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ANALYTICAL SOLUTIONS FOR OPEN NONSHALLOW SPHERICAL SHELL VIBRATIONS

by

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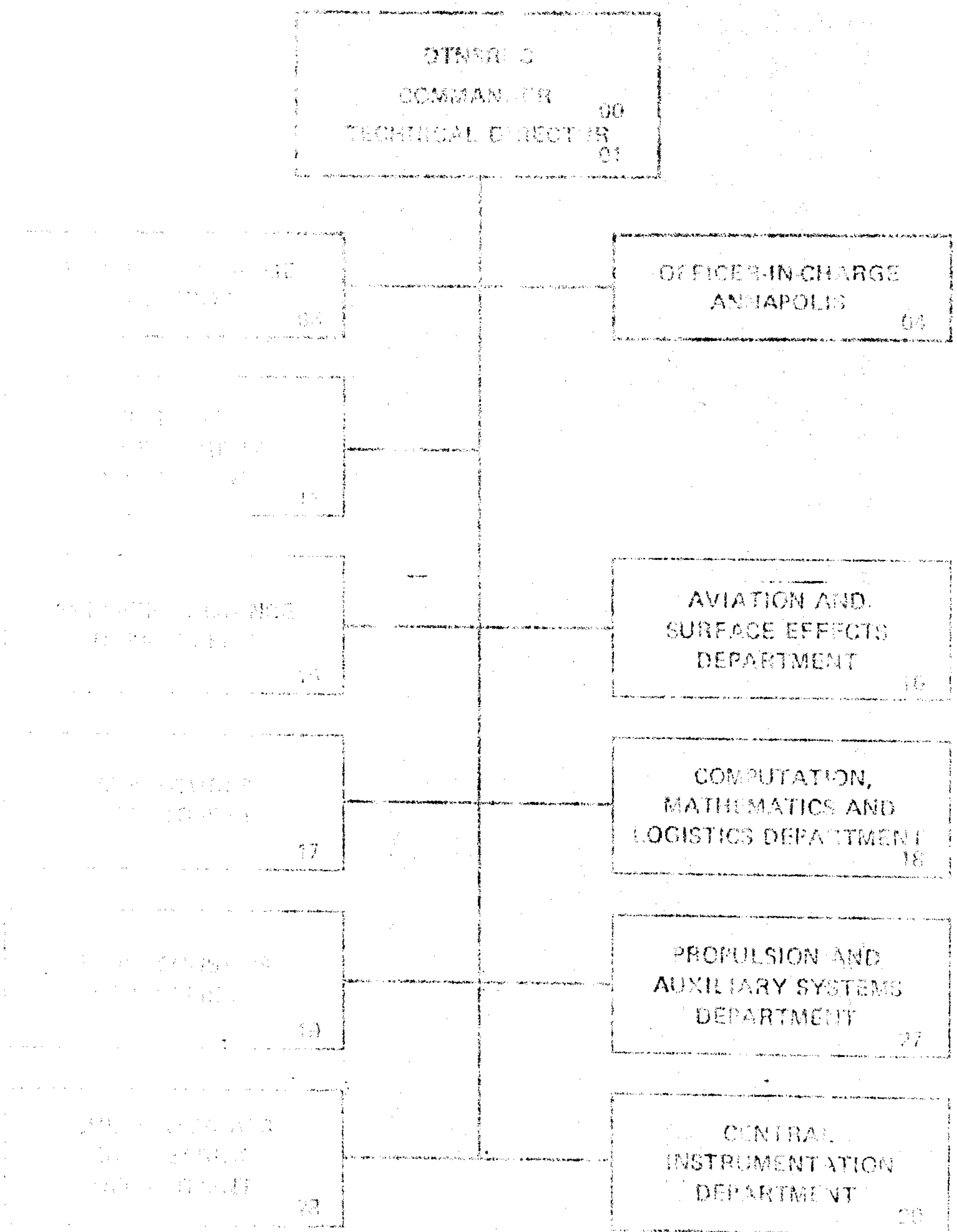

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This report is concerned with axisymmetric as well as nonsymmetric vibrations of open nonshallow thin elastic spherical shells. Without employing the usual auxiliary variables for reduction of shell motion equations introduced by Van der Neut and Berry, independent analytical solutions for middle-surface displacements are obtained and explicitly expressed in terms of associated Legendre functions. In order to gain physical insights into

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the free-vibration characteristics of an open nonshallow shell, theoretical calculations together with asymptotic descriptions are made of natural frequencies and mode shapes of a hemispherical shell with a free edge. The numerical predictions obtained herein compare very well with the experimental results obtained previously by Hwang and recently at the David W. Taylor Naval Ship Research and Development Center, Bethesda, Maryland. Essential features of shell dynamics are ultimately displayed by normalized frequency (Ω) and nondimensional shell thickness parameter (β). Five families of natural frequencies, i.e., low Rayleigh bending, mixed bending-membrane, torsional, bending and membrane frequencies, are found. The corresponding mode shapes exhibit distinctive displacement patterns.

TABLE OF CONTENTS

LIST OF FIGURES

Page

Page

LIST OF TABLES

ABSTRACT

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ADMINISTRATIVE INFORMATION

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INTRODUCTION

Shells of revolution have long been of practical interest in the field of engineering applied mechanics since the 19th century. Their widespread applications such as those in buildings, ships, aircraft and missiles have ever since motivated many investigators to devote themselves to studying the behavior of such shell structures. Considerable progress has been made towards the static analysis of shells of revolution; however, due to the complexity inherent in the mathematical formulation of the dynamic shell problems, relatively less advancement has been achieved pertinent to the dynamic aspect of shell behavior and thus much work still needs to be done in this regard.

In order to explore the shell dynamic characteristics, spherical shells, which offer the convenience of a constant curvature, will be adopted in this report as a simple theoretical model for the analysis purpose. Interestingly enough, the dynamics of open nonshallow spherical shells, which are admittedly simple in configuration, has confronted many researchers in the past. The fundamental aspects of vibration solutions of open nonshallow shells are generally either not analytically exact or not explicitly expressed. This has resulted in few reliable determinations of the shell vibration frequencies and their associated mode shapes, particularly for the more complicated nonsymmetrical vibrations of the shell. Several examples will be cited in the course of following discussions.

Dynamic analysis of spherical shells dated back as early as 1882 when Lamb[1*] investigated the vibrations of a closed spherical shell based on membrane (extensional) energy formulation. Some years later, a famous debate between Rayleigh[2] and Love[3] took place trying to establish a correct theory to predict the vibration of a thin shell. Rayleigh claimed that the low-frequency vibration modes of a thin shell be flexural while Love disputed Rayleigh's hypothesis by explaining that the extensional energy must be the dominant one since the flexural energy is

*A complete listing of references is given on page 69.

proportional to a higher order of shell thickness. This historical debate, according to Kalnins[4], was finally regarded as settled by Ross[5], who disclosed that low frequency inextensional modes are found, by making use of the asymptotic approximations to estimate the general solutions to the shell vibration equations at low frequencies, only when the edge conditions impose no force tangential to the shell surface.

The general shell theory has been considered adequate since the end of the 19th century. Yet it was as late as 1962 that Naghdi and Kalnins[6] first attacked the free vibration problem of an open nonshallow spherical shell using a technique introduced by Van der Neut[7], Havers[8], Federhofer[9] and Berry[10], etc. However, as regards the calculation of natural frequencies of asymmetric vibrations, they erroneously obtained the lowest frequency computed for m=2 (m= circumferential wave number), which can be shown to be impossible by using membrane theory alone for the case of an open spherical shell closed at one end and with a free edge at the other. In fact, several other frequencies are spurious, which are also pointed out by Ross[11].

Due to the inadequacies of the approximation in Ross's[12] earlier paper, he mistakenly concluded that the natural bending vibrations of spherical shells are impossible for $\bar{\Omega} < 1$ ($\bar{\Omega} = \frac{\omega R}{\sqrt{E/\rho_s}}$, where ω = circular frequency, R = radius of curvature of the spherical middle surface of the shell, E = Young's modulus of shell material, ρ_s = mass density of shell material). This statement is not valid for axisymmetric as well as nonsymmetric case, as is evidenced from the exact analysis of the problem.

Hwang[13] successfully carried out some experimental work on the natural frequencies of a hemispherical shell with a free edge, however, he failed to predict analytically the significant Rayleigh low frequency inextensional modes. The reason is simply due to the incorrect description of the solution. The numerical difficulties associated with it further worsen his results.

The relatively unfamiliar and complicated general spherical wave functions, which play the central role in the vibration solutions of the spherical shells, often lead to many pitfalls. For example, an inherent property of the associated Legendre function $P_n^m(x)$ is that it vanishes identically when $m > n$ (m,n = positive integers or zero). Thus, a solution connected with this particular form is, in general, not a true solution. Many spurious results[6,14] are due to this, as discussed in the work of Martynenko and Shpakova[15].

Errors sometimes could arise from the incorrectly posed problems[16] or the results based on incomplete solutions[17] (see References 15 and 18). Generally speaking, not only are the analytical studies concerned susceptible to fallacy, but the numerical techniques as well. For example, Cohen[19] pointed out by his numerical computer technique that the third mode was missing for the axisymmetric vibration of a 60° spherical shell with a fixed-hinged edge in the papers previously studied analytically and numerically by Kalnins[20, 21]. Another example can be found in the illustrative problem of test case No. 5 cited in a numerical computer program called BOSOR 4[22], in which the first three natural frequencies and their associated mode shapes of a vibrating steel hemispherical shell ($\frac{h}{R} = 0.01$, $\bar{\nu} = 0.3$, $E = 10^7$ psi, $\rho = 2.535\text{x}10^{-4}$ lb-sec^2/in^4) with a free edge were given for m = 0, 1, 2. Unfortunately, the results for axisymmetrical case (m = 0) as well as the first frequency for m = 1 are suspicious, as compared with those by the present exact analysis according to the classical thin shell theory[23]. In fact the ambiguous requirements of boundary condition codes at the shell apex[24] and the constrained conditions for prevention of rigid body modes associated with zero and one circumferential wave numbers may reflect the insufficiency of the program. The last example can be directed to the NASTRAN[25] program. To solve the natural frequencies of the open spherical shells, three types of axisymmetric solid element have been available; namely, triangular (TRIARG), trapezoidal (TRAPRG) and (TRZAXX) ring elements. Experience of several trial runs with these ring elements seems to indicate that either the reliability is questionable or the elements are such poor models that they should be refined and the number of them greatly increased if one expects to obtain the desired results. This may result in lengthy and expensive computer time.

In addition to the numerical programs just mentioned above, there are still many other computer oriented methods available. A numerical computer technique, when it comes to practical applications, is of course the most powerful means to generate the desired results for the dynamic problems of open shells. However, the limitations imposed on the computer storage as well as the expensive computer time usually would render the parametric study of the shell problems difficult. In contrast, the physical insight into the shell dynamic characteristics can often be gained by an analytical study of a simple and yet representative model such as an open nonshallow spherical shell.

