}{\ell.i.m.} S_n$$

we call S the integral over [A,B] of X(t) with respect to Z(t) and denote it by

$$S = \int_A^B X(t)\, dZ(t) \tag{A.6}$$

It can be shown that if X(t) and Z(t) are independent $\forall t \in T$, the integral exists if and only if

$$E|S|^2 = \int_A^B \int_A^B R_z(s,t) R_x(s,t) ds dt < \infty \tag{A.7}$$

where R_z and R_x are the respective covariance functions.

Special cases of the integral occur when either X(t) or Z(t) degenerates into a deterministic function. The integral can be defined for an infinite interval by the standard limiting procedure used for Riemann-Stieltjes integrals. The stochastic integral is linear and homogeneous in the integrand and additive in the domain of integration.

It is often desirable to perform integrations with respect to processes possessing orthogonal increments. A process $\{X_t | t \in T\}$ is said to have orthogonal increments if

$$\text{(a)} \quad E|X_t - X_s|^2 < \infty \quad \forall s,t \in T \tag{A.8}$$

(b) If the two intervals $[t_1,t_2]$, $[t_3,t_4]$ do not overlap, then the increments $\left(X_{t_2}-X_{t_1}\right)$ and $\left(X_{t_4}-X_{t_3}\right)$ are orthogonal, i.e.,

$$E\left(X_{t_4}-X_{t_3}\right)\overline{\left(X_{t_2}-X_{t_1}\right)} = 0 \qquad \text{(A.8) (cont.)}$$

where the bar denotes complex conjugation. The importance of processes possessing orthogonal increments is due largely to the celebrated Spectral Representation Theorem. This theorem states that any wide sense stationary, continuous stochastic process Y(t) can be represented as the Fourier-Stieltjes integral

$$Y(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, dX(\omega) \qquad \text{(A.9)}$$

where X(t) is a process with orthogonal increments and $E|dX(\omega)|^2 = dG(\omega)$. Further, X(t) when properly normalized, is uniquely specified by this relationship.

The order of expectation and integration can be interchanged in the stochastic integral so that

$$EY(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, d(EX(\omega)) \qquad \text{(A.10)}$$

whenever the right hand side of Equation (A.10) exists. Since $X(\omega)$ is a random variable for each ω, Y(t) consists of an infinite number of time histories over the range $(-\infty,\infty)$, and each time history is defined by a set of simultaneous outcomes of the random variables $\{X(\omega)|\omega\varepsilon(-\infty,\infty)\}$. Such time histories are called sample functions or realizations of the process Y(t). If Y(t) is a continuous (ℓ.i.m.) process, each sample function will be a deterministic function and will be continuous except possibly on sets of zero probability.

A stochastic derivative can also be defined for a second-order process, Y(t). Analogous to the usual derivative definition, we define

$$Y'(t) = \underset{\tau \to 0}{\ell.i.m.} \frac{Y(t+\tau) - Y(t)}{\tau}$$

whenever the limit exists. If R(s,t) is the covariance function of Y, Y(t) exists whenever

$$\left[\frac{\partial^2 R(s,t)}{\partial s \partial t}\right]_{s=t}$$

exists. The function Y'(t) is called the quadratic mean derivative of Y(t).

APPENDIX B

STOCHASTIC INTEGRAL REPRESENTATION OF THE RANDOM PHASE MODEL

The mathematical form of the random phase wave surface model proposed by St. Denis and Pierson may be unfamiliar to the reader. The model involves the representation of a random function by what appears to be an usual integration process, but which contains a square root symbol over the differential frequency element. Appendix A contains the background material required to properly interpret the integral of a stochastic function. In this appendix, the spectral representation theorem is employed to derive the random phase model in terms of a random Fourier-Stieltjes integral which, notationally, does not require use of the square root symbol. However, the square root remains implicitly and its importance to the representation is shown. It will be shown that physically, the square root implies that the wave amplitude is proportional to the square root of cumulative wave energy.

Assume that the sea surface is in a stationary state and we desire to represent the wave surface elevation $\eta(t)$ as a stationary Gaussian process. According to the spectral representation theorem, it must be possible to represent the real process $\eta(t)$ in the form

$$\eta(t) = \mathrm{Re} \int_{-\infty}^{\infty} e^{i\omega t}\, dZ(\omega) \tag{B.1}$$

where $Z(\omega)$ is a process having orthogonal increments. We shall construct $Z(\omega)$ as follows. Let $f(\omega)$, $-\infty < \omega < \infty$, denote the two-sided wave energy spectral density function. Consider the sequence $\left\{ e^{-i\varepsilon_k} \right\}$ where each ε_k is a random variable uniformly distributed over $[0,2\pi)$, i.e.,

$$P_r\ [0 \le \varepsilon_k < 2\pi] = \frac{1}{2\pi} \tag{B.2}$$

Define the random set function $\tilde{Z}$ on intervals of the real line by

$$\tilde{Z}(\alpha,\beta) = \ell.i.m. \sum_{\substack{\omega_k \varepsilon (\alpha,\beta) \\ \max|\omega_k - \omega_{k-1}| \to 0}} e^{-i\varepsilon k} \left[f_\eta(\omega_k^*)(\omega_k - \omega_{k-1}) \right] \tag{B.3}$$

where the ω_k comprise a sequence of partitions of the frequency axis and $\omega_k^* \varepsilon (\alpha,\beta]$. Then define the random point function $Z(\omega)$ by

$$Z(\omega) = \tilde{Z}(-\infty,\omega]) \tag{B.4}$$

and denote

$$Z(\omega) = \int_{-\infty}^{\omega} e^{-i\varepsilon(\alpha)} \sqrt{f_\eta(\alpha)}\, d\alpha \tag{B.5}$$

