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SYSTEMS OF FREQUENCY CURVES GENERATED BY TRANSFORMATIONS  
OF LOGISTIC VARIABLES

TECHNICAL REPORT

P.R. TADIKAMALLA AND N.L. JOHNSON

JUNE, 1979



U.S. ARMY RESEARCH OFFICE  
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Systems of Frequency Curves Generated by Transformations  
of Logistic Variables

by

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1. Introduction

The systems of distributions (for a random variable  $Y$ ) defined by

$$S_L: Z = \gamma + \delta \ln Y \quad (0 < Y) \quad (1.1)$$

$$S_B: Z = \gamma + \delta \ln\{Y/(1-Y)\} \quad (0 < Y < 1) \quad (1.2)$$

$$S_U: Z = \gamma + \delta \sinh^{-1} Y \quad (1.3)$$

with  $\delta > 0$ , and  $Z$  a unit normal variable, have been described by Johnson (1949); and with  $Z$  a standard Laplace (double exponential) variable, ( $S'_L$ ,  $S'_B$ ,  $S'_U$ ) by Johnson (1954). In the present paper we study analogous systems, generated by ascribing to  $Z$  a standard logistic distribution with probability density function (pdf)

$$f_Z(z) = e^z(1 + e^z)^{-2} \quad (1.4)$$

or, equivalently, with cumulative distribution function (cdf)

$$F_Z(z) = (1 + e^{-z})^{-1} \quad (1.5)$$

The percentile function (inverse of cdf) is

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$$z = \ln\{F_Z(z)/(1 - F_Z(z))\}. \quad (1.6)$$

We will denote these new systems of distributions by  $L_L$ ,  $L_B$ ,  $L_U$  according as transformations (1.1), (1.2), (1.3), respectively, are used. In view of the closeness of the shapes of the logistic and normal distributions, it is to be expected that the new systems will exhibit some similarity in shape to  $S_L$ ,  $S_B$ , and  $S_U$ .

In fact since (Johnson and Kotz (1970), p.6)

$$| \{1 + \exp(-\pi x/\sqrt{3})\}^{-1} - \Phi(\frac{16x}{15}) | < 0.01 \quad \text{for all } x \quad (1.7a)$$

where  $\Phi(u) = (\sqrt{2\pi})^{-1} \int_{-\infty}^u e^{-\frac{1}{2}t^2} dt$  the difference between the cdf of  $S_{L,B,U}$  with parameters  $\gamma$ ,  $\delta$  and  $L_{L,B,U}$  (with the same values of  $\xi$  and  $\lambda$ ) with parameters  $\psi\gamma$ ,  $\psi\delta$ , where

$$\psi = \frac{\pi}{\sqrt{3}} \cdot \frac{15}{16} \doteq 1.7 \quad (1.7b)$$

cannot exceed 0.01. Although this gives good agreement in the central part of the distribution, there can be gross disparities in percentile points in the tails.

If an S curve and an L curve are fitted using the same first four moments, we do not find that (i) the ratios (L/S) for  $\gamma$  values and for  $\delta$  values will each be about 1.7, and (ii) the values of  $\xi$  and  $\lambda$  will be the same. Section 5.4 (Table 4) contains an example wherein this is clearly illustrated. There can be very considerable disparity between the percentiles in the tails of the distributions even though (1.7a) is valid.

The very simple formula (1.6) for  $z$  in terms of  $F_Z(z)$  is an obvious practical advantage of the new systems. Percentile points of fitted distributions can be obtained very simply.

## 2. The Log-Logistic ( $L_L$ ) Distribution

This distribution has been studied by Shah and Dave (1963). The pdf is

$$f_Y(y) = \delta e^{-\gamma} y^{\delta-1} (1 + e^{-\gamma} y^{\delta})^{-2} \quad (y \geq 0; \delta > 0) \quad (2.1)$$

from which it can be seen that it is a special case of Burr's (1942) Type XII system of distributions. Dubey (1966) called it the Weibull-exponential distribution, and fitted it to some business failure data presented by Lomax (1954).

The pdf is unimodal. The mode is at  $y = 0$  for  $\delta \leq 1$  (giving a reversed J-shaped curve); it is at  $y = e^{-\gamma}(\delta-1)/(\delta+1)$  if  $\delta > 1$ . The cdf is

$$F_Y(y) = 1 - (1 + e^{-\gamma} y^{\delta})^{-1} \quad (2.2)$$

and the inverse cdf is

$$y = e^{\Omega} [F_Y(y)/(1-F_Y(y))]^{1/\delta} \quad (2.3)$$

where  $\Omega = \gamma/\delta$ .

From (1.1) and (1.4), the  $r$ -th moment of  $Y$  about zero is

$$\mu'_r = \int_{-\infty}^{\infty} e^{r(z-\gamma)/\delta} e^z (1 + e^z)^{-2} dz.$$

Putting  $u = e^z (1 + e^z)^{-1}$  so that  $du/dz = e^z(1 + e^z)^{-2}$  and  $e^z = u/(1-u)$

$$\begin{aligned} \mu'_r &= e^{-r\Omega} \int_0^1 u^{r/\delta} (1-u)^{-r/\delta} du = e^{-r\Omega} B(1+r\delta^{-1}, 1-r\delta^{-1}) \\ &= e^{-r\Omega} r\theta \operatorname{cosec} r\theta \end{aligned} \quad (2.4)$$

with  $\theta = \pi/\delta$ , provided  $r < \delta$ . (If  $r \geq \delta$ ,  $\mu'_r$  is infinite.)

In particular

$$E[Y] = e^{-\Omega} \theta \operatorname{cosec} \theta \quad (2.5)$$



and

$$\text{Var}(Y) = e^{-2\Omega} \theta \operatorname{cosec}^2 \theta (\tan \theta - \theta) . \quad (2.6)$$

The  $r$ -th central moment,  $\mu_r$ , is a multiple (depending on  $\theta$ , but not on  $\Omega$ ) of  $\exp(-r\Omega)$ , and so the moment-ratios  $\mu_r/\mu_2^{r/2}$  - in particular  $\sqrt{\beta_1} = \mu_3/\mu_2^{3/2}$  and  $\beta_2 = \mu_4/\mu_2^2$  - do not depend on  $\Omega$ , but only on  $\theta$ . The  $(\sqrt{\beta_1}, \beta_2)$  points lie on a line which passes through the 'logistic point' (0,4.2) (corresponding to  $\theta \rightarrow 0$  ( $\delta \rightarrow \infty$ )). Figure 1 shows that  $L_L$  ('log-logistic') line, and also the  $S_L$  (lognormal) line.

The  $(\sqrt{\beta_1}, \beta_2)$  points for  $L_B(L_U)$  distributions lie 'above' ('below') the  $L_L$  line in Figure 1. This is analogous to the situation for  $S_B$ ,  $S_U$  and the lognormal line.

Table 1 gives values of  $\beta_2$  for selected values of  $\sqrt{\beta_1}$ , for points on the  $L_L$  line.

### 3. The $L_B$ System

If transformation (1.2) is valid, with  $Z$  standard logistic, then the pdf of  $Y$  is

$$f_Y(y) = \delta e^\gamma y^{\delta-1} (1-y)^{\delta-1} \{(1-y)^\delta + e^\gamma y^\delta\}^{-2} \quad (0 < y < 1) . \quad (3.1)$$

If  $\delta = 1$ , we get a 'power function' pdf

$$f_Y(y) = e^\gamma (1 - y + e^\gamma y)^{-2} \quad (0 < y < 1) . \quad (3.2)$$

If, in addition,  $\gamma = 0$  we have a standard uniform distribution.

In contrast to the situations when  $Z$  has a normal or Laplace distribution,  $L_B$  curves cannot be multimodal. There is a single mode (if  $\delta > 1$ ) or antimode (if  $\delta < 1$ ) at the unique value of  $y$  between 0 and 1 satisfying the equation

$$e^{\gamma} = \left(\frac{1-y}{y}\right)^{\delta} \cdot \frac{\delta-1+2y}{\delta+1-2y} \quad (3.3)$$

The right hand side of (3.3) is equal to 1 when  $y = \frac{1}{2}$ , whatever the value of  $\delta$ .

The derivative of the logarithm of the right hand side is

$$\delta(1-\delta^2) y^{-1}(1-y)^{-1} \{\delta^2 - (1-2y)^2\} \quad (3.4)$$

If  $\delta > 1$ , this is negative, and so the mode is at  $y \geq \frac{1}{2}$  according as  $\gamma \geq 0$ .

For  $\delta < 1$ , the curve is U-shaped, and the antimode is between  $\frac{1}{2}(1-\delta)$  and  $\frac{1}{2}(1+\delta)$ . Since (3.4) is positive for  $\frac{1}{2}(1-\delta) < y < \frac{1}{2}(1+\delta)$  it follows that the antimode is at  $\frac{1}{2}(1-\delta) < y < \frac{1}{2}$  for  $\gamma < 0$  and  $\frac{1}{2} < y < \frac{1}{2}(1+\delta)$  for  $\gamma > 0$ .

TABLE 1: The Log-logistic Line

| $\sqrt{\beta_1}$ | $\beta_2$ | $\delta$ | $\sqrt{\beta_1}$ | $\beta_2$ | $\delta$ | $\sqrt{\beta_1}$ | $\beta_2$ | $\delta$ |
|------------------|-----------|----------|------------------|-----------|----------|------------------|-----------|----------|
| 0.05             | 4.21      | 174.2    | 0.55             | 4.96      | 16.26    | 1.1              | 7.47      | 8.73     |
| 0.10             | 4.22      | 87.14    | 0.60             | 5.11      | 14.98    | 1.2              | 8.16      | 8.13     |
| 0.15             | 4.26      | 58.16    | 0.65             | 5.27      | 13.90    | 1.3              | 8.94      | 7.63     |
| 0.20             | 4.30      | 43.69    | 0.70             | 5.45      | 12.98    | 1.4              | 9.81      | 7.21     |
| 0.25             | 4.35      | 35.02    | 0.75             | 5.65      | 12.18    | 1.5              | 10.79     | 6.86     |
| 0.30             | 4.42      | 29.26    | 0.80             | 5.86      | 11.49    | 1.6              | 11.88     | 6.55     |
| 0.35             | 4.50      | 25.15    | 0.85             | 6.08      | 10.88    | 1.7              | 13.11     | 6.28     |
| 0.40             | 4.60      | 22.08    | 0.90             | 6.32      | 10.35    | 1.8              | 14.47     | 6.04     |
| 0.45             | 4.71      | 19.70    | 0.95             | 6.58      | 9.87     | 1.9              | 16.00     | 5.84     |
| 0.50             | 4.83      | 17.80    | 1.00             | 6.86      | 9.45     | 2.0              | 17.71     | 5.65     |

(For  $\sqrt{\beta_1} = 0$ ,  $\beta_2 = 4.2$  and  $\delta = \infty$ )



The  $(\sqrt{\beta_1}, \beta_2)$  line corresponding to  $\delta = 1$ , which is the 'lower' boundary of the region with U-shaped curves, is shown in Figure 1. Apart from (3.2), there are no J - or reverse-J shaped curves in the  $L_B$  system.

Following an analysis similar to that leading to (2.4) we obtain

$$\begin{aligned} u'_r &= \int_{-\infty}^{\infty} \{1 + e^{-(z-\gamma)/\delta}\}^{-r} e^z (1 + e^z)^{-2} dz \\ &= \int_0^1 \{1 + e^{\Omega} (1-u)^{1/\delta} u^{-1/\delta}\}^{-r} du \end{aligned} \quad (3.5)$$

with  $\Omega = \gamma/\delta$ , as in Section 2.

Generally, the integral in (3.5) must be evaluated by quadrature. (Of course, for some special cases - such as  $\delta = 1$  - explicit solutions can be obtained.) Expansion of the integrand as a power series in  $(1-u^{-1})^{1/\delta}$  cannot be valid over the whole range of integration. Dichotomy of the interval  $(0,1)$  according as  $(1-u^{-1})^{1/\delta} \gtrless e^{\Omega}$  (i.e.  $u \lesseqgtr (1+e^{-\Omega})^{-1}$ ) leads to valid expansions but the resulting incomplete beta functions must themselves (in general) be evaluated by quadrature.

#### 4. The $L_U$ System

If the transformation (1.3) is valid, the pdf of Y is

$$f_Y(y) = \frac{\delta e^{\gamma}}{(y^2+1)^{1/2}} \cdot \frac{(y+\sqrt{(y^2+1)})^{\delta}}{\{1+e^{\gamma}(y+\sqrt{(y^2+1)})^{\delta}\}^2} \quad (4.1)$$

This curve is unimodal, with mode at the unique value of y satisfying the equation

$$\delta\{1 - e^{\gamma}(y + \sqrt{(y^2+1)})\} = y(y^2+1)^{-1/2} \quad (4.2)$$

Note that, since  $\delta > 0$ , the left hand side of (4.2) is a monotonic decreasing function of y; the right hand side is monotonic increasing.

The  $r$ -th moment of  $Y$  about zero is

$$\begin{aligned}\mu_r' &= 2^{-r} \int_{-\infty}^{\infty} \{e^{(z-\gamma)/\delta} - e^{-(z-\gamma)/\delta}\}^r e^z (1 + e^z)^{-2} dz \\ &= 2^{-r} \sum_{j=0}^r (-1)^j \binom{r}{j} e^{-(r-2j)\Omega} \int_{-\infty}^{\infty} e^{(r-2j)z/\delta} e^z (1 + e^z)^{-2} dz \\ &= 2^{-r} \sum_{j=0}^r (-1)^j \binom{r}{j} e^{-(r-2j)\Omega} (r-2j)\theta \operatorname{cosec}(r-2j)\theta\end{aligned}\quad (4.3)$$

provided  $r < \delta$ . If  $r \geq \delta$ ,  $\mu_r'$  is infinite. When  $r = 2j$ , ' $0 \operatorname{cosec} 0$ ' is interpreted as  $\lim_{\theta \rightarrow 0} \theta \operatorname{cosec} \theta = 1$ . (As in Section 2,  $\Omega = \gamma/\delta$  and  $\theta = \pi/\delta$ .)