The previous representative analytical investigations of the axisymmetric free vibrations of open nonshallow shells may be found from Kalnins[20]

Ross[12], and Lizarev[18]. For the nonsymmetrical vibrations of the same shells, reference may be made to the papers by Prasad[26], Wilkinson and Kalnins[27], Ross[5], Hammel[28], Martynenko and Shpakova[15]. The common feature of these previous studies is the use of the technique introduced by Van der Nent[7] and Berry[10] to reduce the shell motion equations to a manageable system of separable equations. Then the displacement solutions obtained, except the normal displacement explicitly expressed by associated Legendre functions, are expressed in terms of those previously introduced auxiliary variables which are in turn expressed by associated Legendre functions. The rather complicated solution forms for the tangential displacements have precluded the exploration of the free-vibration characteristics in a satisfactory way, let alone the extensive application of these basic solutions to more advanced research work.

In view of the insufficiency of the prior studies on the dynamic problems of spherical shells, the purpose of the present study is to derive independently the analytical solutions to the axisymmetrical as well as nonsymmetrical free vibrations of an open nonshallow spherical shell. Possible sets of homogeneous boundary conditions will also be deduced.

The approach will be first to start from the Hamilton's variational principle by which the partial differential equations of shell motion in terms of middle-surface displacements of the spherical shell together with the possible homogenous boundary conditions can be derived. Then a direct reduction of these equations will be carried out, as used by Flügge[29], to result in one single sixth order differential equation involving a single unknown normal displacement (w) and two other coupled equations in middle surface tangential displacements. Without the introduction of the auxiliary variables or stress functions at the very onset of reduction operation as done by prior investigators, the solutions for tangential displacements can be obtained in a straightforward manner. The final analytical solutions for the middle-surface displacements are explicitly expressed in terms of associated Legendre functions P_ν^m (cos ϕ) with complex degree (ν) in general.

It has been well known that the Legendre functions with integer degree (ν), which occurs in the physical problems such as the spherical wave propagation in free space or the vibrations of a closed (complete) spherical shell, are polynomial in cos ϕ and those properties have been thoroughly studied. However, the boundary conditions of an open spherical shell, when used to solve for natural frequencies, would introduce a frequency characteristic equation which is actually a transcendental equation in the complex degree ν[20] of Legendre functions. The numerical values of associated Legendre functions with complex degree, though not widely tabulated, can be evaluated by hypergeometric series representations.

As an illustrative example, the theoretical calculations are made of the natural frequencies and mode shapes of a hemispherical shell with a free edge. The numerical predictions obtained herein compare very well with the experimental results obtained previously by Hwang[13] and recently at the David W. Taylor Naval Ship R&D Center (DTNSRDC), Bethesda, Maryland. The display of the characteristic parameters of shell dynamics is also explored by the asymptotic descriptions of the solutions. The analytical solutions presented in this report offer the simplicity in their compact forms, thus providing an analytical path towards the further researches of a similar nature.

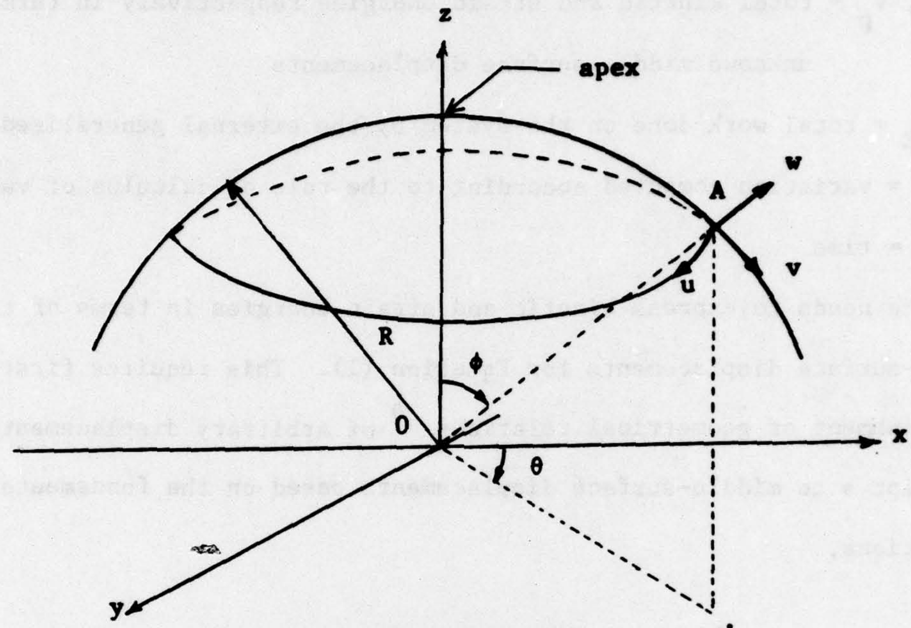
DESCRIPTION OF PROBLEM

Consider a thin, elastic open, nonshallow spherical shell of uniform thickness h, mass density ρ_s, Young's modulus E, Poisson's ratio $\bar{\nu}$ and mean radius R, and refer to a system of spherical polar coordinates as shown in Figure 1. In this study, the shell material is taken to be homogeneous and isotropic. The fundamental assumptions of the analysis has been within the framework of the classical linear shell theory; namely,

1. $\frac{h}{R} \ll 1$,
2. displacements are small compared to the thickness h, and strains small compared to unity
3. stress normal to the middle surface are small compared with other components of stresses
4. plane cross sections normal to the middle surface before deformation remain normal to the middle surface after deformation
5. variation in normal displacements within the shell thickness is negligible during deformation,
6. shell material obeys Hooke's law i.e., the stress is proportional to strain

With the assumptions just mentioned, the complete description of the shell deformation can be fully specified by the middle surface displacements denoted by u, v, and w (Figure 1).

It is noted here that both flexural (bending) and extensional (membrane) strain energy terms are included in the following vibration analysis of open spherical shells.



ϕ (Meridional Angle)

θ (Azimuthal Angle)

Figure 1 - Spherical Coordinates and Middle Surface Displacements (w,v,u), R = Radius of Middle Surface of the Shell

$x = R \sin \phi \cos \Theta$

$y = R \sin \phi \sin \Theta$

$z = R \cos \phi$

EQUATIONS OF SPHERICAL SHELL MOTION AND BOUNDARY CONDITIONS

The derivations of equations for spherical shell motion will be made by using the Hamilton's variational principle, given by:

$$\delta \int_{t_o}^{t_1} (T - V_p)\, dt + \int_{t_o}^{t_1} \delta W_E\, dt = 0 \qquad (1)$$

where:

T, V_p = total kinetic and strain energies respectively in terms of unknown middle-surface displacements

W_E = total work done on the system by the external generalized forces

δ = variation computed according to the rule of calculus of variation

t = time

One needs to express kinetic and strain energies in terms of the middle-surface displacements for Equation (1). This requires first the establishment of geometrical relations [29] of arbitrary displacements with subscript s to middle-surface displacements based on the fundamental assumptions,

$$\begin{aligned} w_s &= w \\ v_s &= \left(\frac{R+s}{R}\right) v - \frac{s\dot{w}}{R} \\ u_s &= \left(\frac{R+s}{R}\right) u - \frac{sw'}{R \sin\phi} \end{aligned} \tag{2}$$

where: $\frac{\partial(\ )}{\partial\phi} = (\dot{\ }),\ \frac{\partial(\ )}{\partial\theta} = (\ )'$,

s = arbitrary distance in radial direction within shell thickness, together with a general strain (ε) and displacement relations:

$$\begin{aligned} \varepsilon_\phi &= \frac{\dot{v}_s + w_s}{R + s} \\ \varepsilon_\theta &= \frac{u_s' + v_s \cos\phi + w_s \sin\phi}{(R + s)\sin\phi} \\ \varepsilon_{\phi\theta} &= \frac{v_s'}{(R + s)\sin\phi} + \frac{\dot{u}_s}{R + s} - \frac{\cot\phi}{R + s} u_s \end{aligned} \tag{3}$$

Then the linear stress (σ) and strain relations need to be developed based on the Hooke's law:

$$\sigma_\phi = \frac{E}{1 - \bar{\nu}^2} (\varepsilon_\phi + \bar{\nu}\, \varepsilon_\theta)$$

$$\sigma_\theta = \frac{E}{1 - \bar{\nu}^2} (\varepsilon_\theta + \bar{\nu}\, \varepsilon_\phi) \qquad (4)$$

$$\sigma_{\phi\theta} = \frac{E}{2(1 + \bar{\nu})} \varepsilon_{\phi\theta}$$

With these basic relations at hand, the total strain energy as well as total kinetic energy can be expressed as the following:

$$V_p = \frac{1}{2}\iiint\left(\sigma_\phi\, \varepsilon_\phi + \sigma_\theta\, \varepsilon_\theta + \sigma_{\phi\theta}\, \varepsilon_{\phi\theta}\right) (R + s)^2 \sin\phi\, d\phi\, d\theta\, ds \qquad (5)$$

$$= \frac{E}{2(1-\bar{\nu}^2)} \iiint\left(\varepsilon_\phi{}^2 + \varepsilon_\theta{}^2 + 2\,\bar{\nu}\, \varepsilon_\phi\, \varepsilon_\theta + \frac{1 - \bar{\nu}}{2}\, \varepsilon_{\phi\theta}{}^2\right) \qquad (6)$$

$$\times\ (R + s)^2 \sin\phi\, d\phi\, d\theta\, ds,$$

$$T = \frac{\rho_s}{2}\iiint\left[\left(\frac{\partial w_s}{\partial t}\right)^2 + \left(\frac{\partial v_s}{\partial t}\right)^2 + \left(\frac{\partial u_s}{\partial t}\right)^2\right](R+s)^2 \sin\phi\, d\phi\, d\theta\, ds \qquad (7)$$

Here, the integration extends over the entire volume of the body under consideration (i.e., from $\phi = 0^\circ$ to $\phi = \phi_o$ (at open edge); from $\theta = 0^\circ$ to $\theta = 2\pi$; from $s = -\frac{h}{2}$ to $s = \frac{h}{2}$).

For the present mathematical model, the contributions of the rotatory inertia to the kinetic energy in Equation (7) and the effects of transverse shear deformation and the rotatory inertia on the spherical shell are neglected. The method of analysis is based on the classical linear thin shell theory.

