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ON THE THEORY AND OPTIMIZATION OF GLOBAL POINT-MASS EXPANSIONS --ETC(U)
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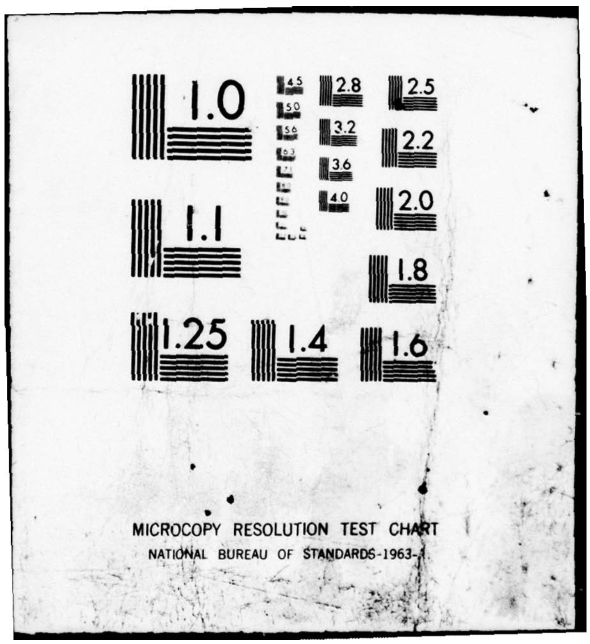
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The relationship between point-mass and spherical harmonic expansions of anomalous gravity is analyzed. In particular, the spherical harmonic expansion of a point-mass set is determined and used to discuss the nature of point-mass solutions to the problem of modeling anomalous gravity. Also, a "global error", which arises when a finite spherical harmonic expansion is replaced by a point-mass set, is first defined and then determined, both for the anomalous potential and its gradient. Defining an optimal point-mass

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20. ABSTRACT (Cont'd)

set as one which minimizes the global gradient error leads to a system of equations (linear in the masses, but non-linear in their positions) whose solution is the optimal point-mass set. Methods of solution are also discussed.

FOREWORD

This work was done in the Fire Control Presetting Analysis Branch of the FBM Geoballistics Division, Strategic Systems Department, Naval Surface Weapons Center (NAVSWC). It is a product of the ongoing effort at NAVSWC to develop and implement efficient methods for calculating the effects of anomalous gravity on missile trajectories. The work was authorized under Strategic Systems Project Office Task Assignment 37430.

This technical report was reviewed and approved by H. J. Boyles, D. L. Owen, and J. R. Fallin of the FBM Geoballistics Division.

The author wishes to credit Alan Rufty of the Fire Control Presetting Analysis Branch with priority in the derivation of equation (8).

Released by:

H. P. CASTER, Acting Head
Strategic Systems Department

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TABLES

INTRODUCTION

Sets of point-masses have been recognized in recent years as viable models for describing the anomalous gravity field of the earth.[1,2] The value of point-masses stems from the fact that, in particular circumstances, computation of the anomalous potential, its gradient (the gravity disturbance vector), or other related quantities are more rapidly achieved by using point-masses instead of various integral equation methods or spherical harmonic expansions. Point-masses have also proved useful in describing local variations of planetary[3,4] and lunar[5] gravity fields.

In this report, emphasis will be placed on the use of point-mass sets to describe the global anomalous gravity field. First, the spherical harmonic expansion of a point-mass set will be developed. This will allow the definition of "global errors" (in the potential and in its gradient) which arise when a finite spherical harmonic expansion is replaced by a set of point-masses. Then, exact forms of these global errors will be determined.

The development of analytic expressions for the global error in the anomalous potential and its gradient will permit the establishment of a means by which a point-mass set may be optimized. Defining an "optimal" point-mass set as one which minimizes global gradient error will lead to a system of optimizing equations. The structure of these equations, as well as methods for their solution, will also be discussed.

Thus, the primary concern of this report is to provide a theoretical basis for the construction of optimal point-mass sets from a finite spherical harmonic expansion. Point-mass sets can be constructed, of course, by methods which do not explicitly depend on spherical harmonic representations; however, since the coefficients of a spherical harmonic expansion can be derived from many sources, including satellite orbits, gravity anomalies, and geoid heights, the results presented here provide a general prescription for using a knowledge of any of these basic sources to create an optimal point-mass set.

THE SPHERICAL HARMONIC EXPANSION OF A POINT-MASS SET

An important purpose of point-mass modeling is to provide a description of the variations in gravity close to the surface of the earth. These variations are, in general, due to mass distributions both inside and outside the earth; however, since external masses are easily identified (the atmosphere, the moon, etc.), their effects can readily be dealt with (i.e., subtracted from geophysical measurements), leaving data which reflects only the influence of internal masses. (In this report, the term "geophysical data" will be used to represent these subtracted measurements.) Thus, the point-mass sets under consideration will all be internal to the earth.

The anomalous potential is usually defined as the small difference between the actual potential and some suitably chosen reference potential.[6] In the ubiquitous "spherical approximation"* it may be represented by a spherical harmonic expansion, an expansion whose coefficients can be derived from a knowledge of gravity anomalies, geoid heights, or other sources. The anomalous potential T may also be

* See Reference 6, p. 87

represented by a set of point-masses, in which case it takes the form

$$T(\vec{r}) = -k \sum_{i=1}^{I} \frac{M_i}{|\vec{r} - \vec{r}_i|} \tag{1}$$

where

- k: the gravitational constant
- M_i: the value of the i^{th} point-mass
- $\vec{r}_i$: the position vector of the i^{th} point-mass
- $\vec{r}$: the position vector of the computation point
- I: total number of point-masses

The geometric relation between $\vec{r}$ and one of the $\vec{r}_i$ is shown in Figure 1. In that figure, P is the computation point and P_i is the location of the i^{th} point-mass.

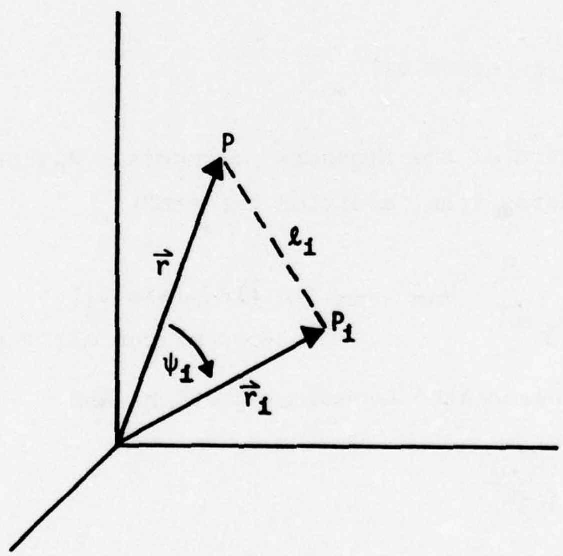

Figure 1. Geometry of Position Vectors

The coordinates of the points P and P_i are (r, ϕ, λ) and (r_i, ϕ_i, λ_i), respectively, in a geocentric coordinate system where r is the radial distance from the center, ϕ is the latitude, and λ is the longitude. In Figure 1, ψ_i is the angle between the vectors $\vec{r}$ and $\vec{r}_i$ and the relationship between ψ_i and the geocentric coordinates of $\vec{r}$ and $\vec{r}_i$ is

$$\cos \psi_i = \sin\phi \sin\phi_i + \cos\phi \cos\phi_i \cos(\lambda-\lambda_i) \tag{2}$$

