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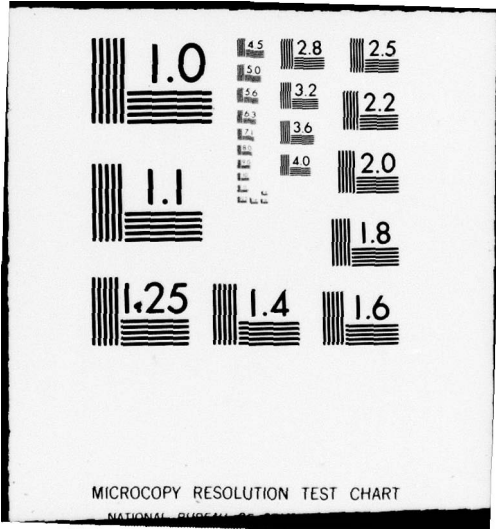
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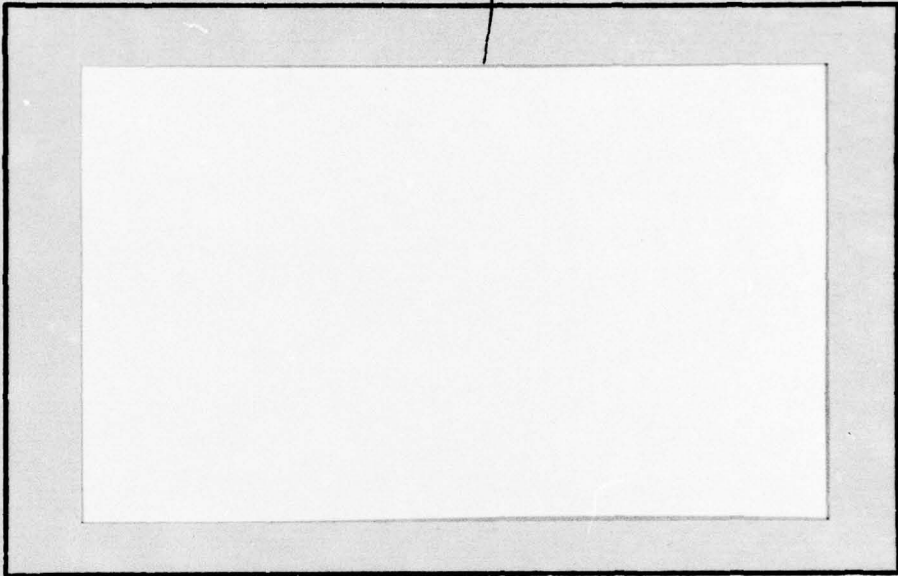
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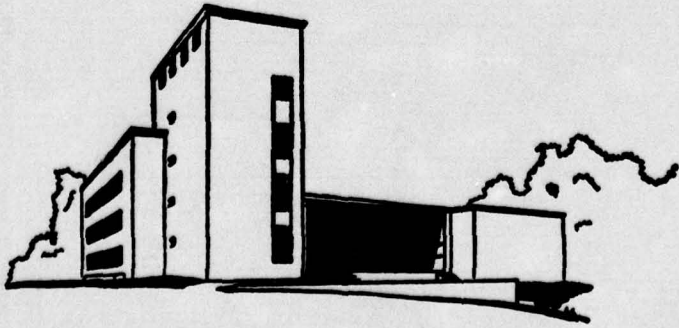
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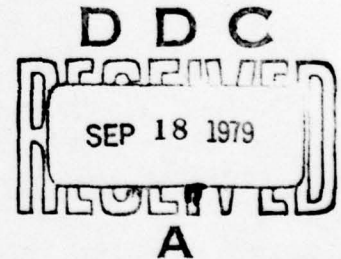
Management Sciences Research Report No. 435

SOME n BY dn LINEAR COMPLEMENTARITY
PROBLEMS SOLVABLE BY A PRINCIPAL
PIVOTING ALGORITHM WITH APPLICATIONS

by

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April 1979



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ABSTRACT

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The purpose of this paper is to study some recent applications of the n by dn LCP solvable by a parametric principal pivoting algorithm (PPP algorithm). First, it is shown that by analyzing the n by dn LCP we could study the problem of solving a system of equations and the (nonlinear) complementarity problem when the function involved is separable. Next, we examine conditions under which the PPP algorithm is applicable to a general LCP, and then present examples of LCP's arising from various applications satisfying the conditions; included among them is the n by dn LCP with a certain P -property. Finally we study a special class of n by dn LCP's which do not possess the P -property but to which the PPP algorithm is still applicable; a major application of this class of problems is a certain economic spatial equilibrium model with piecewise linear prices.

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1. INTRODUCTION

Given a positive integer d , an n -vector q , n by n matrices M^1, \dots, M^d and positive n -vectors f^1, \dots, f^{d-1} , the n by dn linear complementarity problem (n by dn LCP) is that of finding n -vectors w, x^1, \dots, x^d satisfying the conditions:

$$(1.1a) \quad w = q + \sum_{i=1}^d M^i x^i$$

$$(1.1b) \quad w \geq 0, \quad 0 \leq x^i \leq f^i, \quad i=1, \dots, d-1, \quad x^d \geq 0$$

$$(1.1c) \quad w^T x^1 = 0, \quad (f^i - x^i)^T x^{i+1} = 0, \quad i=1, \dots, d-1.$$

When $d=1$, (1.1) reduces to the (ordinary) LCP, which is to find m -vectors w and x such that

$$(1.2) \quad w = r + Lx, \quad w \geq 0, \quad x \geq 0, \quad w^T x = 0,$$

where r and L are given m -vector and m by m matrix, respectively. On the other hand, (1.1) can be converted into an ordinary LCP (1.2) with a dn by dn L (see [12]).

The n by dn LCP was first studied by one of the authors in [12] who characterized the existence and uniqueness of the solution of the problem by a certain P -property (see Section 2). It was also shown that when the problem has this property, then the solution can be obtained by applying a parametric principal pivoting algorithm (abbreviated as PPP algorithm in the following: see Section 3) to the LCP converted from the original n by dn LCP. A noteworthy point here is that the matrix L in this LCP is neither a P -matrix

nor positive semi-definite but still the PPP algorithm is shown to be applicable. As noted in [12] applications leading to an n by dn LCP with the ρ -property include a nonlinear analysis of certain reinforced concrete and other structures (see Kaneko [13]), and a strictly convex quadratic program with bounded variables which by itself has many applications (see [6, 25]).

The purposes of this paper are (i) to examine further the applicability of the PPP algorithm to the n by dn LCP, and (ii) to study some recent applications of the n by dn LCP, solvable by the PPP algorithm.

In the next section, we shall show that two closely related problems, i.e., (i) the problem of solving a system of piecewise linear equations defined on a cubic subdivision and (ii) the piecewise linear complementarity problem defined on a cubic subdivision, are equivalent to the n by dn LCP. This implies, in particular, that one could analyze these problems by considering the corresponding n by dn LCP.