Since  $(-\alpha)\operatorname{cosec}(-\alpha) = \alpha\operatorname{cosec} \alpha$ , (4.3) can be simplified.

For  $r$  even:

$$\mu_r' = 2^{-r} (-1)^{\frac{1}{2}r} \binom{r}{\frac{1}{2}r} + 2^{-(r-1)} \sum_{j=0}^{\frac{1}{2}r-1} (-1)^j \binom{r}{j} (r-2j)\theta \operatorname{cosec}(r-2j)\theta \cosh(r-2j)\Omega \quad (4.4)$$

For  $r$  odd:

$$\mu_r' = -2^{-(r-1)} \sum_{j=0}^{\frac{1}{2}(r-1)} (-1)^j \binom{r}{j} (r-2j)\theta \operatorname{cosec}(r-2j)\theta \sinh(r-2j)\Omega \quad (4.5)$$

In particular

$$\begin{aligned}\mu_1' &= -\theta \operatorname{cosec} \theta \sinh \Omega \\ \mu_2' &= \theta \operatorname{cosec} 2\theta \cosh 2\Omega - \frac{1}{2} \\ \mu_3' &= -\frac{3}{4} \theta (\operatorname{cosec} 3\theta \sinh 3\Omega - \operatorname{cosec} \theta \sinh \Omega) \\ \mu_4' &= \frac{1}{2} \theta (\operatorname{cosec} 4\theta \cosh 4\Omega - 2 \operatorname{cosec} 2\theta \cosh 2\Omega) + \frac{3}{8}\end{aligned}\quad (4.6)$$

so that

$$\begin{aligned}E[Y] &= -\theta \operatorname{cosec} \theta \sinh \Omega \\ \operatorname{Var}(Y) &= \frac{1}{2} \{(\theta \operatorname{cosec} \theta)^2 - 1\} + \frac{1}{2} \theta \operatorname{cosec}^2 \theta (\tan \theta - \theta) \cosh 2\Omega\end{aligned}\quad (4.7)$$

## 5. Fitting the Distributions

5.1. General. The methods of fitting described by Johnson (1949) for the S systems are also applicable to the L systems, with the advantage that the cdf of the logistic is simpler than that of the normal distribution.

Introducing the location and scale parameters  $\xi$ ,  $\lambda$  respectively, by the transformation  $Y = (X - \xi)/\lambda$  we obtain a three parameter family for  $L_L$ :

$$Z = \gamma + \delta \ln (X - \xi) \quad (X > \xi) \quad (5.1)$$

and four parameter families

$$Z = \gamma + \delta \ln \left( \frac{X - \xi}{\xi + \lambda - X} \right) \quad (\xi < X < \xi + \lambda) \quad (5.2)$$

$$Z = \gamma + \delta \ln h^{-1} \left( \frac{X - \xi}{\lambda} \right) \quad (5.3)$$

for  $L_B$ ,  $L_U$  respectively.

We consider fitting by the methods of moments, percentile points and maximum likelihood.

Moments. If four parameters are to be fitted, the first four moments of the distribution of  $X$  are equated to those of the fitted curve. It is convenient to use them in the form of mean ( $\mu_1'$ ), the variance ( $\mu_2$ ) and the moment ratios  $\sqrt{\beta_1}$  ( $= \mu_3/\mu_2^{3/2}$ ) and  $\beta_2$  ( $= \mu_4/\mu_2^2$ ). Since  $\sqrt{\beta_1}$  and  $\beta_2$  are determined by  $\gamma$ ,  $\delta$  and (for  $L_B$  and  $L_U$ , as for  $S_B$  and  $S_U$ ) conversely, the first step is to determine  $\gamma$ ,  $\delta$  from the specified values of  $\sqrt{\beta_1}$  and  $\beta_2$ . This could be done using special tables (Table 1.4), or of course a computer program. Once these values are found the values of  $\xi$  and  $\lambda$  are determined from the equations

$$\begin{aligned} E[X|\gamma, \delta] &= \xi + \lambda E[Y|\gamma, \delta] \\ \text{var}(X|\gamma, \delta) &= \lambda^2 \text{var}(Y|\gamma, \delta) \end{aligned} \quad (5.4)$$



Percentile Points. If four parameters are to be fitted, then (using (1.6)), four equations of form

$$z_{p_j} = \ln\left(\frac{p_j}{1-p_j}\right) = \tilde{\gamma} + \tilde{\delta} f\left(\frac{\hat{x}_{p_j} - \tilde{\xi}}{\tilde{\lambda}}\right) \quad (j = 1, 2, 3, 4) \quad (5.5)$$

where  $\hat{x}_{p_j}$  is the estimated 100  $p_j$ % point of the distribution of  $X$ , have to be solved for  $\tilde{\gamma}$ ,  $\tilde{\delta}$ ,  $\tilde{\xi}$  and  $\tilde{\lambda}$ .

Solutions of especially simple form are obtained by taking symmetric percentiles with  $p_3 = 1-p_2$ ;  $p_4 = 1-p_1$ .

It is interesting to note that with this choice of  $p$ 's, provided the distribution of  $Z$  is symmetrical about zero then the solutions of (5.5) for  $\tilde{\xi}$  and  $\tilde{\lambda}$  must be linked by the same equation

$$f\left(\frac{\hat{x}_{p_1} - \tilde{\xi}}{\tilde{\lambda}}\right) + f\left(\frac{\hat{x}_{1-p_1} - \tilde{\xi}}{\tilde{\lambda}}\right) = f\left(\frac{\hat{x}_{p_2} - \tilde{\xi}}{\tilde{\lambda}}\right) + f\left(\frac{\hat{x}_{1-p_2} - \tilde{\xi}}{\tilde{\lambda}}\right) \quad (5.6)$$

whatever the actual distribution of  $Z$ .

In fact, from

$$\tilde{\gamma} + \tilde{\delta} f\left(\frac{\hat{x}_{p_j} - \tilde{\xi}}{\tilde{\lambda}}\right) = -z_{1-p_j} = -\tilde{\gamma} - \tilde{\delta} f\left(\frac{\hat{x}_{1-p_j} - \tilde{\xi}}{\tilde{\lambda}}\right) \quad (j = 1, 2)$$

it follows that the common value of each side of (5.6) is  $-2\tilde{\gamma}$ .

Another relation is

$$z_{1-p_1} \left\{ f\left(\frac{\hat{x}_{1-p_2} - \tilde{\xi}}{\tilde{\lambda}}\right) - f\left(\frac{\hat{x}_{p_2} - \tilde{\xi}}{\tilde{\lambda}}\right) \right\} = z_{1-p_2} \left\{ f\left(\frac{\hat{x}_{1-p_1} - \tilde{\xi}}{\tilde{\lambda}}\right) - f\left(\frac{\hat{x}_{p_1} - \tilde{\xi}}{\tilde{\lambda}}\right) \right\}. \quad (5.7)$$

This relation does depend on the actual distribution of  $Z$ . Equations (5.6) and (5.7) suffice to determine  $\tilde{\xi}$  and  $\tilde{\lambda}$ .

In certain cases (e.g. for  $L_B$  - see below) equation (5.6) can be solved explicitly giving  $\tilde{\lambda}$  as a function of  $\tilde{\xi}$ .



If  $\xi$  and  $\lambda$  are known, two percentile points suffice to determine  $\gamma$  and  $\delta$ . Knowledge of  $\xi$  and/or  $\lambda$  is more usual when a bounded curve is being fitted. Fitting of  $f_g$  in these cases is discussed in Section 5.3.

Maximum Likelihood. If  $\xi$  and  $\lambda$  are known, the transformed values of  $f((x-\xi)/\lambda)$  can be used, and methods appropriate to fitting logistic distributions (e.g. Johnson and Kotz (1970, Chapter 23)) can be applied. If  $\xi$  and  $\lambda$  are not known, a series of pairs of values of these parameters can be used and the corresponding maximized likelihoods calculated, the maximum likelihood estimates being found by trial and error - or, of course, a computer program which directly maximizes the likelihood function can be used. Except in special cases, maximum likelihood calculations are rather lengthy, but not impracticably so.

5.2. Fitting the  $L_L$  System. If  $\xi$  is known, the transformed values

$$Y_i = \log(X_i - \xi)$$

can be used, and the parameters  $\gamma$  and  $\delta$  fitted by methods appropriate to fitting a logistic distribution.

When  $\xi$  is not known there are three parameter values to be fitted ( $\xi, \gamma$ , and  $\delta$ ). With the method of *percentile points*, it is natural to use the sample median  $\hat{x}_{0.5}$  and the upper and lower sample 100 P% points,  $\hat{x}_{1-p}$  and  $\hat{x}_p$  respectively. Noting (1.6), this leads to the equations

$$\begin{aligned} 0 &= \hat{\gamma} + \hat{\delta} \ln(\hat{x}_{0.5} - \tilde{\xi}) \\ -\ln\{P/(1-P)\} &= \tilde{\gamma} + \tilde{\delta} \ln(\hat{x}_{1-p} - \tilde{\xi}) \\ \ln\{P/(1-P)\} &= \tilde{\gamma} + \tilde{\delta} \ln(\hat{x}_p - \tilde{\xi}) \end{aligned} \tag{5.8}$$

which have the unique solutions

$$\begin{aligned}\tilde{\xi} &= \frac{\hat{x}_{0.5}^2 - \hat{x}_{1-P} \hat{x}_P}{2\hat{x}_{0.5} - \hat{x}_{1-P} - \hat{x}_P} \\ \tilde{\delta} &= \frac{-2 \ln\{P/(1-P)\}}{\ln\{(\hat{x}_{1-P} - \tilde{\xi})(\hat{x}_P - \tilde{\xi})\}} \\ \tilde{\gamma} &= -\tilde{\delta} \ln(\hat{x}_{0.5} - \tilde{\xi})\end{aligned}\tag{5.9}$$

provided  $\tilde{\xi} < \hat{x}_P$ .

Of course, several values of  $P$  may be used. The resulting estimates  $\tilde{\xi}$ ,  $\tilde{\gamma}$  and  $\tilde{\delta}$  should be reasonably consistent with each other if an  $L_L$  distribution is suitable. If only one set is to be used,  $P = 0.10$  (i.e. the sample median, and upper and lower sample deciles) would seem to be a good choice.

We now come to the method of *moments*. The skewness parameter  $\sqrt{\beta_1} = \mu_3/\mu_2^{3/2}$  is

$$\begin{aligned}\sqrt{\beta_1}(\theta) &= \frac{3\theta \operatorname{cosec} 3\theta - 6\theta^2 \operatorname{cosec} \theta \operatorname{cosec} 2\theta + 3\theta^3 \operatorname{cosec}^3 \theta}{\{\theta \operatorname{cosec}^2 \theta (\tan \theta - \theta)\}^{3/2}} \\ &= \frac{3}{\sqrt{\theta}} \cdot \frac{\operatorname{cosec} 3\theta - \theta \operatorname{cosec}^2 \theta \sec \theta + \theta^2 \operatorname{cosec}^3 \theta}{\operatorname{cosec}^3 \theta (\tan \theta - \theta)^{3/2}}\end{aligned}\tag{5.10}$$

Equating (5.3) to  $\sqrt{b_1} = m_3/m_2^{3/2}$ , the sample skewness ratio, and solving for  $\theta$  gives the moment estimator,  $\tilde{\theta}$ , of  $\theta$ . The function (5.3) is an increasing function of  $\theta$ , and tends to infinity as  $\theta \rightarrow \infty$ , so provided  $\sqrt{b_1} > 0$ , there will be a unique value for  $\tilde{\theta}$ . The estimator for  $\delta$  is

$$\tilde{\delta} = \pi/\tilde{\theta}.$$

**Example 1.** As can be seen from Figure 1, the  $L_L$  line is in the Pearson Type IV region. Suppose we want to use an  $L_L$  curve to approximate a Type IV with  $\sqrt{\beta_1} = 0.4$ ,  $\beta_2 = 4.6$ . From Table 1 (with extra decimal places),  $\delta = 22.0803$

yields  $\sqrt{\beta_1} = 0.4$ ,  $\beta_2 = 4.5991$  with  $E[Y|\gamma=0, \delta] = 1.0034$ ,  $\delta(Y|\gamma=0, \delta) = 0.0828$  whence  $\gamma = -55.0$ ,  $\xi = -12.1184$ . Standardized percentiles of the fitted distribution are compared with those of the Type IV distribution (obtained by interpolation in the tables of Bouver and Bargmann (1974)) in Table 2.

