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ON THE UNIFORMLY MINIMUM VARIANCE UNBIASED ESTIMATORS OF THE VA--ETC(U)
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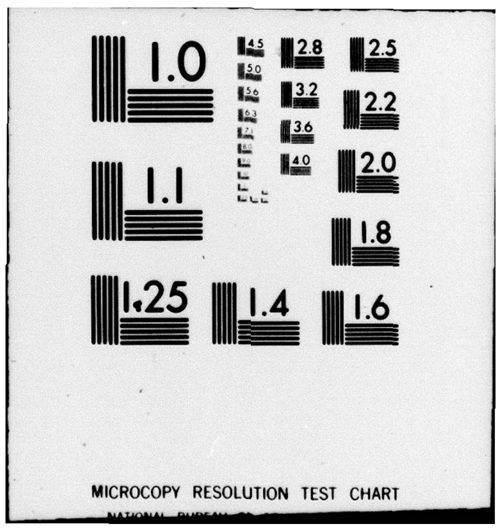
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(6) ON THE UNIFORMLY MINIMUM VARIANCE UNBIASED ESTIMATORS OF THE VARIANCE AND ITS RECIPROCAL OF AN INVERSE GAUSSIAN DISTRIBUTION[1]

by

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(15)

On the uniformly minimum variance unbiased estimators
of the variance and its reciprocal of
an Inverse Gaussian distribution

Abstract

The two-parameter Inverse Gaussian distribution is found to have useful applications in a wide variety of fields. The uniformly minimum variance estimator of its mean is known and is the sample mean. However, no such estimator of the variance is reported in literature. Here the uniformly minimum variance unbiased estimators of the variance and its reciprocal are derived.

KEY WORDS: Inverse Gaussian distribution; reciprocal of the variance; variance; uniformly minimum variance unbiased estimator.

On the uniformly minimum variance unbiased estimators of the variance and its reciprocal of an Inverse Gaussian distribution

1. INTRODUCTION

The Inverse Gaussian distribution was introduced by Schrödinger (1915), and has been studied extensively by many authors, pioneered by Tweedie (1945, 1956, 1957a, 1957b), and Wald (1945). The distribution is found to have useful applications in a wide variety of fields, such as biology, economics, medicine, and reliability and life testing. See the recent review article by Folks and Chhikara (1978) and discussion on it.

Using Tweedie's notation the probability density function of an Inverse Gaussian random variable X is given by

$$f(x;\, \mu,\lambda) = (\lambda/2\pi x^3)^{\frac{1}{2}} \exp\{-\lambda(x-\mu)^2/2\mu^2 x\}, \quad x > 0,$$

$$= 0 \quad , \quad \text{otherwise} \tag{1.1}$$

where μ,λ are both positive. As far as estimation of the parameters μ and λ is concerned, it is known that the sample mean is the uniformly minimum variance unbiased estimator of the mean μ of X.

However, no such estimator has been obtained in literature for the variance μ^3/λ . See Folks and Chhikara (1978). Theorem 1 below gives this estimator, and Theorem 2 gives the uniformly minimum variance unbiased estimator of the reciprocal λ/μ^3 of the variance.

2. THE ESTIMATORS AND THEIR DERIVATIONS

Let $X_1,\ldots,X_n (n \geq 2)$ be a random sample form (1.1), and $\bar{X}$ be the sample mean, and $V = \sum_{i=1}^{n} \left(\frac{1}{X_i} - \frac{1}{\bar{X}} \right)$. Then

Theorem 1. The uniformly minimum variance unbiased estimator of the variance of (1.1) is given by for $n \geq 3$

$$T(X_1,\ldots X_n) = n\bar{X}^2 (1 + U) \tag{2.1}$$

where U is given by

$$U = \frac{1}{B\left(\frac{1}{2}, \frac{n-2}{2}\right)} \left\{ \frac{n}{V\bar{X}} \sum_{r=1}^{k} \left(\frac{V\bar{X} + n}{V\bar{X}} \right)^{r-1} B\left(\frac{1}{2}, \frac{n-2-2r}{2}\right) - \left(\frac{V\bar{X} + n}{V\bar{X}}\right)^{\frac{n-4}{2}} \sqrt{\frac{n}{V\bar{X}}}\, W \right\}$$

where $k = (n-3)/2$ if n is odd $= (n-4)/2$ if n even, and

$W = 2\tan^{-1}\sqrt{(V\bar{X})/n}$ if n is even and $= \pi$ if n is odd, and

where $B(p,q)$ is the usual beta integral. And for $n = 2$ the estimator is given

$$T(X_1, X_2) = \frac{2\bar{X}^3 V}{V\bar{X} + 2} \quad .$$

Proof: It is known that the statistic $(\bar{X}, V)$ is complete and sufficient for the family (1.1). The sample variance $S^2 = \sum_1^n (X_i - \bar{X})^2/(n-1)$ is unbiased for the variance $\sigma^2 = \mu^3/\lambda$. To finish the proof for the case $n = 2$, we apply the Lehmann-Scheffé Theorem to S^2 and note that $S^2 = 2\bar{X}^3 V/(V\bar{X} + 2)$.