After discussing the applicability of the PPP algorithm in some general terms in Section 3, we shall study in the fourth and final section another application of the n by dn LCP arising from the computation of equilibrium prices in an economic spatial model. In this model, a certain commodity is transshipped among various regions where its prices are given as piecewise linear functions of the local net imports. (Of course, the piecewise linearity could arise as an approximation of more general nonlinearities.) One wants to determine a set of prices and commodity flows such that certain spatial equilibrium conditions are satisfied. In most practical cases, the n by dn LCP obtained in this model does not possess the ρ -property; however, we shall show that the problem can still be solved by the PPP algorithm.

The special case of the equilibrium model where the prices are linear has

been studied by many researchers (see [1, 10, 28, 31]). In particular, one of the authors has shown in [28] that the PPP algorithm can be applied to solve the linear model and that the computational procedure is greatly simplified by taking advantage of some special structures of the problem. The linear model has a number of applications (see Polito [30], Takayama and Judge [32] and references therein) but suffers the obvious limitation because of the linear assumption on the prices. Presumably, the piecewise linear model offers a broader scope of applicability.

To our knowledge, there has been no explicit study of the piecewise linear model. The general model (where prices are general nonlinear functions) has been treated in [29] where necessary and sufficient conditions for equilibrium are derived in terms of a generalized complementarity problem over polyhedral cones, and in [20] where a fixed point algorithm is suggested to compute the equilibrium prices under some general setting.

2. THE n BY dn LCP AND RELATED PROBLEMS

In this section we shall study some relationships of the n by dn LCP (1.1) with the following two problems:

(2.1) Find $x \in \mathbb{R}^n$ such that $h = G(x)$, where $h \in \mathbb{R}^n$, and $G: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous and piecewise linear on a cubic subdivision of \mathbb{R}^n ;

(2.2) Find $w \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$ such that

$$w = s + F(x), \quad w \geq 0, \quad x \geq 0, \quad w^T x = 0,$$

where $s \in \mathbb{R}^n$, and $F: \mathbb{R}_+^n \rightarrow \mathbb{R}^n$ is continuous and piecewise linear on a cubic subdivision of \mathbb{R}_+^n .

After giving a definition of a cubic subdivision of \mathbb{R}^n (or \mathbb{R}_+^n), we shall show that the three problems, (1.1), (2.1) and (2.2) are equivalent in the sense that any one of them can be transformed into either of the other two.

The requirement that a function be piecewise linear on a cubic subdivision is highly restrictive; for instance, a (nonlinear) function must virtually be separable in order that it can be approximated by a piecewise linear function on a cubic subdivision. Thus, the scope of application of the problems (2.1) and (2.2), is admittedly severely limited. However, investigation of these problems seems to be justified because (i) in many practical situations nonlinear functions representing the problem are in fact separable (see e.g., problems described in [9, 21, 34, 35] in addition to well-known engineering problems), and (ii) the simple structure of the problem leads to sharp theoretical results and more efficient computation of the solution. Some results can be found in recent papers by Kojima [16] and Todd [33] on the

computation of a solution of the system of equations (2.1) using fixed point algorithms.

Let positive integers d_1, \dots, d_n scalars f_j^0 , $j=1, \dots, n$, and positive scalars f_j^i , $i=1, \dots, d_j - 1$, $j=1, \dots, n$ be given. Let Γ be the set of all γ of the form $\gamma = (\gamma_1, \dots, \gamma_n)$ where $\gamma_j \in \{0, 1, \dots, d_j\}$, $j=1, \dots, n$. For each $\gamma \in \Gamma$, define

$$(2.3) \quad C(\gamma) = \{x \in \mathbb{R}^n : \sum_{i=0}^{\gamma_j-1} f_j^i \leq x_j \leq \sum_{i=0}^{\gamma_j} f_j^i \text{ if } \gamma_j \geq 1, \text{ and } x_j \leq f_j^0 \text{ if } \gamma_j = 0\}$$

where by convention we set $f_j^j = +\infty$ for each j . We shall call the partition $\{C(\gamma) : \gamma \in \Gamma\}$ of \mathbb{R}^n a cubic subdivision of \mathbb{R}^n .

A cubic subdivision of \mathbb{R}_+^n , the nonnegative orthant of \mathbb{R}^n , can be similarly defined. Let d_1, \dots, d_n be positive integers and f_j^i positive scalars, $i=1, \dots, d_j - 1$, $j=1, \dots, n$. Let Γ_+ be the set of $\gamma = (\gamma_1, \dots, \gamma_n)$ with $\gamma_j \in \{1, \dots, d_j\}$, $j=1, \dots, n$. For each $\gamma \in \Gamma_+$ define

$$(2.4) \quad D(\gamma) = \{x \in \mathbb{R}_+^n : \sum_{i=0}^{\gamma_j-1} f_j^i \leq x_j \leq \sum_{i=0}^{\gamma_j} f_j^i, j=1, \dots, n\},$$

where by convention $f_j^0 = 0$ and $f_j^j = \infty$ for each j .

A function $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$ (or $F : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$) is called piecewise linear on a cubic subdivision if it is affine on each cell $C(\gamma)$ (or $D(\gamma)$, respectively) of the subdivision. For the sake of simplicity, we shall assume^{1/} in the following that $d = d_j$ for all j . We shall denote by f^i the n -vector

¹ There is no loss of generality in doing so; in general, let d be the largest d_j and for k with $d > d_k$, one may add $d - d_k$ dummy subdivision(s) with respect to the k -th coordinate.

whose j -th component is f_j^i , $i=0, \dots, d$.

Before proceeding further, we would like to observe the fact that the complementarity conditions (along with nonnegativity) in the problem (1.1) provide an equivalent statement of the condition $x \in D(\gamma)$ in the problem (2.2).

To be more specific, let $\{D(\gamma) : \gamma \in \Gamma_+\}$ be a cubic subdivision of R_+^n . If $x \in D(\gamma)$, then by setting $x_j^i = f_j^i$, $i=1, \dots, \gamma_j-1$, $x_j^{\gamma_j} = x_j - \sum_{i=0}^{\gamma_j-1} f_j^i$ and

$x_j^i = 0$, $i = \gamma_j+1, \dots, d$, for each $j=1, \dots, n$, it follows that $\{x^1, \dots, x^d\}$,

where $x^i = (x_1^i, \dots, x_n^i)^T$, $i=1, \dots, d$, satisfies the conditions (1.1b) - (1.1c)

(except for $w \geq 0$ and $w^T x^1 = 0$). Conversely, if $\{x^1, \dots, x^d\}$ satisfies

(1.1b) - (1.1c), then $x = x^1 + \dots + x^d \in D(\gamma)$, where for each $j=1, \dots, n$, $\gamma_j - 1$

is the largest index i such that $x_j^i = f_j^i$. Similarly, if $\{C(\gamma) : \gamma \in \Gamma\}$ is

a cubic subdivision of R^n , then $z \in C(\gamma)$ is characterized by the following

conditions: $z = z^0 + \dots + z^d$ and

$$z^0 \leq f^0, \quad 0 \leq z^i \leq f^i, \quad i=1, \dots, d-1, \quad z^d \geq 0, \quad (2.5)$$

$$(f^i - z^i)^T z^{i+1} = 0, \quad i = 0, 1, \dots, d-1.$$

2.1 Proof of Equivalence of Problems (1.1), (2.1) and (2.2)

First of all, we shall show the equivalence between (1.1) and (2.2).