Note that $Z(\omega)$ exists for every ω because

$$E|Z(\omega)|^2 = EZ(\omega)\ \overline{Z(\omega)}$$

$$= \int_{-\infty}^{\omega} f_\eta(\alpha)\, d\alpha < \infty$$

It follows easily that $Z(\omega)$ has orthogonal increments and that $E|dZ(\omega)|^2 = dG(\omega)$, where $G(\omega)$ is the spectral distribution function, defined by

$$G(\omega) = \int_{-\infty}^{\omega} f_\eta(\alpha)\, d\alpha \tag{B.6}$$

Now, although $Z(\omega)$ is not differentiable, because of the manner in which $Z(\omega)$ was constructed, we can write

$$dZ(\omega) = e^{-i\varepsilon(\omega)} \sqrt{f_\eta(\omega)d\omega} \tag{B.7}$$

Thus, the complex form of the random phase model is

$$\eta(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, dZ(\omega) \tag{B.8}$$

with $Z(\omega)$ given by Equation (B.5). The real random phase model is then defined by

$$\eta(t) = \mathrm{Re} \int_{-\infty}^{\infty} e^{i\omega t}\, dZ(\omega) \tag{B.9}$$

Equations (B.8) and (B.9) formally represent the random phase model in the usual manner for Fourier-Stieltjes integrals. However, the square root remains an essential part of $Z(\omega)$ as is seen from Equations (B.5) and (B.7). Equation (B.9) then yields the integral form used in the simulation program. Because

$$\eta(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, dZ(\omega) \qquad \text{by (B.9)}$$

$$= \mathrm{Re} \int_{-\infty}^{\infty} e^{i\omega t}\, e^{-i\varepsilon(\omega)} \sqrt{f_\eta(\omega)d\omega} \qquad \text{by (B.6)}$$

$$= \int_{-\infty}^{\infty} \cos\,(\omega t - \varepsilon(\omega))\ \sqrt{f_\eta(\omega)d\omega} \quad \text{by definition}$$

$$\eta(t) = 2 \int_{0}^{\infty} \cos\,(\omega t - \varepsilon(\omega))\ \sqrt{f_\eta(\omega)d\omega} \quad \text{by symmetry,}$$

$$= \int_{0}^{\infty} \cos\,(\omega t - \varepsilon(\omega))\ \sqrt{2S(\omega)d\omega}$$

by definition of $S(\omega)$ as the one-sided wave spectral density.

The necessity of the square root also follows from the spectral representation theorem which requires

$$dG_\eta(\omega) = E\left|dZ(\omega)\right|^2 \tag{B.10}$$

where $G_\eta(\omega)$ is the wave process spectral distribution function, called "cumulative energy density" in the St. Denis-Pierson paper. In most practical applications, $G_\eta(\omega)$ is absolutely continuous so that

$$dG_\eta(\omega) = f_\eta(\omega)d\omega \tag{B.11}$$

Thus,

$$E\left|dZ(\omega)\right|^2 = f_\eta(\omega)d\omega \tag{B.12}$$

which implies that $dZ(\omega)$, and hence $\eta(t)$, in Equation (B.1) must be proportional to $(d\omega)^{1/2}$. Physically, this result means that the squared wave amplitude must be proportional to the cumulative wave energy.

APPENDIX C

STATISTICAL PROPERTIES OF THE RANDOM PHASE MODEL

We wish to show that the random phase wave surface elevation model represents a stationary Gaussian process having zero mean. Let $\eta(t)$ denote the time varying wave surface elevation and let $S(\omega)$ denote the wave energy spectrum. Then

$$\eta(t) = \int_0^\infty \cos\ (\omega t - \varepsilon(\omega))\ \sqrt{2S(\omega)d\omega} \tag{C.1}$$

$$= \text{Re} \int_0^\infty e^{i[\omega t - \varepsilon(\omega)]}\ \sqrt{2S(\omega)d\omega} \tag{C.2}$$

where $\varepsilon(\omega)$ is an uniformly distributed random variable over $[0,2\pi)$. The expected value of $\eta(t)$ is then

$$E\eta(t) = E\ \text{Re} \int_0^\infty e^{i[\omega t - \varepsilon(\omega)]}\ \sqrt{2S(\omega)d\omega}$$

$$= \text{Re} \int_0^\infty e^{i\omega t}\ E\{e^{-i\varepsilon(\omega)}\}\ \sqrt{2S(\omega)d\omega}$$

The characteristic function of $\varepsilon(\omega)$ is

$$\varphi^{(t)}_{\varepsilon(\omega)} = E\{e^{i\varepsilon(\omega)t}\} = \frac{e^{2\pi i t} - 1}{2\pi i t} \tag{C.3}$$

which implies

$$\varphi_{\varepsilon(\omega)}^{(-t)} = E\{e^{-i\varepsilon(\omega)t}\}$$

$$= \frac{1 - e^{-2\pi it}}{2\pi it}$$

Thus,

$$\varphi_{\varepsilon(\omega)}^{(-1)} = 0$$

$$E\,\eta(t) = \mathrm{Re}\int_0^\infty e^{i\omega t} \cdot 0 \cdot \sqrt{2S(\omega)d\omega} = 0 \qquad \text{(C.4)}$$

and the process has the mean value zero.