Table 2. Comparison of Standardized Type IV and  $L_L$  Percentiles ( $\beta_1 = 0.4$ ,  $\beta_2 = 4.6$ )

| %       | 0.1    | 0.25   | 0.5    | 1      | 2.5    | 5      | 10     | 25     | 50     |
|---------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| Type IV | -3.315 | -2.887 | -2.574 | .2.262 | -1.854 | -1.532 | -1.186 | -0.641 | -0.046 |
| $L_L$   | -3.285 | -2.910 | -2.615 | -2.310 | -1.887 | -1.548 | -1.185 | -.627  | -.041  |
| %       | 75     | 90     | 95     | 97.5   | 99     | 99.5   | 99.75  | 99.9   |        |
| Type IV | 0.587  | 1.237  | 1.686  | 2.127  | 2.717  | 3.182  | 3.674  | 4.346  |        |
| $L_L$   | 0.575  | 1.223  | 1.682  | 2.139  | 2.753  | 3.231  | 3.723  | 4.395  |        |

### 5.3. Fitting the $L_B$ System.

Moments. If all four parameters  $(\xi, \lambda, \gamma, \delta)$  have to be estimated, it is necessary to have special tables (i) to determine  $\gamma$ ,  $\delta$  from  $\sqrt{\beta_1}$  and  $\beta_2$  and (ii) giving values of  $E[Y|\gamma, \delta]$ , and  $S.D(Y|\gamma, \delta)$  for these values of  $\gamma$  and  $\delta$ . Such tables are in preparation but as yet they are not sufficiently extensive for practical use.

Percentile Points. For  $L_B$  curves, introducing the notation  $\tilde{\xi}_p = \hat{\lambda}_p - \xi$ , equations (5.6) and (5.7) become

$$\frac{\tilde{\xi}_{p_1} \tilde{\xi}_{1-p_1}}{(\tilde{\lambda} - \tilde{\xi}_{p_1})(\tilde{\lambda} - \tilde{\xi}_{1-p_1})} = \frac{\tilde{\xi}_{p_2} \tilde{\xi}_{1-p_2}}{(\tilde{\lambda} - \tilde{\xi}_{p_2})(\tilde{\lambda} - \tilde{\xi}_{1-p_2})} \quad (5.11)$$

and

$$z_{1-p_1} \ln \left| \frac{\tilde{\xi}_{p_2} (\tilde{\lambda} - \tilde{\xi}_{1-p_2})}{(\tilde{\lambda} - \tilde{\xi}_{p_2}) \tilde{\xi}_{1-p_2}} \right| = z_{1-p_2} \ln \left| \frac{\tilde{\xi}_{p_1} (\tilde{\lambda} - \tilde{\xi}_{1-p_1})}{(\tilde{\lambda} - \tilde{\xi}_{p_1}) \tilde{\xi}_{1-p_1}} \right| \quad (5.12)$$



respectively. From (5.11)

$$\tilde{\lambda} = \frac{\tilde{\xi}_{p_1} \tilde{\xi}_{1-p_1} (\tilde{\xi}_{p_2} + \tilde{\xi}_{1-p_2}) - \tilde{\xi}_{p_2} \tilde{\xi}_{1-p_2} (\tilde{\xi}_{p_1} + \tilde{\xi}_{1-p_1})}{\tilde{\xi}_{p_1} \tilde{\xi}_{1-p_1} - \tilde{\xi}_{p_2} \tilde{\xi}_{1-p_2}} \quad (5.13)$$

These equations may be solved by a process of trial and error, illustrated in Example 2(a) below.

Techniques of solving similar equations for fitting  $S_B$  curves, devised by Bukac<sup>v</sup> (1972), Mage (1978) and Shapiro (1978), may also be used for  $L_B$ . Since  $p_1$  and  $p_2$  may be chosen arbitrarily, it is possible to arrange that  $z_{1-p_1}/z_{1-p_2}$  is an integer. We do this in Example 2(a), though it is not necessary to do so.

Devices for simplifying estimation of the four parameters for  $S_B$  (Bukac<sup>v</sup> (1972), Mage (1978), Slifker and Shapiro (1979)) can equally be applied to fitting  $L_B$ . We do not pursue these matters here.

In Example 2(b) and 2(c) we fit the same data as in 2(a), when both  $\xi$  and  $\lambda$  are assumed known, and when  $\xi$  (but not  $\lambda$ ) is assumed known.

If the range of variation (and so both  $\xi$  and  $\lambda$ ) be known, then only two sample percentiles are needed to estimate  $\gamma$  and  $\delta$ . It is convenient to choose them symmetrically and use the equations

$$\begin{aligned} -Z_{1-p} &= \tilde{\gamma} + \tilde{\delta} \ln \left( \frac{\xi_p}{\lambda - \xi_p} \right) \\ Z_{1-p} &= \tilde{\gamma} + \tilde{\delta} \ln \left( \frac{\xi_{1-p}}{\lambda - \xi_{1-p}} \right) \end{aligned} \quad (5.14)$$

where  $\xi_p = \hat{x}_p - \xi$ , whence

$$\tilde{\delta} = 2 Z_{1-p} \left[ \ln \frac{\xi_{1-p} (\lambda - \xi_p)}{\xi_p (\lambda - \xi_{1-p})} \right]^{-1} \quad (5.15)$$

and

$$\tilde{\gamma} = -\frac{1}{2} \tilde{\delta} \ln \frac{\xi_p \xi_{1-p}}{(\lambda - \xi_p)(\lambda - \xi_{1-p})} \quad (5.16)$$



If only one end point - say the lower ( $\xi$ ) - is known, then three sample percentiles are needed, from which to estimate  $\lambda$ ,  $\gamma$  and  $\delta$ . It is convenient to take the sample median  $\hat{\chi}_{0.5}$  in addition to  $\hat{\chi}_p$  and  $\hat{\chi}_{1-p}$ . From the resulting equations we obtain

$$\left(\frac{\xi_{0.5}}{\tilde{\lambda} - \xi_{0.5}}\right)^2 = \frac{\xi_p \xi_{1-p}}{(\tilde{\lambda} - \xi_p)(\tilde{\lambda} - \xi_{1-p})}$$

whence

$$\tilde{\lambda} = \frac{\xi_{0.5} \{ \xi_{0.5} (\xi_p + \xi_{1-p}) - 2 \xi_p \xi_{1-p} \}}{\xi_{0.5}^2 - \xi_p \xi_{1-p}} \quad (5.17)$$

$\tilde{\gamma}$  and  $\tilde{\delta}$  are then calculated from (5.15) and (5.16).

Example 2. (a) We will use the data in Table 6-4 of Hahn and Shapiro (1967).

For fitting four parameters we use, in addition to the values  $\hat{\chi}_{.09} = 0.84$ ,

$\hat{\chi}_{.91} = 1.42$  employed by Hahn and Shapiro for fitting  $S_B$ , the values  $\hat{\chi}_{.3162} = 0.97$ ,

$\hat{\chi}_{.6838} = 1.18$ . (These values are chosen to make

$$Z_{.91} = \ln \frac{0.91}{0.09} = 2.3136 = 3 \ln \frac{0.6838}{0.3162} = 3 Z_{.6838})$$

The trial and error calculations are set out briefly below.

| $\xi$ | $\tilde{\lambda}(\xi)$ | (1)<br>$g(0.09)$ | (2)<br>$g(0.3162)$ | (1)/(2) | $g(P) = \ln \left  \frac{\xi_p(\tilde{\lambda} - \xi_{1-p})}{\xi_{1-p}(\tilde{\lambda} - \xi_p)} \right $ |
|-------|------------------------|------------------|--------------------|---------|-----------------------------------------------------------------------------------------------------------|
| 0.5   | 6.32                   | -1.097           | -0.424             | 2.59    |                                                                                                           |
| 0.75  | 0.95                   | -3.130           | -1.009             | 3.10    | We want $g(0.09)/g(0.3162) = 3$ .                                                                         |
| 0.72  | 1.12                   | -2.631           | -0.886             | 2.97    |                                                                                                           |
| 0.73  | 1.06                   | -2.779           | -0.924             | 3.01    |                                                                                                           |

Hence we obtain  $\tilde{\xi} = 0.73$ ,  $\tilde{\lambda} = 1.06$ . These lead to estimates  $\tilde{\gamma} = 1.276$ ,  $\tilde{\delta} = 1.665$ .

Percentage distribution is shown in column (3) of Table 3.

(b) If we take  $\xi = 0.5$ ,  $\lambda = 1.5$  and use  $\hat{\chi}_{0.09}$  and  $\hat{\chi}_{0.91}$ , we obtain

$$\tilde{\gamma} = 1.049, \quad \tilde{\delta} = 2.740.$$

Since we are using the same values for  $\xi$  and  $\lambda$  as Hahn and Shapiro used in fitting an  $S_B$  curve to the same data (and so the same  $\hat{x}_{0.09}$  and  $\hat{x}_{0.91}$ ), the values are each in the same ratio ( $2.3136/1.3408 = 1.726$  - note that  $\Phi(1.3408) = 0.91$ ) to the corresponding values for their fitted  $S_B$ .

Percentage distributions of the fitted  $L_B$  are shown in column (5) of Table 3. Column (4) contains values obtained by Hahn and Shapiro for their fitted  $S_B$ .

(c) If  $\hat{x}_{0.5} = 1.07$  is used in addition to  $\hat{x}_{0.09}$  and  $\hat{x}_{0.91}$  we obtain from (5.17)

$$\tilde{\lambda} = 4.36$$

which, of course, agrees with the value obtained by Hahn and Shapiro when fitting an  $S_B$ . The values for  $\gamma$  and  $\delta$ ,

$$\tilde{\gamma} = 7.614 \quad \text{and} \quad \tilde{\delta} = 4.019$$

are again in the same ratio (1.726) to Hahn and Shapiro's  $S_B$  values.

Percentage distributions of the fitted  $L_B$  are shown in column (7) of Table 3. Column (6) contains values obtained by Hahn and Shapiro for their fitted  $S_B$ .

Comparing the three  $L_B$  fits (columns (3), (5) and (7)) it is noteworthy how similar shaped curves can be obtained with quite different values for  $\gamma$  and  $\delta$ , by appropriate adjustment of  $\xi$  and  $\lambda$  (and conversely).

Table 3 - Comparison of Fitted  $S_B$  and  $L_B$  Curves

| (1)<br>Production<br>Time<br>(min) | (2)<br>Actual<br>% | (3)<br>$L_B$ | (4)<br>$S_B$ | (5)<br>$L_B$ | (6)<br>$S_B$ | (7)<br>$L_B$ |
|------------------------------------|--------------------|--------------|--------------|--------------|--------------|--------------|
| $\leq 0.695$                       | 0.9                | -            | 0.9          | 1.5          | 0.4          | 0.9          |
| 0.695-0.795                        | 3.7                | 3.7          | 4.7          | 4.2          | 4.5          | 4.2          |
| 0.795-0.895                        | 12.6               | 14.0         | 10.3         | 8.8          | 12.8         | 11.0         |
| 0.895-0.995                        | 18.4               | 18.8         | 15.2         | 14.6         | 18.7         | 18.4         |
| 0.995-1.095                        | 18.8               | 18.6         | 17.4         | 18.4         | 19.1         | 20.6         |
| 1.095-1.195                        | 15.8               | 15.3         | 16.9         | 18.1         | 15.7         | 16.8         |
| 1.195-1.295                        | 12.2               | 11.3         | 13.9         | 14.3         | 11.3         | 11.2         |
| 1.295-1.395                        | 7.6                | 7.8          | 10.2         | 9.4          | 7.4          | 6.8          |
| 1.395-1.495                        | 5.0                | 5.1          | 6.1          | 5.5          | 4.5          | 4.0          |
| 1.495-1.595                        | 2.8                | 3.1          | 3.1          | 2.9          | 2.6          | 2.4          |
| 1.595-1.695                        | 1.1                | 1.7          | 1.0          | 1.4          | 1.4          | 1.4          |
| 1.695-1.795                        | 0.9                | 0.6          | 0.3          | 0.6          | 0.8          | 0.9          |
| $\geq 1.795$                       | 0.2                | -            | 0.0          | 0.2          | 0.8          | 1.5          |
|                                    | $\xi$              | 0.73*        |              | 0.5          |              | 0.5          |
|                                    | $\lambda$          | 1.06*        |              | 1.5          |              | 4.36*        |
|                                    | $\gamma$           | 1.276*       | 0.608*       | 1.049*       | 4.411*       | 7.614*       |
|                                    | $\delta$           | 1.665*       | 1.587*       | 2.740*       | 2.328*       | 4.019*       |

\*estimated



### 5.4. Fitting $L_U$ Distributions

Moments: Table 4 gives values of  $\delta$  and  $\Omega (= \gamma/\delta)$ , corresponding to  $\sqrt{\beta_1} = 0.00(0.05)1(.1)2.0$  combined (for each  $\sqrt{\beta_1}$ ) with twenty values of  $\beta_2$  increasing by intervals of 0.2, starting from a value just 'below' the  $L_L$  line. The simultaneous nonlinear equations

$$\begin{aligned}\sqrt{\beta_1(\gamma, \delta)} &= \sqrt{\beta_1} \\ \beta_2(\gamma, \delta) &= \beta_2\end{aligned}\tag{5.18}$$

were solved for  $\gamma$  and  $\delta$  using Brown's (1967) algorithm. The search for values of  $\Omega$  and  $\delta$  continued until  $|\sqrt{\beta_1(\gamma, \delta)} - \sqrt{\beta_1}| \leq 10^{-6}$  and  $|\beta_2(\gamma, \delta) - \beta_2| \leq 10^{-6}$ . The use of the table is illustrated by the following example.

Example 3. We will compare a standardized Pearson Type IV curve with  $\sqrt{\beta_1} = 0.9$ ,  $\beta_2 = 8.6$  against an  $L_U$  curve having the same first four moments.