To prove that the n by dn LCP (1.1) can be written in the form (2.2),

it suffices to show that $q + \sum_{i=1}^d M^i x^i$ is a continuous function of

$x = x^1 + \dots + x^d$ which is affine on $D(\gamma)$ for each γ in Γ_+ . This is obvious because we have ^{2/}

2 For a matrix K , $K_{\cdot j}$ denotes its j -th column.

$$\begin{aligned}
 q + \sum_{i=1}^d M^i x^i &= q + \sum_{i=1}^d \sum_{j=1}^n M_{\cdot j}^i x_j^i = q + \sum_{j=1}^n \sum_{i=1}^d M_{\cdot j}^i x_j^i \\
 (2.6) \quad &= q + \sum_{j=1}^n \left[\sum_{i=1}^{\gamma_j^{-1}} M_{\cdot j}^i f_j^i + M_{\cdot j}^{\gamma_j} (x_j - \sum_{i=1}^{\gamma_j^{-1}} f_j^i) \right] \\
 &= (q + \sum_{j=1}^n \sum_{i=1}^{\gamma_j^{-1}} (M_{\cdot j}^i - M_{\cdot j}^{\gamma_j}) f_j^i) + [M_{\cdot 1}^{\gamma_1}, \dots, M_{\cdot n}^{\gamma_n}] x.
 \end{aligned}$$

To show the other implication, let the piecewise linear complementarity problem (2.2) be given and let $\{D(\gamma) : \gamma \in \Gamma_+\}$ be the associated cubic subdivision of R_+^n . For each $\gamma \in \Gamma_+$, let

$$(2.7) \quad F(x) = a(\gamma) + A(\gamma)x,$$

for some $a(\gamma) \in R^n$ and $A(\gamma) \in R^{n \times n}$ for each $x \in D(\gamma)$. Further let

$M^i = A(\gamma^i)$, where $\gamma^i = (i, \dots, i)$, for $i=1, \dots, d$. Then, it is not difficult to show (see [14]) that for each γ in Γ_+ , we have that

$$(2.8) \quad A(\gamma) = [M_{\cdot 1}^{\gamma_1}, \dots, M_{\cdot n}^{\gamma_n}] .$$

Also it follows from some elementary calculations (see [14]) that

$$(2.9) \quad a(\gamma) = b + \sum_{j=1}^n \sum_{i=1}^{\gamma_j^{-1}} [M_{\cdot j}^i - M_{\cdot j}^{\gamma_j}] f_j^i,$$

where $b = a(\gamma^1)$. By (2.8), (2.9), and by reversing the argument in (2.6), we can conclude that (2.2) is converted to the n by dn LCP (1.1) with $q = s + b$.

It is well-known that a complementarity problem can be transformed into the problem of solving a system of equations; i.e., the transformation

from (2.2) into (2.1) is known. However, the transformation must be such that the resulting equations are piecewise linear on a cubic subdivision of R^n . This can be achieved by using the device found in Eaves [7] and Megiddo and Kojima [23]. To be specific, let the problem (2.2) be given and let $\{D(\gamma) : \gamma \in \Gamma_+\}$ be the associated cubic subdivision of R_+^n . Then the problem is transformed into (2.1) with $h = -s$ and

$$(2.10) \quad G(z) = F(P(z)) - P(z) + z, \quad z \in R^n,$$

where $P(\cdot)$ is the projection of R^n onto R_+^n , i.e., $P(z)$ is the "plus part" of z . It is not difficult to see that G is piecewise linear on a cubic subdivision

$\{C(\gamma) : \gamma \in \Gamma\}$, with $C(\gamma)$ as defined in (2.3), where for $j=1, \dots, n, f_j^0 = 0$

and f_j^i is the same as that defining $\{D(\gamma) : \gamma \in \Gamma_+\}$ for $i=1, \dots, d-1$.

Finally, to complete the proof of equivalence of the three problems, we show that (2.1) can be transformed into the form (1.1). Let the problem (2.1) be given and let the associated cubic subdivision of R^n be $\{C(\gamma) : \gamma \in \Gamma\}$. Further, let the function G be expressed as

$$(2.11) \quad G(z) = a(\gamma) + A(\gamma)z$$

for each $z \in C(\gamma)$. Then, using the same argument as before showing that (2.2) is transformed into (1.1), the problem (2.1) can be rewritten as

$$h = b + \sum_{i=0}^d N^i z^i$$

$$(2.12) \quad z^0 \leq f^0, \quad 0 \leq z^i \leq f^i, \quad i=1, \dots, d-1, \quad z^d \geq 0$$

$$(f^i - z^i)^T z^{i+1} = 0, \quad i = 0, \dots, d-1,$$

where $b = a(\gamma^0)$ and $N^i = A(\gamma^i)$ with $\gamma^i = (i, \dots, i)^T$, $i=0, \dots, d$. We need to assume that N^0 (or N^d) is nonsingular. Then by letting

$$w = f^0 - z^0, \quad x^i = z^i, \quad i=1, \dots, d \quad (2.13)$$

$$q = (N^0)^{-1}(b-h) + f^0 \quad \text{and} \quad M^i = (N^0)^{-1}N^i, \quad i=1, \dots, d,$$

we see that (2.12) is transformed into the n by dn LCP (1.1).^{3/}

2.2 Existence and Uniqueness of Solution

Let $M^i \in \mathbb{R}^{n \times n}$, $i=1, \dots, d$. The n by dn matrix (M^1, \dots, M^d) is said to have the P-property if for each $\gamma \in \Gamma_+$, the matrix $A(\gamma)$ as defined in (2.8) is a P-matrix (i.e., all of its principal minors are positive). The following was proved in [12].

Theorem 1. Given $f^i \in \mathbb{R}^n$, $f^i > 0$, $i=1, \dots, d-1$, the n by dn LCP (1.1) has a unique solution for every $q \in \mathbb{R}^n$ if and only if (M^1, \dots, M^d) has the P-property.

Recently some results have been obtained on the existence and uniqueness of the solution of a system of piecewise linear equations (see e.g., [17, 18, 19, 22]). When the underlying subdivision is cubic, then the following necessary and sufficient condition holds.

Theorem 2 (Kojima [17]). The system of piecewise linear equations (2.1) on a cubic subdivision has a unique solution for every $h \in \mathbb{R}^n$ if and only if the Jacobians of G on all the cells in the subdivision are nonzero and of the same sign.