The distance between the points P and P_i is ℓ_i, where

$$\ell_i = |\vec{r}-\vec{r}_i| = (r^2-2rr_i\cos\psi_i + r_i^2)^{\frac{1}{2}} \tag{3}$$

The potential T can be written in a spherical harmonic expansion by using (1) along with the following results. First, the expansion of ℓ_i^{-1} in Legendre polynomials

$$\frac{1}{\ell_i} = \sum_{n=o}^{\infty} \frac{r_i^n}{r^{n+1}} P_n(\cos \psi_i) \tag{4}$$

Second, the expansion of the Legendre polynomials $P_n(\cos \psi)$ in terms of geocentric coordinates (the "addition theorem")

$$P_n (\cos \psi_i) = \sum_{m=o}^{n} N_{nm} P_{nm}(\sin \phi)P_{nm}(\sin \phi_i) \cdot (\cos m\lambda \cos m\lambda_i + \sin m\lambda \sin m\lambda_i) \tag{5}$$

where the P_{nm} are associated Legendre functions and

$$N_{nm} = \varepsilon_m^{-1} \frac{(n-m)!}{(n+m)!}$$

$$\varepsilon_m = \begin{Bmatrix} 1, & m=o \\ \frac{1}{2}, & m\neq o \end{Bmatrix} \tag{6}$$

Now, if (5) is put into (4) and the result of this is put into (1), then the following spherical harmonic expansion ensues:

$$T(\vec{r}) = \frac{-km}{R} \sum_{n=o}^{\infty} \sum_{m=o}^{n} \left(\frac{R}{r}\right)^{n+1} P_{nm} (\sin \phi) \left[A_{nm} \cos m\lambda + B_{nm} \sin m\lambda\right] \quad (7)$$

where

$$\begin{Bmatrix} A_{nm} \\ B_{nm} \end{Bmatrix} = N_{nm} \sum_{i} \frac{M_i}{M} \left(\frac{r_i}{R}\right)^n P_{nm} (\sin \phi_i) \begin{Bmatrix} \cos m\lambda_i \\ \sin m\lambda_i \end{Bmatrix} \quad (8)$$

(Here, R: mean earth radius, and M: earth mass. These parameters appear in (7) and (8) in order to make the coefficients A_{nm} and B_{nm} dimensionless.)

Expression (8) shows explicitly how to construct the spherical harmonic expansion equivalent to a given point-mass set. The main concern of point-mass modeling, however, is the converse problem: given a spherical harmonic expansion, find the corresponding point-mass set. In practice, this means: given a finite set of coefficients $\{A_{nm}, B_{nm}\}$, invert (8) to find the corresponding point-mass parameters $\{M_i, r_i, \phi_i, \lambda_i\}$. The nature of exact and approximate solutions to this problem will be discussed in the next section.

NATURE OF SOLUTIONS

It is clear that any practical point-mass set is defined by a finite, bounded set of real numbers $\{M_i, r_i, \phi_i, \lambda_i\}$. Since this bounded set gives rise, via (8), to an infinite number of non-zero coefficients $\{A_{nm}, B_{nm}\}$, the spherical harmonic expansion corresponding to any practical point-mass set will, in general, be infinite. This leads to a basic problem in point-mass modeling, a problem whose origins will now be elucidated.

The discreteness of geophysical data imposes a practical limit* on our knowledge of the fine structure of the global gravity field. This inherent limitation prohibits (long before any computational or time constraints would) the construction of spherical harmonic expansions beyond some finite maximum degree and order. Thus, by attempting to construct a practical point-mass set from geophysical data, we are, in effect, trying to match an infinite orthogonal expansion with a finite one. This is a fundamental difficulty since an exact match is impossible (because all the coefficients cannot possibly match).

In the foregoing, the term "practical" has been emphasized in discussing point-mass sets, since it is only those point-mass sets defined by bounded parameters which can be used for practical purposes. In addition, there also exist "impractical" point-mass sets (which will be presently defined as the limit of a sequence of practical sets). These impractical sets are conceptually important as they provide exact (though not physically realizable) solutions to the problem of point-mass modeling; in comparison, practical point-mass sets give physically realizable, though approximate, solutions. Both cases will now be considered in more detail.

EXACT SOLUTIONS (IMPRACTICAL SETS)

From (8), the spherical harmonic expansion coefficients associated with a point-mass set are given by

$$\begin{Bmatrix} A_{nm} \\ B_{nm} \end{Bmatrix} = N_{nm} \sum_i \mu_i \, \rho_i^n \, P_{nm} \, (\sin \phi_i) \begin{Bmatrix} \cos m\lambda_i \\ \sin m\lambda_i \end{Bmatrix} \tag{9}$$

where

$$\mu_i = \frac{M_i}{M} \quad , \quad \rho_i = \frac{r_i}{R}$$

* In this regard see Reference 7.

Consider the process of taking the limits $\rho_i \to 0$, $\mu_i \to \infty$ such that

$$\mu_i \rho_i^n = C_i^n \tag{10}$$

where the C_i^n are finite constants; assume that all the ρ_i are taken to zero at the same time, i.e., $\rho_i = \delta$ and $\delta \to 0$. Then (10) can be written

$$\mu_i = C_i^n \delta^{-n} \tag{11}$$

The μ_i then diverge for any continuous or discrete sequence whereby $\delta \to 0$.

For a particular value of n, say n=k, (11) becomes

$$\mu_i = C_i^k \delta^{-k} \tag{12}$$

Putting this into (9) (using $\rho_i = \delta$) gives

$$\begin{Bmatrix} A_{nm} \\ \\ B_{nm} \end{Bmatrix} = N_{nm} \delta^{n-k} \sum_i C_i^k P_{nm} (\sin \phi i) \begin{Bmatrix} \cos m\lambda_i \\ \\ \sin m\lambda_i \end{Bmatrix} \tag{13}$$

Equation (13) can be broken into three cases:

1. <u>$n < k$, $k-n = \ell > 0$</u>

$$\begin{Bmatrix} \delta^\ell A_{nm} \\ \\ \delta^\ell B_{nm} \end{Bmatrix} = N_{nm} \sum_i C_i^k P_{nm} (\sin \phi_i) \begin{Bmatrix} \cos m\lambda_i \\ \\ \sin m\lambda_i \end{Bmatrix} \tag{14}$$

2. <u>$n = k$</u>

$$\begin{Bmatrix} A_{km} \\ \\ B_{km} \end{Bmatrix} = N_{km} \sum_i C_i^k P_{km} (\sin \phi_i) \begin{Bmatrix} \cos m\lambda_i \\ \\ \sin m\lambda_i \end{Bmatrix} \tag{15}$$

3. <u>$n > k$, $n - k = \ell' > 0$</u>

$$\begin{Bmatrix} A_{nm} \\ \\ B_{nm} \end{Bmatrix} = \delta^{\ell'} N_{nm} \sum_i C_i^k P_{nm} (\sin \phi_i) \begin{Bmatrix} \cos m\lambda_i \\ \\ \sin m\lambda_i \end{Bmatrix} \tag{16}$$