The total work done by external force, say the outward normal pressure P_a, will be given by

$$W_E = \int_0^{\phi_o} \int_0^{2\pi} (P_a w_s R^2 \sin \phi \, d\phi \, d\theta \tag{8}$$

After substituting Equations (6), (7), and (8) into Equation (1) and manipulating according to the rules of calculus of variation, one is able to obtain the following equations of shell motion:

$$\beta^2 \Bigg\{ \Bigg[- \ddddot{w} - 2 \cot \phi \, \dddot{w} + (1+\bar{\nu}+\cot^2\phi) \, \ddot{w} + \cot \phi \, (\bar{\nu}-2-\cot^2\phi) \, \dot{w}$$

$$+ \dddot{v} + 2 \cot \phi \, \ddot{v} - (1+\bar{\nu}+\cot^2\phi) \, \dot{v} + \cot \phi \, (2-\nu+\cot^2\phi) \, v \Bigg]$$

$$+ \Bigg[- \frac{w''''}{\sin^4\phi} - 2 \frac{\ddot{w}''}{\sin^2\phi} + 2 \frac{\cot \phi}{\sin^2\phi} \dot{w}'' + \frac{\bar{\nu} - 3 - 4 \cot^2\phi}{\sin^2\phi} w''$$

$$+ \frac{\dot{v}''}{\sin^2\phi} + \frac{\cot \phi}{\sin^2\phi} v'' + \frac{u'''}{\sin^3\phi} - \frac{\cot \phi}{\sin \phi} \dot{u}' + \frac{\ddot{u}'}{\sin \phi}$$

$$+ (2-\bar{\nu}+\cot^2\phi) \frac{u'}{\sin \phi} \Bigg] \Bigg\}$$

$$+ \Bigg[- (1+\bar{\nu}) \, \dot{v} - (1+\bar{\nu}) \, v \cot \phi - 2 \, (1+\bar{\nu}) \, w - \frac{1 + \bar{\nu}}{\sin \phi} u'$$

$$- \frac{R^2}{c_p^2} \frac{\partial^2 w}{\partial t^2} \Bigg] = - P_a \frac{(1-\bar{\nu}^2)R^2}{Eh} \tag{9a}$$

$$\beta^2 \left\{ \left[-\dddot{w} - \cot\phi\, \ddot{w} + (\bar{\nu}+\cot^2\phi)\, \dot{w} + \ddot{v} + \cot\phi\, \dot{v} - (\bar{\nu}+\cot^2\phi)\, v \right] \right.$$

$$+ \left(- \frac{\dot{w}''}{\sin^2\phi} + 2\,\frac{\cot\phi}{\sin^2\phi}\, w'' + \frac{1-\bar{\nu}}{2}\,\frac{v''}{\sin^2\phi} + \frac{1+\bar{\nu}}{2}\,\frac{\dot{u}'}{\sin\phi} \right.$$

$$\left.\left. + \frac{\bar{\nu}-3}{2}\,\frac{\cot\phi}{\sin\phi}\, u' \right) \right\} + \left[\ddot{v} + \cot\phi\, \dot{v} - (\bar{\nu}+\cot^2\phi)\, v \right.$$

$$+ \frac{1-\bar{\nu}}{2}\,\frac{v''}{\sin^2\phi} + (1+\bar{\nu})\, \dot{w}$$

$$+ \frac{\bar{\nu}-3}{2}\,\frac{\cot\phi}{\sin\phi}\, u' + \frac{1+\bar{\nu}}{2}\,\frac{\dot{u}'}{\sin\phi}$$

$$\left. - \frac{R^2}{c_p^2}\,\frac{\partial^2 v}{\partial t^2} \right] = 0 \qquad \text{(9b)}$$

$$\beta^2 \left\{ \left(\frac{1-\bar{\nu}}{2}\right) \left[\ddot{u} + \cot\phi\, \dot{u} + (1-\cot^2\phi)\, u \right] \right.$$

$$+ \left(- \frac{w'''}{\sin^3\phi} - \frac{\ddot{w}'}{\sin\phi} - \frac{\cot\phi}{\sin\phi}\, \dot{w}' + \frac{\bar{\nu}-1}{\sin\phi}\, w' + \frac{1+\bar{\nu}}{2}\,\frac{\dot{v}'}{\sin\phi} \right.$$

$$\left.\left. + \frac{3-\bar{\nu}}{2}\,\frac{\cot\phi}{\sin\phi}\, v' + \frac{u''}{\sin^2\varphi} \right) \right\}$$

$$+ \left[\frac{u''}{\sin^2\phi} + \frac{1-\bar{\nu}}{2}\cot\phi\, \dot{u} + \frac{1-\bar{\nu}}{2}(1-\cot^2\phi)\, u + \frac{1-\bar{\nu}}{2}\, \ddot{u} \right.$$

$$+ \frac{3-\bar{\nu}}{2}\,\frac{\cot\phi}{\sin\phi}\, v' + \frac{1+\bar{\nu}}{\sin\phi}\, w' + \frac{1+\bar{\nu}}{2}\,\frac{\dot{v}'}{\sin\phi}$$

$$\left. - \frac{R^2}{c_p^2}\,\frac{\partial^2 u}{\partial t^2} \right] = 0 \qquad \text{(9c)}$$

where

$$\beta^2 = \frac{h^2}{12R^2} \qquad c_p = \sqrt{\frac{E}{(1 - \bar{\nu}^2)\,\rho_s}}$$

The sixteen possible sets of boundary conditions along each open edge ($\phi = \phi_o$) of the shell are furnished by the independent vanishing of the terms of line integrals through the manipulation of Equation (1). They are either stress or displacement boundary conditions. The required four boundary conditions at each open edge can be of the following combinations (totaling sixteen possible sets):

(i) $\sin\phi\,(RN_\phi + M_\phi) = 0$ or $v = 0$

(ii) $\sin\phi\,(RN_{\phi\theta} + M_{\phi\theta})$ or $u = 0$

(iii) $R\sin\phi\,Q_\phi + \frac{\partial M_{\phi\theta}}{\partial\theta} = 0$ or $w = 0$

(iv) $\sin\phi\,M_\phi = 0$ or $\frac{dw}{d\phi} = 0$ (10)

where:

$$N_\phi = \frac{Eh}{1 - \bar{\nu}^2}(\varepsilon_\phi + \bar{\nu}\,\varepsilon_\theta)$$

$$N_\theta = \frac{Eh}{1 - \bar{\nu}^2}(\varepsilon_\theta + \bar{\nu}\,\varepsilon_\phi)$$

$$N_{\phi\theta} = N_{\theta\phi} = \frac{Eh}{2(1 + \bar{\nu})}\varepsilon_{\phi\theta}$$

$$M_\phi = D\,(K_\phi + \bar{\nu}\,K_\theta)$$

$$M_\theta = D\,(K_\theta + \bar{\nu}\,K_\phi)$$

$$M_{\phi\theta} = M_{\theta\phi} = D\left(\frac{1-\bar{\nu}}{2}\right) K_{\phi\theta}$$

$$R \sin\phi\; Q_\phi = \frac{\partial(M_\phi \sin\phi)}{\partial\phi} + \frac{\partial M_{\phi\theta}}{\partial\theta} - M_\theta \cos\phi \qquad (11)$$

Here, $(\varepsilon_\phi, \varepsilon_\theta, \varepsilon_{\phi\theta})$ refer to the strains in the middle surface of the shell. The flexural rigidity of the shell (D) and the changes of curvatures $(K_\phi, K_\theta, K_{\phi\theta})$ are defined as follows:

$$D = \frac{Eh^3}{12\,(1-\bar{\nu}^2)}\,, \qquad K_\phi = \frac{1}{R^2}\,(\dot{v} - \ddot{w})$$

$$K_\theta = \frac{1}{R^2}\left(v \cot\phi + \frac{u'}{\sin\phi} - \frac{w''}{\sin^2\phi} - \dot{w}\cot\phi\right)$$

$$K_{\phi\theta} = \frac{1}{R^2}\left(\frac{2\,w'\cos\phi}{\sin^2\phi} - \frac{2\,\dot{w}'}{\sin\phi} + \dot{u} - u\cot\phi + \frac{v'}{\sin\phi}\right) \qquad (12)$$

REDUCTION OF EQUATIONS OF MOTION

In order to achieve a solution, one must simplify the equations of shell motion into a manageable system. This can be accomplished through the use of the spherical harmonic operator (∇^2), which is defined as

$$\nabla^2(\;) = \frac{\partial^2(\;)}{\partial\phi^2} + \cot\phi\,\frac{\partial(\;)}{\partial\phi} - \frac{m^2(\;)}{\sin^2\phi}\,, \qquad (13)$$

together with the expansion of displacements in the following forms:

$$w = W(\phi) \cdot \cos m\theta \cdot e^{i\omega t}$$

$$v = V(\phi) \cdot \cos m\theta \cdot e^{i\omega t}$$

$$u = U(\phi) \cdot \sin m\theta \cdot e^{i\omega t} \tag{14}$$

where

ω = circular frequency

m = circumferential wave number

$i = \sqrt{-1}$

By substituting Equation (14) into Equations (9a, b, c), one can first obtain the following simplified equation by combining Equations (9b) and (9c):

$$\nabla^2\eta + (2+\varepsilon)\,\eta = 0 \tag{15}$$

where

$$\eta = \frac{mV}{\sin\phi} + \dot{U} + \cot\phi U$$

$$\varepsilon = \left(\frac{1}{1+\beta^2}\right)\frac{2\,\Omega^2}{1-\bar{\nu}}$$

$$\Omega = \frac{\omega R}{c_p} = \frac{f}{c_p/(2\pi R)} \quad (f = \frac{\omega}{2\pi}) \tag{16}$$

Then, another simplified differential equation can be made by using the relation in Equation (15) and eliminating V in Equation (9a):

$$m\,\{\nabla^4 W + 2\nabla^2 W + [(1-\bar{\nu}^2)\frac{1+\beta^2}{\beta^2} - \frac{\Omega^2}{\beta^2}]\,W - \frac{(1-\bar{\nu}^2)R^2 p_a}{Eh}\}$$

$$+ \{\nabla^4 (U \sin\phi) + 2\nabla^2 (U \sin\phi) + [(1-\bar{\nu}^2)\frac{1+\beta^2}{\beta^2} + (\frac{1+\bar{\nu}}{\beta^2})\,\Omega^2](U \sin\phi)\}$$

$$+ \{-2\nabla^2(\dot{\eta} \sin\phi) + [\frac{1}{2}(1-\bar{\nu}^2)\frac{1+\beta^2}{\beta^2} - \frac{2\Omega^2}{(1+\beta^2)(1-\bar{\nu})}](\dot{\eta} \sin\phi)\} = 0 \tag{17}$$

STEADY-STATE SOLUTIONS OF OPEN SPHERICAL SHELL VIBRATIONS

In what follows, the procedures for arriving at the solutions of unknown middle-surface displacements, which are the key solutions of the free open spherical shells, are briefly described.

First, by setting $p_a = 0$ and eliminating V and U between Eqs. (15) and (17), a single sixth-order uncoupled partial differential equation in normal displacement (W) will result:

$$\nabla^6 W + c_2 \nabla^4 W + c_1 \nabla^2 W + c_o W = 0 \tag{18}$$

where

$$c_2 = 4 + \Omega^2$$

$$c_1 = 4 + (1+\frac{1}{\beta^2})(1-\bar{\nu}^2) - (\bar{\nu}+\frac{1}{\beta^2})\,\Omega^2$$

$$c_o = 2\left(1+\frac{1}{\beta^2}\right)(1-\bar{\nu}^2) + \left[\frac{1}{\beta^2}(1+3\bar{\nu}) + \bar{\nu}-1\right]\Omega^2 - \frac{\Omega^4}{\beta^2}$$

The familiar expression in Equation (18) is also given by several authors[20,26,27].