The zero mean value implies that the covariance function of $\eta(t)$ is

$$\mathrm{COV}\,(\eta(t+\tau),\eta(t)) = E\,\eta(t+\tau)\,\eta(t)$$

$$= 1/2\ \mathrm{Re}\int_0^\infty \int_0^\infty e^{i[\omega(t+\tau)-\theta\cdot t]} \cdot E\{e^{-i[\varepsilon(\omega)+\varepsilon(\theta)]}\}\sqrt{2S(\omega)2S(\theta)d\omega d\theta}$$

$$+ 1/2\ \mathrm{Re}\int_0^\infty \int_0^\infty e^{i[\omega(t+\tau)+\theta\cdot t]} \cdot E\{e^{-i[\varepsilon(\omega)-\varepsilon(\theta)]}\}\sqrt{2S(\omega)2S(\theta)d\omega d\theta} \qquad \text{(C.5)}$$

Now for $\omega \neq \theta$, $\varepsilon(\omega)$, and $\varepsilon(\theta)$ are statistically independent, by definition of the random phase. Thus the characteristic function of their sum is

$$\varphi_{\varepsilon(\omega)+\varepsilon(\theta)}^{(t)} = \varphi_{\varepsilon(\omega)}^{(t)} \cdot \varphi_{\varepsilon(\theta)}^{(t)}$$

$$= \left(\frac{e^{2\pi it}-1}{2\pi it}\right)^2, \quad \omega \neq \theta \qquad \text{(C.6)}$$

and

$$E\{e^{-i[\varepsilon(\omega)+\varepsilon(\theta)]}\} = \varphi^{(-1)}_{\varepsilon(\omega)+\varepsilon(\theta)} = 0, \quad \omega \neq \theta \tag{C.7}$$

Similarly,

$$\varphi^{(t)}_{\varepsilon(\omega)-\varepsilon(\theta)} = \varphi^{(t)}_{\varepsilon(\omega)} \cdot \varphi^{(t)}_{-\varepsilon(\theta)}$$

$$= \frac{\cos(2\pi t) - 1}{-2\pi t^2}, \quad \omega \neq \theta \tag{C.8}$$

and

$$E\{e^{-i[\varepsilon(\omega)-\varepsilon(\theta)]}\} = \varphi^{(-1)}_{\varepsilon(\omega)-\varepsilon(\theta)} = 0, \quad \omega \neq \theta \tag{C.9}$$

Thus, applying Equations (C.7) and (C.9) to Equation (C.5) yields

$$\text{COV}\,(\eta(t+\tau)\eta(t)) = 1/2\ \text{Re} \int_0^\infty E\{e^{-2i\varepsilon(\omega)}\}\ 2S(\omega)d\omega$$

$$+ 1/2\ \text{Re} \int_0^\infty e^{i\omega\tau}\ E\{1\}2S(\omega)d\omega$$

But

$$E\{e^{-2i\varepsilon(\omega)}\} = \varphi^{(-1)}_{2\varepsilon(\omega)} = 0$$

so that

$$\text{COV } (\eta(t+\tau)\eta(t)) = 1/2 \text{ Re } \int_0^\infty e^{i\omega\tau} \, 2S(\omega)d\omega$$

$$= \int_0^\infty \cos(\omega\tau) S(\omega) d\omega \qquad \text{(C.10)}$$

The covariance function is thus a function of τ and is independent of t. Hence, the wave process is wide sense stationary. We write

$$R(\tau) = \text{COV } (\eta(t+\tau)\eta(t)) = \int_0^\infty S(\omega) \cos(\omega\tau) d\omega \qquad \text{(C.11)}$$

The variance of the process is then, by definition,

$$\sigma^2 = R(0) = \int_0^\infty S(\omega) d\omega \qquad \text{(C.12)}$$

The probability distribution function of $\eta(t)$ can be inferred from the definition of the integral of Equation (C.2) by expressing the integral as a limiting sum and properly applying the central limit theorem. We write

$$\eta(t) = \underset{\substack{N \to \infty \\ \Delta\omega_k \to 0}}{\text{l.i.m.}} \sum_{k=1}^{N} \cos(\omega_k^* t - \varepsilon_k) \sqrt{2S(\omega_k^*)\Delta\omega_k}$$

$$= \underset{\substack{N \to \infty \\ \Delta\omega_k \to 0}}{\text{l.i.m.}} \sum_{k=1}^{N} A_k X_k \qquad \text{(C.13)}$$

$$= \underset{\substack{N \to \infty \\ \Delta\omega_k \to 0}}{\ell.i.m.} \sum_{k=1}^{N} Y_k \qquad (C.14)$$

with

$$A_k = \sqrt{2S(\omega_k^*)\Delta\omega_k} \qquad (C.15a)$$

$$X_k = \cos\ (\omega_k^* t - \varepsilon_k) \qquad (C.15b)$$

and

$$Y_k = A_k\ X_k \qquad (C.16)$$

We first note that the usual central limit theorem for identically distributed random variables is not applicable to the sequence $\{Y_k\}$. Fortunately, a second version of the central limit theorem can be used to obtain the desired result provided certain regularity conditions are satisfied. This theorem states that the sum of a sequence of independently but not necessarily identically distributed second order random variables with means μ_k and variances σ_k^2 is asymtotically normally distributed. More exactly, the sum converges in distribution to a normal distribution. A sufficient condition for convergence to the normal law in this case is given[12] by a theorem due to Liapounoff which requires that the sequence of third absolute moments exist for the random variables and that

$$\underset{n \to \infty}{\ell.i.m.}\ S_n = 0$$

where

$$S_n = \frac{\left(\sum_{k=1}^{n} \rho_k^3\right)^{1/3}}{\left(\sum_{k=1}^{n} \sigma_k^2\right)^{1/2}} \qquad (C.17)$$

It is not obvious by inspection that the condition of Equation (C.17) holds for any particular application of the central limit theorem. The condition, therefore, must be verified for the random phase model components.