In the limit $\delta \to 0$, equations (14), (15), and (16) become:

1. $n < k$

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = N_{nm} \sum_i C_i^k P_{nm} (\sin \phi_i) \begin{Bmatrix} \cos m\lambda_i \\ \sin m\lambda_i \end{Bmatrix} \tag{17}$$

2. $n = k$

$$\begin{Bmatrix} A_{km} \\ B_{km} \end{Bmatrix} = N_{km} \sum_i C_i^k P_{km} (\sin \phi_i) \begin{Bmatrix} \cos m\lambda_i \\ \sin m\lambda_i \end{Bmatrix} \tag{18}$$

3. $n > k$

$$\begin{Bmatrix} A_{nm} \\ B_{nm} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{19}$$

Up to and including degree k, there are $(k+1)^2$ coefficients, thus (17) and (18) comprise $(k+1)^2$ equations. Since (10) defines the relationship between μ_i and ρ_i, there are three independent variables for each point-mass. Thus, if I is the total number of masses, then there are 3I independent variables in the point-mass set. Therefore, the minimum number of masses needed in order that the equations (17) and (18) be solvable is the minimum number which contains at least $(k+1)^2$ independent variables. This number is the integer I, where

$$\frac{(k+1)^2}{3} \leq I < \frac{(k+1)^2}{3} + 1$$

For specific values of k, I is given in Table 1.

Table 1. Number of Masses (I) for a Particular Degree (k)

k	0	1	2	3	4	5	6	7	8	9	10
I	1	2	3	6	9	12	17	22	27	34	41

APPROXIMATE SOLUTIONS (PRACTICAL SETS)

If we wish to use point-mass sets to approximate a finite spherical harmonic expansion, then the radii of the point-masses must be non-zero; i.e., the limiting procedure $\rho_i = \delta \to 0$ is only a heuristic theoretical device for understanding multipoles. We can, however, create point-mass sets whose corresponding coefficients [via (9)] exactly match those of the finite spherical harmonic expansion (FSHE) for degree $n \leq N$ (where N = maximum degree of the FSHE) and are sufficiently small for $n > N$. This can be done in the following manner.

First, (17) and (18) can be solved for each degree $k \leq N$ to produce a point-mass set $\{C_i^k, \phi_i^k, \lambda_i^k\}$. Since $C_i^k = \mu_i^k \delta^k$, a $\delta > 0$ can be chosen, and using $\mu_i^k = C_i^k \delta^{-k}$, the corresponding mass can be found. Now, although $\delta > 0$, equations (17) and (18) are still satisfied; the A_{nm} and B_{nm} for $n > k$ are no longer identically zero, however, but instead

$$\underline{n > k}, \quad \begin{Bmatrix} A_{nm} \\ B_{nm} \end{Bmatrix} = \delta^{n-k} N_{nm} \sum_i C_i^n P_{nm}(\sin \phi_i^k) \begin{Bmatrix} \cos m\lambda_i^k \\ \sin m\lambda_i^k \end{Bmatrix} \qquad (20)$$

By choosing δ properly, the coefficients in equation (20) can be made sufficiently small; the only limit comes from attempting to use the point-mass sets so defined in numerical simulations because of round-off error, etc. Let the coefficients defined by equation (20) for each degree k be written A_{nm}^k, B_{nm}^k, and let them be called the residual coefficients of degree k.

Now, a procedure for constructing point-mass sets, whose intrinsic coefficients for $n \leq N$ exactly match the given FSHE coefficients and for $n > N$ are sufficiently small, can be outlined. Starting with $k = 0$ solve equations (17) and (18); using the point-mass set of the solution, use equation (20) to generate the A_{nm}^0, B_{nm}^0. Subtract these from the remaining FSHE coefficients $(0 < n \leq N)$, A_{nm}, B_{nm}, and now use these subtracted coefficients to solve for the k=1 point-mass set, generating A'_{nm}, B'_{nm} from the solution. Then, subtract these from the remaining (already once-subtracted) coefficients and use the twice-subtracted coefficients to solve for the point-mass set corresponding to k=2, then k=3,...then k=N.

The number of point-masses needed to describe an FSHE of maximum degree N is then the sum of the number needed for each degree $k \leq N$; using Table 1, this number (call it S) can be determined. The result is given in Table 2.

Table 2. Total Number of Masses (S) Needed for Expansion up to Degree and Order N

N	0	1	2	3	4	5	6	7	8	9	10
S	1	3	6	12	21	33	50	72	99	133	174

This result is exact in the limit that all $\delta_k \rightarrow 0$. (δ_k means that for each k, δ is chosen independently of the other k's.)

It may be possible to use another method for finding a point-mass set which adequately represents a given FSHE, and, at the same time, has less masses than the corresponding sets given in Table 2. One way is to use (9) directly; in this method, the ρ_i would have to be small also, but could be picked independently of one another. Again, there would be 3 degrees of freedom for each point-mass $\{\mu_i, \phi_i, \lambda_i\}$. The ρ_i, though independent of one another, are not independent in the general sense because they are constrained to be small (i.e., picked beforehand).

Thus, if the set of non-linear equations (9) is to be solvable, there must be enough variables $\{\mu_i, \phi_i, \lambda_i\}$ in the point-mass set; that is, 3I (I: number of point-masses) must be greater than the number of coefficients on the left-hand side of (9). This number has already been determined, however, in Table 1. Therefore, it may be possible to use 41 masses to adequately describe a (10x10) FSHE to within an acceptable error; the minimum amount of error is determined by numerical limitations (i.e., interaction between smallness of the ρ_i and a particular computer's round-off error, etc. One thing that happens as the $\rho_i \to 0$ in (9) is that the $\mu_i \to \infty$. The combination of small ρ_i and large μ_i may tend to further increase the computational inaccuracy).

Since any practical point-mass set will, by necessity, only provide an approximate model of the geopotential, it would be useful to have a method for determining how "good" the approximation really is. An appropriate method is defined and developed in the next section.

GLOBAL ERROR

The geopotential may be represented, as has already been remarked, by a finite spherical harmonic expansion

$$T_G = \frac{-kM}{R} \sum_{n=0}^{N} \sum_{m=0}^{n} \left(\frac{R}{r}\right)^{n+1} P_{nm} (\sin \phi) \left[a_{nm} \cos m\lambda + b_{nm} \sin m\lambda\right] \quad (21)$$

where the a_{nm} and b_{nm} are derived from geopotential data, and N is the maximum degree and order of the expansion.

The difference between the geopotential given by (21) and the point-mass potential given by (7) is

$$\begin{aligned} \delta T &= T_G - T \\ &= \frac{-kM}{R} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{R}{r}\right)^{n+1} P_{nm} (\sin \phi) \Big[(a_{nm} - A_{nm}) \cos m\lambda \\ &\qquad + (b_{nm} - B_{nm}) \sin m\lambda\Big] \end{aligned} \quad (22)$$

where the a_{nm} and b_{nm} are all zero for $n > N$.