3. Eaves and Lemke [8] have recently announced a scheme by which one can transform a system of piecewise linear equations into a LCP; it is not clear to the authors of this paper whether or not their scheme produces the same LCP (or n by dn LCP) as that resulting from our procedure described above.

It turns out, as one might expect, that these two results are equivalent in the sense that they imply each other; we shall prove this equivalence in the remainder of this section.

Let the n by dn LCP (1.1) be given and let $\{D(\gamma) : \gamma \in \Gamma_+\}$ be the associated cubic subdivision of R_+^n . Transform (1.1) into the problem (2.1) via the problem (2.2) as we described in the previous subsection. Namely, first convert (1.1) into (2.2); the function F in the resulting problem is of the form $F(x) = a(\gamma) + A(\gamma)x$ for $x \in D(\gamma)$, where $A(\gamma)$ is as given in (2.8). The problem (2.1) is then obtained by using the function G in (2.10): this G is piecewise linear on a cubic subdivision $\{C(\gamma) : \gamma \in \Gamma\}$. To prove that Theorem 2 implies Theorem 1, we need to show that the Jacobians of the G on all the cells have the same, nonzero sign if and only if the matrix (M^1, \dots, M^d) in (1.1) has the \mathcal{P} -property.

Now, for each $\gamma \in \Gamma$, let γ^+ be the element of Γ_+ which is obtained from γ by replacing each zero component by unity. To be more specific, let β be the set of j with $\gamma_j = 0$ and δ its complement in $\{1, \dots, n\}$. Thus $\gamma^+ = (\gamma_1^+, \dots, \gamma_n^+)$ satisfies: $\gamma_j^+ = 1$ for $j \in \beta$ and $\gamma_j^+ = \gamma_j$ for $j \in \delta$. Then, the Jacobian matrix of G on $C(\gamma)$ is written as $\frac{4}{4}$ (after an appropriate principal rearrangement):

$$(2.14) \quad \begin{bmatrix} I_{\beta\beta} & A(\gamma^+)_{\beta\delta} \\ 0 & A(\gamma^+)_{\delta\delta} \end{bmatrix},$$

where I is the n by n identity matrix. Clearly, the determinant of this matrix is the same as that of $A(\gamma^+)_{\delta\delta}$ if $\delta \neq \emptyset$ and unity if $\delta = \emptyset$. This implies that the set of Jacobians of G consists of the determinant of $A(\gamma^+)$ and all of its principal minors for every $\gamma^+ \in \Gamma_+$, plus unity. The condition

4. For a matrix K and index sets ξ and η , $K_{\xi\eta}$ denotes the submatrix of K with rows and columns indexed by ξ and η respectively.

in Theorem 2 is then equivalent to that $A(\gamma^+)$ is a P-matrix for each γ^+ in Γ_+ , or that (M^1, \dots, M^d) has the P-property.

To show that Theorem 1 implies Theorem 2, let the problem (2.1) with a cubic subdivision $\{C(\gamma) : \gamma \in \Gamma\}$ of R^n be given. Let $G(z)$ be expressed as in (2.11). To apply Theorem 1, we need to transform (2.1) into an n by dn LCP. Recall that this process required that the matrix $A(\gamma^0)$ with $\gamma^0 = (0, \dots, 0)$ be nonsingular. To prove Theorem 2, however, we can assume that all $A(\gamma)$ are nonsingular, "for free". This is because it follows from a well-known result (see e.g., [19]) that if some $A(\gamma)$ is singular, then the problem (2.1) can not have a unique solution for all $h \in R^n$.

We shall consider only the case where $\det A(\gamma^0) > 0$; the other case is similar. Let $N^i = A(\gamma^i)$ with $\gamma^i = (i, \dots, i)$ for $i = 0, \dots, d$ and let $M^i = (N^0)^{-1} N^i$, $i = 1, \dots, d$. Then, M^1, \dots, M^d define the n by dn LCP obtained from (2.1). The condition in Theorem 2 is

$$(2.15) \quad \det A(\gamma) > 0 \quad \text{all } \gamma \in \Gamma,$$

and we need to show that this is equivalent to that (M^1, \dots, M^d) has the P-property.

To do so, for any given $\gamma \in \Gamma$, let γ^+ be the element of Γ_+ associated with γ in the sense that γ^+ is obtained from γ by replacing each zero component by unity. Note that if we let $\beta = \{j : \gamma_j = 0\}$ and δ its complement, then

$$\begin{aligned} \det (N^0)^{-1} A(\gamma) &= \det \begin{bmatrix} I_{\beta\beta} & ((N^0)^{-1} A(\gamma^+))_{\beta\delta} \\ 0 & ((N^0)^{-1} A(\gamma^+))_{\delta\delta} \end{bmatrix} \\ &= \det ((N^0)^{-1} A(\gamma^+))_{\delta\delta}. \end{aligned}$$

From this relationship and the fact that $\det (N^0)^{-1} > 0$, it follows that (2.15) is equivalent to

$$(2.16) \quad \text{For each } \gamma \in \Gamma, \det((N^0)^{-1}A(\gamma^+))_{\delta\delta} > 0, \text{ where } \delta = \{j : \gamma_j \geq 1\}$$

and $\gamma_j^+ = \gamma_j$, for $j \in \delta$, and $\gamma_j^+ = 1$, for $j \notin \delta$.

Now, it is not difficult to show that (2.16) is indeed equivalent to the fact that the matrix $(N^0)^{-1}A(\gamma^+)$ is a P-matrix for every $\gamma^+ \in \Gamma$, or that (M^1, \dots, M^d) has the P-property.

3. A PARAMETRIC PRINCIPAL PIVOTING ALGORITHM

In this section we shall examine the applicability of a parametric version of Graves' principal pivoting algorithm (PPP algorithm; see Graves [11] for the original version and Cottle [3] for its parametric version) to the (ordinary) LCP and some n by n LCP's.

For m -vectors r and p , m by m matrix L the parametric linear complementarity problem is that of finding m -vectors w and x such that

$$(3.1) \quad w = r + \lambda p + Lx, \quad w \geq 0, \quad x \geq 0, \quad w^T x = 0,$$

for each value of scalar λ in a given interval. One approach for solving an ordinary LCP (1.2) is to choose an appropriate p and apply the PPP algorithm to the parametric LCP (3.1); if the algorithm generates a solution to (3.1) for $\lambda = 0$, then the original LCP is solved.

It is well-known (see [11]) that this approach is successful (in the sense that it terminates either with a solution to (1.2) or with the conclusion that the problem is infeasible) if the matrix L in the given LCP is either a P-matrix or positive semi-definite. Obviously, the matrix L in the LCP (1.2) need not be a P-matrix or positive semi-definite in order for the LCP algorithm to be applicable. In particular, if the problem has the property that every pivot candidate on the diagonal during the application of the algorithm is positive (this is certainly the case when L is a P-matrix), then the LCP has a solution and the algorithm computes it. Similarly, if the problem has the property that whenever a pivot candidate is non-positive, we can find an "appropriate" two by two pivot, or else we could declare the infeasibility; then the PPP algorithm processes the LCP without L being positive semi-definite. The notion of the applicability of

the PPP algorithm may be extended further by considering pivots, but we shall not consider it in this paper.