The solution of Equation (18) can be concisely expressed in terms of the associated Legendre functions of the first and second kinds, i.e., $P_\nu^m(\cos\phi)$, $Q_\nu^m(\cos\phi)$, of the order m and general complex degree ν:

$$W = \sum_{j=1}^{3}\left[A_j^m P_{\nu j}^m(\cos\phi) + B_j^m Q_{\nu j}^m(\cos\phi)\right] \tag{19}$$

where

$$\nu_j = -\frac{1}{2} + \sqrt{\frac{1}{4} + \lambda_j} \tag{20}$$

Here, the parameter λ_j must satisfy the following characteristic equation:

$$\lambda^3 - c_2\lambda^2 + c_1\lambda - c_o = 0 \tag{21}$$

It should be noted here that the choice of either sign, positive or negative, before the square root in Equation (20) is sufficient without repetition for the determination of the parameter ν_j, as can be perceived from one property of associated Legendre function, i.e., $P_\nu^m = P_{\nu-1}^m$ and $Q_{\nu-1}^m = Q_\nu^m - \pi P_\nu^m \cot\nu\pi$. Also, one of the roots in Equation (21) is always real and positive, while the other two roots are either complex conjugate or real.

As to the solutions for middle-surface displacements, they can be obtained in a straightforward manner, namely, by eliminating the normal **displacement** (W) between Equations (17) and (18), and remembering that the solution of the variable η is readily given by

$$\eta = G_q^m P_q^m (\cos\phi) + H_q^m Q_q^m (\cos\phi) \tag{22}$$

where

$$q = -\frac{1}{2} + \sqrt{\frac{1}{4} + (2+\varepsilon)} \tag{23}$$

The solution for U then is easily obtained. Once U is obtained, Equation (15) will provide the solution for V.

The final exact steady-state solutions for the middle-surface displacements w, v, and u are explicitly obtained in terms of associated Legendre functions.

$$w = \sum_{m=0}^{\infty} \sum_{j=1}^{3} \left[A_j^m P_{\nu j}^m (\cos\phi) + B_j^m Q_{\nu j}^m (\cos\phi) \right] \cos m\theta \, e^{i\omega t} \tag{24a}$$

$$v = \sum_{m=0}^{\infty} \left\{ \sum_{j=1}^{3} (-D_j) \left[A_j^m \dot{P}_{\nu j}^m (\cos\phi) + B_j^m \dot{Q}_{\nu j}^m (\cos\phi) \right] + \frac{m^2}{(2+\varepsilon)\sin\phi} \left[G_q^m P_q^m (\cos\phi) + H_q^m Q_q (\cos\phi) \right] \right\} \cos m\theta \, e^{i\omega t} \tag{24b}$$

$$u = \sum_{m=0}^{\infty} \left\{ \sum_{j=1}^{3} \frac{m\, D_j}{\sin\phi} \left[A_j^m\, P_{\nu j}^m (\cos\phi) + B_j^m\, Q_{\nu j}^m (\cos\phi) \right] \right.$$

$$\left. - \frac{1}{2+\epsilon} \left[G_q^m\, \dot{P}_q^m (\cos\phi) + H_q^m\, \dot{Q}_q^m (\cos\phi) \right] \right\} \sin m\theta\, e^{i\omega t} \tag{24c}$$

where

$$D_j = \frac{1 + \bar{\nu} + \beta^2(\lambda_j + \bar{\nu} - 1)}{\Omega^2 - (1+\beta^2)(\lambda_j + \bar{\nu} - 1)} \tag{25}$$

and the notations (A_j^m, B_j^m, G_q^m, H_q^m) represent eight independent arbitrary (complex in general) constants.

The analytical solutions given in Equation (24) constitute a convenient mathematical tool for exploration of the characteristics of parameters played in the dynamic problems of a general open spherical shell. However, considerations should be given to several aspects of the solution forms, such as mentioned in the following:

1. In order to determine the motion of an open spherical shell completely, one has to evaluate the eight independent arbitrary constants (A_j^m, B_j^m, G_q^m, H_q^m) for each circumferential wave number (m) by imposing the eight boundary conditions at both edges of the shell, which is in general bounded by two concentric openings, four at each edge of the spherical zone.

2. For the study of free vibrations of a spherical shell closed at one pole and open at the other, the four arbitrary constants (B_j^m, H_q^m) have

to be dropped to insure the finiteness of the displacements at the apex, which has replaced the application of four boundary conditions at the point.

3. The character of the solution, as discussed in detail by Kalnins[20] and Wilkinson[27], depends strongly on that of the degrees ν_j and q, of which q and one of ν_j are always real and positive, while the other two of ν_j either complex conjugate or real. When any one of the degrees (say, ν_1) is a natural number (positive integer or zero), and $m>\nu_1$, where m is a positive integer, then the particular solution corresponding to this degree is $A_1^m P_{\nu 1}^{-m} (\cos \phi)$[30] and not $A_1^m P_{\nu 1}^m (\cos \phi)$ since $P_{\nu 1}^m (\cos \phi) = 0$ for this case.

4. The multiple roots of the parameter λ_j (say, $\lambda_2=\lambda_3=\alpha$) could arise for particular structural parameters and frequency from the characteristic Equation (21). In this instance, the original two subsolutions, i.e., $A_2^m P_{\nu 2}^m$ and $A_3^m P_{\nu 3}^m$, are collapsed into one independent subsolution $A_2^m P_{\nu 2}^m$ ($\nu_2=\nu_3$). This renders the solution forms insufficient. The other independent subsolution should be sought from other means, for example, the method of variation of parameters.

5. It can be shown by limiting process as $\phi \to 0^o$ that the displacement solutions given in Equation (24), for the case of no opening (hole) at the apex, are bounded at this point ($\phi=0^o$).

NATURAL FREQUENCIES AND MODE SHAPES FOR AN OPEN SPHERICAL SHELL CLOSED AT ONE POLE

As previously stated, the vibration analysis of a spherical shell closed at one pole and open at the other requires four homogeneous boundary conditions at the open edge for each wave number (m), by which one can form a (4 x 4) determinant. The vanishing of the determinant, which would result in a characteristic frequency equation

$$|D_{ij}| = 0, \quad (i,j = 1, 2, 3, 4) \tag{26}$$

will give rise to the natural frequencies for that particular shell. The determinant elements D_{ij} are, of course, determined by the nature of associated boundary conditions. Upon the determination of the natural frequencies, the associated mode shapes can be readily obtained from Equation (24).

In general, the calculated torsionless natural frequencies of an open classical thin shell can be mainly classified as the following two types:

1. Membrane (extensional, stretching) natural frequencies:

   These frequencies are very nearly independent of shell thickness and can be closely determined by using membrane theory alone. They will be designated as predominantly membrane frequencies (or membrane frequencies). For an open torsionless membrane spherical shell, there are two (low and high) branches of membrane frequencies. Since the singularity at $\Omega = \sqrt{1-\bar{\nu}^2}$ associated with a membrane shell, which results in the formation of an infinite low branch of membrane frequency, is unrealistically created by neglecting the bending stiffness of the shell, only the high branch of membrane frequency together with the first few frequencies in the low circumferential wave number of the low branch would survive as the predominantly membrane frequencies for a general open spherical shell. It should be pointed out here that the appropriate boundary conditions for an open membrane shell can be chosen from the combinations of

$$\text{a.} \quad v = 0 \quad \text{or} \quad \sin\phi\,(RN_{\phi} + M_{\phi}) = 0$$

$$\text{b.} \quad u = 0 \quad \text{or} \quad \sin\phi\,(RN_{\phi\theta} + M_{\phi\theta}) = 0$$

at an open edge of the membrane shell ($\phi=\phi_o$). Thus, in a general open spherical shell, where bending stiffness as well as membrane stretching are accounted for, the predominantly membrane frequencies are expected to be influenced by the boundary conditions involving the tangential displacements.

2. Bending (inextensional, flexural) natural frequencies:

These frequencies depend strongly on the dimensionless shell thickness and with the mode number. They are termed as predominantly bending frequencies (or bending frequencies).

In addition to the two main categories of the torsionless frequency just described above, there is another group of frequencies that are the products of combined contributions from comparable membrane and bending energies within the shell. They usually occur in the intermediate frequency range such as $0.82<\Omega< \sqrt{2(1+\bar{\nu})}$, where the strong coupling between membrane and bending effects is expected to take place. These frequencies can be termed as combined bending-membrane frequencies for future reference.

NUMERICAL EXAMPLE

As an illustrative example, the axisymmetric as well as nonsymmetric free-vibration analysis of a hemispherical shell closed at the apex with a free open edge is demonstrated in detail. The boundary conditions for this particular shell are directly obtained from Equation (10) at $\phi_o = \frac{\pi}{2}$

$$N_\phi = 0 \tag{27a}$$

$$RN_{\phi\theta} + M_{\phi\theta} = 0 \tag{27b}$$

$$RQ_\phi + \frac{\partial M_{\phi\theta}}{\partial\theta} = 0 \tag{27c}$$

$$M_\phi = 0 \tag{27d}$$

The explicit expressions for the elements (D_{ij}) in the frequency determinant, Equation (26), are given by substituting displacement solutions in Equations (24) into Equations (27):

$$D_{1j} = [1 + \bar{\nu} + \lambda_j D_j - m^2 (1-\bar{\nu}) D_j] P^m_{\nu j}$$

$$D_{2j} = [(1+\beta^2)D_j + \beta^2] \dot{P}^m_{\nu j}$$

$$D_{3j} = (1+D_j) [(\lambda_j+\bar{\nu}-1) + m^2 (1-\bar{\nu})] \dot{P}^m_{\nu j}$$

$$D_{4j} = (1+D_j) [\lambda_j - m^2 (1-\bar{\nu})] P^m_{\nu j}$$

$$D_{14} = D_{44} = \frac{m^2 (1-\bar{\nu})}{2 + \varepsilon} \dot{P}^m_q$$

$$D_{24} = (1+\beta^2)\left(\frac{1}{2} - \frac{m^2}{2 + \varepsilon}\right) P^m_q$$

$$D_{34} = \frac{m^2 (1-m^2)(1-\bar{\nu})}{2 + \varepsilon} \dot{P}^m_q \tag{28}$$

where j = 1, 2, 3.

Natural Frequencies

Numerical calculations of natural frequencies are carried out for axisymmetrical ($m=0$) and nonsymmetrical vibration ($m \neq 0$) from the characteristic frequency Equation (26) with the elements expressed in Equation (28). The calculated frequencies are given in Table 1, and plots of the frequencies versus circumferential wave number (m) are shown in Figures (2) and (3).

Torsional frequencies can be easily obtained by requiring one boundary condition, $N_{\phi\theta}\ (\phi_o = \frac{\pi}{2}) = 0$, with the circumferential displacement

$$U = - \frac{\dot{P}_n^m\ (\cos\ \phi)}{2 + \varepsilon} B,$$

and are given from Legendre polynomial $P_n\ (o) = 0$, where $n = 1, 3, 5...$odd. The Legendre polynomial $P_n\ (\cos\ \phi)$ exhibits n nodal points from $\phi = o^o$ to 180^o along the meridional direction. This allows one to drive the following simple formula for torsional frequencies:

$$\Omega_n = \sqrt{\frac{1}{2}\ (1+\beta^2)(1-\bar{\nu})[n(n+1)-2]} \quad (n=1,\ 3,\ 5...\text{odd}) \qquad (29)$$

Like membrane frequencies, torsional frequencies arise at relatively high frequencies and are very little influenced by the shell thickness ($\beta^2<<1$). To aid in visualizing the relative positions between torsional and membrane frequencies, a plot of calculated torsional and axisymmetric torsionless natural frequencies versus the nodal numbers (n) is made in Figure (4) for a free-edged hemispherical membrane shell.