From Table 4 we obtain  $\delta = 6.0151$  and  $\Omega = -0.5250$ , where  $\gamma = 6.0151 \times (-0.5250) = -3.1580$ . Using (4.7) we obtain

$$E[Y] = 0.5753; \quad \text{S.D.}[Y] = 0.3712.$$

Using (5.4) (with 'observed' mean zero, and standard deviation 1) we obtain  $\lambda = (0.3712)^{-1} = 2.6940$ ,  $\xi = 0 - (2.6940 \times 0.5753) = -1.5498$ . So the fitted  $L_U$  curve is defined by the statement that

$$Z = -3.1580 + 6.0151 \sinh^{-1} \{(X + 1.5498/2.6940)\}$$



Table 4: To Facilitate Fitting  $L_U$  Curves

| $\sqrt{\beta_1} = 0.00$ |          |          |        |          | $\sqrt{\beta_1} = 0.05$ |          |          |        |          |
|-------------------------|----------|----------|--------|----------|-------------------------|----------|----------|--------|----------|
| $\beta_2$               | $\delta$ | $\Omega$ | $\mu$  | $\sigma$ | $\beta_2$               | $\delta$ | $\Omega$ | $\mu$  | $\sigma$ |
| 4.40                    | 16.1153  | 0.0000   | 0.0000 | 0.1136   | 4.40                    | 16.2968  | -0.0909  | 0.0916 | 0.1127   |
| 4.60                    | 11.7442  | 0.0000   | 0.0000 | 0.1571   | 4.60                    | 11.8071  | -0.0638  | 0.0646 | 0.1565   |
| 4.80                    | 9.8648   | 0.0000   | 0.0000 | 0.1884   | 4.80                    | 9.8988   | -0.0520  | 0.0529 | 0.1879   |
| 5.00                    | 8.7752   | 0.0000   | 0.0000 | 0.2131   | 5.00                    | 8.7974   | -0.0451  | 0.0461 | 0.2128   |
| 5.20                    | 8.0510   | 0.0000   | 0.0000 | 0.2337   | 5.20                    | 8.0668   | -0.0404  | 0.0414 | 0.2334   |
| 5.40                    | 7.5295   | 0.0000   | 0.0000 | 0.2512   | 5.40                    | 7.5415   | -0.0369  | 0.0380 | 0.2510   |
| 5.60                    | 7.1336   | 0.0000   | 0.0000 | 0.2665   | 5.60                    | 7.1432   | -0.0343  | 0.0354 | 0.2663   |
| 5.80                    | 6.8216   | 0.0000   | 0.0000 | 0.2800   | 5.80                    | 6.8294   | -0.0322  | 0.0333 | 0.2798   |
| 6.00                    | 6.5687   | 0.0000   | 0.0000 | 0.2920   | 6.00                    | 6.5752   | -0.0304  | 0.0316 | 0.2918   |
| 6.20                    | 6.3590   | 0.0000   | 0.0000 | 0.3028   | 6.20                    | 6.3647   | -0.0290  | 0.0302 | 0.3026   |
| 6.40                    | 6.1822   | 0.0000   | 0.0000 | 0.3126   | 6.40                    | 6.1870   | -0.0277  | 0.0289 | 0.3124   |
| 6.60                    | 6.0308   | 0.0000   | 0.0000 | 0.3215   | 6.60                    | 6.0351   | -0.0266  | 0.0279 | 0.3214   |
| 6.80                    | 5.8998   | 0.0000   | 0.0000 | 0.3297   | 6.80                    | 5.9036   | -0.0257  | 0.0270 | 0.3296   |
| 7.00                    | 5.7850   | 0.0000   | 0.0000 | 0.3373   | 7.00                    | 5.7884   | -0.0249  | 0.0262 | 0.3372   |
| 7.20                    | 5.6837   | 0.0000   | 0.0000 | 0.3442   | 7.20                    | 5.6868   | -0.0241  | 0.0254 | 0.3441   |
| 7.40                    | 5.5935   | 0.0000   | 0.0000 | 0.3507   | 7.40                    | 5.5963   | -0.0235  | 0.0248 | 0.3506   |
| 7.60                    | 5.5127   | 0.0000   | 0.0000 | 0.3567   | 7.60                    | 5.5153   | -0.0229  | 0.0242 | 0.3566   |
| 7.80                    | 5.4398   | 0.0000   | 0.0000 | 0.3623   | 7.80                    | 5.4422   | -0.0223  | 0.0236 | 0.3623   |
| 8.00                    | 5.3738   | 0.0000   | 0.0000 | 0.3676   | 8.00                    | 5.3760   | -0.0218  | 0.0232 | 0.3675   |
| 8.20                    | 5.3137   | 0.0000   | 0.0000 | 0.3725   | 8.20                    | 5.3157   | -0.0214  | 0.0227 | 0.3725   |

| $\sqrt{\beta_1} = 0.10$ |          |          |        |          | $\sqrt{\beta_1} = 0.15$ |          |          |        |          |
|-------------------------|----------|----------|--------|----------|-------------------------|----------|----------|--------|----------|
| $\beta_2$               | $\delta$ | $\Omega$ | $\mu$  | $\sigma$ | $\beta_2$               | $\delta$ | $\Omega$ | $\mu$  | $\sigma$ |
| 4.40                    | 16.8772  | -0.1905  | 0.1927 | 0.1103   | 4.40                    | 18.0083  | -0.3121  | 0.3188 | 0.1065   |
| 4.60                    | 12.0026  | -0.1306  | 0.1325 | 0.1549   | 4.60                    | 12.3522  | -0.2040  | 0.2076 | 0.1523   |
| 4.80                    | 10.0035  | -0.1057  | 0.1076 | 0.1867   | 4.80                    | 10.1867  | -0.1628  | 0.1661 | 0.1846   |
| 5.00                    | 8.8648   | -0.0912  | 0.0933 | 0.2118   | 5.00                    | 8.9813   | -0.1396  | 0.1430 | 0.2100   |
| 5.20                    | 8.1148   | -0.0815  | 0.0837 | 0.2325   | 5.20                    | 8.1972   | -0.1243  | 0.1277 | 0.2311   |
| 5.40                    | 7.5779   | -0.0745  | 0.0767 | 0.2502   | 5.40                    | 7.6401   | -0.1132  | 0.1167 | 0.2489   |
| 5.60                    | 7.1720   | -0.0690  | 0.0713 | 0.2656   | 5.60                    | 7.2211   | -0.1048  | 0.1094 | 0.2645   |
| 5.80                    | 6.8530   | -0.0647  | 0.0671 | 0.2792   | 5.80                    | 6.8931   | -0.0981  | 0.1017 | 0.2782   |
| 6.00                    | 6.5950   | -0.0612  | 0.0636 | 0.2913   | 6.00                    | 6.6285   | -0.0926  | 0.0963 | 0.2904   |
| 6.20                    | 6.3815   | -0.0582  | 0.0607 | 0.3021   | 6.20                    | 6.4102   | -0.0881  | 0.0918 | 0.3013   |
| 6.40                    | 6.2017   | -0.0557  | 0.0582 | 0.3120   | 6.40                    | 6.2266   | -0.0842  | 0.0880 | 0.3113   |
| 6.60                    | 6.0481   | -0.0535  | 0.0560 | 0.3210   | 6.60                    | 6.0698   | -0.0809  | 0.0847 | 0.3203   |
| 6.80                    | 5.9150   | -0.0516  | 0.0542 | 0.3292   | 6.80                    | 5.9344   | -0.0779  | 0.0818 | 0.3286   |
| 7.00                    | 5.7987   | -0.0499  | 0.0525 | 0.3368   | 7.00                    | 5.8161   | -0.0754  | 0.0793 | 0.3362   |
| 7.20                    | 5.6961   | -0.0485  | 0.0510 | 0.3438   | 7.20                    | 5.7118   | -0.0731  | 0.0770 | 0.3433   |
| 7.40                    | 5.6048   | -0.0471  | 0.0497 | 0.3503   | 7.40                    | 5.6191   | -0.0711  | 0.0750 | 0.3498   |
| 7.60                    | 5.5230   | -0.0459  | 0.0485 | 0.3564   | 7.60                    | 5.5361   | -0.0692  | 0.0732 | 0.3559   |
| 7.80                    | 5.4494   | -0.0448  | 0.0474 | 0.3620   | 7.80                    | 5.4614   | -0.0676  | 0.0715 | 0.3616   |
| 8.00                    | 5.3826   | -0.0438  | 0.0464 | 0.3673   | 8.00                    | 5.3938   | -0.0661  | 0.0700 | 0.3669   |
| 8.20                    | 5.3219   | -0.0429  | 0.0455 | 0.3722   | 8.20                    | 5.3322   | -0.0647  | 0.0686 | 0.3719   |

$\sqrt{\beta_1} = 0.20$ 

| $\beta_2$ | $\delta$ | $\Omega$ | $\mu$  | $\sigma$ |
|-----------|----------|----------|--------|----------|
| 4.40      | 20.0817  | -0.4873  | 0.5089 | 0.1019   |
| 4.60      | 12.9064  | -0.2896  | 0.2966 | 0.1486   |
| 4.80      | 10.4634  | -0.2259  | 0.2313 | 0.1817   |
| 5.00      | 9.1539   | -0.1917  | 0.1967 | 0.2077   |
| 5.20      | 8.3178   | -0.1696  | 0.1745 | 0.2290   |
| 5.40      | 7.7305   | -0.1539  | 0.1589 | 0.2472   |
| 5.60      | 7.2920   | -0.1421  | 0.1471 | 0.2629   |
| 5.80      | 6.9509   | -0.1327  | 0.1378 | 0.2768   |
| 6.00      | 6.6767   | -0.1252  | 0.1302 | 0.2891   |
| 6.20      | 6.4512   | -0.1189  | 0.1240 | 0.3002   |
| 6.40      | 6.2621   | -0.1135  | 0.1187 | 0.3102   |
| 6.60      | 6.1009   | -0.1089  | 0.1141 | 0.3194   |
| 6.80      | 5.9620   | -0.1049  | 0.1101 | 0.3277   |
| 7.00      | 5.8408   | -0.1014  | 0.1067 | 0.3354   |
| 7.20      | 5.7341   | -0.0983  | 0.1036 | 0.3425   |
| 7.40      | 5.6393   | -0.0955  | 0.1008 | 0.3491   |
| 7.60      | 5.5546   | -0.0930  | 0.0983 | 0.3552   |
| 7.80      | 5.4784   | -0.0907  | 0.0960 | 0.3610   |
| 8.00      | 5.4095   | -0.0887  | 0.0940 | 0.3663   |
| 8.20      | 5.3468   | -0.0868  | 0.0921 | 0.3713   |

 $\sqrt{\beta_1} = 0.25$ 

| $\beta_2$ | $\delta$ | $\Omega$ | $\mu$  | $\sigma$ |
|-----------|----------|----------|--------|----------|
| 4.40      | 24.2825  | -0.8445  | 0.9511 | 0.1035   |
| 4.60      | 13.7515  | -0.3975  | 0.4116 | 0.1444   |
| 4.80      | 10.8595  | -0.2985  | 0.3073 | 0.1782   |
| 5.00      | 9.3936   | -0.2494  | 0.2567 | 0.2047   |
| 5.20      | 8.4828   | -0.2188  | 0.2257 | 0.2265   |
| 5.40      | 7.8528   | -0.1975  | 0.2042 | 0.2450   |
| 5.60      | 7.3875   | -0.1816  | 0.1882 | 0.2609   |
| 5.80      | 7.0280   | -0.1692  | 0.1758 | 0.2750   |
| 6.00      | 6.7408   | -0.1592  | 0.1658 | 0.2875   |
| 6.20      | 6.5055   | -0.1509  | 0.1575 | 0.2987   |
| 6.40      | 6.3090   | -0.1439  | 0.1506 | 0.3089   |
| 6.60      | 6.1420   | -0.1380  | 0.1446 | 0.3181   |
| 6.80      | 5.9983   | -0.1328  | 0.1395 | 0.3266   |
| 7.00      | 5.8732   | -0.1282  | 0.1349 | 0.3344   |
| 7.20      | 5.7633   | -0.1242  | 0.1309 | 0.3416   |
| 7.40      | 5.6659   | -0.1206  | 0.1273 | 0.3482   |
| 7.60      | 5.5789   | -0.1174  | 0.1241 | 0.3544   |
| 7.80      | 5.5008   | -0.1145  | 0.1212 | 0.3602   |
| 8.00      | 5.4301   | -0.1118  | 0.1185 | 0.3656   |
| 8.20      | 5.3659   | -0.1094  | 0.1161 | 0.3707   |

 $\sqrt{\beta_1} = 0.30$ 

| $\beta_2$ | $\delta$ | $\Omega$ | $\mu$  | $\sigma$ |
|-----------|----------|----------|--------|----------|
| 4.60      | 15.0745  | -0.5508  | 0.5833 | 0.1407   |
| 4.80      | 11.4198  | -0.3870  | 0.4018 | 0.1742   |
| 5.00      | 9.7192   | -0.3157  | 0.3266 | 0.2012   |
| 5.20      | 8.7013   | -0.2736  | 0.2831 | 0.2234   |
| 5.40      | 8.0126   | -0.2451  | 0.2540 | 0.2423   |
| 5.60      | 7.5109   | -0.2242  | 0.2329 | 0.2586   |
| 5.80      | 7.1269   | -0.2081  | 0.2166 | 0.2729   |
| 6.00      | 6.8225   | -0.1953  | 0.2037 | 0.2856   |
| 6.20      | 6.5745   | -0.1847  | 0.1931 | 0.2970   |
| 6.40      | 6.3683   | -0.1759  | 0.1842 | 0.3073   |
| 6.60      | 6.1938   | -0.1683  | 0.1766 | 0.3167   |
| 6.80      | 6.0440   | -0.1618  | 0.1701 | 0.3252   |
| 7.00      | 5.9140   | -0.1561  | 0.1643 | 0.3331   |
| 7.20      | 5.8000   | -0.1510  | 0.1593 | 0.3404   |
| 7.40      | 5.6992   | -0.1466  | 0.1548 | 0.3472   |
| 7.60      | 5.6093   | -0.1425  | 0.1508 | 0.3534   |
| 7.80      | 5.5286   | -0.1389  | 0.1471 | 0.3593   |
| 8.00      | 5.4558   | -0.1356  | 0.1438 | 0.3647   |
| 8.20      | 5.3898   | -0.1326  | 0.1408 | 0.3699   |
| 8.40      | 5.3295   | -0.1299  | 0.1381 | 0.3747   |