GLOBAL POTENTIAL ERROR

The "global error" in the potential will be called E_T and will be defined by

$$E_T^2 = \frac{1}{V} \iiint_V |\delta T|^2 \, dv$$

$$= \frac{1}{V} \int_a^b S(r) \, r^2 dr \qquad (23)$$

where

$$S(r) = \iint_\sigma |\delta T|^2 \, d\sigma \qquad (24)$$

is the integral of $|\delta T|^2$ over a spherical surface at radius r. The integration volume V in (23) is the volume in which the point-mass method is to be used; this volume will be taken to be the space bounded by two concentric spheres, the outer one of radius r=b and the inner one of radius r=a, where $b > a \geq R$. Thus, the global error in the potential is seen to be the root-mean-square value of (22) in the aforementioned volume V, where

$$V = 4\pi(b^3 - a^3)/3 \qquad (25)$$

Now, placing (22) into (24) and using the orthogonality properties of spherical harmonics (see Appendix A) gives

$$S(r) = 4\pi\left(\frac{kM}{R}\right)^2 \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{2n+2} \sum_{m=0}^{n} \frac{N_{nm}^{-1}}{2n+1} \left[(a_{nm} - A_{nm})^2 + (b_{nm} - B_{nm})^2\right] \qquad (26)$$

where N_{nm} is defined in (6), and where the a_{nm} and b_{nm} are zero for $n > N$.

Substituting the expressions for A_{nm} and B_{nm} given in (8) into (26) and using the addition theorem (5) yields

$$S(r) = C \sum_{n=0}^{N} \left(\frac{R}{r}\right)^{2n+2} \sum_{m=0}^{n} \frac{N_{nm}^{-1}}{2n+1} \left[a_{nm}^{-2} + b_{nm}^{-2} - 2(a_{nm}A_{nm} + b_{nm}B_{nm}) \right]$$

$$+ C \sum_{i} \sum_{j} \frac{M_i M_j}{M^2} \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^2 \left(\frac{r_i r_j}{r^2}\right)^n \frac{P_n(\cos \psi_{ij})}{2n+1} \qquad (27)$$

where

$$C = 4\pi \left(\frac{kM}{R}\right)^2 \qquad (28)$$

and where ψ_{ij} is the angle between the position vectors of the i^{th} and j^{th} point-masses.

Expression (27) for the surface integral S(r) can be put into closed-form by utilizing the first result of Appendix B; thus

$$S(r) = C \sum_{n=0}^{N} \left(\frac{R}{r}\right)^{2n+2} \sum_{m=0}^{n} \frac{N_{nm}^{-1}}{2n+1} \left[a_{nm}^{2} + b_{nm}^{2} - 2(a_{nm}A_{nm} + b_{nm}B_{nm}) \right]$$

$$+ \tfrac{1}{2} C \sum_{i} \sum_{j} \frac{M_i M_j}{M^2} \frac{R^2}{r\sqrt{r_i r_j}} F(\alpha_{ij}, \beta_{ij}) \qquad (29)$$

where $F(\alpha, \beta)$ is the elliptic integral of the first kind (see Appendix C) with

$$\alpha_{ij} = \arccos \left(\frac{r^2 - r_i r_j}{r^2 + r_i r_j} \right) \qquad (30)$$

$$\beta_{ij} = \cos \left(\tfrac{1}{2}\psi_{ij}\right)$$

The global potential error can now be evaluated. Recalling (23)

$$E_T^2 = \frac{1}{V} \int_a^b S(r) r^2 dr$$

and placing (27) into it [(27) is more useful in the present context than (29)] yields

$$E_T^2 = \frac{CR^3}{V} \sum_{n=0}^{N} L_1(a,b,n) \sum_{m=0}^{n} \frac{N_{nm}^{-1}}{2n+1} \left[a_{nm}^2 + b_{nm}^2 (a_{nm}A_{nm} + b_{nm}B_{nm})\right]$$

$$+ \frac{CR^3}{V} \sum_i \sum_j \frac{M_i M_j}{M^2} \; L_2(a,b,n) \tag{31}$$

where

$$L_1(a,b,n) = \frac{1}{2n-1} \left[\left(\frac{R}{a}\right)^{2n-1} - \left(\frac{R}{b}\right)^{2n-1}\right] \tag{32}$$

$$L_2(a,b,n) = \sum_{n=0}^{\infty} \frac{\sqrt{r_i r_j}}{R} \left[\left(\frac{r_i r_j}{a}\right)^{2n-1} - \left(\frac{\sqrt{r_i r_j}}{b}\right)^{2n-1}\right] \frac{P_n(\cos \psi_{ij})}{(2n-1)(2n+1)} \tag{33}$$

Since

$$\frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

(33) can be written

$$L_2(a,b,n) = \frac{\sqrt{r_i r_j}}{2R} \left[\frac{r^2}{r_i r_j} K_1\left(\frac{r}{\sqrt{r_i r_j}}, \psi_{ij}\right) - K_2\left(\frac{r}{\sqrt{r_i r_j}}, \psi_{ij}\right)\right] \Bigg|_a^b \tag{34}$$

where the functions K_1 and K_2 are given in Appendix B, and

$$f(r) \Big|_x^y = f(y) - f(x) \tag{35}$$

Equation (31) is a rather complicated expression for the global potential error, primarily because of the function L_2. In practice, however, (31) will be of minimal importance since the usual aim of point-mass modeling is to match the gradient of the geopotential rather than the potential itself. To this end, a "global gradient error" will now be derived.

GLOBAL GRADIENT ERROR

The global gradient error E_G will be defined as follows

$$E_G^2 = \frac{1}{V} \iiint_V |\delta \nabla T|^2 \, dV \tag{36}$$

where the difference in the gradients of the point-mass potential (7) and the geopotential (21) is

$$\delta \nabla T = \nabla T_G - \nabla T = \nabla (T_G - T) = \nabla \delta T \tag{37}$$

Thus, (36) can be written

$$E_G^2 = \frac{1}{V} \iiint_V \nabla \delta T \cdot \nabla \delta T \, dV \tag{38}$$

[Here, the volume V is the same as was used in defining the global potential error (23).]

Equation (38) can be put in a more useful form by utilizing Green's First Identity (Reference 6, p. 11): if U and W are any two scalar functions (potentials), then

$$\iiint_V (U \nabla^2 W + \nabla U \cdot \nabla W) dV = \oiint_S U \, \vec{n} \cdot \nabla W dS \tag{39}$$

where S is the closed surface bounding V and $\vec{n}$ is a unit vector normal to the surface S and directed out of the volume V. Setting $U = W = \delta T$ and using $\nabla^2 \delta T = 0$ (since T_G and T in (22) are both harmonic in V) allows (38) to be written

$$E_G^2 = \frac{1}{V} \oiint_S \delta T \, \vec{n} \cdot \nabla \delta T \, dS \tag{40}$$

The surface S in (40) consists of the inner and outer spheres which bound V; thus, the normal on the outer sphere (r = b) is $\vec{n}_b = \hat{r} = \vec{r}/|\vec{r}|$, while on the inner sphere (r = a) the normal is $n_a = -\hat{r}$.