The following conditions ensure the applicability

Theorem 3. Consider a LCP (1.2) with a P_0 -matrix^{5/} L . The problem can be solved by the PPP algorithm using solely the one by one diagonal pivots if there exists an m -vector p satisfying the following conditions:

$$(3.2) \quad r + \bar{\lambda}p > 0 \text{ for some } \bar{\lambda} > 0 ;$$

(3.3) for every index set $\gamma \subset \{1, \dots, m\}$ for which $L_{\gamma\gamma}$ is nonsingular,

$$L_{kk}^{(\gamma)} = 0, \quad r_k^{(\gamma)} < 0 \quad \text{and} \quad p_k^{(\gamma)} > 0$$

hold for no $k \in \{1, \dots, m\}$, where (γ) denotes the principal pivotal transform with respect to $L_{\gamma\gamma}$.

The proof of the above result is straightforward and is outlined as follows: Set up the parametric LCP (3.1) using the p ; the condition (3.2) ensures that one may initiate the PPP algorithm at $\lambda = \bar{\lambda}$ (the value of the parameter λ is to be decreased to zero). The condition (3.3) eliminates the possibility that the k -th diagonal element is zero when it is a candidate for the next pivot. The finiteness of the algorithm may be guaranteed by using a lexicographic scheme (see Cottle [3]) or in some cases by using the "least-index" rule without lexicography (see Chang [2] and Kaneko [14]).

⁵ A real square matrix is P_0 if all of its principal minors are nonnegative.

We could state a set of conditions ensuring the applicability of the PPP algorithm for a possibly infeasible LCP using two by two as well as one by one pivots; instead of discussing it in a fully general framework we shall consider, in the next section, a specific example of such a LCP arising in a certain economic equilibrium model. In the remainder of this section we shall identify LCP's arising in various applications satisfying the conditions in Theorem 3.

Example 1: Consider the n by dn LCP (1.1), where we assume that (M^1, \dots, M^d) has the P -property, and the ordinary LCP obtained from it. It was shown in Kaneko [12] that the matrix L is a P_0 -matrix (but not a P -matrix) and for p such that $p_j = 1$, $j = 1, \dots, n$, $p_j = 0$, $j \geq n+1$, it holds that $L_{kk}^{(\gamma)} = 0$ implies that $p_k^{(\gamma)} = 0$, for each γ with $\det L_{\gamma\gamma} \neq 0$. Thus, the conditions in Theorem 3 hold. We note also that the PPP algorithm applied to the converted LCP can be "condensed" so that only n (rather than dn) equations need be dealt with.

It follows from the discussion in Section 2.2 that if the condition in Theorem 2 is satisfied, then the solution of the system of equations (2.1) can be computed by using the PPP algorithm (applied to the corresponding n by dn LCP). In this connection we would like to point out, without elaboration, that the PPP algorithm applied to the n by dn LCP generates the same solution path as Katzenelson's algorithm (see [15]) applied to the corresponding system of equations.

Example 2: Consider the strictly convex quadratic program:

$$\text{minimize } f^T x + \frac{1}{2} x^T F x \quad \text{subject to } Ax \leq b, \quad x \geq 0,$$

where $f \in \mathbb{R}^n$, F is an n by n symmetric, positive definite matrix, $A \in \mathbb{R}^{m \times n}$ and b is a nonnegative m -vector. Notice that not every feasible, strictly convex quadratic program can be put into this form. It has been shown in Pang [27] that the LCP corresponding to the Kuhn-Tucker conditions of the above problem satisfies the conditions in Theorem 3.

Example 3: The least distance program (see Wolfe [36] and Cottle and Djang [5]) is the problem of finding a point, with the smallest distance from the origin, in a convex polyhedron given as a convex hull of m points in \mathbb{R}^n . This problem can be formulated as the quadratic program:

$$\text{minimize } \frac{1}{2} x^T P^T P x \quad \text{subject to } e^T x = 1 ; x \geq 0,$$

where P is an m by n matrix and e is the n -vector of one's. We note that this program is not a special case of the quadratic program in the previous example (which requires, among other things, that $x = 0$ be feasible). After a certain two by two principal pivot performed in the Kuhn-Tucker conditions, the problem can be transformed into a LCP with an $n - 1$ by $n - 1$ matrix (see [5]). By using some special structures of the problem and some results in [5], it can be shown (see [14]) that this LCP satisfies the conditions in Theorem 3 (with any positive p): in fact, it holds that $L_{kk}^{(\gamma)} = 0$ implies $r_k^{(\gamma)} \geq 0$ for every γ with $\det L_{\gamma\gamma} \neq 0$.

Example 4: Consider the economic spatial equilibrium problem with linear prices mentioned in Section 1. This can be formulated as a convex program of the form

$$(3.4) \quad \text{minimize } \frac{1}{2} x^T A^T D A x + q^T x \quad \text{subject to } x \geq 0$$

where A is the node-arc incidence matrix of a digraph, D is a positive diagonal matrix and the vector q is defined in terms of the transportation costs and some other constants. We note that this program is not a special case of the last two examples. Under the assumption that the transportation costs satisfy certain triangle inequalities it was proved in Pang and Lee [28] that the LCP arising from this problem satisfies the conditions:

$$L_{kk}^{(\gamma)} = 0 \quad \text{implies} \quad r_k^{(\gamma)} \geq 0$$

for every γ with $\det L_{\gamma\gamma} \neq 0$. It will be shown in the next section that this property carries over to the case where we allow the prices to be piecewise linear.

4. A SPECIAL n BY dn LCP

In this section, we study the following linear complementarity problem: given a positive integer ℓ , an n by n matrix M ; m by n matrices A, N^2, \dots, N^ℓ ; m by m matrices B^2, \dots, B^ℓ ; an n -vector r and m -vectors a^2, a^3, \dots, a^ℓ with $a^i > 0$ for $i=3, \dots, \ell$, find an n -vector x and m -vectors y^2, \dots, y^ℓ such that

$$(4.1) \quad u = r + Mx + \sum_{i=2}^{\ell} (N^i)^T y^i \geq 0 \quad x \geq 0$$

$$v^2 = a^2 + Ax + \sum_{i=2}^{\ell} B^i y^i \geq 0 \quad y^2 \geq 0$$

$$v^i = a^i - y^{i-1} \geq 0 \quad y^i \geq 0 \quad i = 3, \dots, \ell$$

$$u^T x = (v^i)^T y^i = 0 \quad i = 2, \dots, \ell .$$

By decomposing the vector x into $(\ell - 1)$ pieces, it is easy to see that this problem is an $(m+n)$ by $(\ell - 1)(m+n)$ LCP with the matrices

$$(4.2) \quad M^i = \begin{pmatrix} M & (N^i)^T \\ A & B^i \end{pmatrix} \quad i=2, \dots, \ell .$$

Conversely, an n by dn LCP of the form (1.1) is equivalent to a problem of the form (4.1).