TABLE 1 - NATURAL FREQUENCY (Ω) FOR A TORSIONLESS HEMISPHERICAL SHELL ($\phi=\pi/2, \bar{\nu}=0.3$) WITH A FREE-EDGE FOR DIFFERENT CIRCUMFERENTIAL WAVE NUMBERS (m)

m	0	1	2	3	4	5	6	7	8	9	10
h/R =0.01			0.012	0.034	0.064	0.102	0.146	0.197	0.253	0.315	0.381
h/R = $\frac{3/16}{16}$	0.831	0.836	0.014	0.040	0.074	0.118	0.169	0.227	0.292	0.362	0.438
	0.910	0.910	0.876	0.903	0.922	0.939	0.955	0.974	0.997	1.025	1.060
	0.940	0.942	0.926	0.941	0.957	0.976	0.999	1.028	1.063	1.106	1.158
	0.960	0.975	0.957	0.975	0.998	1.027	1.063	1.106	1.158	1.219	1.289
	0.997	1.024	0.997	1.026	1.061	1.104	1.156	1.217	1.287	1.367	1.457
	1.059	1.101	1.059	1.102	1.153	1.214	1.284	1.364	1.453	1.552	1.660
	1.151	1.210	1.151	1.211	1.281	1.360	1.449	1.548	1.656	1.773	1.900
	1.278	1.356	1.156	1.358	1.446	1.544	1.652	1.769	1.895	2.030	2.174
	1.443	1.403	1.279	1.542	1.649	1.765	1.891	2.026	2.169	2.321	2.482
	1.646	1.540	1.444	1.753	1.888	2.022	2.165	2.317	2.477	2.646	2.822
	1.886	1.762	1.647	1.763	2.163	2.314	2.473	2.641	2.818	3.002	3.194
	1.975	2.019	1.886	2.020	2.301	2.638	2.814	2.998	3.189	3.388	3.595
	2.160	2.311	2.161	2.312	2.471	2.838	3.186	3.385	3.591	3.805	4.027
	2.469	2.432	2.204	2.636	2.811	2.995	3.372	3.802	4.023	4.252	4.488
	2.809	2.635	2.469	2.987	3.183	3.382	3.588	3.905	4.440	4.727	
	3.181	2.786	2.810	2.993	3.586	3.799	4.020	4.248	4.484		
	3.584	2.992	3.032	3.380	3.696	4.246	4.481	4.724			
	3.635	3.379	3.182	3.705	4.018	4.339					
	4.016	3.738	3.584	3.797	4.454	4.722					
	4.477	3.746	3.667	4.244	4.479						

Several points of interest can be observed from Figures (2), (3) and (4).

a. First of all, an excellent agreement is indicated between the theoretical predictions of a sequence of low frequency (Ω<<1) by the present solutions and the experimental results previously obtained by Hwang[(13)] and recently at the David W. Taylor Naval Ship R&D Center (DTNSRDC), Bethesda, Maryland*. The display of this distinctive sequence of low bending frequencies associated with each m ($\geq$2) as shown in Figures (2) and (3) is characteristic of the natural vibrations of an open spherical shell with a free edge. Since this phenomenon can be approximately determined by the inextensional theory (especially for lower m) developed by Rayleigh, this sequence of frequency will be referred to as low Rayleigh bending frequencies for future reference.

b. Most frequencies in the frequency range $0.82<\Omega<\sqrt{2(1+\bar{\nu})}$ are considered as the mixed membrane-bending frequencies, as they are the combined products of comparable membrane and bending energies. In the frequency range $\sqrt{1-\bar{\nu}^2}<\Omega<\sqrt{2(1+\bar{\nu})}$ Kalnins[20] concluded that the membrane theory of shells can predict no natural frequencies. However, two frequencies (Ω=1.403 for m=1 and Ω=1.151 for m=2) are very closely determined by the membrane theory. They are considered as predominantly membrane frequencies here.

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*High frequencies (Ω>1) generated by experimental work in DTNSRDC are sparse and incomplete in the sense that not all natural frequencies can be excited within the specified frequency range and also the results cannot yet be systematically and definitely identified according to their associated wave numbers. Thus, the comparison of the higher frequencies between theory and experiment is not made here.

c. Except for a few membrane modes at lower number of m (such as Ω=0.831, 0910 for m=0; Ω=0.836, 0.910 for m=1; Ω=0.876 for m=2; 0.903 for m=3), which occur in the frequency range $0.82<\Omega<\sqrt{1-\bar{\nu}^2}$ and are considered to be degenerate bending modes according to Kalnins, most membrane frequencies are sparsely scattered at relatively high frequency range (Ω>2) in the Ω-m diagram, with larger variation with m as compared to the bending modes. In other words, each membrane frequency occurs alternately with a group of bending frequencies at the same wave number m.

d. Mixed membrane-bending frequencies $[0.82<\Omega<\sqrt{2(1+\bar{\nu})}]$ as well as bending frequencies $[\Omega>\sqrt{2(1+\bar{\nu})}]$ have denser distribution in the Ω-m diagram especially around Ω=1 and for lower wave number. The distribution of the higher bending frequencies acquires a regular pattern for large m.

e. The lowest mode of vibration of a free-edged hemispherical shell is indeed related to m=2 as also confirmed by other authors, which cannot be obtained by membrane theory alone.

f. Both torsional and torsionless axisymmetric natural frequencies of a free-edged hemispherical membrane shell are varying with odd nodal mode numbers (n) which represent (n) nodal points between $\phi=0^\circ$ to 180° along the meridional direction, as shown in Figure (4). A sequence of torsional frequency, though shown in the high frequency range of Ω-n diagram, is lower than that of membrane frequency in the upper branch.

In order to explore the analytic nature of the axisymmetric as well as nonsymmetric free-vibration of a hemispherical shell with a free-edge, the asymptotic technique is employed. In what follows, the approximate analysis to the determination of natural frequencies will be treated for two different frequency ranges, i.e., $\Omega<<1$ and $\Omega>>1$.

(i) Low frequency range ($\Omega << 1$)

Three roots of the characteristic Equation (21) are approximately given by

$$\lambda_1 \simeq 2.0 + \frac{3\Omega^2}{1-\bar{\nu}}$$

$$\lambda_2 \simeq 1 - \frac{1}{2}\left(\frac{2+\bar{\nu}}{1-\bar{\nu}}\right)\Omega^2 + \bar{b}^2 i$$

$$\lambda_3 \simeq 1 - \frac{1}{2}\left(\frac{2+\bar{\nu}}{1-\bar{\nu}}\right)\Omega^2 - \bar{b}^2 i \qquad (30)$$

where

$$\bar{b}^2 = \sqrt{1-\bar{\nu}^2-\Omega^2}/\beta$$
$$i = \sqrt{-1} \qquad (31)$$

The corresponding degrees ν_j (j=1, 2, 3) and q of the associated Legendre functions $P^m_{\nu j}$ and P^m_q are

$$\nu_1 \simeq 1 + \frac{\Omega^2}{1-\bar{\nu}}$$

$$\nu_2 \simeq -\frac{1}{2} + \bar{b}\, e^{i\alpha}$$

$$\nu_3 \simeq -\frac{1}{2} + \bar{b}\, e^{-i\alpha}$$

$$q \simeq 1 + \frac{2\,\Omega^2}{3\,(1-\bar{\nu})} \qquad (32)$$

where

$$\alpha = \frac{\pi}{4} - \frac{5\beta}{8} \qquad (33)$$

Based on the large parameter $\bar{b}$ contained in the degrees ν_2 and ν_2, the asymptotic forms for the associated Legendre functions can be expressed as

$$P^m_{\nu_2}(o) = \frac{1}{\sqrt{2\pi}} (\bar{b})^{m-\frac{1}{2}} e^{(\pi\bar{b}\sin\alpha/2)} e^{i(\gamma-\alpha/2)}$$

$$P^m_{\nu_3}(o) = \frac{1}{\sqrt{2\pi}} (\bar{b})^{m-\frac{1}{2}} e^{(\pi\bar{b}\sin\alpha/2)} e^{-i(\gamma-\alpha/2)}$$

$$\dot{P}^m_{\nu_2}(o) = \frac{-i}{\sqrt{2\pi}} (\bar{b})^{m+\frac{1}{2}} e^{(\pi\bar{b}\sin\alpha/2)} e^{i(\gamma+\alpha/2)}$$

$$\dot{P}^m_{\nu_3}(o) = \frac{i}{\sqrt{2\pi}} (\bar{b})^{m+\frac{1}{2}} e^{(\pi\bar{b}\sin\alpha/2)} e^{-i(\gamma+\alpha/2)} \qquad (34)$$

where

$$\gamma = m\alpha - \frac{\pi}{2}\left(m+\bar{b}\cos\alpha - \frac{1}{2}\right) \qquad (35)$$

By substituting the expressions given by Equations (30), (32) and (34) into the frequency characteristic Equation (26) with the determinant elements in Equation (28), a simplified equation involving the low Rayleigh bending frequencies will be obtained:

$$[\Omega^2 + (1-\bar{\nu})(1+\beta^2)(1-m^2)]^2 P^m_{\nu 1}(o) P^m_q(o) - m^2(1-\bar{\nu})^2(1+\beta^2)$$

$$\times \dot{P}^m_{\nu 1}(o) \dot{P}^m_q(o) + (1-\bar{\nu})(m^2-1) m^2 \beta^2 [\Omega^2 + (1-\bar{\nu})(1+\beta^2)(1-m^2)]$$

$$\times P^m_{\nu 1}(o) P^m_q(o) / (1+\beta^2) - (1-\bar{\nu}) m^4 \beta^2 \dot{P}^m_{\nu 1}(o) \dot{P}^m_q(o) = 0 \qquad (36)$$

The very low Rayleigh bending frequencies (Ω<<1) of a hemispherical shell with a free edge are finally derived from Equation (36):

$$\Omega = \beta \sqrt{(1-\bar{\nu})(m^2-1)(2m^2-1)/[C_m(1+\beta^2)]} \quad (m \geq 2), \tag{37}$$

where

$$C_m = \frac{5(m^2-1)}{3}\left[\ln\left(\frac{m}{m-1}\right) + \frac{2m-1}{3m^2(m-1)^2} - \frac{1}{m+1}\right] - 2 \tag{38}$$

It is of interest to note that the low natural frequency formula, Equation (37), is very nearly a linear function of dimensionless shell thickness parameter β, when β<<1, and approximately varying with m^s, where $s=2 \sim 2.5$. The explicit function of m, in addition to the linear dependence of β, introduced in Equation (37) clearly constitutes a simple method for the estimation of low natural frequencies of a free-edge hemispherical shell.

A comparison of low natural frequencies obtained by exact Equation (26), approximate Equation (37), and Rayleigh inextensional methods is made in Table 2 for h/R = 0.01 and $\bar{\nu}$ = 0.3.