Therefore,

$$\vec{n}_b \cdot \nabla = -\vec{n}_a \cdot \nabla = \hat{r} \cdot \nabla = \frac{\partial}{\partial r} \tag{41}$$

and (40) becomes

$$E_G^2 = \frac{1}{V} \left[\iint_{\sigma_b} \delta T \left(\frac{\partial}{\partial r} \delta T \right) b^2 d\sigma - \iint_{\sigma_a} \delta T \left(\frac{\partial}{\partial r} \delta T \right) a^2 d\sigma \right]$$

$$= \frac{1}{V} \iint_{\sigma} \delta T \left(\frac{\partial}{\partial r} \delta T \right) r^2 d\sigma \Big|_a^b$$

$$= \frac{r^2}{2V} \frac{\partial}{\partial r} \iint_{\sigma} |\delta T|^2 \, d\sigma \Big|_a^b \tag{42}$$

Finally, using (24), the square of the global gradient error becomes

$$E_G^2 = \frac{r^2}{2V} \frac{\partial}{\partial r} S(r) \Big|_a^b \tag{43}$$

where S(r) is given by (29).

Evaluating the partial derivatives in equation (43) gives

$$E_G^2 = \frac{CR}{V} \sum_{n=0}^{N} G_n(a,b) \sum_{m=0}^{n} N_{nm}^{-1} \left[a_{nm}^2 + b_{nm}^2 - 2(a_{nm}A_{nm} + b_{nm}B_{nm}) \right]$$

$$+ \frac{CR}{4V} \sum_i \sum_j \frac{M_i M_j}{M^2} H_{ij}(a,b) \tag{44}$$

where

$$G_n(a,b) = \frac{n+1}{2n+1} \left[\left(\frac{R}{a} \right)^{2n+1} - \left(\frac{R}{b} \right)^{2n+1} \right] \tag{45}$$

and where

$$H_{ij}(a,b) = \left[\frac{R}{\sqrt{r_i r_j}} F(\alpha_{ij},\beta_{ij}) - \frac{rR}{\sqrt{r^4 + (r_i r_j)^2 - 2r^2 r_i r_j \cos\psi_{ij}}} \right]_b^a \quad (46)$$

Here $F(\alpha,\beta)$ is again the elliptic integral of the first kind (Appendix C) with

$$\alpha_{ij} = \arccos\left(\frac{r^2 - r_i r_j}{r^2 + r_i r_j}\right) \quad (47)$$

$$\beta_{ij} = \cos\left(\tfrac{1}{2}\psi_{ij}\right)$$

and where, as in (42), use has been made of (35).

The analytic expression (44) for the square of the global gradient error can be written in a more compact form by again using the following change of parameters

$$\mu_i = \frac{M_i}{M} \quad , \quad \rho_i = \frac{r_i}{R} \quad (48)$$

(the latitude and longitude parameters ϕ_i and λ_i remain unchanged). Using (48), (44) becomes

$$E_G^2 = C'\left(f - \sum_i \mu_i g_i + \frac{1}{2} \sum_i \sum_j \mu_i \mu_j h_{ij}\right) \quad (49)$$

where

$$C' = \frac{CR}{V} = \frac{3G^2R^3}{b^3 - a^3} \quad (50)$$

(from (25), (28) and $G = kM/R^2$, where G: mean gravity on the earth's surface) and where

$$f = \sum_{n=0}^{N} G_n(a,b) \sum_{m=0}^{n} N_{nm}^{-1}\left(a_{nm}^2 + b_{nm}^2\right) \quad (51)$$

$$g_i = 2 \sum_{n=0}^{N} G_n(a,b)\rho_i^n \sum_{m=0}^{n} P_{nm}(\sin \phi_i)\left(a_{nm} \cos m\lambda_i + b_{nm} \sin m\lambda_i\right) \tag{52}$$

$$h_{ij} = \frac{1}{2}\left[\left(\rho_i\rho_j\right)^{\frac{1}{2}} F(\alpha_{ij}, \beta_{ij}) - \frac{\rho}{\{\rho^4+(\rho_i\rho_j)^2 - 2\rho^2\rho_i\rho_j \cos \psi_{ij}\}^{\frac{1}{2}}}\right]\Bigg|_{\rho=b/R}^{a/R} \tag{53}$$

with

$$\alpha_{ij} = \arccos\left(\frac{\rho^2-\rho_i\rho_j}{\rho^2+\rho_i\rho_j}\right) \tag{54}$$

$$\beta_{ij} = \cos\left(\tfrac{1}{2}\psi_{ij}\right) \tag{55}$$

Here $G_n(a,b)$ is given by (45) and $F(\alpha,\beta)$ is again an elliptic integral of the first kind.

OPTIMIZATION PROCEDURE

Now that an analytic expression (49) has been obtained for the global gradient error, it will be possible to develop an optimization procedure for point-mass sets. The obvious requirement for optimization is that the global gradient error, E_G, be a minimum with respect to the independent parameters of the point-mass set. Since E_G is always positive, its minima occur for the same values as those of E_G^2; thus, the optimum point-mass parameters are to be found by solving the following equations:

$$\frac{\partial E_G^2}{\partial \mu_k} = 0 \quad , \quad \nabla_k E_G^2 = 0 \tag{56}$$

where the gradient with respect to the coordinates of the k^{th} point-mass is

$$\nabla_k = \hat{r}_k \frac{\partial}{\partial r_k} + \hat{\phi}_k \frac{1}{r_k} \frac{\partial}{\partial \phi_k} + \hat{\lambda}_k \frac{1}{r_k \cos \phi_k} \frac{\partial}{\partial \lambda_k}$$

$$= \frac{1}{R}\left(\hat{r}_k \frac{\partial}{\partial \rho_k} + \hat{\phi}_k \frac{1}{\rho_k} \frac{\partial}{\partial \phi_k} + \hat{\lambda}_k \frac{1}{\rho_k \cos \phi_k} \frac{\partial}{\partial \lambda_k}\right) \qquad (57)$$

where $\hat{r}_k$, $\hat{\phi}_k$, and $\hat{\lambda}_k$ are unit vectors in the direction of greatest increase of r_k, ϕ_k, and λ_k, respectively.

Applying the conditions (56) to the expression (49) for E_G^2 gives

$$g_k = \sum_j \mu_j h_{kj} \qquad (58)$$

$$\nabla_k g_k = \sum_j \mu_j \nabla_k h_{kj} \qquad (59)$$

(*where it must* be understood that $\nabla_k\, h_{kk} = \lim(j \to k)\ \nabla_k\, h_{kj}$ and <u>not</u> $\nabla_k\, h_{kk} = \nabla_k \lim(j \to k) h_{kj}$). Assuming that $\{h_{kj}\}$ is a non-singular matrix allows equation (58) to be inverted to give

$$\mu_j = \sum_k h_{jk}^{-1}\, g_k \qquad (60)$$

Placing (60) into (59) gives an optimizing equation which does not contain any explicit masses μ_k

$$\nabla_k g_k = \sum_j \sum_l h_{jl}^{-1} \left[\nabla_k h_{kj}\right] g_l \qquad (61)$$

[Remember that (61) represents three equations for each mass, because of (57). Expressions for $\nabla_k h_{kj}$ are given in Appendix E.]

The derivation of the optimizing equations (58) through (61) is much more straight-forward than their solution. If equation (61) could be solved exactly to give the optimal positions, then (60) could be easily used to complete the set by producing the optimum mass values. (61), however, is highly non-linear and a solution can probably be obtained only by an extensive numerical search algorithm; the speed of computation and accuracy of a final solution are difficult to estimate, primarily because of the presence of h^{-1}_{ij} in (61).