Rice,[3] in his classic paper, proved that $S_n^3 \to 0$ for any finite range of integration in the random phase model of Equation (C.14). His proof can also be used to show that $S_n \to 0$ for any finite frequency range $(0,\omega_c)$. From Equation (C.14), the sequency of independent random variables is defined by

$$Y_k = A_k \cos\ (\omega_k^* t - \varepsilon_k),\ k = 1,2,\ \ldots \qquad (C.18)$$

The variances are then

$$\sigma_k^2 = EY_k^2 = \frac{A_k^2}{2} \qquad (C.19)$$

and the third absolute moments are

$$\rho_k^3 = E|Y_k|^3$$

$$= A_k^3\ E|\cos^3\ (\omega_k^* t - \varepsilon_k)|^3$$

$$= \frac{4}{3\pi}\ A_k^3 \qquad (C.20)$$

Then from Equations (C.17), (C.19), and (C.20),

$$S_n^3 = \frac{\left(\sum_{k=1}^{n} \rho_k^3\right)}{\left(\sum_{k=1}^{n} \sigma_k^2\right)^{3/2}}$$

$$= \left(\sum_{k=1}^{n} \frac{4}{3\pi} A_k^3\right) \cdot \left(\sum_{k=1}^{n} \frac{A_k^2}{2}\right)^{-3/2}$$

$$= \frac{8(2^{1/2})}{3\pi} (\Delta\omega)^{1/2} \sum_{k=1}^{n} (S(\omega_k^*))^{3/2} \cdot \Delta\omega \cdot \left(\sum_{k=1}^{n} S(\omega_k^*)\Delta\omega\right)^{-3/2}$$

Assume $\Delta\omega \to 0$, $n \to \infty$, $\ni$ $n\Delta\omega = \omega_c$. Then, for $\Delta\omega$ sufficiently small,

$$S_n < 2(\Delta\omega)^{1/6} \cdot \left(\int_0^{\omega_c} S(\omega)^{3/2}\, d\omega\right)^{1/3} \cdot \left(\int_0^{\omega_c} S(\omega)\, d\omega\right)^{-1/2}$$

Thus, since $n\Delta\omega = \omega_c$, we have $S_n \to 0$ as $n \to \infty$ as long as ω_c is finite. However, because ω_c can be chosen as large as desired, we say that the random phase model from Equation (C.1) converges in distribution to a normal (Gaussian) distribution function.

In practice, it is convenient to employ probability density functions in addition to probability distribution functions. It is thus of interest to know whether or not the sequence of probability density functions for the random phase model components converges to a normal density function. von Mises[13] has given the following sufficient conditions for the component random variables:

1. The variances satisfy $0 < \sigma_k^2 \leq C < \infty \ \forall k$

2. The third central moments of absolute values are uniformly bounded.

3. The characteristic functions $\varphi_k(t)$ never approach the value 1 at points other than t = 0.

4. For large $|t|$, the $\varphi_n(t)$ tend to zero like some negative power of $|t|$, and uniformly for all k.

From Equation (C.18) and the fact that the ε_k are uniformly distributed, it follows that probability density functions for the Y_k are given by

$$f_k(x) = \frac{1}{\pi\sqrt{A_k^2 - x^2}} \qquad -A_k < x < A_k$$

Thus

$$\sigma_k^2 = \frac{A_k^2}{2} \leq S_\omega u\rho \ S(\omega) < \infty \text{ for every } k$$

and

$$\rho_k^3 = \frac{4}{3\pi} A_k^3 \leq \frac{4}{3\pi} S_\omega u\rho \ (S(\omega))^{3/2} \text{ for every } k$$

so that the first two conditions are fulfilled. Condition 3 is also satisfied because

$$\varphi_{y_k}(t) = \int_{-\infty}^{\infty} \frac{1}{\pi\sqrt{A_k^2 - x^2}} e^{ixt} \, dx$$

$$= J_0(A_k t)$$

where J_0 is the zeroth order Bessel function of the first kind which equals 1 only at $t = 0$. For large $|t|$ $J_0(A_k t)$ tends to zero like $(tA_k)^{-1/2}$ but not necessarily uniformly in k. The convergence will be uniform in k if it is assumed that the A_k's are bounded below by a positive constant. The assumption is valid for any frequency range [a,b] satisfying $0 < a < b < \infty$. Then, since a and b can be selected arbitrarily close to zero and infinity, respectively, we say that condition 4 is satisfied for the random phase model and conclude that the sequence of density functions converges to a normal density for each t.