TABLE 2 - COMPARISONS OF LOW NATURAL FREQUENCIES OF FREE-EDGED HEMISPHERICAL SHELL OBTAINED BY DIFFERENT METHODS (h/R=0.01 AND $\bar{\nu}$=0.3)

m	Exact Equation (26)	Approximate Equation (37)	Rayleigh
2	0.012	0.011	0.013
3	0.034	0.034	0.036
4	0.064	0.067	0.069
5	0.102	0.111	0.112
6	0.146	0.164	0.165

The computed approximate frequencies seem to check well with the exact frequencies, especially in the low wave number region. Also, the approximate frequencies are seen to be closer to the exact values than those by Rayleigh inextensional theory.

(ii) High frequency range ($\Omega >> 1$)

Three roots of the characteristic Equation (21) are approximately given by:

$$\lambda_1 \simeq \Omega^2 - \bar{\nu}\,(3+\bar{\nu}) - \frac{3\,(1+\bar{\nu})(1-\bar{\nu}^2)}{\Omega^2}$$

$$\lambda_2 \simeq 2 + \frac{3\bar{\nu} + \bar{\nu}^2}{2} - b^2$$

$$\lambda_3 \simeq 2 + \frac{3\bar{\nu} + \bar{\nu}^2}{2} + b^2 \qquad (39)$$

where

$$b^2 = \frac{\sqrt{\Omega^2 + \bar{\nu}^2 - 1}}{\beta} \qquad (40)$$

The corresponding degrees ν_j and q are

$$\nu_1 \simeq \Omega - 0.5 + \left(\frac{0.25-3\bar{\nu}-\bar{\nu}^2}{2}\right)\frac{1}{\Omega}$$

$$\nu_2 \simeq -\frac{1}{2} + bi$$

$$\nu_3 \simeq -\frac{1}{2} + b$$

$$q = -\frac{1}{2} + \left(\frac{9}{4} + \frac{2\Omega^2}{1-\nu}\right)^{\frac{1}{2}} \qquad (41)$$

The asymptotic forms of associated Legendre functions are

$$P^m_{\nu 2}\ (o) \simeq \frac{1}{\sqrt{2\pi}}\ (b)^{m - \frac{1}{2}}\ e^{(b\pi/2)}$$

$$P^m_{\nu 3}\ (o) \simeq \frac{1}{\sqrt{2\pi}}\ (b)^{m - \frac{1}{2}}\ \sin\left(a_o + \frac{\pi}{4}\right)$$

$$\dot{P}^m_{\nu 2}\ (o) \simeq \frac{1}{\sqrt{2\pi}}\ (b)^{m + \frac{1}{2}}\ e^{(b\pi/2)}$$

$$\dot{P}^m_{\nu 3}\ (o) \simeq \frac{1}{\sqrt{2\pi}}\ (b)^{m + \frac{1}{2}}\ \cos\left(a_o + \frac{\pi}{4}\right) \qquad (42)$$

where

$$a_o = \frac{\pi}{2}\ (m+b) \qquad (43)$$

Upon the substitution of the expressions given in Equations (39), (41), and (42) into the frequency characteristic Equation (26) and neglecting the small quantities of higher order terms, a simplified equation will yield:

$$G(\Omega)\cdot H(\Omega) \simeq 0, \qquad (44)$$

where

$$G(\Omega) = [\Omega^2 + (1-\bar{\nu})(1+\beta^2)(1-m^2)]^2\ \ P^m_{\nu 1}\ (o)\ P^m_q\ (o)$$

$$- m^2\ (1-\bar{\nu})^2(1+\beta^2)\ \dot{P}^m_{\nu 1}\ (o)\ \dot{P}^m_q\ (o) \qquad (45)$$

and

$$H(\Omega) = \sin a_o \left[2b^4 + 2b^2\left(1+ \frac{\bar{\nu}+\bar{\nu}^2}{2}\right) +2(1-\bar{\nu})^2\ m^4\right]$$

$$+ \cos a_o \left[-4\ (1-\bar{\nu})m^2\ b^2\right] \tag{46}$$

The product of two functions G (Ω) and H (Ω) in Equation (44) implies that at least two families of natural frequency with different characters might result. Indeed, this turns out to be the case. Function G (Ω), which is almost independent of shell thickness ($\beta^2 \ll 1$), will determine torsional and torsionless membrane frequencies. In contrast, function H (Ω) strongly depends on the shell thickness and this will generate bending frequencies.

The solution of Equation (44) is obviously determined by setting either function G (Ω) or H (Ω) to zero. Two different families of natural frequencies are obtained as follows:

(i) Membrane frequencies: $G(\Omega)=0$ (47)

If the very small quantity β^2 is neglected, the expression G (Ω) would be the frequency equation that results from the membrane theory[6] by satisfying the boundary conditions $N_\phi\left(\phi_o = \frac{\pi}{2}\right) = N_{\phi\theta}\left(\phi_o = \frac{\pi}{2}\right) = 0$, which imply the free movement of the middle-surface tangential displacements at the open edge of a hemispherical shell. Thus, the natural frequencies generated from Equation (47) are the membrane frequencies.

For the torsionless axisymmetric (m=0) membrane vibrations, only one boundary condition $N_\phi\left(\phi_o = \frac{\pi}{2}\right) = 0$ is necessary to yield the desired natural frequencies. In other words,

$$P_\nu(o) = 0 \quad (\nu = n = 1,3,5\ldots\text{odd}) \tag{48}$$

where

$$\nu = -0.5 + \sqrt{\lambda + 0.25} \tag{49}$$

Here, $\lambda = \left(\Omega^2 + 1 - \bar{\nu}\right)\left(\Omega^2 - 2 - 2\bar{\nu}\right) \Big/ \left(\Omega^2 + \bar{\nu}^2 - 1\right)$

For higher axisymmetric torsionless membrane frequencies ($\Omega \gg 1$), Equation (48) can be approximated to be

$$\Omega_n \simeq (n+0.5) + \frac{3\bar{\nu} + \bar{\nu}^2 - 0.25}{2(n+0.5)}, \quad n=1,3,5\ldots\text{odd}. \tag{50}$$

Table 3 shows a comparison of exact and approximate axisymmetric membrane frequencies by Equation (50).

TABLE 3 - A COMPARISON OF EXACT AND APPROXIMATE AXISYMMETRIC TORSIONLESS MEMBRANE FREQUENCIES (m=0) BY EQUATION (50) FOR A FREE-EDGED HEMISPHERICAL MEMBRANE SHELL ($\bar{\nu}$=0.3)

Ω	
Exact	Approximate
1.98	1.75
3.63	3.61
5.58	5.57
7.55	7.55
9.54	9.54
11.53	11.53

Regarding the nonsymmetric membrane vibrations ($m \neq 0$), the following approximations for torsional and torsionless membrane frequencies (in the high branch) are possible:

When $\Omega >> m$, Equation (47) can be simplified to

$$\cot \frac{\pi}{2} (m+\nu) \cdot \cot \frac{\pi}{2} (m+q) = (1-\bar{\nu}^2)\, m^2 \sqrt{(\nu^2-m^2)(q^2-m^2)} \Big/ [\Omega^2 + (1-\bar{\nu}) \times (1-m^2)]^2 \qquad (51)$$

where definitions of ν and q are given previously in Equations (49) and (23), respectively.

Since the right hand side of Equation (51) is a small quantity of order $O\left(\frac{m^2}{\Omega^2}\right)$, two sets of natural frequency are approximately derived:

$$m + \nu = 2k + 1 \quad (k=1,2,3,\ldots.) \quad \text{(torsionless frequency)} \qquad (52)$$

$$m + q = 2k + 1 \quad (k=1,2,3,\ldots.) \quad \text{(torsional frequency)} \qquad (53)$$

Equations (52) and (53) can be further simplified under the condition $\Omega >> 1$ to yield two explicit forms of natural frequency:

$$\Omega \simeq (2k+1.5-m) + \left(\frac{0.25-3\bar{\nu}-\bar{\nu}^2}{2}\right)\frac{1}{2k+1.5-m}, \quad (k=1,2,3,\ldots \text{ and } m \geq 1) \qquad (54)$$

$$\Omega \simeq \sqrt{0.5(1-\bar{\nu})(2k-m)(2k-m+3)}, \quad (k=m+1, m+2, \ldots \text{and } m \geq 1) \qquad (55)$$

Equation (54) yields the high frequency approximation of torsionless membrane frequency under consideration, while Equation (55) will give the torsional membrane frequency. The latter is equivalent to Equation (29) when β^2 is neglected.

A comparison of membrane frequencies, between the results by exact membrane solutions and those by Equations (54) and (55) for m = 1,2,3, is made in Table 4, where for completeness the results of axisymmetric case (m=0) shown in Table 3 and Figure (4) are also included.

TABLE 4 - COMPARISONS OF MEMBRANE FREQUENCIES BY EXACT MEMBRANE SOLUTIONS AND BY APPROXIMATE FORMULAS EQUATIONS (54) AND (55), FOR m=1,2,3, FOR A FREE-EDGED HEMISPHERICAL MEMBRANE SHELL ($\bar{\nu}$=0.3)

m	Torsionless Membrane Frequency		Torsional Membrane Frequency	
	Exact	Approximate Equation (54)	Exact	Approximate Equation (55)
0	1.98	1.75	1.87	1.87
	3.63	3.61	3.13	3.13
	5.58	5.57	4.35	4.35
	7.55	7.55	5.55	5.55
1	2.78	2.35	2.43	2.51
	4.58	4.42	3.74	3.74
	6.57	6.44	4.97	4.95
	8.49	8.46	6.14	6.15
2	3.67	3.39	3.03	3.13
	5.72	5.43	4.37	4.35
	7.53	7.45	5.40	5.55
	9.55	9.46	6.74	6.75
3	2.99	2.35	3.70	3.74
	4.48	4.42	5.08	4.95
	6.61	6.44	6.06	6.15
	8.69	8.46	7.36	7.34

(ii) Bending frequencies: $H(\Omega)=0$ (56)

The expression of $H(\Omega)$ given in Equation (46) depends strongly on the large parameter b and thus represents a high frequency bending solution. Under the condition of $\Omega>>1$, Equation (56) can be rearranged as:

$$\tan\frac{\pi}{2}(m+b) \simeq \frac{2(1-\bar{\nu})m^2\beta}{\sqrt{\Omega^2+\bar{\nu}^2-1}}\left[\frac{1}{1+(1-\bar{\nu})^2\, m^4/b^4}\right] \tag{57}$$

Since the right hand side of Equation (57) is normally a small quantity, the high bending frequencies can be approximately deduced from Equation (57) by replacing the right hand side expression with zero. By doing so, an explicit expression of natural frequency is obtained as follows:

$$\Omega = \sqrt{(1-\bar{\nu}^2) + \beta^2\,(2k-m)^4} \tag{58}$$

where k is a large positive integer.

As can be seen from Equation (58), the high bending frequency repeats itself for the even or odd circumferential wave number m.

With m=0, Equation (58) becomes the formula for high bending frequency of axisymmetric case, which coincides with the approximation obtained by Ross[12].