Another approach is to try to find the minimum of E^2_G numerically, with the aid of the optimizing equations, rather than trying to solve (61) directly (either numerically or analytically). In particular, for an arbitrary set of positions, the h_{ij} can be calculated and (assuming the matrix $\{h_{ij}\}$ is non-singular) (60) can be readily utilized to compute the optimum choice of masses for the arbitrary positions. (That these masses yield a minimum for E^2_G follows from the fact that E^2_G is positive and quadratic in the μ_k. Let the masses which satisfy (60) for arbitrary locations be denoted by $\bar{\mu}_k$; then E^2_G can be written

$$E^2_G(\mu_k) = E^2_G(\bar{\mu}_k) + Q \qquad (62)$$

where Q is the quadratic form

$$Q = \sum_i \sum_j h_{ij}\varepsilon_i\varepsilon_j \qquad (63)$$

and $\mu_k = \bar{\mu}_k + \varepsilon_i$. Since $E^2_G \geq 0$, then $Q \geq - E^2_G(\bar{\mu}_k)$; and since Q can always be written in canonical form[8], i.e.,

$$Q = \sum_i a_i\, \zeta^2_i \qquad (64)$$

then if any $a_i < 0$, there exists a direction ζ_i in which Q will grow unboundedly large and negative exceeding the lower bound $- E^2_G(\bar{\mu}_k)$. Therefore, all a_i must be non-negative from which it follows that $Q \geq 0$, and thus the $\bar{\mu}_k$ specify the minimum.)

Once the $\bar{\mu}_k$ have been found, via (60), for a given set of positions, a gradient descent method can be utilized to seek out the minimum of E_G^2 with respect to the position parameters. (Since the gradient descent method is thoroughly discussed elsewhere[9], it will not be considered in detail here.) The standard gradient descent algorithms need only be appended in a simple manner: at the end of each iteration use (60) to determine optimum masses for the new (iterated) positions: then the descent algorithm need work on 3I variables instead of 4I, where I is the number of masses.

It is important to note that care must be taken in choosing the number of masses for use with this optimization procedure. As was shown earlier, each finite spherical harmonic expansion of degree and order N has associated with it an exact solution (as a limiting case), and each exact solution has its own particular number of masses S. Therefore, if the degree and order of the FSHE, which is being approximated, is N and the point-mass set, which initializes the numerical optimization procedure, consists of S or more masses, the search algorithm may try to reach the limiting values $r_i \to 0$ and $M_i \to \infty$ $(i = 1,\ldots,S)$; this will clearly cause problems.

In practice, however, it is often useful to have as few masses as necessary. In particular, it has been determined at NAVSWC that the time required to compute a gravity vector (on the Mk 98 Mod 0 fire control computer) is the same for a 9x9 FSHE as for a point-mass expansion consisting of about 50 masses. In this case, it is obvious that fewer than 50 masses are needed if computation time is to be conserved (an important consideration in fire control). Thus, the availability of an efficient point-mass optimization procedure can greatly enhance existing fire control capabilities.

DISCUSSION

In order to give some credence to the theoretical claims contained in this report, the following numerical work was done. Since equation (8) is linear in the mass values M_i, it can be inverted to yield these mass values provided certain conditions are met: 1) there are as many masses as coefficients A_{nm}, B_{nm}, 2) the positions (r_i, ϕ_i, λ_i) are chosen beforehand and in such a manner that the matrix which must be inverted is non-singular. Also, in order to achieve a form appropriate for inversion, the double indices n,m appearing on both sides of equation (8) had to be "stretched" into a single index (suitable for describing a column vector).

The coefficients used for A_{nm}, B_{nm} came from the NASA GEM-6 set (n=0,1 terms equal to zero) with maximum n,m equal to nine; thus, there were 100 coefficients. The angular positions of the required 100 masses were chosen at random and the radii of these point-mass positions was the same for all masses (except for one which was always placed at r=0). The value of the common radius was set at .15, .30, .45, .60, .75, and .90 earth radii (R). For each of these values, equation (8) was inverted and the corresponding mass values were found (the angular positions were the same for all radii).

To check how well each of these point-mass sets approximated the original finite spherical harmonic expansion (9x9), both expansions (point-mass and FSHE) were used to compute the values of the gravity disturbance vector at 100 randomly chosen positions between 1.0 and 1.01 earth radii. The root-mean-square (RMS) differences were then calculated for each of the point-mass set radii .15,..., .90, and the results are shown in Table 3.

Table 3. RMS Difference in Components of Gravity Disturbance Vector $\vec{\delta}$

	RMS of $\vec{\delta}$ Components (Milligals)		
Radius (R)	δr	$\delta \phi$	$\delta \lambda$
.15	4.22	2.68	14.3
.30	18.9	13.7	19.2
.45	64.0	42.2	37.4
.60	165.	99.1	106.
.75	1060	774.	905.
.90	40400	29700	24500

Although the analytic expression for the global gradient error was not used, these results still provide quantitative verification of some of the theoretical developments of this report. In particular, they support the contention contained in the section on "Nature of Solutions", that the point-mass sets provide for better approximations the closer the masses are to the center of the earth. The optimal point-masses should, however, have radial values which do not tend completely to zero (though small) values. (Otherwise, these optimal point-mass sets would not exist, in a practical sense, as has already been discussed.)

CONCLUSION

In this report, the basic theory of global point-mass modeling has been discussed. Since all the information obtainable from discrete global geophysical data can be essentially incorporated into the coefficients of a finite spherical harmonic expansion, the independent parameters of a point-mass expansion are effectively functions of these coefficients. The implicit functional form was determined and is given by (8). This functional relationship showed that a practical point-mass expansion will have an associated infinite spherical harmonic expansion. This, in turn, exposed a basic problem in point-mass modeling, that of matching an infinite with a finite orthogonal expansion. The functional relationship then allowed the nature of solutions (both exact and approximate) for this basic problem to be carefully considered.

In order to better gauge how well a particular point-mass set approximates the global geophysical data, the concept of "global error" was defined and developed. A desire to minimize the global gradient error led, in turn, to a set of simultaneous non-linear equations whose solution allows for the determination of the optimal positions and values of a given number of point-masses.

Although a procedure for solving these equations was sketched, it is well recognized "that the solution of simultaneous non-linear equations is usually a very difficult problem"[10], a problem beyond the scope of the present work. Thus, the implementation of the methods described herein are left as a (hopefully) well-posed numerical problem, contenting ourselves at present with their derivation.

APPENDIX A

ORTHOGONALITY PROPERTIES OF SPHERICAL HARMONICS

ORTHOGONALITY PROPERTIES OF SPHERICAL HARMONICS

The following results will be written in the notation of Heiskanen and Moritz (Ref. 6, p. 29).