The study of the problem(4.1) is motivated by its application to the piecewise linear spatial equilibrium model which is of considerable importance as briefly outlined in the Introduction. In this application, M is an arc-arc weighted adjacency matrix of the (weighted) digraph G with node set V

denoting the various regions in the market, arc set E denoting the routes between regions, and (positive) weights $\{b_{\alpha 1}\}_{\alpha \in V}$ on the nodes; (see [28] for some basic properties of such an arc-arc matrix M); each B^i is an identity matrix.

$$N^i = (D^1 - D^i)A$$

where D^j is the diagonal matrix with positive diagonal entries $\{b_{\alpha j}\}_{\alpha \in V}$; A is the node-arc incidence matrix of the digraph G ; r is an arc-vector such that for each arc $(\alpha, \beta) \in E$

$$(4.3) \quad r_{\alpha\beta} = a_\alpha - a_\beta + c_{\alpha\beta}$$

where a_α is a positive scalar and $c_{\alpha\beta}$ is the nonnegative unit transportation cost from region α to region β . Each component of the vector $\sum_{i=1}^l y^i$ where

$y^1 = a^2 - v^2$, represents the total net import at a particular region. The price of the commodity at region α is given by

$$p_\alpha = a_\alpha - \sum_{i=1}^l b_{\alpha i} y_\alpha^i$$

which is a piecewise linear function of the total local net import

$y_\alpha = \sum_{i=1}^l y_\alpha^i$ by the complementarity conditions on the y^i 's. (cf. (2.5) and

the discussion there.) Each a_α is thus the equilibrium price of the commodity in the absence of imports and exports.

When $\ell = 1$, the above piecewise linear model reduces to the linear model (3.4). Notice, however, that problem (4.1) does not in general correspond to a convex quadratic program.

Notations. Throughout the section, we denote by \tilde{M} the matrix

$$\begin{pmatrix} M & (N^2)^T & \dots & (N^\ell)^T \\ A & B^2 & \dots & B^\ell \end{pmatrix}.$$

For $\alpha \subseteq \{1, \dots, n\}$ and each sequence $\{\beta_2, \dots, \beta_\ell\}$ of disjoint subsets of $\{1, \dots, m\}$ we denote the corresponding "principal" submatrix of \tilde{M}

$$\begin{pmatrix} M_{\alpha\alpha} & (N_{\beta_2\alpha}^2)^T & \dots & (N_{\beta_\ell\alpha}^\ell)^T \\ A_{\beta\alpha} & B_{\beta\beta_2}^2 & \dots & B_{\beta\beta_\ell}^\ell \end{pmatrix}$$

by $B(\alpha, \beta_2, \dots, \beta_\ell)$. Here $\beta = \bigcup_{k=2}^{\ell} \beta_k$.

Definition 1. We say that the matrix \tilde{M} is consistent with M in determinantal signs if for all $\alpha, \beta_2, \dots, \beta_\ell$, we have

$$\text{sgn det } B(\alpha, \beta_2, \dots, \beta_\ell) = \begin{cases} \text{sgn det } M_{\alpha\alpha} & \text{if } \alpha \neq \emptyset \\ 1 & \text{otherwise} \end{cases}$$

where for a scalar x ,

$$\text{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$

This property of determinantal consistency is motivated by the piecewise linear equilibrium model where it is satisfied easily. In general, if (B^2, \dots, B^l) has the \mathcal{P} -property, then the matrix \tilde{M} is consistent with M in determinantal signs if and only if for each nonempty $\alpha \subseteq \{1, \dots, n\}$

$$\text{sgn det } (B(\alpha, \beta_2, \dots, \beta_l) / (B_{\beta_2 \beta_2}^2, \dots, B_{\beta_l \beta_l}^l)) = \text{sgn det } M_{\alpha \alpha}$$

where $(B(\alpha, \beta_2, \dots, \beta_l) / (B_{\beta_2 \beta_2}^2, \dots, B_{\beta_l \beta_l}^l))$ denotes the Schur complement of $(B_{\beta_2 \beta_2}^2, \dots, B_{\beta_l \beta_l}^l)$ in $B(\alpha, \beta_2, \dots, \beta_l)$ (see Cottle [4]), i.e.

$$\begin{aligned} & (B(\alpha, \beta_2, \dots, \beta_l) / (B_{\beta_2 \beta_2}^2, \dots, B_{\beta_l \beta_l}^l)) \\ &= M_{\alpha \alpha} - ((N_{\beta_2 \alpha}^2)^T, \dots, (N_{\beta_l \alpha}^l)^T) (B_{\beta_2 \beta_2}^2, \dots, B_{\beta_l \beta_l}^l)^{-1} A_{\beta \alpha} \end{aligned}$$

By some easy calculation, it can be shown that in the piecewise linear equilibrium model, this Schur complement is again an arc-arc weighted adjacency matrix whose associated digraph has precisely the same structure as that of $M_{\alpha \alpha}$; only the weights on some of the nodes are suitably changed. According to a result established in [28], the sign of the determinant of an arc-arc matrix depends only on the structure of the graph. Therefore the desired property of determinantal consistency follows.

Notice that if \tilde{M} is consistent with M in determinantal signs, then the matrix (B^2, \dots, B^l) has the \mathcal{P} -property. The converse is not always true. However, we have

Proposition 1. If for $i = 2, \dots, \ell$, M^i is given by (4.2), then (M^2, \dots, M^ℓ) has the P-property if and only if M is a P-matrix and \tilde{M} is consistent with M in determinantal signs.

Combined with Theorem 1, the last proposition gives the following existence and uniqueness theorem for problem (4.1).

Theorem 4. The problem (4.1) has a unique solution for every $r, a^2, a^3, \dots, a^\ell$ with $a^i > 0$ for $i \geq 3$ if and only if M is a P-matrix and \tilde{M} is consistent with M in determinantal signs.

Remark. Since the property of determinantal consistency is always satisfied in the piecewise linear equilibrium model, the model therefore has a unique solution for all constant vectors if and only if the arc-arc matrix is P, or in other words, if and only if the network is a forest (see [28]).

As a consequence of Proposition 1, it follows that if M is a P-matrix and if the property of determinantal consistency is satisfied, then the PPP algorithm will always compute the unique solution to the problem (4.1) by performing only the 1×1 diagonal pivots. We would like to point out nevertheless, that in the application of the algorithm, there is no need to introduce the matrices M^i . In fact, it is easy to develop a condensed version of the algorithm which operates on the matrix \tilde{M} . For more details, see [12, 26].