It follows by a theorem given by Lukacs and Laha[14] that for any times $t_1 \ldots t_n$, the distribution of $(\eta(t_1) \ldots \eta(t_n)) = \tilde{\eta}(t)$ is multivariate normal. Since each $\eta(t)$ has mean zero and covariance function $R(\tau)$, $\tilde{\eta}(t)$ will have mean vector 0 and covariance matrix $M = (R(t_i - t_j))$. The finite dimensional distributions of the process $\eta(t)$ depend only on time differences and are independent of t. Therefore, the process is really strict sense stationary. A strict sense Gaussian process is ergodic if and only if its spectral distribution function has no points of discontinuity as expressed by Doob.[10] This condition will always be satisfied in the random phase model for the following reason. We have assumed that the spectral density function $f_\eta(\omega)$ exists for the random phase model. Then, from Equation (B.11), the spectral distribution function $G_\eta(\omega)$ is given by

$$G_\eta(\omega) = \int_{-\infty}^{\omega} f_\eta(\theta)d\theta$$

$$= \int_{0}^{\omega} S(\theta)d\theta$$

Thus, the spectral distribution is an indefinite integral. Equivalently, this means that $G_\eta(\omega)$ is an absolutely continuous function, and thus, it is certainly a continuous function.

APPENDIX D

DIGITAL GENERATION OF RANDOM NUMBERS

Random numbers are required for the digital simulation of stochastic processes. It is usually sufficient to obtain uniformly distributed random numbers since most other distributions of interest can be obtained as elementary transformations of the uniform distribution.

Ideally, a random number represents an observation from some phenomenon whose outcomes indicate no deterministic regularity. That is, distinct outcomes may be different even when observed under the same conditions. Thus a sequence of random numbers will never repeat itself and it can be identified only through its statistical properties.

In the past, random numbers were obtained from a number of diverse sources such as roulette wheels, census tables, and electronic noise generators. With the advent of high speed digital computers most of the random number sources were rejected in favor of simple numerical algorithms. Such numerical algorithms produce sequences of numbers that satisfy the statistical properties of randomness. However, the same sequence of numbers always results when an algorithm is exercised twice under exact conditions. Because of this reproducibility property, digitally generated numbers are called "pseudorandom numbers."

There are various algorithms for digitally generating pseudorandom numbers. The midsquare method appeared quite popular at one time. Its principle is as follows. Let X_0 be a p digit number. Its square gives roughly 2p digits; the middle p digits are taken as X_1. The value of X_1 is then squared and the process is repeated. Thus, if $X_0 = 0.76$, $X_0^2 = 0.5776$ and the sequence 0.76, 0.77, 0.92, 0.46, 0.11, 0.12 ... is obtained. The sequence eventually repeats itself. Also for certain choices of X_0, such sequences converge quickly to zero, and are thus biased toward lower values.

Today, most computer systems possess algorithms for generating pseudorandom numbers that are based on the number theory concept of binomial congruence, as first proposed by Lehmer.[15] Such algorithms are defined by

$$X_{n+1} = KX_n + b \bmod(2^p) \tag{D.1}$$

where p is the word size in bits for the employed binary machine. The indicated congruence relation means that if the difference $(X_{n+1}-KX_n)$ were divided by 2^p the remainder would be b. When b = 0, the generator is called multiplicative; otherwise it is called mixed. Sequences generated by such congruence algorithms also have finite periods (cycle lengths). The constant K is chosen to maximize the cycle length. The initial number, X_0, is usually taken as any odd number.

Random numbers generated by the system routine on CDC 6000 series computers are multiplicative and are generated according to

$$X_{n+1} = (2^{17}+55205)\ X_n \bmod(1) \tag{D.2}$$

where mod(1) implies that the numbers in Equation (D.1) have been divided by 2^p to obtain pseudorandom numbers that assume values between zero and one. The CDC 6000 computer series uses sixty bit words such that p = 60. As implemented by CDC, the generator is designated as RANF and has a full period of 2^{48}.

Given a finite sequence of numbers generated by a numerical algorithm, it is desirable to determine whether the numbers are distinguishable from a random sample or from a real random process. The standard technique for this determination has been the use of standard "goodness-of-fit" tests. Canavos[16] has performed a comprehensive set of goodness-to-fit tests involving the CDC 6000 series generator, RANF. His tests included analyses of runs, serial correlations, moments, chi-squares, and Kolmogorov-Smirnov statistics. Each method was tested against a set of ten sequences of random numbers and for sample sizes of 200, 1,000, and 10,000. Although his results indicated slightly different results for the different tests, the overall results indicated that RANF does generate uniformly distributed pseudorandom numbers over the interval (0,1). To further investigate the quality of numbers obtained from RANF, twelve sequences

of 10,000 numbers each were generated on the DTNSRDC CDC 6700 system and chi-square tests were performed at the 0.10 significance level. The number of class intervals k was selected optimally in accordance with the procedure of Mann and Wald[8] as detailed by Williams.[9] Thus, 165 intervals were employed which gave 164 degrees of freedom for the tests and yielded a critical value of $\chi^2_{164,0.10} = 187.56$.

Based on the assumed statistical hypothesis, no more than one out of ten samples would be expected to yield a chi-square value that exceeded the critical value if the samples were representative of an uniform distribution. The chi-square values computed from the 12 samples are shown in Table 6. As indicated in this table, none of the computed values exceeded the critical value. Thus, it was concluded, that pseudorandom numbers generated using RANF exhibit the characteristics of an uniformly distributed random variable.

TABLE 6 - RESULTS OF THE GOODNESS-OF-FIT TESTS FOR 12 SEQUENCES OF 10,000 NUMBERS OBTAINED FROM THE CDC GENERATOR RANF

SEQUENCE NUMBER	1	2	3	4	5	6	7	8	9	10	11	12
CHI-SQUARE VALUE OF SEQUENCE	154.13	150.93	176.28	172.78	158.69	185.75	180.86	182.97	160.70	146.94	147.04	142.38

NOTE:
Critical value for the 0.10 probability level with 164 degrees of freedom is 187.56.