1. Spherical Harmonics

$$R_{nm}(\theta, \lambda) = P_{nm}(\cos\theta)\cos m\lambda \tag{65}$$

$$S_{nm}(\theta, \lambda) = P_{nm}(\cos\theta)\sin m\lambda \tag{66}$$

2. Orthogonality Relations

$$\iint_\sigma R_{nm}(\theta, \lambda)\, R_{rs}(\theta, \lambda)\, d\sigma = \frac{4\pi}{2n+1}\,\varepsilon_m \frac{(n+m)!}{(n-m)!}\,\delta_{nr}\,\delta_{ms} \tag{67}$$

$$\iint_\sigma S_{nm}(\theta, \lambda)\, S_{rs}(\theta, \lambda)\, d\sigma = \frac{4\pi}{2n+1}\,\varepsilon_m \frac{(n+m)!}{(n-m)!}\,\delta_{nr}\,\delta_{ms} \tag{68}$$

$$\iint_\sigma R_{nm}(\theta, \lambda)\, S_{rs}(\theta, \lambda)\, d\sigma = 0 \quad \text{, always} \tag{69}$$

Here, δ_{nr} is the Kronecker delta,

$$\delta_{nr} = \begin{cases} 1, & n = r \\ 0, & n \neq r \end{cases} \tag{70}$$

and $\varepsilon_m = \frac{1}{2}(1 + \delta_{mo})$.

The coordinate system employed in the above definitions and results is a spherical polar coordinate system:

$$\iint_\sigma d\sigma = \int_{\lambda=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta\, d\theta\, d\lambda \tag{71}$$

The relation of spherical polar coordinates θ and λ and the geocentric coordinates ϕ and λ is

$$\phi = \theta - \frac{\pi}{2} \tag{72}$$

with the azimuthal angle λ being the same in both systems. Thus,

$$\cos\theta = \sin\phi \tag{73}$$

and

$$\iint_{\sigma} d\sigma = \int_{\lambda=0}^{2\pi} \int_{\phi=\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos\phi \, d\phi \, d\lambda \tag{74}$$

APPENDIX B

CLOSED FORM EXPRESSIONS FOR CERTAIN INFINITE SERIES OF LEGENDRE POLYNOMIALS

CLOSED FORM EXPRESSIONS FOR CERTAIN INFINITE SERIES OF LEGENDRE POLYNOMIALS

Result:

$$K_1(z, \psi) = \sum_{n=0}^{\infty} \frac{1}{z^{2n+1}} \frac{P_n(\cos \psi)}{2n+1} = \tfrac{1}{2}F(\alpha, \beta) \tag{75}$$

where $F(\alpha, \beta)$ is the elliptic integral of the first kind (see Appendix C) with

$$\alpha = \arccos\left(\frac{z^2-1}{z^2+1}\right)$$

$$\beta = \cos(\psi/2)$$

Derivation:

$$\sum_{n=0}^{\infty} \frac{1}{z^{2n+1}} \frac{P_n(\cos \psi)}{2n+1} = \int_z^{\infty} \sum_{n=0}^{\infty} \frac{1}{x^{2n+2}} P_n(\cos \psi)\, dx$$

$$= \int_z^{\infty} \frac{dx}{(1-2x^2 \cos \psi + x^4)^{\frac{1}{2}}} \tag{76}$$

where (3) and (4) have been used.

Utilizing formula 3.165-2 of Reference 11 leads to (75).

Result:

$$K_2(z, \psi) = \sum_{n=0}^{\infty} \frac{1}{z^{2n+1}} \frac{P_n(\cos \psi)}{2n - 1}$$

$$= -1 + z\left[1 - z^2(1-2z^2 \cos \psi + z^4)^{-\frac{1}{2}}\right]$$

$$+2\left[\cos \psi\, J_4\,(z, 1, -\cos \psi) - J_2\,(z, 1, -\cos \psi)\right] \tag{77}$$

where J_2 and J_4 are given by (97) and (99), respectively.

Derivation:

$$K_2(z, \psi) = \sum_{n=0}^{\infty} \frac{1}{z^{2n-1}} \frac{P_n(\cos \psi)}{2n - 1}$$

$$= -1 + \int_z^{\infty} \sum_{n=1}^{\infty} \frac{1}{x^{2n}} P_n(\cos \psi)\, dx$$

$$= -1 + \int_z^{\infty} \left[x^2 (1 - 2x^2 \cos \psi + x^4)^{-\frac{1}{2}} - 1\right] dx \qquad (78)$$

where the last step follows from (3) and (4).

Integrating (78) by parts gives

$$K_2(z, \psi) = -1 + z \left[1 - z^2 (1 - 2z^2 \cos \psi + z^4)^{-\frac{1}{2}}\right]$$

$$+ \int_z^{\infty} \frac{x^4 \cos \psi - x^2}{(1 - 2x^2 \cos \psi + x^4)^{3/2}}\, dx \qquad (79)$$

using (97) and (99) of Appendix D then gives the result (77).

APPENDIX C

SOME PROPERTIES OF ELLIPTIC INTEGRALS OF THE FIRST AND SECOND KINDS

SOME PROPERTIES OF ELLIPTIC INTEGRALS OF THE FIRST AND SECOND KINDS

The following results come from Section 8.1 of Reference 11.

1. Elliptic integral of the first kind

$$F(\phi, k) = \int_0^{\phi} \frac{d\alpha}{\sqrt{1-k^2 \sin^2 \alpha}} \tag{80}$$

2. Elliptic integral of the second kind

$$E(\phi, k) = \int_0^{\phi} \sqrt{1-k^2 \sin^2 \alpha}\, d\alpha \tag{81}$$

3. Derivatives

$$\frac{\partial F}{\partial \phi} = \frac{1}{\sqrt{1-k^2 \sin^2 \phi}} \tag{82}$$

$$\frac{\partial E}{\partial \phi} = \sqrt{1-k^2 \sin^2 \phi} \tag{83}$$

$$\frac{\partial F}{\partial k} = \frac{1}{k'^2}\left(\frac{E - k'^2 F}{k} - \frac{k \sin \phi \cos \phi}{\sqrt{1-k^2 \sin^2 \phi}}\right) \tag{84}$$

$$\frac{\partial E}{\partial k} = \frac{E - F}{k} \tag{85}$$

where $k' = \sqrt{1-k^2}$

Elliptic integrals of the first and second kinds also have the following special values.[12]

$$F(\phi, o) = E(\phi, o) = \phi \tag{86}$$

$$F(\phi, 1) = \ln(\sec \phi + \tan \phi) \tag{87}$$

$$E(\phi, 1) = \sin \phi \tag{88}$$

APPENDIX D

EVALUATION OF CERTAIN INTEGRALS WHOSE INTEGRANDS CONTAIN THE FACTOR $(x^4+2b^2x^2+a^2)^{-\frac{1}{2}}$

EVALUATION OF CERTAIN INTEGRALS WHOSE INTEGRANDS CONTAIN THE FACTOR $(x^4+2b^2x^2+a^2)^{-\frac{1}{2}}$

Formula 3.165-2 of Reference 11 is

$$I_1(u,a^2,\ b^2) = \int_u^\infty \frac{dx}{\sqrt{x^4+2b^2x^2+a^4}} = \frac{1}{2a} F(\alpha,\ r) \tag{89}$$

$$\left[a^2 > b^2 > -\infty,\ a^2 > 0,\ U \geq 0\right]$$

where $F(\alpha,\ r)$ is an elliptic integral of the first kind (see Appendix B) and

$$\alpha = \arccos\left(\frac{u^2 - a^2}{u^2 + a^2}\right) \tag{90}$$

$$r = \frac{\sqrt{a^2 - b^2}}{a\sqrt{2}} \tag{91}$$

Consider the following:

$$J_1(u,a^2,\ b^2) = \int_u^\infty \frac{dx}{(x^4+2b^2x^2+a^4)^{3/2}}$$

$$= -\frac{1}{2a^3}\ \frac{\partial I_1}{\partial a}$$

$$= \frac{1}{4a^5} F(\alpha,\ r) - \frac{1}{4a^4}\ \frac{\partial}{\partial a}\ F(\alpha,\ r)$$

$$= \frac{1}{4a^4}\left[\frac{1}{a} F(\alpha,\ r) - \frac{\partial F}{\partial \alpha}\ \frac{\partial \alpha}{\partial a} - \frac{\partial F}{\partial r}\ \frac{\partial r}{\partial a}\right] \tag{92}$$