 $\sqrt{\beta_1} = 0.35$ 

| $\beta_2$ | $\delta$ | $\Omega$ | $\mu$  | $\sigma$ |
|-----------|----------|----------|--------|----------|
| 4.60      | 17.3274  | -0.8266  | 0.9291 | 0.1439   |
| 4.80      | 12.2247  | -0.5035  | 0.5309 | 0.1705   |
| 5.00      | 10.1592  | -0.3954  | 0.4123 | 0.1975   |
| 5.20      | 8.9876   | -0.3366  | 0.3500 | 0.2200   |
| 5.40      | 8.2178   | -0.2984  | 0.3103 | 0.2393   |
| 5.60      | 7.6672   | -0.2711  | 0.2823 | 0.2559   |
| 5.80      | 7.2512   | -0.2505  | 0.2612 | 0.2704   |
| 6.00      | 6.9244   | -0.2341  | 0.2446 | 0.2834   |
| 6.20      | 6.6602   | -0.2208  | 0.2311 | 0.2950   |
| 6.40      | 6.4416   | -0.2098  | 0.2199 | 0.3055   |
| 6.60      | 6.2575   | -0.2004  | 0.2195 | 0.3150   |
| 6.80      | 6.1001   | -0.1923  | 0.2023 | 0.3237   |
| 7.00      | 5.9638   | -0.1853  | 0.1953 | 0.3317   |
| 7.20      | 5.8448   | -0.1791  | 0.1890 | 0.3391   |
| 7.40      | 5.7396   | -0.1736  | 0.1835 | 0.3459   |
| 7.60      | 5.6461   | -0.1687  | 0.1786 | 0.3523   |
| 7.80      | 5.5624   | -0.1643  | 0.1741 | 0.3582   |
| 8.00      | 5.4869   | -0.1603  | 0.1701 | 0.3637   |
| 8.20      | 5.4186   | -0.1566  | 0.1664 | 0.3689   |
| 8.40      | 5.3563   | -0.1533  | 0.1631 | 0.3738   |

$\sqrt{\beta_1} = 0.40$ 

| $\beta_2$ | $\delta$ | $\Omega$ | $\mu$  | $\sigma$ |
|-----------|----------|----------|--------|----------|
| 4.80      | 13.4343  | -0.6780  | 0.7379 | 0.1699   |
| 5.00      | 10.7623  | -0.4972  | 0.5254 | 0.1941   |
| 5.20      | 9.3632   | -0.4118  | 0.4316 | 0.2166   |
| 5.40      | 8.4802   | -0.3597  | 0.3761 | 0.2360   |
| 5.60      | 7.8638   | -0.3238  | 0.3385 | 0.2529   |
| 5.80      | 7.4056   | -0.2972  | 0.3109 | 0.2677   |
| 6.00      | 7.0499   | -0.2765  | 0.2896 | 0.2809   |
| 6.20      | 6.7648   | -0.2599  | 0.2726 | 0.2927   |
| 6.40      | 6.5307   | -0.2462  | 0.2585 | 0.3034   |
| 6.60      | 6.3345   | -0.2346  | 0.2468 | 0.3131   |
| 6.80      | 6.1676   | -0.2248  | 0.2368 | 0.3219   |
| 7.00      | 6.0238   | -0.2162  | 0.2281 | 0.3301   |
| 7.20      | 5.8984   | -0.2087  | 0.2205 | 0.3376   |
| 7.40      | 5.7880   | -0.2021  | 0.2138 | 0.3445   |
| 7.60      | 5.6901   | -0.1961  | 0.2078 | 0.3510   |
| 7.80      | 5.6026   | -0.1908  | 0.2024 | 0.3570   |
| 8.00      | 5.5239   | -0.1860  | 0.1976 | 0.3626   |
| 8.20      | 5.4527   | -0.1816  | 0.1931 | 0.3679   |
| 8.40      | 5.3881   | -0.1776  | 0.1891 | 0.3728   |
| 8.60      | 5.3290   | -0.1740  | 0.1854 | 0.3775   |

 $\sqrt{\beta_1} = 0.45$ 

| $\beta_2$ | $\delta$ | $\Omega$ | $\mu$  | $\sigma$ |
|-----------|----------|----------|--------|----------|
| 4.80      | 15.4222  | -1.0249  | 1.2225 | 0.1874   |
| 5.00      | 11.6174  | -0.6403  | 0.6934 | 0.1932   |
| 5.20      | 9.8638   | -0.5068  | 0.5378 | 0.2137   |
| 5.40      | 8.8176   | -0.4330  | 0.4563 | 0.2329   |
| 5.60      | 8.1110   | -0.3848  | 0.4044 | 0.2499   |
| 5.80      | 7.5968   | -0.3501  | 0.3677 | 0.2649   |
| 6.00      | 7.2035   | -0.3238  | 0.3401 | 0.2783   |
| 6.20      | 6.8918   | -0.3029  | 0.3185 | 0.2903   |
| 6.40      | 6.6380   | -0.2859  | 0.3009 | 0.3012   |
| 6.60      | 6.4269   | -0.2717  | 0.2863 | 0.3110   |
| 6.80      | 6.2482   | -0.2596  | 0.2739 | 0.3200   |
| 7.00      | 6.0950   | -0.2492  | 0.2633 | 0.3283   |
| 7.20      | 5.9619   | -0.2402  | 0.2541 | 0.3359   |
| 7.40      | 5.8452   | -0.2322  | 0.2459 | 0.3430   |
| 7.60      | 5.7419   | -0.2251  | 0.2387 | 0.3496   |
| 7.80      | 5.6499   | -0.2187  | 0.2323 | 0.3557   |
| 8.00      | 5.5673   | -0.2130  | 0.2264 | 0.3614   |
| 8.20      | 5.4928   | -0.2078  | 0.2212 | 0.3667   |
| 8.40      | 5.4252   | -0.2031  | 0.2164 | 0.3717   |
| 8.60      | 5.3635   | -0.1988  | 0.2120 | 0.3764   |

 $\sqrt{\beta_1} = 0.50$ 

| $\beta_2$ | $\delta$ | $\Omega$ | $\mu$  | $\sigma$ |
|-----------|----------|----------|--------|----------|
| 5.00      | 12.9053  | -0.8811  | 1.0095 | 0.2023   |
| 5.20      | 10.5493  | -0.6366  | 0.6906 | 0.2132   |
| 5.40      | 9.2579   | -0.5250  | 0.5602 | 0.2306   |
| 5.60      | 8.4243   | -0.4577  | 0.4850 | 0.2472   |
| 5.80      | 7.8343   | -0.4116  | 0.4348 | 0.2622   |
| 6.00      | 7.3916   | -0.3775  | 0.3984 | 0.2757   |
| 6.20      | 7.0457   | -0.3510  | 0.3704 | 0.2879   |
| 6.40      | 6.7669   | -0.3298  | 0.3481 | 0.2989   |
| 6.60      | 6.5370   | -0.3122  | 0.3299 | 0.3089   |
| 6.80      | 6.3438   | -0.2975  | 0.3146 | 0.3180   |
| 7.00      | 6.1789   | -0.2849  | 0.3016 | 0.3265   |
| 7.20      | 6.0365   | -0.2740  | 0.2903 | 0.3342   |
| 7.40      | 5.9121   | -0.2644  | 0.2805 | 0.3414   |
| 7.60      | 5.8025   | -0.2559  | 0.2718 | 0.3480   |
| 7.80      | 5.7050   | -0.2484  | 0.2641 | 0.3543   |
| 8.00      | 5.6178   | -0.2416  | 0.2571 | 0.3601   |
| 8.20      | 5.5393   | -0.2355  | 0.2509 | 0.3655   |
| 8.40      | 5.4681   | -0.2299  | 0.2452 | 0.3706   |
| 8.60      | 5.4034   | -0.2248  | 0.2400 | 0.3754   |
| 8.80      | 5.3443   | -0.2201  | 0.2352 | 0.3799   |

 $\sqrt{\beta_1} = 0.55$ 

| $\beta_2$ | $\delta$ | $\Omega$ | $\mu$  | $\sigma$ |
|-----------|----------|----------|--------|----------|
| 5.00      | 15.0669  | -1.6001  | 2.3931 | 0.3150   |
| 5.20      | 11.5340  | -0.8409  | 0.9553 | 0.2211   |
| 5.40      | 9.8481   | -0.6492  | 0.7078 | 0.2309   |
| 5.60      | 8.8274   | -0.5493  | 0.5897 | 0.2456   |
| 5.80      | 8.1321   | -0.4854  | 0.5174 | 0.2600   |
| 6.00      | 7.6233   | -0.4402  | 0.4676 | 0.2734   |
| 6.20      | 7.2325   | -0.4061  | 0.4307 | 0.2855   |
| 6.40      | 6.9219   | -0.3792  | 0.4020 | 0.2966   |
| 6.60      | 6.6683   | -0.3574  | 0.3789 | 0.3067   |
| 6.80      | 6.4570   | -0.3393  | 0.3598 | 0.3160   |
| 7.00      | 6.2779   | -0.3239  | 0.3438 | 0.3246   |
| 7.20      | 6.1240   | -0.3107  | 0.3300 | 0.3324   |
| 7.40      | 5.9902   | -0.2992  | 0.3181 | 0.3397   |
| 7.60      | 5.8728   | -0.2891  | 0.3076 | 0.3465   |
| 7.80      | 5.7689   | -0.2801  | 0.2983 | 0.3528   |
| 8.00      | 5.6761   | -0.2721  | 0.2900 | 0.3587   |
| 8.20      | 5.5928   | -0.2648  | 0.2826 | 0.3642   |
| 8.40      | 5.5176   | -0.2583  | 0.2758 | 0.3694   |
| 8.60      | 5.4493   | -0.2523  | 0.2697 | 0.3742   |
| 8.80      | 5.3870   | -0.2469  | 0.2641 | 0.3788   |



$\sqrt{B_1} = 0.60$ 

| $B_2$ | $\delta$ | $\Omega$ | $\mu$  | $\sigma$ |
|-------|----------|----------|--------|----------|
| 5.20  | 13.0664  | -1.3000  | 1.7148 | 0.2789   |
| 5.40  | 10.6714  | -0.8392  | 0.9550 | 0.2396   |
| 5.60  | 9.3592   | -0.6725  | 0.7381 | 0.2471   |
| 5.80  | 8.5111   | -0.5784  | 0.6252 | 0.2593   |
| 6.00  | 7.9114   | -0.5160  | 0.5536 | 0.2719   |
| 6.20  | 7.4611   | -0.4708  | 0.5031 | 0.2837   |
| 6.40  | 7.1090   | -0.4362  | 0.4651 | 0.2947   |
| 6.60  | 6.8252   | -0.4086  | 0.4353 | 0.3048   |
| 6.80  | 6.5911   | -0.3861  | 0.4111 | 0.3141   |
| 7.00  | 6.3943   | -0.3672  | 0.3910 | 0.3228   |
| 7.20  | 6.2264   | -0.3511  | 0.3740 | 0.3307   |
| 7.40  | 6.0813   | -0.3372  | 0.3595 | 0.3381   |
| 7.60  | 5.9545   | -0.3251  | 0.3467 | 0.3450   |
| 7.80  | 5.8427   | -0.3144  | 0.3355 | 0.3513   |
| 8.00  | 5.7433   | -0.3049  | 0.3256 | 0.3573   |
| 8.20  | 5.6543   | -0.2963  | 0.3168 | 0.3629   |
| 8.40  | 5.5742   | -0.2886  | 0.3087 | 0.3682   |
| 8.60  | 5.5017   | -0.2816  | 0.3015 | 0.3731   |
| 8.80  | 5.4357   | -0.2753  | 0.2949 | 0.3778   |
| 9.00  | 5.3753   | -0.2694  | 0.2888 | 0.3822   |

 $\sqrt{B_1} = 0.65$ 

| $B_2$ | $\delta$ | $\Omega$ | $\mu$  | $\sigma$ |
|-------|----------|----------|--------|----------|
| 5.40  | 11.8992  | -1.2277  | 1.5784 | 0.2891   |
| 5.60  | 10.0875  | -0.8593  | 0.9848 | 0.2578   |
| 5.80  | 9.0064   | -0.7040  | 0.7793 | 0.2624   |
| 6.00  | 8.2756   | -0.6121  | 0.6670 | 0.2723   |
| 6.20  | 7.7438   | -0.5496  | 0.5939 | 0.2830   |
| 6.40  | 7.3367   | -0.5037  | 0.5416 | 0.2935   |
| 6.60  | 7.0138   | -0.4681  | 0.5020 | 0.3034   |
| 6.80  | 6.7508   | -0.4396  | 0.4707 | 0.3126   |
| 7.00  | 6.5318   | -0.4161  | 0.4451 | 0.3212   |
| 7.20  | 6.3464   | -0.3963  | 0.4238 | 0.3292   |
| 7.40  | 6.1873   | -0.3794  | 0.4058 | 0.3366   |
| 7.60  | 6.0492   | -0.3647  | 0.3902 | 0.3435   |
| 7.80  | 5.9279   | -0.3519  | 0.3766 | 0.3500   |
| 8.00  | 5.8206   | -0.3406  | 0.3646 | 0.3560   |
| 8.20  | 5.7249   | -0.3304  | 0.3540 | 0.3617   |
| 8.40  | 5.6390   | -0.3213  | 0.3444 | 0.3670   |
| 8.60  | 5.5615   | -0.3131  | 0.3358 | 0.3720   |
| 8.80  | 5.4911   | -0.3056  | 0.3290 | 0.3768   |
| 9.00  | 5.4269   | -0.2988  | 0.3239 | 0.3812   |
| 9.20  | 5.3681   | -0.2926  | 0.3144 | 0.3854   |