The numerical data of high bending natural frequencies for the case of $\left(\frac{h}{R} = \frac{3/16}{16} \text{ and } \bar{\nu} = 0.3\right)$ are calculated according to the approximate formula (58) for m = 0, 1, 2, and 3 and are presented in Table 5, where the corresponding exact bending frequencies by exact analysis are taken from Table 1, and included here for comparison.

TABLE 5 - COMPARISON OF HIGH BENDING FREQUENCIES BY APPROXIMATE FORMULA (58) WITH THOSE BY EXACT ANALYSIS ($\phi_o=\pi/2, h/R=3/256, \bar{\nu}=0.3$)

Approximate Equation (58)		Exact			
m=even	m=odd	m=0	m=1	m=2	m=3
1.656	1.771	1.646	1.762	1.647	1.763
1.895	2.028	1.886	2.019	1.886	2.020
2.170	2.320	2.160	2.311	2.161	2.312
2.478	2.644	2.469	2.635	2.469	2.636
2.819	3.001	2.809	2.992	2.810	2.993
3.191	3.388	3.181	3.379	3.182	3.380
3.593	3.806	3.584	3.796	3.584	3.797
4.025	4.252	4.016	4.243	4.016	4.244
4.487	4.728	4.477	4.719	4.478	4.720

In Table 5, the first few frequencies for each m are expected to have more coupling effect between membrane and bending than those higher frequencies.

Mode Shapes

The mode shapes of a hemispherical shell with a free edge $\left(\frac{h}{R} = \frac{3/16}{16};\ \bar{\nu}=0.3\right)$ are numerically evaluated according to a few representative natural frequencies from the exact analytical solutions, Equations (24). The results are plotted from Figures (5) to (19). Several points of interest are disclosed as follows:

a. It is indeed a very interesting phenomenon that while the lowest natural bending frequencies associated with each circumferential wave number ($m \geq 2$) for a hemispherical shell with a free edge can be closely derived by Rayleigh's inextensional theory, their corresponding lowest asymmetrical vibration modes belong to the predominantly membrane-type deformations (Figure 5). The features of such low frequency vibration modes are such that the magnitudes of the meridional displacement (V) and that of the circumferential displacement (U) are approximately equal, and the ratio of either tangential displacement to the radial displacement (W) is equal to $\left(\frac{-\sin\phi}{m+\cos\phi}\right)$. All displacements start from zero at the apex and gradually increase to maximum values at the free edge. As the wave number (m) increases the displacements become more pronounced towards the open edge of the shell.

b. The modes of displacement associated with the mixed bending-membrane frequencies approximately in the middle frequency range of $0.82<\Omega<\sqrt{2(1+\nu)}$ (Figures 6, 7, 11, 14, and 17) are primarily of membrane-type deformations, except at the open free edge, where the bending effect plays an important role in order to satisfy the boundary conditions (edge effect). Those radial displacements around the middle frequency range are normally larger than the tangential displacements. For the nonsymmetrical case, the meridional and circumferential displacements are about the same order of magnitude (Figures 11, 14, and 17).

c. In the high frequency range ($\Omega>>1$), mode shapes corresponding to the membrane frequencies are quite different in nature from those of the bending frequencies. Modes associated with membrane frequencies (Figures 8, 10, 13, 16, and 19) are predominantly of membrane-type deformations and possess slowly varying displacement patterns with relatively few nodal points.

The tangential (meridional or circumferential) displacement is comparable with the radial displacement. The ratio $\frac{V}{W}$ or $\frac{U}{W}$ becomes particularly large especially for larger m and higher frequencies (Figure 19). As to the mode shapes corresponding to the bending frequencies, unlike the previous case, displacements are generated predominantly by the bending energy of the shell and exhibit the rapidly changing displacement patterns with relatively many nodal points. The radial displacement dramatically dominates over either tangential displacement (Figures 9, 12, 15, and 18).

In an attempt to understand the dynamic behavior of modal displacements, similar asymptotic technique again will be applied in different frequency ranges to derive the approximate formulae for the mode shapes of a hemispherical shell with a free edge.

(i) Mode shapes in low frequency range ($\Omega << 1$)

The approximate lowest asymmetric vibration modes associated with each low Rayleigh bending frequency can be asymptotically obtained from exact solutions, Equations (24), by dropping the small quantities of higher order terms. They are given by:

$$W = A\, P_1^{-m} (\cos \phi) \tag{59a}$$

$$V = U = A \left[\dot{P}_1^{-m} (\cos \phi) - \frac{m\, P_1^{-m} (\cos \phi)}{\sin \phi} \right], \quad (m \geq 2) \tag{59b,c}$$

The formulae for mode shapes in Equations (59) are, in fact, equivalent to the approximate solutions given by Rayleigh inextensional principle; namely,

$$W = B\,(m + \cos\phi)\,\tan^{m}(\phi/2) \tag{60a}$$

$$V = U = -B \sin\phi \cdot \tan^{m}(\phi/2) \tag{60b,c}$$

As can be seen from Equations (59) or (60), meridional displacement (V) is asymptotically equal to the circumferential displacement (U) and the ratio between either tangential displacement and the radial one is a simple relation

$$\frac{V}{W} = \frac{U}{W} = \frac{-\sin\phi}{m + \cos\phi} \tag{61a,b}$$

(ii) Mode shapes in high frequency range ($\Omega >> 1$):

Under the conditions of $b^2 >> (1-\bar{\nu})\, m^2$ and $\phi > 0^{o}$, the high-frequency mode shapes can be approximately obtained in the following two classes:

a. Mode shapes associated with membrane frequencies:

$$W = A \left\{ P^{m}_{\nu 1}(\cos\phi) - \frac{E_{21}\, b^{-2}}{\sin a_o\,(2\sin\phi)^{1/2}} \left[\sin\left(a + \frac{\pi}{4}\right) + e^{b\,(\phi - \pi/2)} \cos\left(a_o + \frac{\pi}{4}\right) \right] \right\} \tag{62a}$$

$$V = A \left\{ -D \left[\dot{P}^{m}_{\nu 1}(\cos\phi) + s_m\, \dot{P}^{m}_{q}(o) \right] + \frac{E_{21}\,(1+\bar{\nu})\, b^{-3}}{\sin a_o\,(2\sin\phi)^{1/2}} \times \left[-\cos\left(a + \frac{\pi}{4}\right) + e^{b\,(\phi - \pi/2)} \cos\left(a_o + \frac{\pi}{4}\right) \right] \right\} \tag{62b}$$

$$U = A\left\{ mD_1 \left[\frac{P^m_{\nu 1} (\cos\phi)}{\sin\phi} + q_m \dot{P}^m_{\nu 1}(o) \right] + \frac{m(1+\bar{\nu})\,E_{21}}{\sin a_o (2)^{1/2} (\sin\phi)^{3/2}} \right.$$

$$\left. \times \left[\sin\left(a + \frac{\pi}{4}\right) - e^{b(\phi - \pi/2)} \cos\left(a_o + \frac{\pi}{4}\right) \right] \right\} \qquad (62c)$$

where $E_{21} = \left[\lambda_1 - 1 - \bar{\nu} - (1-\bar{\nu})\, m^2 \right] P^m_{\nu 1}(o)$

$$a = \frac{m\pi}{2} + b\phi$$

$$a_o = \frac{\pi}{2}\ (m+b)$$

$$s_m = \frac{(1-\bar{\nu})(1+\beta^2))m^2}{\sin\phi\ [\Omega^2 + (1-\bar{\nu})(1+\beta^2)(1-m^2)]} \cdot \frac{P^m_q (\cos\phi)}{P^m_q (o)}$$

$$q_m = \frac{(1-\bar{\nu})(1+\beta^2)}{\Omega^2 + (1-\bar{\nu})(1+\beta^2)(1-m^2)}\ \frac{\dot{P}^m_q (\cos\phi)}{P^m_q (o)}$$

A = undetermined coefficient

For the region close to the apex of the shell ($\phi \sim 0^o$), different asymptotic representations should be employed to rederive the mode shapes. As $\phi \to 0^o$ with $m \geq 0$ and $\nu >> 1$, the associated Legendre function is approximated as:

$$P^m_\nu (\cos\phi) = (-1)^m \left[\left(\nu + \frac{1}{2}\right) \cos\frac{\phi}{2} \right]^m J_m \left[(2\nu+1) \sin\frac{\phi}{2} \right]$$

$$+\ O\left(\sin^4 \frac{\phi}{2} \right) \qquad (63)$$

where $J_m(x)$ is the Bessel function of the first kind with integer order m.

By using the expansion (63), the mode shapes when ϕ is sufficiently close to 0^o are simplified as:

$$W = A \left[P_{\nu 1}^{m} (\cos \phi) - \frac{(-1)^m E_{21} (\pi)^{1/2} (b)^{-3/2}}{2 \sin a_o} J_m (b\phi) \right] \quad (64a)$$

$$V = A \left\{ -D_1 \left[\dot{P}_{\nu 1}^{m} (\cos \phi) + s_m \dot{P}_{\nu 1}^{m} (o) \right] - \frac{(-1)^m (1+\bar{\nu}) E_{21} (\pi)^{1/2} (b)^{-5/2}}{2 \sin a_o} \dot{J}_m (b\phi) \right\} \quad (64b)$$

$$U = A \left\{ mD_1 \left[\frac{P_{\nu 1}^{m} (\cos \phi)}{\sin \phi} + q_m \dot{P}_{\nu 1}^{m} (o) \right] + \frac{m(-1)^m (1+\bar{\nu}) E_{21} (\pi)^{1/2} (b)^{-7/2}}{2 \sin a_o \cdot \sin \phi} J_m (b\phi) \right\} \quad (64c)$$

In Equations (62) or (64), the first term or collective term associated with the brackets in each displacement expression denotes the membrane contribution, while the second term or collective term represents the bending contribution. As can be seen from these formulas, the modes of displacement are dominated by the membrane terms, which display the relatively long-wavelength simple modes of vibration, with small-amplitude, short-wavelength fluctuation due to the presence of bending contributions superimposed on them.

The order-of-magnitude relationships between maximum tangential displacements and maximum radial displacements for $\Omega \gg m$, when the torsionless membrane frequency is not close to a torsional membrane frequency, can be estimated from Equations (62):

$$\frac{V_{max}}{W_{max}} = O\left(\frac{\sqrt{(\Omega-0.5)^2-m^2}}{1+\bar{\nu}}\right) \tag{65a}$$

$$\frac{U_{max}}{W_{max}} = O\left(\frac{1+\sqrt{2(1-\bar{\nu})}}{1+\bar{\nu}}\; m\right) \tag{65b}$$

The relation given in Equation (65a) indicates that meridional displacement gradually dominates the radial displacement as frequency (Ω) increases. From relation shown in Equation (65b) one can expect that the circumferential displacement will become more important than the radial displacement for higher wave number (m). These order-of-magnitude relations are characteristic of the modes of displacement according to the high torsionless membrane frequencies when $\Omega \gg m$.