APPENDIX E

PROGRAM LISTING OF WAVESIM

```
      PROGRAM WAVESIM (INPUT,OUTPUT,TAPE6=OUTPUT,TAPE7)
C***PROGRAM TO GENERATE RANDOM WAVE PROCESS USING PIERSON-MOSKOWITZ
C     AMPLITUDE SPECTRUM BY RAE HURWITZ  CODE 1524    MARCH 7,1973
C      N IS THE NUMBER OF CASES
C      DELT IS THE TIME INCREMENT IN SECONDS
C      DELW IS THE FREQUENCY INCREMENT IN RADIANS/SECOND
C      M IS THE NUMBER OF FREQUENCY SUBDIVISIONS
C      MM IS THE TOTAL NUMBER OF FREQUENCIES, MM= M + 1
C      NO IS THE TOTAL NUMBER OF POINTS COMPUTED FOR WAVE PROCESS
C      OMEGMIN IS THE MINIMUM FREQUENCY IN RADIANS/SECOND
C      OMEGMAX IS THE MAXIMUM FREQUENCY IN RADIANS/SECOND
C      TINMIN IS THE LENGTH OF THE WAVE PROCESS IN MINUTES
C      TINSEC IS THE LENGTH OF THE WAVE PROCESS IN SECONDS
C      IFPLOT = 1  PLOT OUTPUT
C      IFPLOT = 2  DO NOT PLOT OUTPUT
      COMMON W(105),WSTAR(105),STARRAN(105),S(105),PMS(105),
     + RANDOM(105),XX(105),WW(105),TIME(1805),WAVE(1805),A,AREAPMS,
     + DELT,DELW,P,MM,N,NN,NO,OMEGMAX,OMEGMIN,PI,RMEAN,
     + TINMIN,TINSEC,VARANCE,VK
  500 READ 100,N,M,MM,DELT,OMEGMIN,OMEGMAX,TINMIN
      READ 120,IFPLOT
      DELW = (OMEGMAX - OMEGMIN)/FLOAT(M)
      TINSEC = 60. * TINMIN
      ONO = CNO + 1.
      NO = ONO
      PRINT 116
      PRINT 102,N,NO,MM,OMEGMIN,OMEGMAX,DELW,DELT,TINMIN,TINSEC
      CALL SPECTRA
      PRINT 116
      PRINT 104
      PRINT 106,(W(I),WSTAR(I),PMS(I),S(I),I=1,MM)
      CALL WAVES
      PRINT 116
      PRINT 108,(XX(I),I=1,MM)
      CALL ARAMEAN
      DO 510 I=1,NO
      WAVE(I) = WAVE(I) - RMEAN
  510 CONTINUE
      PRINT 116
      PRINT 110
      PRINT 112,(TIME(J),WAVE(J),TIME(J+1),WAVE(J+1),TIME(J+2),
     + WAVE(J+2),TIME(J+3),WAVE(J+3),J=1,NO,4)
      CALL ARAMEAN
      CALL VAR
      PRINT 116
      PRINT 114,RMEAN,VARANCE,AREAPMS
      WMIN = 0.0
      WMAX = 0.0
      DO 515 I=1,NO
      IF(WAVE(I) .GT. WMAX) WMAX = WAVE(I)
      IF (WAVE(I) .LT. WMIN) WMIN = WAVE(I)
```

```
515 CCNTINUE
    PRINT 116
    PRINT 117,WMIN,WMAX
    IF (IFPLOT.EQ.1)535,540
535 CALL PLOTW
540 IF (N.EQ.1) 550,500
100 FORMAT(3I5,4F10.2)
102 FORMAT (20X,*NUMBER OF INPUT CASES                                    *,
   +I3//20X,*TOTAL NUMBER OF POINTS COMPUTED FOR WAVE PROCESS  *,I5//
   +20X,*NUMBER OF FREQUENCIES USED                          *,I7//
   +20X,*MINIMUM FREQUENCY IN RADIANS/SECOND                *,F8.3//
   +20X,*MAXIMUM FREQUENCY IN RADIANS/SECOND                *,F8.3//
   +20X,*FREQUENCY INCREMENT IN RADIANS/SECOND               *,F6.3//
   +20X,*TIME INCREMENT IN SECONDS                           *,F6.3//
   +20X,*LENGTH OF WAVE PROCESS IN MINUTES                   *,F6.3//
   +20X,*LENGTH OF WAVE PROCESS IN SECONDS                  *,F8.3//)
104 FORMAT (2X,*FREQUENCY SUBDIVISION*,5X,*ACTUAL FREQUENCY*,5X,
   + *SPECTRAL ORDINATE BASED*/48X,
   + *ON ACTUAL FREQUENCY*//)
106 FORMAT (8X,F6.3,18X, F6.3,16X,F8.3,20X,F8.3)
110 FORMAT(10X,*TIME*,7X,*WAVE*,10X,*TIME*,7X,*WAVE*,10X,*TIME*,
   +7X,*WAVE*,10X,*TIME*,7X,*WAVE*//)
112 FORMAT (8X,F8.2,3X,F8.2,6X,F8.2,3X,F8.2,6X,F8.2,3X,F8.2,6X,F8.2,
   +3X,F8.2)
114 FORMAT (20X,*MEAN OF THE GENERATED WAVE PROCESS
   +               *,F10.5//20X,*VARIANCE OF THE COMPUTED WAVE PROCESS
   +                          *,F10.5//20X,
   +          //20X,*AREA BASED ON THE CONTINUOUS FORM OF THE PIERSON-
   +MOSKOWITZ SPECTRUM *,F10.5//)
116 FORMAT (1H1)
117 FORMAT (2X,*WMIN = *,F10.5,10X,*WMAX = *,F10.5)
120 FORMAT (I5)
550 STOP
    END
```

```
      SUBROUTINE SPECTRA
C     THIS SUBROUTINE CALCULATES THE SPECTRAL DENSITY FUNCTION USING
C          THE PIERSON-MOSKOWITZ FORMULA
C     RANF(DUM) IS A CDC EXTERNAL FUNCTION WHICH RETURNS VALUES UNIFORMLY
C        DISTRIBUTED OVER THE RANGE(0,1),RANDOM NUMBER GENERATOR
C     RANSET(A) IS A CDC SUBROUTINE WHICH INITIALIZES THE VALUE(A) OF THE
C               RANDOM NUMBER GENERATOR
C            A IS A FLOATING POINT NUMBER
C     PMS IS THE PIERSON-MOSKOWITZ SPECTRAL ORDINATE
C               BASED ON FREQUENCY SUBDIVISION
C     S IS THE PIERSON-MOSKOWITZ SPECTRAL ORDINATE
C               BASED ON ACTUAL FREQUENCY
C     S(J) ARE THE SPECTRAL ORDINATE VALUES
C     STARRAN IS THE RANDOM NUMBER OBTAINED FROM RANF