In the remainder of this section, we drop the assumption that M is a P-matrix and study the solution of (4.1). To start, we give an example to show that the matrix

$$K = \begin{pmatrix} M_{\alpha\alpha} & (N_{\beta_3\alpha}^2)^T & (N_{\beta_2\alpha}^2)^T & (N_{\gamma\alpha}^3)^T & (N_{\beta_2\alpha}^3)^T \\ A_{\beta_3\alpha} & B_{\beta_3\beta_3}^2 & B_{\beta_3\beta_2}^2 & B_{\beta_3\gamma}^3 & B_{\beta_3\beta_3}^3 \\ A_{\beta_2\alpha} & B_{\beta_2\beta_3}^2 & B_{\beta_2\beta_2}^2 & B_{\beta_2\gamma}^3 & B_{\beta_2\beta_3}^3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -I_{\beta_3\beta_3} & 0 & 0 & 0 \end{pmatrix}$$

where $\alpha \subseteq \{1, \dots, n\}$ and γ, β_2 and β_3 are mutually disjoint subsets in $\{1, \dots, m\}$. Clearly $\det K$ is zero unless $\gamma = \emptyset$. If $\gamma = \emptyset$, then by expanding $\det K$ with respect to $-I_{\beta_3\beta_3}$, we have

$$\det K = \det B(\alpha, \beta_2, \beta_3) \geq 0.$$

by the assumptions.

We now establish that under certain assumptions which are satisfied in the piecewise linear equilibrium model, the PPP algorithm will successfully process the problem (4.1).

Theorem 5. Suppose that the conditions below are satisfied:

(4.5) $M = E^T D^1 E$ where $E \in \mathbb{R}^{m \times n}$ and $D^1 \in \mathbb{R}^{m \times m}$ with D^1 being positive definite;

(4.6) $A = CE$ and $N^i = D^i E$ for $i = 2, \dots, \ell$ where $C \in \mathbb{R}^{m \times m}$ and $D^i \in \mathbb{R}^{m \times m}$;

(4.7) The matrix \tilde{M} is consistent with M in determinantal signs.

Then by choosing the parametric vector ^{5/} to be $(e_n^T, e_m^T, 0, \dots, 0)^T$, the PPP algorithm will in a finite number of pivots, either terminate with a complementary solution to the problem (4.1), or indicate that the problem has no feasible solution.

Before proving this theorem, we point out several remarks. First, assumption (4.5) implies that M is positive semi-definite but not necessarily symmetric or positive definite. Second, as Example 5 above shows, these assumptions do not imply that the matrix L is positive semi-definite or copositive. Finally, except for the assumed property on D^1 , the matrices D^j , C and E are otherwise arbitrary.

We need the lemma below to prove the theorem. It follows from Property 2.3 in [12].

Lemma 1. Under the assumptions of Theorem 5, the following assertion holds: If $\alpha \subseteq \{1, \dots, n\}$ is nonempty such that $M_{\alpha\alpha}$ is nonsingular, and if $\{\beta_2, \dots, \beta_\ell\}$ is a family of mutually disjoint subsets of $\{1, \dots, m\}$, then for $2 \leq i < j < \ell$, the matrix

$$B(\alpha, \beta_2, \dots, \beta_\ell)^{-1} \bar{B}(\alpha, \beta_2, \dots, \beta_1, \dots, \beta_j, \dots, \beta_\ell)$$

is P. Here, for $\beta = \bigcup_{k=2}^{\ell} \beta_k$,

^{5/} By e_k we mean the k -vector of ones.

$$\bar{B}(\alpha, \beta_2, \dots, \beta_i, \dots, \beta_j, \dots, \beta_\ell) =$$

$$\begin{pmatrix} M_{\alpha\alpha} & (N_{\beta_2\alpha}^2)^T & \dots & (N_{\beta_i\alpha}^j)^T & \dots & (N_{\beta_j\alpha}^i)^T & \dots & (N_{\beta_\ell\alpha}^\ell)^T \\ A_{\beta\alpha} & B_{\beta\beta_2}^2 & \dots & B_{\beta\beta_i}^j & \dots & B_{\beta\beta_j}^i & \dots & B_{\beta\beta_\ell}^\ell \end{pmatrix}.$$

Proof of Theorem 5. To simplify the notations, we shall prove the theorem for $\ell = 3$. The general case can be treated in a similar way. Let $\alpha \subseteq \{1, \dots, n\}$, β_2 and $\beta_3 \subseteq \{1, \dots, m\}$ be index sets such that

$$(i) \quad \beta_2 \cap \beta_3 = \emptyset$$

$$(ii) \quad x_\alpha, y_{\beta_2 \cup \beta_3}^2 \text{ and } y_{\beta_3}^3 \text{ are currently basic.}$$

The fact that the set of basic y^2 -variables always contains that of basic y^3 -variables may be justified easily. (See [12, 26] e.g.) The current canonical system has the form

	Constant Column	Parametric Column	u_α	x_δ	$v_{\beta_3}^3$	$v_{\beta_2}^2$	y_γ^2	$v_{\beta_3}^2$	$y_{\beta_2}^3$	y_γ^3
x_α			$\hat{M}_{\alpha\alpha}$	$\hat{M}_{\alpha\delta}$						
u_δ			$-\hat{M}_{\delta\alpha}$	$\hat{M}_{\delta\delta}$						
$y_{\beta_3}^3$				$\hat{A}_{\beta_3\delta}$	$-\hat{B}_{\beta_3\beta_3}^2$					
$y_{\beta_2}^2$				$\hat{A}_{\beta_2\delta}$		$\hat{B}_{\beta_2\beta_2}^2$				
v_γ^2				$\hat{A}_{\gamma\delta}$			$\hat{B}_{\gamma\gamma}^2$			
$y_{\beta_3}^2$		0		0				0		
$v_{\beta_2}^3$				$-\hat{A}_{\beta_2\delta}$					$-\hat{B}_{\beta_2\beta_2}^3$	
v_γ^3		0		0						0

Here δ and γ are the complements of α and $\beta_2 \cup \beta_3$ respectively. The following are P-matrices

(i) $\hat{M}_{\alpha\alpha}$, $\hat{B}_{\beta_2\beta_2}^2$ and $\hat{B}_{\gamma\gamma}^2$ (by assumptions (4.7) and (4.5))

(ii) $-\hat{B}_{\beta_3\beta_3}^2$ and $-\hat{B}_{\beta_2\beta_2}^3$ (by Lemma 1).