Differentiating (90) and (91) with respect to a yields

$$\frac{\partial \alpha}{\partial a} = \frac{2u}{u^2+a^2} \tag{93}$$

$$\frac{\partial r}{\partial a} = \frac{b^2}{\sqrt{2}\, a^2 \sqrt{a^2-b^2}} \tag{94}$$

Now, using (93) and (94) along with (82) and (84) of Appendix C, (92) becomes

$$J_1(u,a^2,b^2) = \frac{1}{4a^3} F(\alpha,r)$$

$$- \frac{1}{4a^4} \left\{ \frac{1}{\sqrt{1-r^2 \sin^2\alpha}} \cdot \frac{2u}{(u^2+a^2)} + \right.$$

$$\left. \frac{b^2}{\sqrt{2}\, r'^2\, a^2 \sqrt{a^2-b^2}} \left[\frac{F(\alpha, r) - r'^2 F(\alpha, r)}{r} - \frac{r \sin\alpha \cos\alpha}{\sqrt{1-r^2 \sin^2\alpha}} \right] \right\} \tag{95}$$

where $r' = \sqrt{1-r^2}$.

Using (90) and (91), (95) can also be written

$$J_1(u,a^2,b^2) = \frac{1}{4a^3} \left[\frac{1}{a^2-b^2} F(\alpha, r) - \frac{2b^2}{(a^4-b^4)} E(\alpha, r) \right.$$

$$\left. - \frac{2ua(u^2+2b^2+a^2)}{(a^2+b^2)(u^2+a^2)\sqrt{u^4+2b^2u^2+a^4}} \right] \tag{96}$$

Formula 3.165-6 of Reference 11 gives

$$J_2(u,a^2,b^2) = \int_u^\infty \frac{x^2 dx}{(x^4+2b^2x^2+a^2)^{3/2}}$$

$$= \frac{a}{2(a^4-b^4)} E(\alpha, r) - \frac{1}{4a(a^2-b^2)} F(\alpha, r)$$

$$- \frac{u(u^2-a^2)}{2(a^2+b^2)(u^2+a^2)\sqrt{u^4+2b^2u^2+a^2}} \tag{97}$$

$$\left[a^2 > b^2 > -\infty,\ a^2 > 0,\ u \geq 0\right]$$

Also, formula 3.165-7 of Reference 11 is

$$J_3(u,a^2,b^2) = \int_u^{\infty} \frac{(x^2-a^2)^2\,dx}{(x^4+2b^2x^2+a^4)^{3/2}}$$

$$= \frac{a}{a^2-b^2}\left[F(\alpha,\ r) - E(\alpha,\ r)\right]$$

$$+ \frac{u^2-a^2}{u^2+a^2}\ \frac{u}{\sqrt{u^4+2b^2u^2+a^4}} \qquad (98)$$

$$\left[|b^2| < a^2,\ u \geq 0\right]$$

Here, and in (97), α and r are still defined by (90) and (91), respectively.

Finally, (96), (97), and (98) can be used to write

$$J_4(u,a^2,b^2) = \int_u^{\infty} \frac{x^4\,dx}{(x^4+2b^2x^2+a^4)^{3/2}}$$

$$= J_3 + 2a^2J_2 - a^4J_1$$

$$= \frac{3}{4}\,\frac{a}{a^2-b^2}\,F(\alpha,\ r) \quad - \quad \frac{b^2(a+2b)}{2(a^4-b^4)}\,E(\alpha,\ r)$$

$$+ \frac{u\left[b^2(u^2-2a^2) - \tfrac{1}{2}a^2(u^2+a^2)\right]}{(a^2+b^2)(u^2+a^2)\sqrt{u^4+2bu^2+a^4}} \qquad (99)$$

$$\left[|b^2| < a^2,\ u \geq 0\right]$$

APPENDIX E

DERIVATIVES OF THE FUNCTIONS h_{ij}

DERIVATIVES OF THE FUNCTIONS h_{ij}

$$\frac{\partial h_{ij}}{\partial \rho_i} = \frac{1}{2\rho_i}\left\{-\tfrac{1}{2}(\rho_i\rho_j)^{-\frac{1}{2}}\; F(\alpha_{ij}, \beta_{ij}) + \frac{\rho\left[\rho^4 + 2(\rho_i\rho_j)^2 - 3\rho^2\rho_i\rho_j \cos\psi_{ij}\right]}{\left[\rho^4 + (\rho_i\rho_j)^2 - 2\rho^2\rho_i\rho_j \cos\psi_{ij}\right]^{3/2}}\right\}\Bigg|_{\rho=b/R}^{a/R} \tag{100}$$

$$\frac{\partial h_{ij}}{\partial \phi_i} = D_{ij}\left[\cos\phi_i \sin\phi_j - \sin\phi_i \cos\phi_j \cos(\lambda_i - \lambda_j)\right] \tag{101}$$

$$\frac{\partial h_{ij}}{\partial \lambda_i} = D_{ij}\left[-\cos\phi_i \cos\phi_j \sin(\lambda_i - \lambda_j)\right] \tag{102}$$

where

$$\alpha_{ij} = \arccos\left[\frac{\rho^2 - \rho_i\rho_j}{\rho^2 + \rho_i\rho_j}\right] \tag{103}$$

$$\beta_{ij} = \cos\left(\tfrac{1}{2}\psi_{ij}\right) \tag{104}$$

$$\cos\psi_{ij} = \sin\phi_i \sin\phi_j + \cos\phi_i \cos\phi_j \cos(\lambda_i - \lambda_j) \tag{105}$$

$$D_{ij} = \frac{1}{2}\left\{\frac{(\rho_i\rho_j)^{-\frac{1}{2}}}{4(1-\beta_{ij}^2)}\left[\frac{E(\alpha_{ij}, \beta_{ij}) - (1-\beta_{ij}^2)\, F(\alpha_{ij}, \beta_{ij})}{\beta_{ij}^2} - \frac{\sin\alpha_{ij}\cos\alpha_{ij}}{\sqrt{1-\beta_{ij}^2 \sin^2\alpha_{ij}}}\right] - \frac{\rho^3\rho_i\rho_j}{\left[\rho^4 + (\rho_i\rho_j)^2 - 2\rho^2\rho_i\rho_j\cos\psi_{ij}\right]^{3/2}}\right\}\Bigg|_{\rho=b/R}^{a/R} \tag{106}$$

Note: the relation

$$\frac{\partial h_{ii}}{\partial \rho_i} = \lim_{j \to i} \frac{\partial h_{ij}}{\partial \rho_i} \tag{107}$$

(and similarly for ϕ_i and λ_i derivatives) is to be used, when necessary, in the evaluation of expressions such as equations (59) and (61).

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