The proof for the case $n \geq 3$ is however more involved. For an application of the Lehmann-Scheffé Theorem using S^2, we need to compute the conditional expectation of X_1^2 given $(\bar{X}, V)$. The conditional density $h(x_1|\bar{x}, v)$ of X_1 given $(\bar{X}, V)$ is derived in Chhikara and Folks (1974) in a different context. It is given by, for $n \geq 3$,

$$h(x_1|\bar{x},v) = \frac{\sqrt{n}\,(n-1)}{B\left(\frac{1}{2}, \frac{n-2}{2}\right)} \left\{\frac{\bar{x}^3}{vx_1^3\,(n\bar{x} - x_1)^3}\right\}^{\frac{1}{2}} \left\{1 - \frac{n(x_1 - \bar{x})^2}{v\bar{x}x_1\,(n\bar{x}-x_1)}\right\}^{\frac{n-4}{2}} \qquad (2.2)$$

where $L < x_1 < U$ and L and U are the two roots of the quadratic equation $n(x_1 - \bar{x})^2 = vx_1\bar{x}(n\bar{x} - x_1)$. Now letting $u = \sqrt{n}(x_1-\bar{x})/\{v\bar{x}x_1\,(n\bar{x} - x_1)\}^{\frac{1}{2}}$ in (2.2) we obtain after some simplifications,

$$E(X_1^2|\bar{x},v) = n\bar{x}^2 - \frac{2(n-1)n\bar{x}^2}{B\left(\frac{1}{2}, \frac{n-2}{2}\right)} I \qquad (2.3)$$

where

$$I = \int_0^1 \frac{1}{n+v\bar{x}u^2}\,(1-u^2)^{\frac{n-4}{2}}\,du \quad . \qquad (2.4)$$

To evaluate the integral I in (2.4), first write the integrand as

$$-\frac{1}{v\bar{x}}\sum_{r=1}^{k}\left(\frac{v\bar{x}+n}{v\bar{x}}\right)^{r-1}(1-u^2)^{k-r} + \left(\frac{v\bar{x}+n}{v\bar{x}}\right)^{k}\frac{1}{n+v\bar{x}u^2}\text{, n even}$$

$$(2.5)$$

$$-\frac{1}{v\bar{x}}\sum_{r=1}^{k}\left(\frac{v\bar{x}+n}{v\bar{x}}\right)^{r-1}(1-u^2)^{k-r-\frac{1}{2}} + \left(\frac{v\bar{x}+n}{v\bar{x}}\right)^{k}\frac{(1-u^2)^{-\frac{1}{2}}}{n+v\bar{x}\,u^2}\text{, n odd}$$

where $k = (n-4)/2$ if n is even and $= (n-3)/2$ if n is odd. (In deriving (2.3) use is made of the fact that $x_1 = n\bar{x}(2+v\bar{x}u^2)/2(n+v\bar{x}u^2) + g(u)$, where g is an odd function of u.) Now to finish the proof use (2.5) in (2.4) and standard integrals, and note that $E(S^2|\bar{x},v) = n\,E(X_1^2|\bar{x},v)/(n-1) - n^2\bar{x}^2/(n-1)$.

Theorem 2: The uniformly minimum variance unbiased estimator of the reciprocal of the variance of (1) is given by, for $n \geq 4$,

$$T(X_1,\ldots,X_n) = \frac{(n-3)}{V\bar{X}^3} - \frac{6}{n\bar{X}^2} + \frac{3}{n^2\bar{X}}\frac{V}{(n-1)}\,. \qquad (2.6)$$

Proof: The unbiasedness of (2.6) follows from the expressions for negative moments about zero of an Inverse Gaussian distribution given by Tweedie (1957a) and the facts that (i) $\bar{X}$ and V are independent and that (ii) $\bar{X}$ has an Inverse Gaussian distribution with parameters $(\mu, n\lambda)$, (iii) λV has a chi-square distribution with $(n-1)$ degrees of freedom. An application of the Lehmann-Scheffé Theorem to (2.6) using the fact that $(\bar{X}, V)$ is complete as well as sufficient finishes the proof.

REFERENCES

Chhikara, R. S., and Folks, J. L. (1974), "Estimation of the Inverse Gaussian Distribution Function", Journal of the American Statistical Association, 69, 250-4.

Folks, J. L., and Chhikara, R. S. (1978), "The Inverse Gaussian Distribution and its Application - a Review", Journal of the Royal Statistical Society, Ser. B, 40, 263-89.

Schrödinger, E. (1915), "Zur Theorie der Fall-und Steig versuche an Teilchen mit Brownscher Bewegung", Physikalische Zeitschrift, 16, 289-95.

Tweedie, M. C. K. (1945), "Inverse Statistical Variates", Nature, 155,453

_______, (1956), "Some Statistical Properties of Inverse Gaussian Distributions", Virginia Journal of Science, 7, 160-5.

_______, (1957a), "Statistical Properties of Inverse Gaussian Distributions I", Annals of Mathematical Statistics, 28, 362-77.

_______, (1957b), "Statistical Properties of Inverse Gaussian Distributions II", Annals of Mathematical Statistics, 28, 696-705.

Wald ,A. (1947), Sequential Analysis, New York: Wiley.

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The two-parameter Inverse Gaussian distribution is found to have useful applications in a wide variety of fields. The uniformly minimum variance estimator of its mean is known and is the sample mean. However, no such estimator of the variance is reported in literature. Here the uniformly minimum variance unbiased estimators of the variance and its reciprocal are derived.

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