 $\sqrt{B_1} = 0.70$ 

| $B_2$ | $\delta$ | $\Omega$ | $\mu$  | $\sigma$ |
|-------|----------|----------|--------|----------|
| 5.60  | 11.1440  | -1.2297  | 1.5848 | 0.3102   |
| 5.80  | 9.6753   | -0.8950  | 1.0375 | 0.2765   |
| 6.00  | 8.7476   | -0.7431  | 0.8311 | 0.2776   |
| 6.20  | 8.0996   | -0.6506  | 0.7153 | 0.2850   |
| 6.40  | 7.6174   | -0.5867  | 0.6389 | 0.2939   |
| 6.60  | 7.2427   | -0.5392  | 0.5838 | 0.3030   |
| 6.80  | 6.9420   | -0.5022  | 0.5419 | 0.3118   |
| 7.00  | 6.6949   | -0.4723  | 0.5086 | 0.3202   |
| 7.20  | 6.4878   | -0.4476  | 0.4813 | 0.3281   |
| 7.40  | 6.3114   | -0.4268  | 0.4586 | 0.3355   |
| 7.60  | 6.1592   | -0.4089  | 0.4392 | 0.3424   |
| 7.80  | 6.0265   | -0.3934  | 0.4225 | 0.3489   |
| 8.00  | 5.9096   | -0.3798  | 0.4079 | 0.3549   |
| 8.20  | 5.8059   | -0.3677  | 0.3950 | 0.3606   |
| 8.40  | 5.7131   | -0.3569  | 0.3836 | 0.3660   |
| 8.60  | 5.6296   | -0.3472  | 0.3733 | 0.3711   |
| 8.80  | 5.5541   | -0.3384  | 0.3640 | 0.3758   |
| 9.00  | 5.4854   | -0.3304  | 0.3556 | 0.3804   |
| 9.20  | 5.4226   | -0.3231  | 0.3479 | 0.3846   |
| 9.40  | 5.3650   | -0.3164  | 0.3408 | 0.3887   |

 $\sqrt{B_1} = 0.75$ 

| $B_2$ | $\delta$ | $\Omega$ | $\mu$  | $\sigma$ |
|-------|----------|----------|--------|----------|
| 5.80  | 10.6310  | -1.2778  | 1.6793 | 0.3397   |
| 6.00  | 9.3805   | -0.9454  | 1.1133 | 0.2966   |
| 6.20  | 8.5581   | -0.7901  | 0.8948 | 0.2931   |
| 6.40  | 7.9694   | -0.6943  | 0.7713 | 0.2978   |
| 6.60  | 7.5240   | -0.6277  | 0.6896 | 0.3048   |
| 6.80  | 7.1737   | -0.5778  | 0.6305 | 0.3125   |
| 7.00  | 6.8900   | -0.5388  | 0.5854 | 0.3203   |
| 7.20  | 6.6551   | -0.5073  | 0.5495 | 0.3278   |
| 7.40  | 6.4571   | -0.4811  | 0.5202 | 0.3349   |
| 7.60  | 6.2876   | -0.4590  | 0.4957 | 0.3417   |
| 7.80  | 6.1409   | -0.4400  | 0.4748 | 0.3481   |
| 8.00  | 6.0124   | -0.4235  | 0.4567 | 0.3542   |
| 8.20  | 5.8989   | -0.4089  | 0.4410 | 0.3599   |
| 8.40  | 5.7979   | -0.3960  | 0.4270 | 0.3652   |
| 8.60  | 5.7073   | -0.3845  | 0.4147 | 0.3703   |
| 8.80  | 5.6257   | -0.3741  | 0.4035 | 0.3751   |
| 9.00  | 5.5517   | -0.3647  | 0.3935 | 0.3797   |
| 9.20  | 5.4842   | -0.3561  | 0.3844 | 0.3839   |
| 9.40  | 5.4225   | -0.3483  | 0.3761 | 0.3880   |
| 9.60  | 5.3659   | -0.3411  | 0.3684 | 0.3919   |



$\sqrt{\beta_1} = 0.80$ 

| $\beta_2$ | $\delta$ | $\Omega$ | $\mu$  | $\sigma$ |
|-----------|----------|----------|--------|----------|
| 6.00      | 10.2742  | -1.3685  | 1.8665 | 0.3813   |
| 6.20      | 9.1702   | -1.0114  | 1.2165 | 0.3196   |
| 6.40      | 8.4219   | -0.8461  | 0.9731 | 0.3100   |
| 6.60      | 7.8763   | -0.7443  | 0.8369 | 0.3113   |
| 6.80      | 7.4581   | -0.6733  | 0.7472 | 0.3162   |
| 7.00      | 7.1260   | -0.6202  | 0.6826 | 0.3224   |
| 7.20      | 6.8551   | -0.5786  | 0.6333 | 0.3289   |
| 7.40      | 6.6295   | -0.5449  | 0.5943 | 0.3355   |
| 7.60      | 6.4384   | -0.5169  | 0.5623 | 0.3419   |
| 7.80      | 6.2741   | -0.4933  | 0.5356 | 0.3481   |
| 8.00      | 6.1314   | -0.4729  | 0.5129 | 0.3540   |
| 8.20      | 6.0061   | -0.4552  | 0.4933 | 0.3595   |
| 8.40      | 5.8951   | -0.4396  | 0.4761 | 0.3649   |
| 8.60      | 5.7961   | -0.4258  | 0.4610 | 0.3699   |
| 8.80      | 5.7072   | -0.4134  | 0.4475 | 0.3747   |
| 9.00      | 5.6269   | -0.4023  | 0.4355 | 0.3792   |
| 9.20      | 5.5540   | -0.3922  | 0.4246 | 0.3835   |
| 9.40      | 5.4875   | -0.3830  | 0.4147 | 0.3876   |
| 9.60      | 5.4265   | -0.3745  | 0.4056 | 0.3915   |
| 9.80      | 5.3704   | -0.3668  | 0.3973 | 0.3952   |

 $\sqrt{\beta_1} = 0.85$ 

| $\beta_2$ | $\delta$ | $\Omega$ | $\mu$  | $\sigma$ |
|-----------|----------|----------|--------|----------|
| 6.20      | 10.0236  | -1.5141  | 2.1995 | 0.4462   |
| 6.40      | 9.0243   | -1.0969  | 1.3577 | 0.3479   |
| 6.60      | 8.3289   | -0.9132  | 1.0707 | 0.3293   |
| 6.80      | 7.8142   | -0.8017  | 0.9149 | 0.3263   |
| 7.00      | 7.4158   | -0.7246  | 0.8138 | 0.3285   |
| 7.20      | 7.0971   | -0.6670  | 0.7416 | 0.3329   |
| 7.40      | 6.8355   | -0.6221  | 0.6869 | 0.3381   |
| 7.60      | 6.6166   | -0.5857  | 0.6436 | 0.3437   |
| 7.80      | 6.4305   | -0.5555  | 0.6084 | 0.3493   |
| 8.00      | 6.2700   | -0.5300  | 0.5790 | 0.3547   |
| 8.20      | 6.1302   | -0.5080  | 0.5541 | 0.3601   |
| 8.40      | 6.0071   | -0.4889  | 0.5326 | 0.3652   |
| 8.60      | 5.8979   | -0.4721  | 0.5138 | 0.3701   |
| 8.80      | 5.8003   | -0.4572  | 0.4973 | 0.3748   |
| 9.00      | 5.7125   | -0.4439  | 0.4826 | 0.3793   |
| 9.20      | 5.6330   | -0.4315  | 0.4694 | 0.3835   |
| 9.40      | 5.5608   | -0.4210  | 0.4575 | 0.3876   |
| 9.60      | 5.4948   | -0.4111  | 0.4467 | 0.3915   |
| 9.80      | 5.4343   | -0.4020  | 0.4368 | 0.3952   |
| 10.00     | 5.3786   | -0.3936  | 0.4278 | 0.3987   |

 $\sqrt{\beta_1} = 0.90$ 

| $\beta_2$ | $\delta$ | $\Omega$ | $\mu$  | $\sigma$ |
|-----------|----------|----------|--------|----------|
| 6.40      | 9.8674   | -1.7795  | 2.9282 | 0.5809   |
| 6.60      | 8.9309   | -1.2101  | 1.5599 | 0.3865   |
| 6.80      | 8.2718   | -0.9950  | 1.1960 | 0.3529   |
| 7.00      | 7.7789   | -0.8689  | 1.0096 | 0.3437   |
| 7.20      | 7.3944   | -0.7830  | 0.8920 | 0.3424   |
| 7.40      | 7.0849   | -0.7195  | 0.8094 | 0.3445   |
| 7.60      | 6.8298   | -0.6701  | 0.7475 | 0.3482   |
| 7.80      | 6.6155   | -0.6303  | 0.6988 | 0.3525   |
| 8.00      | 6.4327   | -0.5974  | 0.6594 | 0.3572   |
| 8.20      | 6.2747   | -0.5696  | 0.6267 | 0.3619   |
| 8.40      | 6.1367   | -0.5457  | 0.5991 | 0.3666   |
| 8.60      | 6.0151   | -0.5250  | 0.5753 | 0.3712   |
| 8.80      | 5.9069   | -0.5068  | 0.5546 | 0.3756   |
| 9.00      | 5.8101   | -0.4906  | 0.5363 | 0.3800   |
| 9.20      | 5.7229   | -0.4762  | 0.5201 | 0.3841   |
| 9.40      | 5.6439   | -0.4632  | 0.5056 | 0.3881   |
| 9.60      | 5.5720   | -0.4514  | 0.4926 | 0.3919   |
| 9.80      | 5.5063   | -0.4407  | 0.4807 | 0.3955   |
| 10.00     | 5.4459   | -0.4309  | 0.4700 | 0.3990   |
| 10.20     | 5.3903   | -0.4218  | 0.4601 | 0.4024   |

 $\sqrt{\beta_1} = 0.95$ 

| $\beta_2$ | $\delta$ | $\Omega$ | $\mu$  | $\sigma$ |
|-----------|----------|----------|--------|----------|
| 6.60      | 9.7701   | -2.5098  | 6.2171 | 1.1941   |
| 6.80      | 8.8814   | -1.3688  | 1.8770 | 0.4461   |
| 7.00      | 8.2464   | -1.0976  | 1.3644 | 0.3840   |
| 7.20      | 7.7676   | -0.9490  | 1.1285 | 0.3651   |
| 7.40      | 7.3920   | -0.8506  | 0.9863 | 0.3590   |
| 7.60      | 7.0883   | -0.7790  | 0.8890 | 0.3580   |
| 7.80      | 6.8371   | -0.7238  | 0.8172 | 0.3596   |
| 8.00      | 6.6255   | -0.6797  | 0.7615 | 0.3625   |
| 8.20      | 6.4446   | -0.6434  | 0.7167 | 0.3660   |
| 8.40      | 6.2879   | -0.6128  | 0.6798 | 0.3698   |
| 8.60      | 6.1508   | -0.5866  | 0.6497 | 0.3738   |
| 8.80      | 6.0297   | -0.5640  | 0.6221 | 0.3778   |
| 9.00      | 5.9220   | -0.5441  | 0.5990 | 0.3817   |
| 9.20      | 5.8254   | -0.5264  | 0.5787 | 0.3856   |
| 9.40      | 5.7384   | -0.5107  | 0.5608 | 0.3894   |
| 9.60      | 5.6594   | -0.4965  | 0.5447 | 0.3930   |
| 9.80      | 5.5875   | -0.4837  | 0.5303 | 0.3965   |
| 10.00     | 5.5217   | -0.4720  | 0.5172 | 0.3999   |
| 10.20     | 5.4613   | -0.4614  | 0.5053 | 0.4032   |
| 10.40     | 5.4055   | -0.4516  | 0.4945 | 0.4063   |