C     VK IS THE WIND SPEED IN KNOTS FOR THE PIERSON-MOSKOWITZ FORMULA
C     W ARE THE FREQUENCY SUBDIVISIONS
C     WSTAR ARE THE ACTUAL FREQUENCIES
      COMMON W(105),WSTAR(105),STARRAN(105),S(105),PMS(105),
     + RANDOM(105),XX(105),WW(105),TIME(1805),WAVE(1805),A,AREAPMS,
     + DELT,DELW,M,MM,N,NN,NO,OMEGMAX,OMEGMIN,PI,RMEAN,
     + TINMIN,TINSEC,VARANCE,VK
      READ 100,VK
      PRINT 110,VK
  100 FORMAT (F10.2)
  110 FORMAT (20X,*WIND SPEED IN KNOTS                                  *
     + F6.3//)
      W(1) = OMEGMIN
      DO 120 I=2,MM
  120 W(I) = W(I-1) + DELW
      WSTAR(1) = OMEGMIN
      CALL RANSET(3.12)
      DO 130 I=2,MM
      STARRAN(I) = RANF(DUM)
      WSTAR(I) = W(I) + STARRAN(I) * DELW
  130 CONTINUE
      A = .00810 * (32.174)**2
      B = .74 * (32.174/(VK*1.6878))**4
      PI = 3.1415927
      DO 140 J=1,MM
      S(J) = A/(WSTAR(J)**5)*EXP(-B/(WSTAR(J)**4))
      PMS(J) = A/(W(J)**5)*EXP(-B/(W(J)**4))
  140 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE WAVES
C     THIS SUBROUTINE COMPUTES THE RANDOM WAVE PROCESS
C     RANDOM IS A RANDOM NUMBER BETWEEN 0 AND 1
C     RANF(DUM) IS A CDC EXTERNAL FUNCTION WHICH RETURNS VALUES UNIFORMLY
C        DISTRIBUTED OVER THE RANGE(0,1),RANDOM NUMBER GENERATOR
C     RANSET(A) IS A CDC SUBROUTINE WHICH INITIALIZES THE VALUE(A) OF THE
C                RANDOM NUMBER GENERATOR
C     TIME IS TIME IN SECONDS
C     WAVE IS THE WAVE PROCESS AS A FUNCTION OF TIME
C     XX IS THE RANDOM PHASE ANGLE
      COMMON W(105),WSTAR(105),STARRAN(105),S(105),PMS(105),
     + RANDOM(105),XX(105),WW(105),TIME(1805),WAVE(1805),A,AREAPMS,
     + DELT,DELW,M,MM,N,NN,NO,OMEGMAX,OMEGMIN,PI,RMEAN,
     + TINMIN,TINSEC,VARANCE,VK
C     TIME BEGINS AT ZERO
      TIME (1) = 0.0
      DO 20 JJ=2,NO
   20 TIME(JJ) = TIME(JJ-1) + DELT
      CALL RANSET (7.11)
      DO 25 I=1,MM
      RANDOM(I) = RANF(DUM)
   25 XX(I) = RANDOM(I) * 2. * PI
      DO 30 K=1,NO
   30 WAVE(K) = 0.0
      DO 40 J=1,NO
      DO 35 I=1,MM
      WW(I) = SQRT(2.*S(I)*DELW)*COS(WSTAR(I)*TIME(J) - XX(I))
      WAVE(J) = WAVE(J) + WW(I)
   35 CONTINUE
   40 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE ARAMEAN
C     AREAPMS IS THE AREA BASED ON THE CONTINUOUS FORM OF THE
C            PIERSON-MOSKOWITZ SPECTRUM
C     THIS SUBROUTINE CALCULATES THE MEAN VALUE OF THE GENERATED WAVE PROCESS
C     RMEAN IS THE TIME-AVERAGE OF THE WAVE PROCESS
      COMMON W(105),WSTAR(105),STARRAN(105),S(105),PMS(105),
     + RANDOM(105),XX(105),WW(105),TIME(1805),WAVE(1805),A,AREAPMS,
     + DELT,CELW,M,MP,N,NN,NO,OMEGMAX,OMEGMIN,PI,RMEAN,
     + TINMIN,TINSEC,VARANCE,VK
      CON1 = .00810*(32.174)**2
      CON2 = .74 * (32.174/(VK*1.6878))**4
      AREAPMS = CON1/(4.*CON2)*(EXP(-CON2/OMEGMAX**4)-
     + EXP(-CON2/OMEGMIN**4))
      WMEAN = 0.0
      RNO = FLOAT(NO)
      DO 60 I=1,NO
   60 WMEAN = WAVE(I) + WMEAN
      RMEAN = WMEAN/RNO
      RETURN
      END
```

```
      SUBROUTINE VAR
C     THIS SUBROUTINE CALCULATES THE VARIANCE OF THE COMPUTED
C                    WAVE PROCESS
      COMMON W(105),WSTAR(105),STARRAN(105),S(105),PMS(105),
     + RANDOM(105),XX(105),WW(105),TIME(1805),WAVE(1805),A,AREAPMS,
     + DELT,DELW,M,MP,N,NN,NO,OMEGMAX,OMEGMIN,PI,RMEAN,
     + TINMIN,TINSEC,VARANCE,VK
      VARIAN = 0.0
      RNO = FLOAT(NO)
      DO 70 I=1,NO
   70 VARIAN = VARIAN + WAVE(I) * WAVE(I)
      VARANCE = VARIAN/RNO
      RETURN
      END
```

```
      SUBROUTINE PLOTW
      COMMON W(105),WSTAR(105),STARRAN(105),S(105),PMS(105),
     + RANDOM(105),XX(105),WW(105),TIME(1805),WAVE(1805),A,AREAPMS,
     + DELT,DELW,M,MM,N,NN,NO,OMEGMAX,OMEGMIN,PI,RMEAN,
     + TINMIN,TINSEC,VARANCE,VK

C       PLOT ROUTINE FOR CALCOMP 936
      CALL PLOTS(0.0,0.0,7)
      CALL PLOT (0.0,0.0,-3)
      CALL SCALE(TIME,18.0,1801,1)
      CALL SCALE(WAVE,4.0,1801,1)
      CALL AXIS(0.0,0.0,15HTIME IN SECONDS,-15,18.0,0.0,TIME(1802),
     + TIME(1803))
      CALL AXIS (0.0,0.0,14HWAVE ELEVATION,14,4.0,90.0,WAVE(1802),
     + WAVE(1803))
      CALL LINE (TIME,WAVE,NO,1,0,0)
      CALL SYMBOL(2.0,8.0,.20,12HTIME HISTORY,0,12)
      CALL PLOT(0.0,0.0,-3)
      CALL PLOT(0.0,0.0,999)
      PRINT 650
  650 FORMAT (20H1PLOT IS FINISHED     )
      RETURN
      END
```