Consequently, the next diagonal pivot entry will be positive unless it occurs at a u_δ -row. Suppose now that the next diagonal pivot entry does occur at a u_i -row with $i \notin \alpha$ and \hat{m}_{ii} is zero. Then it follows that

$$\det B(\alpha', \beta_2, \beta_3) = 0$$

where $\alpha' = \alpha \cup \{i\}$. By assumption (4.7), $M_{\alpha'\alpha'}$ is singular. Therefore

$E_{\alpha'}$ has linearly dependent columns. This implies that for some vector

$f_\alpha \neq 0$, we have

$$(4.8) \quad E_{\cdot i} = E_{\alpha'} f_\alpha.$$

By an easy calculation, one can show that

$$\begin{pmatrix} \hat{M}_{\alpha i} \\ \hat{A}_{\beta_3 i} \\ \hat{A}_{\beta_2 i} \end{pmatrix} = B(\alpha, \beta_2, \beta_3)^{-1} \begin{pmatrix} M_{\alpha i} \\ A_{\beta_3 i} \\ A_{\beta_2 i} \end{pmatrix} = \begin{pmatrix} f_\alpha \\ 0 \\ 0 \end{pmatrix}$$

$$\hat{A}_{\gamma i} = A_{\gamma i} - (A_{\gamma \alpha} \quad B_{\gamma \beta_3}^3 \quad B_{\gamma \beta_2}^2) \begin{pmatrix} \hat{M}_{\alpha i} \\ \hat{A}_{\beta_3 i} \\ \hat{A}_{\beta_2 i} \end{pmatrix} = 0$$

and

$$\hat{M}_{\delta i} = M_{\delta i} - (M_{\delta \alpha} (N_{\beta_3 \delta}^3)^T (N_{\beta_2 \delta}^2)^T) \begin{pmatrix} \hat{M}_{\alpha i} \\ \hat{A}_{\beta_3 i} \\ \hat{A}_{\beta_2 i} \end{pmatrix} = 0.$$

Consequently, in the current canonical system, except for those x_α -rows, the entire x_δ -column is zero. Furthermore, we have

$$\hat{M}_{i \alpha} = (M_{i \alpha} (N_{\beta_3 i}^3)^T (N_{\beta_2 i}^2)^T) (B(\alpha, \beta_2, \beta_3)^{-1})_{\cdot \alpha} = f_\alpha = (\hat{M}_{\alpha i})^T$$

Therefore, one can perform a 2×2 block pivot unless $f_\alpha \geq 0$. If $f_\alpha \geq 0$,

then the homogeneous problem

$$\begin{pmatrix} u \\ v^2 \\ v^3 \end{pmatrix} = \begin{pmatrix} N & (N^2)^T & (N^3)^T \\ A & B^2 & B^3 \\ 0 & -I & 0 \end{pmatrix} \begin{pmatrix} x \\ y^2 \\ y^3 \end{pmatrix} \geq 0 \quad \begin{pmatrix} x \\ y^2 \\ y^3 \end{pmatrix} \geq 0$$

$$u^T x = (v^2)^T y^2 = (v^3)^T y^3 = 0$$

has a nonzero complementary solution given by $y^2 = y^3 = 0$ and

$$\tilde{x}_\alpha = f_\alpha, \quad \tilde{x}_i = 1 \quad \text{and} \quad \tilde{x}_j = 0 \quad \text{for} \quad j \notin \alpha'.$$

Now $\tilde{x}^T M \tilde{x} = 0$ implies $N^2 \tilde{x} = N^3 \tilde{x} = 0$ so that if λ^* is the current value of the parameter, we have

$$\begin{aligned}
0 &= \tilde{x}^T u^* = \tilde{x}^T (r + \lambda e) + \tilde{x}^T [Mx^* + \Sigma(N^i)^T (y^i)^*] \\
&= \tilde{x}^T r + \lambda^* e^T \tilde{x} > \tilde{x}^T r
\end{aligned}$$

because λ^* is positive. Consequently the system

$$u = r + Mx + \Sigma(N^i)^T y^i \geq 0, \quad x \geq 0, \quad y^i \geq 0$$

and therefore the problem (4.1), is infeasible. This completes the proof of the theorem.

The last part of the proof also establishes

Corollary 1. Under the assumptions of Theorem 5, if either one of the two systems

$$(i) \quad u = r + Mx \geq 0, \quad x \geq 0$$

$$(ii) \quad u = r + Mx + \Sigma(N^i)^T y^i \geq 0, \quad x \geq 0, \quad y^i \geq 0$$

is consistent, then the PPP algorithm will always compute a solution to problem (4.1).

Remark. Theorem 4.5 remains valid if M is given by

$$M = E^T D^l E + D^{l+1} E$$

provided that the product $D^{l+1} E$ is positive semi-definite.

In general, the conditions in Theorem 5 are not sufficient to guarantee that the PPP will terminate with a solution to the problem (4.1). In what follows we present a sufficient condition which together with those of Theorem 5, will imply that the conditions in Theorem 3 are satisfied. In

particular, the problem (4.1) will then have a solution which can be computed by the PPP algorithm by performing only the 1×1 diagonal pivots.

Theorem 6. Suppose that the assumptions (4.5) - (4.7) in Theorem 5 are satisfied. Furthermore, suppose that there exists an n -vector s satisfying

$$(4.9) \quad r + \bar{\lambda}s \geq 0 \text{ for some scalar } \bar{\lambda} > 0$$

(4.10) For every $\alpha \subseteq \{1, \dots, n\}$ and $i \notin \alpha$ such that the columns in E_α are linearly independent, it holds for no vector f_α such that

$$E_{\cdot i} = E_{\alpha} f_{\alpha} \quad r_i - r_{\alpha}^T f_{\alpha} < 0 \quad \text{and} \quad s_i - s_{\alpha}^T f_{\alpha} > 0 .$$

Then conditions (3.2) and (3.3) in Theorem 3 are satisfied for the problem (4.1) when it is transformed as an ordinary LCP of the form (1.2) with the matrix L given by (4.4).

Proof. Observe that in the current canonical system (given in the proof of Theorem 5), the only place where condition (3.3) could possibly be violated is in a u_i -row with $i \notin \alpha$ and $\hat{m}_{ii} = 0$. By the argument used in the proof of Theorem 5, it is not difficult to show that the current values of the corresponding r - and s -components are given by

$$\bar{r}_i = r_i - r_{\alpha}^T f_{\alpha} \quad \text{and} \quad \bar{s}_i = s_i - s_{\alpha}^T f_{\alpha}$$

respectively. Here f_{α} is given in (4.8). From the given assumption (4.10), the desired conclusion follows readily.

As a concluding remark, we point that in the piecewise linear equilibrium model, the condition

$$E_{\cdot i} = E_{\alpha} f_{\alpha}$$

implies that the arcs indexed by $\alpha \cup \{i\}$ form a minimal cycle. As proved in [28], this cycling condition in turn implies that the current i -component of the r -vector (i.e., \bar{r}_i) is nonnegative, provided that the transportation costs $c_{\alpha\beta}$ (cf. (4.3)) are nonnegative and satisfy the triangle inequality

$$c_{\alpha\beta} + c_{\beta\gamma} \leq c_{\alpha\gamma} .$$

Consequently, under these conditions on the transportation costs, the last theorem implies that the PPP algorithm can be successfully used to compute a solution to the piecewise linear equilibrium model.

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