$\sqrt{B_1} = 1.00$ 

| $\beta_2$ | $\delta$ | $\Omega$ | $\mu$  | $\sigma$ |
|-----------|----------|----------|--------|----------|
| 7.00      | 8.8717   | -1.6214  | 2.4828 | 0.5618   |
| 7.20      | 8.2499   | -1.2330  | 1.6086 | 0.4297   |
| 7.40      | 7.7787   | -1.0475  | 1.2845 | 0.3934   |
| 7.60      | 7.4075   | -0.9306  | 1.1037 | 0.3796   |
| 7.80      | 7.1065   | -0.8477  | 0.9847 | 0.3744   |
| 8.00      | 6.8569   | -0.7849  | 0.8991 | 0.3731   |
| 8.20      | 6.6463   | -0.7351  | 0.8338 | 0.3740   |
| 8.40      | 6.4658   | -0.6944  | 0.7819 | 0.3761   |
| 8.60      | 6.3093   | -0.6603  | 0.7396 | 0.3788   |
| 8.80      | 6.1722   | -0.6313  | 0.7041 | 0.3819   |
| 9.00      | 6.0510   | -0.6063  | 0.6740 | 0.3852   |
| 9.20      | 5.9431   | -0.5843  | 0.6479 | 0.3885   |
| 9.40      | 5.8462   | -0.5650  | 0.6252 | 0.3919   |
| 9.60      | 5.7588   | -0.5477  | 0.6051 | 0.3952   |
| 9.80      | 5.6795   | -0.5322  | 0.5872 | 0.3985   |
| 10.00     | 5.6073   | -0.5182  | 0.5711 | 0.4017   |
| 10.20     | 5.5411   | -0.5055  | 0.5566 | 0.4048   |
| 10.40     | 5.4803   | -0.4938  | 0.5434 | 0.4078   |
| 10.60     | 5.4242   | -0.4831  | 0.5314 | 0.4107   |
| 10.80     | 5.3722   | -0.4733  | 0.5203 | 0.4135   |

 $\sqrt{B_1} = 1.10$ 

| $\beta_2$ | $\delta$ | $\Omega$ | $\mu$  | $\sigma$ |
|-----------|----------|----------|--------|----------|
| 7.60      | 8.3382   | -1.7704  | 2.9200 | 0.6917   |
| 7.80      | 7.8646   | -1.3511  | 1.8503 | 0.5020   |
| 8.00      | 7.4903   | -1.1532  | 1.4690 | 0.4472   |
| 8.20      | 7.1861   | -1.0281  | 1.2587 | 0.4233   |
| 8.40      | 6.9333   | -0.9390  | 1.1212 | 0.4115   |
| 8.60      | 6.7194   | -0.8712  | 1.0225 | 0.4056   |
| 8.80      | 6.5360   | -0.8173  | 0.9474 | 0.4030   |
| 9.00      | 6.3767   | -0.7731  | 0.8879 | 0.4022   |
| 9.20      | 6.2370   | -0.7360  | 0.8393 | 0.4026   |
| 9.40      | 6.1134   | -0.7043  | 0.7987 | 0.4038   |
| 9.60      | 6.0032   | -0.6769  | 0.7642 | 0.4054   |
| 9.80      | 5.9043   | -0.6528  | 0.7344 | 0.4074   |
| 10.00     | 5.8149   | -0.6315  | 0.7083 | 0.4095   |
| 10.20     | 5.7337   | -0.6125  | 0.6853 | 0.4118   |
| 10.40     | 5.6597   | -0.5954  | 0.6648 | 0.4141   |
| 10.60     | 5.5919   | -0.5799  | 0.6464 | 0.4164   |
| 10.80     | 5.5296   | -0.5658  | 0.6298 | 0.4188   |
| 11.00     | 5.4720   | -0.5529  | 0.6147 | 0.4211   |
| 11.20     | 5.4187   | -0.5411  | 0.6010 | 0.4234   |
| 11.40     | 5.3692   | -0.5301  | 0.5883 | 0.4257   |

 $\sqrt{B_1} = 1.20$ 

| $\beta_2$ | $\delta$ | $\Omega$ | $\mu$  | $\sigma$ |
|-----------|----------|----------|--------|----------|
| 8.20      | 8.0376   | -2.4435  | 5.8615 | 1.3851   |
| 8.40      | 7.6442   | -1.5981  | 2.4388 | 0.6482   |
| 8.60      | 7.3250   | -1.3295  | 1.8123 | 0.5333   |
| 8.80      | 7.0601   | -1.1732  | 1.5108 | 0.4861   |
| 9.00      | 6.8364   | -1.0658  | 1.3256 | 0.4616   |
| 9.20      | 6.6446   | -0.9857  | 1.1973 | 0.4477   |
| 9.40      | 6.4783   | -0.9227  | 1.1021 | 0.4394   |
| 9.60      | 6.3323   | -0.8715  | 1.0277 | 0.4344   |
| 9.80      | 6.2033   | -0.8288  | 0.9678 | 0.4315   |
| 10.00     | 6.0883   | -0.7924  | 0.9182 | 0.4300   |
| 10.20     | 5.9851   | -0.7610  | 0.8763 | 0.4295   |
| 10.40     | 5.8919   | -0.7335  | 0.8404 | 0.4296   |
| 10.60     | 5.8073   | -0.7093  | 0.8091 | 0.4302   |
| 10.80     | 5.7301   | -0.6876  | 0.7816 | 0.4311   |
| 11.00     | 5.6595   | -0.6681  | 0.7572 | 0.4323   |
| 11.20     | 5.5945   | -0.6505  | 0.7354 | 0.4336   |
| 11.40     | 5.5345   | -0.6345  | 0.7158 | 0.4350   |
| 11.60     | 5.4790   | -0.6198  | 0.6979 | 0.4365   |
| 11.80     | 5.4274   | -0.6064  | 0.6817 | 0.4381   |
| 12.00     | 5.3794   | -0.5940  | 0.6668 | 0.4397   |

 $\sqrt{B_1} = 1.30$ 

| $\beta_2$ | $\delta$ | $\Omega$ | $\mu$  | $\sigma$ |
|-----------|----------|----------|--------|----------|
| 9.00      | 7.5308   | -2.3441  | 5.3169 | 1.3512   |
| 9.20      | 7.2437   | -1.6612  | 2.6191 | 0.7306   |
| 9.40      | 7.0022   | -1.4063  | 1.9838 | 0.6010   |
| 9.60      | 6.7958   | -1.2527  | 1.6658 | 0.5433   |
| 9.80      | 6.6173   | -1.1453  | 1.4671 | 0.5115   |
| 10.00     | 6.4612   | -1.0640  | 1.3282 | 0.4920   |
| 10.20     | 6.3234   | -0.9996  | 1.2243 | 0.4793   |
| 10.40     | 6.2008   | -0.9468  | 1.1429 | 0.4708   |
| 10.60     | 6.0909   | -0.9024  | 1.0771 | 0.4650   |
| 10.80     | 5.9918   | -0.8644  | 1.0224 | 0.4611   |
| 11.00     | 5.9019   | -0.8315  | 0.9761 | 0.4584   |
| 11.20     | 5.8201   | -0.8026  | 0.9353 | 0.4567   |
| 11.40     | 5.7452   | -0.7769  | 0.9017 | 0.4557   |
| 11.60     | 5.6763   | -0.7539  | 0.8712 | 0.4552   |
| 11.80     | 5.6129   | -0.7332  | 0.8440 | 0.4551   |
| 12.00     | 5.5541   | -0.7144  | 0.8198 | 0.4552   |
| 12.20     | 5.4996   | -0.6973  | 0.7978 | 0.4557   |
| 12.40     | 5.4489   | -0.6816  | 0.7790 | 0.4562   |
| 12.60     | 5.4015   | -0.6671  | 0.7598 | 0.4570   |
| 12.80     | 5.3572   | -0.6538  | 0.7432 | 0.4578   |

$$\sqrt{\beta_1} = 1.40$$

| $\beta_2$ | $\delta$ | $\Omega$ | $\mu$  | $\sigma$ |
|-----------|----------|----------|--------|----------|
| 10.00     | 6.9974   | -1.9166  | 3.4399 | 0.9690   |
| 10.20     | 6.8005   | -1.5807  | 2.4112 | 0.7288   |
| 10.40     | 6.6291   | -1.3962  | 1.9691 | 0.6344   |
| 10.60     | 6.4784   | -1.2717  | 1.7094 | 0.5838   |
| 10.80     | 6.3447   | -1.1791  | 1.5339 | 0.5526   |
| 11.00     | 6.2252   | -1.1065  | 1.4054 | 0.5319   |
| 11.20     | 6.1177   | -1.0473  | 1.3062 | 0.5175   |
| 11.40     | 6.0205   | -0.9979  | 1.2268 | 0.5071   |
| 11.60     | 5.9321   | -0.9557  | 1.1614 | 0.4994   |
| 11.80     | 5.8513   | -0.9191  | 1.1065 | 0.4937   |
| 12.00     | 5.7772   | -0.8870  | 1.0594 | 0.4895   |
| 12.20     | 5.7090   | -0.8585  | 1.0186 | 0.4863   |
| 12.40     | 5.6459   | -0.8331  | 0.9828 | 0.4839   |
| 12.60     | 5.5874   | -0.8102  | 0.9511 | 0.4821   |
| 12.80     | 5.5331   | -0.7894  | 0.9228 | 0.4809   |
| 13.00     | 5.4824   | -0.7704  | 0.8972 | 0.4800   |
| 13.20     | 5.4350   | -0.7530  | 0.8741 | 0.4794   |
| 13.40     | 5.3907   | -0.7370  | 0.8531 | 0.4791   |
| 13.60     | 5.3490   | -0.7223  | 0.8338 | 0.4790   |
| 13.80     | 5.3098   | -0.7085  | 0.8161 | 0.4790   |

$$\sqrt{\beta_1} = 1.60$$

| $\beta_2$ | $\delta$ | $\Omega$ | $\mu$  | $\sigma$ |
|-----------|----------|----------|--------|----------|
| 12.00     | 6.4694   | -2.3521  | 5.4165 | 1.6232   |
| 12.20     | 6.3485   | -1.8711  | 3.3040 | 1.0392   |
| 12.40     | 6.2392   | -1.6431  | 2.5973 | 0.8545   |
| 12.60     | 6.1400   | -1.4956  | 2.2144 | 0.7598   |
| 12.80     | 6.0495   | -1.3881  | 1.9661 | 0.7016   |
| 13.00     | 5.9665   | -1.3045  | 1.7887 | 0.6622   |
| 13.20     | 5.8903   | -1.2366  | 1.6540 | 0.6339   |
| 13.40     | 5.8199   | -1.1799  | 1.5474 | 0.6128   |
| 13.60     | 5.7547   | -1.1315  | 1.4604 | 0.5964   |
| 13.80     | 5.6941   | -1.0896  | 1.3877 | 0.5836   |
| 14.00     | 5.6377   | -1.0528  | 1.3259 | 0.5732   |
| 14.20     | 5.5850   | -1.0201  | 1.2725 | 0.5648   |
| 14.40     | 5.5357   | -0.9908  | 1.2258 | 0.5579   |
| 14.60     | 5.4894   | -0.9643  | 1.1845 | 0.5521   |
| 14.80     | 5.4459   | -0.9403  | 1.1477 | 0.5473   |
| 15.00     | 5.4049   | -0.9183  | 1.1147 | 0.5433   |
| 15.20     | 5.3662   | -0.8982  | 1.0848 | 0.5398   |
| 15.40     | 5.3296   | -0.8796  | 1.0576 | 0.5369   |
| 15.60     | 5.2950   | -0.8623  | 1.0328 | 0.5345   |
| 15.80     | 5.2622   | -0.8463  | 1.0100 | 0.5324   |

$$\sqrt{\beta_1} = 1.50$$

| $\beta_2$ | $\delta$ | $\Omega$ | $\mu$   | $\sigma$ |
|-----------|----------|----------|---------|----------|
| 10.80     | 6.8450   | -3.4154  | 15.7446 | 4.3688   |
| 11.00     | 6.6758   | -1.9659  | 3.6334  | 1.0731   |
| 11.20     | 6.5265   | -1.7229  | 2.6152  | 0.8178   |
| 11.40     | 6.3937   | -1.5440  | 2.1544  | 0.7100   |
| 11.60     | 6.2746   | -1.3508  | 1.8782  | 0.6497   |
| 11.80     | 6.1673   | -1.2582  | 1.6895  | 0.6114   |
| 12.00     | 6.0700   | -1.1848  | 1.5504  | 0.5851   |
| 12.20     | 5.9813   | -1.1246  | 1.4425  | 0.5661   |
| 12.40     | 5.9001   | -1.0740  | 1.3559  | 0.5520   |
| 12.60     | 5.8256   | -1.0306  | 1.2843  | 0.5412   |
| 12.80     | 5.7568   | -0.9928  | 1.2240  | 0.5328   |
| 13.00     | 5.6931   | -0.9595  | 1.1723  | 0.5262   |
| 13.20     | 5.6341   | -0.9299  | 1.1274  | 0.5210   |
| 13.40     | 5.5791   | -0.9033  | 1.0879  | 0.5168   |
| 13.60     | 5.5277   | -0.8793  | 1.0528  | 0.5134   |
| 13.80     | 5.4797   | -0.8575  | 1.0215  | 0.5106   |
| 14.00     | 5.4347   | -0.8375  | 0.9933  | 0.5084   |
| 14.20     | 5.3923   | -0.8192  | 0.9677  | 0.5066   |
| 14.40     | 5.3525   | -0.8022  | 0.9444  | 0.5052   |
| 14.60     | 5.3149   | -0.7866  | 0.9231  | 0.5041   |