REFERENCES

1. St. Denis, M. and W.J. Pierson, "On the Motion of Ships in Confused Seas," Society of Naval Architects and Marine Engineers Transactions, Vol. 61, p. 280 (1953).

2. Rudnick, P., "Correlograms for Pacific Ocean Waves," Proceedings Second Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, p. 627 (1951).

3. Rice, S.O., "Mathematical Analysis of Random Noise," Bell System Technical Journal, Vol. 23, p. 282 (1944), Vol. 24, p. 46 (1945).

4. Levy, P., "A Special Problem of Brownian Motion and a General Theory of Gaussian Random Functions," Proceedings Third Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, p. 133 (1956).

5. Grenander, U. and M. Rosenblatt, "Statistical Analysis of Stationary Time Series," John Wiley and Sons (1957).

6. Levy, P., "Processus Stochastiques et Movement Browien," Paris (1948).

7. Pierson, W.D. and L. Moskowitz, "A Proposed Spectral Form For Fully Developed Wind Seas Based on the Similarity Theory of S.A. Kitaigorodskii," Journal of Geophyiscal Research, Vol. 69 (24), p. 5181 (1964).

8. Mann, H.B. and A. Wald, "On the Choice of the Number of Class Intervals in the Application of the Chi Square Test," The Annals of Mathematical Statistics, Vol. 13, p. 306 (1942).

9. Williams, C.A., "On the Choice of the Number and Width of Classes for the Chi-Square Test of Goodness of Fit," Journal of American Statistical Association, Vol. 45, p. 77 (1950).

10. Doob, J.L., "Stochastic Processes," John Wiley and Sons (1953).

11. Lukacs, E., "Stochastic Convergence," D.C. Heath and Company (1968).

12. Cramer, H., "Mathematical Methods of Statistics," Princeton University Press (1946).

13. von Mises, R., "Mathematical Theory of Probability and Statistics," Academic Press (1964).

14. Lukacs, E. and R.G. Laha, "Applications of Characteristic Functions," Hafner Publishing Company (1964).

15. Lehmer, D.H., "Random Number Generation on the BRL Highspeed Computing Machines," Mathematical Reviews, Vol. 15, p. 559 (1954).

16. Canavos, G.C., "A Comparative Analysis of Two Concepts in the Generation of Pseudo-Random Numbers," Proceedings of 22nd National ACM Conference, ACM Publication P-67 (1967).

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