$$\sqrt{\beta_1} = 1.70$$

| $\beta_2$ | $\delta$ | $\Omega$ | $\mu$  | $\sigma$ |
|-----------|----------|----------|--------|----------|
| 13.20     | 6.2311   | -2.5592  | 6.7043 | 2.0823   |
| 13.40     | 6.1374   | -2.0037  | 3.8047 | 1.2296   |
| 13.60     | 6.0513   | -1.7582  | 2.9453 | 0.9880   |
| 13.80     | 5.9721   | -1.6020  | 2.4942 | 0.8666   |
| 14.00     | 5.8990   | -1.4888  | 2.2059 | 0.7923   |
| 14.20     | 5.8313   | -1.4010  | 2.0018 | 0.7419   |
| 14.40     | 5.7683   | -1.3297  | 1.8478 | 0.7054   |
| 14.60     | 5.7097   | -1.2702  | 1.7253 | 0.6779   |
| 14.80     | 5.6549   | -1.2194  | 1.6274 | 0.6564   |
| 15.00     | 5.6036   | -1.1754  | 1.5450 | 0.6392   |
| 15.20     | 5.5555   | -1.1366  | 1.4750 | 0.6253   |
| 15.40     | 5.5102   | -1.1021  | 1.4146 | 0.6137   |
| 15.60     | 5.4676   | -1.0712  | 1.3619 | 0.6041   |
| 15.80     | 5.4274   | -1.0433  | 1.3153 | 0.5960   |
| 16.00     | 5.3893   | -1.0178  | 1.2738 | 0.5891   |
| 16.20     | 5.3533   | -0.9945  | 1.2366 | 0.5831   |
| 16.40     | 5.3191   | -0.9732  | 1.2029 | 0.5780   |
| 16.60     | 5.2866   | -0.9534  | 1.1723 | 0.5735   |
| 16.80     | 5.2557   | -0.9351  | 1.1444 | 0.5696   |
| 17.00     | 5.2263   | -0.9180  | 1.1187 | 0.5662   |



$$\sqrt{\beta_1} = 1.80$$

| $\beta_2$ | $\delta$ | $\alpha$ | $\mu$  | $\sigma$ |
|-----------|----------|----------|--------|----------|
| 14.60     | 5.9954   | -2.5092  | 6.3957 | 2.0763   |
| 14.80     | 5.9241   | -2.0480  | 3.9966 | 1.3398   |
| 15.00     | 5.8578   | -1.8214  | 3.1585 | 1.0916   |
| 15.20     | 5.7961   | -1.6725  | 2.6989 | 0.9601   |
| 15.40     | 5.7384   | -1.5626  | 2.3988 | 0.8770   |
| 15.60     | 5.6845   | -1.4763  | 2.1835 | 0.8195   |
| 15.80     | 5.6338   | -1.4057  | 2.0196 | 0.7771   |
| 16.00     | 5.5863   | -1.3463  | 1.8896 | 0.7446   |
| 16.20     | 5.5414   | -1.2954  | 1.7833 | 0.7189   |
| 16.40     | 5.4991   | -1.2509  | 1.6944 | 0.6981   |
| 16.60     | 5.4591   | -1.2117  | 1.6186 | 0.6809   |
| 16.80     | 5.4212   | -1.1766  | 1.5531 | 0.6666   |
| 17.00     | 5.3853   | -1.1451  | 1.4957 | 0.6544   |
| 17.20     | 5.3511   | -1.1165  | 1.4449 | 0.6441   |
| 17.40     | 5.3187   | -1.0904  | 1.3997 | 0.6351   |
| 17.60     | 5.2878   | -1.0665  | 1.3590 | 0.6273   |
| 17.80     | 5.2583   | -1.0444  | 1.3221 | 0.6205   |
| 18.00     | 5.2302   | -1.0240  | 1.2886 | 0.6145   |
| 18.20     | 5.2034   | -1.0050  | 1.2579 | 0.6092   |
| 18.40     | 5.1777   | -0.9873  | 1.2298 | 0.6045   |

$$\sqrt{\beta_1} = 1.90$$

| $\beta_2$ | $\delta$ | $\alpha$ | $\mu$  | $\sigma$ |
|-----------|----------|----------|--------|----------|
| 16.20     | 5.7796   | -2.3853  | 5.6597 | 1.9215   |
| 16.40     | 5.7255   | -2.0464  | 4.0034 | 1.3948   |
| 16.60     | 5.6751   | -1.8545  | 3.2811 | 1.1715   |
| 16.80     | 5.6274   | -1.7207  | 2.8595 | 1.0420   |
| 17.00     | 5.5825   | -1.6190  | 2.5580 | 0.9564   |
| 17.20     | 5.5400   | -1.5377  | 2.3430 | 0.8951   |
| 17.40     | 5.4998   | -1.4702  | 2.1765 | 0.8489   |
| 17.60     | 5.4617   | -1.4128  | 2.0429 | 0.8128   |
| 17.80     | 5.4255   | -1.3632  | 1.9326 | 0.7838   |
| 18.00     | 5.3910   | -1.3195  | 1.8396 | 0.7599   |
| 18.20     | 5.3583   | -1.2808  | 1.7599 | 0.7401   |
| 18.40     | 5.3271   | -1.2460  | 1.6907 | 0.7232   |
| 18.60     | 5.2973   | -1.2146  | 1.6299 | 0.7088   |
| 18.80     | 5.2688   | -1.1859  | 1.5759 | 0.6963   |
| 19.00     | 5.2416   | -1.1597  | 1.5275 | 0.6854   |
| 19.20     | 5.2156   | -1.1356  | 1.4840 | 0.6758   |
| 19.40     | 5.1906   | -1.1132  | 1.4444 | 0.6674   |
| 19.60     | 5.1666   | -1.0925  | 1.4084 | 0.6598   |
| 19.80     | 5.1437   | -1.0732  | 1.3753 | 0.6531   |
| 20.00     | 5.1216   | -1.0551  | 1.3449 | 0.6471   |

$$\sqrt{\beta_1} = 2.00$$

| $\beta_2$ | $\delta$ | $\alpha$ | $\mu$  | $\sigma$ |
|-----------|----------|----------|--------|----------|
| 17.80     | 5.6333   | -2.8775  | 9.3328 | 3.2305   |
| 18.00     | 5.5903   | -2.2981  | 5.1969 | 1.8380   |
| 18.20     | 5.5494   | -2.0440  | 4.0066 | 1.4467   |
| 18.40     | 5.5107   | -1.8815  | 3.3860 | 1.2471   |
| 18.60     | 5.4739   | -1.7629  | 2.9905 | 1.1228   |
| 18.80     | 5.4388   | -1.6702  | 2.7107 | 1.0367   |
| 19.00     | 5.4054   | -1.5944  | 2.4997 | 0.9731   |
| 19.20     | 5.3736   | -1.5308  | 2.3333 | 0.9240   |
| 19.40     | 5.3432   | -1.4760  | 2.1979 | 0.8849   |
| 19.60     | 5.3142   | -1.4282  | 2.0850 | 0.8530   |
| 19.80     | 5.2864   | -1.3858  | 1.9891 | 0.8265   |
| 20.00     | 5.2597   | -1.3480  | 1.9063 | 0.8040   |
| 20.20     | 5.2342   | -1.3138  | 1.8340 | 0.7848   |
| 20.40     | 5.2097   | -1.2828  | 1.7701 | 0.7681   |
| 20.60     | 5.1862   | -1.2544  | 1.7131 | 0.7536   |
| 20.80     | 5.1636   | -1.2283  | 1.6620 | 0.7408   |
| 21.00     | 5.1418   | -1.2042  | 1.6157 | 0.7295   |
| 21.20     | 5.1209   | -1.1818  | 1.5736 | 0.7193   |
| 21.40     | 5.1007   | -1.1609  | 1.5351 | 0.7103   |
| 21.60     | 5.0812   | -1.1415  | 1.4998 | 0.7021   |

has a standard logistic distribution. Percentile points of the Type IV curve (from Douver and Bargmann (1974)) and this  $L_U$  curve are compared in Table 5, which also contains values for

- (i) the  $S_U$  curve with the same first four moments, defined by

$$Z^* = -0.4048 + 1.455 \sinh^{-1} \{(X + 0.3842)/1.0765\}$$

having a standard normal distribution, and

- (ii) the  $S_U$  curve defined by  $Z^{**} = Z/1.7$  (see (1.7 a,b)).

We note that (a) the  $L_U$  curve agrees more closely with the Type IV than does the first (moment-fitted)  $S_U$  in the tails and (b) although the probability integral of the second  $S_U$  curve must agree with the  $L_U$  to within 0.01, there is considerable divergence between the percentiles near 0 or 100.

Table 5. Comparison of Type IV,  $L_U$  and  $S_U$  Standardized Percentiles

| %          | 0.1    | 0.25   | 0.5    | 1      | 2.5    | 5      | 10     | 25     | 50     |
|------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| Type IV    | -3.345 | -2.841 | -2.492 | -2.164 | -1.748 | -1.437 | -1.114 | -0.619 | -0.078 |
| $L_U$      | -3.339 | -2.865 | -2.526 | -2.220 | -1.777 | -1.454 | -1.118 | -0.609 | -0.070 |
| $S_U$ (i)  | -3.707 | -3.086 | -2.656 | -2.257 | -1.778 | -1.417 | -1.073 | -0.585 | -0.081 |
| $S_U$ (ii) | -2.507 | -2.281 | -2.100 | -1.908 | -1.628 | -1.387 | -1.109 | -0.632 | -0.070 |
| %          | 75     | 90     | 95     | 97.5   | 99     | 99.5   | 99.75  | 99.9   |        |
| Type IV    | 0.524  | 1.192  | 1.690  | 2.211  | 2.959  | 3.586  | 4.279  | 5.316  |        |
| $L_U$      | 0.520  | 1.178  | 1.677  | 2.203  | 2.967  | 3.609  | 4.318  | 5.375  |        |
| $S_U$ (i)  | 0.490  | 1.162  | 1.686  | 2.244  | 3.051  | 3.721  | 4.451  | 5.513  |        |
| $S_U$ (ii) | 0.546  | 1.166  | 1.574  | 1.955  | 2.432  | 2.781  | 3.123  | 3.571  |        |

(i) moment fit; (ii)  $-1.8576 + 3.5383 \sinh^{-1} \{(X + 1.5498)/2.6940\}$  unit normal

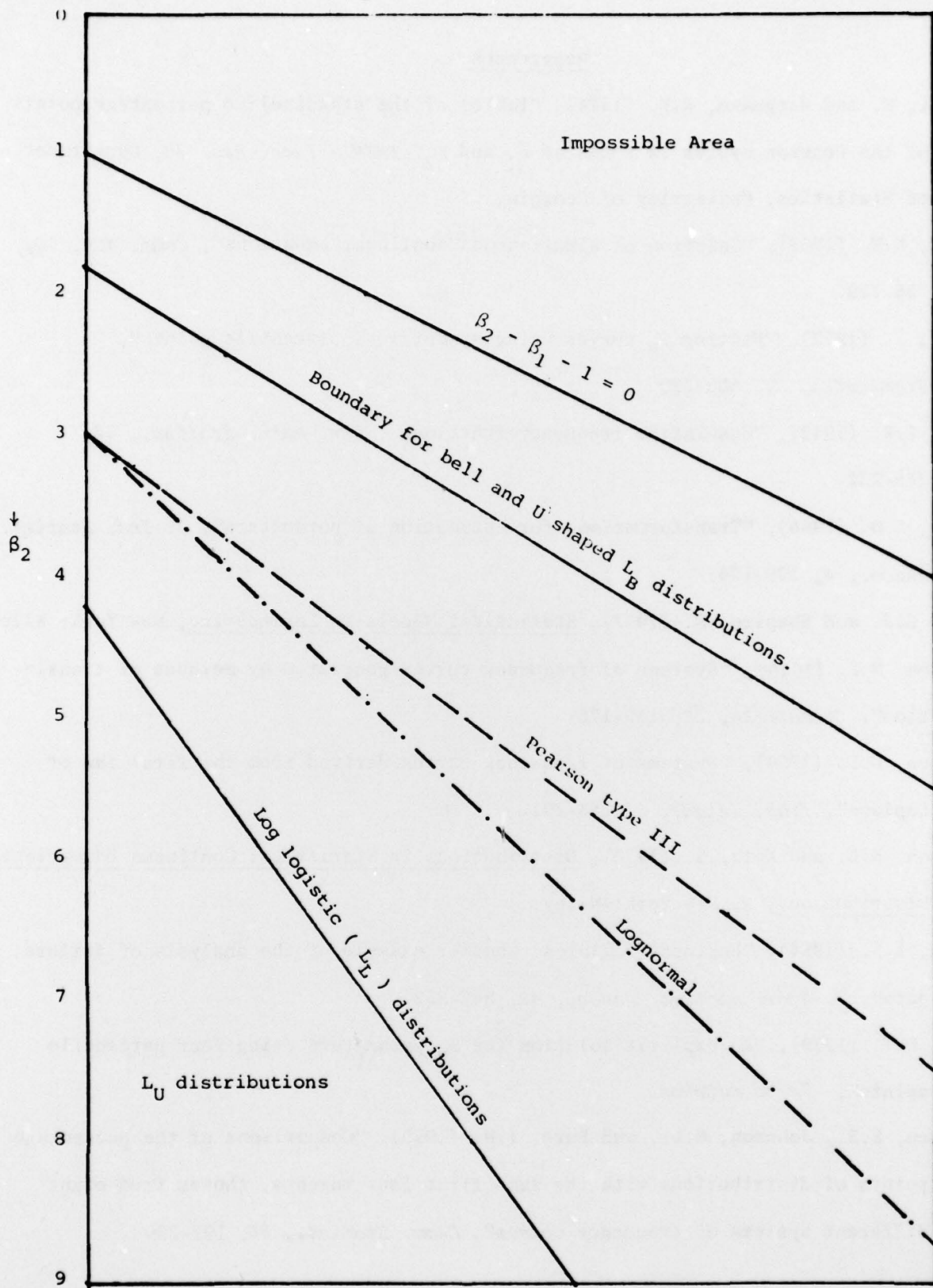


FIGURE A.  $\beta_1, \beta_2$  Region for  $L_U, L_L, L_B$  Distributions



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