(iii) Mode shapes associated with high bending frequencies

By setting $\sin a_o = 0$ in Equations (62) and (64), which is equivalent to the condition for generation of high bending frequency shown previously in Equation (57), the mode shapes associated with the high bending frequencies are readily obtained for $\Omega^2 \gg (1-\bar{\nu})m^2$: For $\phi > 0^o$

$$W = \frac{A}{(\sin\phi)^{1/2}}\left[\sin\left(a+\frac{\pi}{4}\right)+e^{b(\phi-\pi/2)}\cos\left(a_o+\frac{\pi}{4}\right)\right] \tag{66a}$$

$$V = \frac{A(1+\bar{\nu})b^{-1}}{(\sin\phi)^{1/2}} \left[\cos\left(a+\frac{\pi}{4}\right) - e^{b(\phi-\pi/2)} \cos\left(a_o+\frac{\pi}{4}\right) \right] \tag{66b}$$

$$U = \frac{-Am(1+\bar{\nu})b^{-2}}{(\sin\phi)^{3/2}} \left[\sin\left(a+\frac{\pi}{4}\right) - e^{b(\phi-\pi/2)} \cos\left(a_o+\frac{\pi}{4}\right) \right] \tag{66c}$$

For $\phi \sim 0^o$

$$W = AJ_m(b\phi) \tag{67a}$$

$$V = A\,(1+\bar{\nu})\,b^{-1}\,\dot{J}_m\,(b\phi) \tag{67b}$$

$$U = -Am\,(1+\bar{\nu})\,b^{-2}\,\frac{J_m(b\phi)}{\sin\phi} \tag{67c}$$

The order-of-magnitude relations between maximum displacements can be easily obtained from Equations (66) and (67)

$$\frac{V_{max}}{W_{max}} = O\left[(1+\bar{\nu})b^{-1}\right] \tag{68a}$$

$$\frac{U_{max}}{W_{max}} = O\left[\frac{(1+\bar{\nu})mb^{-2}}{\sin\phi}\right] \tag{68b}$$

It is quite obvious as revealed from Equations, (66), (67) and (68) that tangential displacements are quite small compared with radial displacement. Especially when ϕ increases, the circumferential displacement will decay very fast for high bending frequency and not very large wave number (m). Also, since a large parameter b is contained in the argument of the trigonometric and Bessel functions, all displacements undergo rapid fluctuation along the ϕ-direction. The rapidly fluctuating phenomena of the modal displacements with either tangential displacement much smaller than the radial displacement are the salient features of the mode shapes associated with high bending frequencies.

CONCLUSIONS

The main object of this report is to independently derive the analytical solutions for open nonshallow, axisymmetric as well as nonsymmetric spherical shell vibrations. This is accomplished, with the aid of spherical harmonic operator ∇^2, by reducing the shell motion equations to a set of manageable equations in a straightforward manner, from which the solutions for middle-surface displacements are obtained in terms of associated Legendre functions. The analytical solutions derived are of simple compact forms which would provide an analytical path towards the further researches of a similar nature.

An illustrative example of a free-edged hemispherical shell vibration is presented to explore the free-vibration characteristics of an open nonshallow spherical shell. Numerical predictions using exact solutions are made and compare very well with the experimental results, which demonstrate the soundness and the capability of the present analytical solutions. An asymptotic technique is also employed to reveal the salient features of the shell dynamic parameters such as normalized frequency (Ω) and nondimensional shell thickness parameter (β). The following are some conclusions for free-vibration problems of a hemispherical shell with a free edge:

1. In the low frequency region ($\Omega<1$), a sequence of low bending frequency associated with each circumferential wave number (m) occurs. Each such frequency is very nearly a linear function of thickness parameter (β) and varies approximately with m^s, where $s = 2\sim2.5$. The corresponding mode shapes are of membrane-type deformations and are characterized by $P_1^{-m}(\cos\phi)$; i.e.,

$$W = AP_1^{-m}(\cos\phi)$$

$$\frac{V}{W} = \frac{U}{W} = -\frac{\sin\phi}{m+\cos\phi}$$

The computed results show that the lowest mode of vibration of a free-edged open shell is indeed related to m = 2 mode, which is also confirmed by other authors.

2. In the *middle* frequency range $\left[0.82<\Omega<\sqrt{2(1+\bar{\nu})}\right]$, most frequencies except a few membrane modes, are *mixed* bending-membrane frequencies due to the strong coupling of bending with membrane energies in this frequency range.

3. In the high frequency range ($\Omega>>1$), vibration modes eventually become discrete as bending and (torsional and torsionless) membrane modes due to the weak coupling effect of bending stiffness and membrane stretching. Their corresponding mode shapes exhibit distinctive displacement patterns as briefly described in the following:

(i) Torsional membrane modes of vibration are uncoupled from the torsionless vibration modes for the axisymmetric case (m=0), and are involved with the circumferential displacement only: i.e.,

$$U = AP_q^{m}(\cos\phi)$$

A sequence of pure torsional frequency (m=0) in the Ω-n diagram (n = mode number), is lower than, but closely parallel with that of torsionless membrane frequency in the not very high mode number region.

(ii) Torsionless natural frequencies are further distinguished from each other; namely, (predominantly) bending and (predominantly) membrane frequencies:

a. Bending frequencies and their associated mode shapes depend strongly on the frequency-thickness parameter (b), which is a large-parameter of high-frequency, thin shell vibrations. The mode shapes are approximately characterized by the following relations:

$$W \simeq A \cdot \frac{\sin(m\pi/2+\pi/4+b\phi)}{(\sin\phi)^{1/2}}$$

$$V \simeq A \cdot \frac{\left[m(1+\bar{\nu})b^{-1}\right]}{(\sin\phi)^{1/2}} \cos(m\pi/2+\pi/4+b\phi)$$

$$U \simeq A \cdot \frac{-m(1+\bar{\nu})b^{-2}}{(\sin\phi)^{3/2}} \sin(m\pi/2+\pi/4+b\phi)$$

for $\Omega^2 >> (1-\bar{\nu})m^2$ and $\phi > 0^o$.

All displacements undergo rapid fluctuation along the ϕ-direction with either tangential displacement (V or U) much smaller than the radial displacement (W). The circumferential displacement (U) decays even faster than meridional displacement (V) towards larger ϕ.

b. Membrane frequencies, almost independent of shell thickness, are generated through the boundary conditions involving tangential displacement only. The corresponding mode shapes are roughly typified by the following relations:

$$W = AP^m_{\nu l}\ (\cos \phi)$$

$$\frac{V_{max}}{W_{max}} = O\left(\frac{\sqrt{(\Omega-0.5^2-m^2)}}{1+\bar{\nu}}\right)$$

$$\frac{U_{max}}{W_{max}} = O\left[\frac{1+\sqrt{2(1-\bar{\nu})}}{1+\bar{\nu}}\ m\right]$$

for $\Omega >> m$.

Since the degree νl of $P^m_{\nu l}$ is close to the normalized frequency (Ω), the radial displacement is expected to be a slowly varying simple mode of vibration up to the frequency range where the classical thin shell theory is still applicable. The tangential displacements will eventually dominate over the radial displacement for higher frequencies and/or higher circumferential wave number.

c. Only for axisymmetric vibration modes ($m=0$), could there be relatively large normal displacements around apex $\phi=0$ of the shell for either bending or membrane frequencies. For other nonsymmetric modes ($m \neq 0$), normal displacement is always zero at the apex.

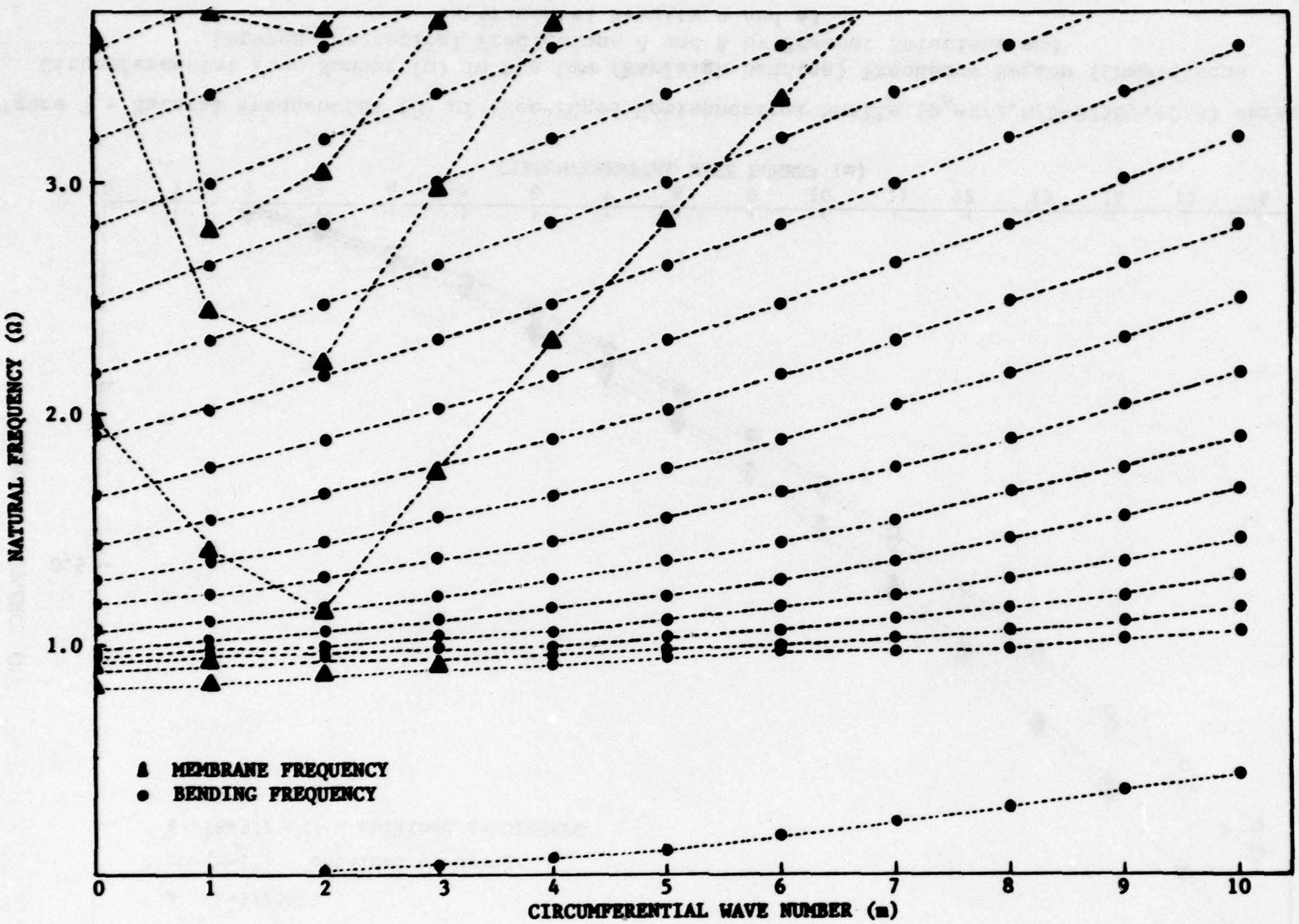

Figure 2 - Natural Frequencies (Ω) for a Torsionless Hemispherical Shell ($\phi_o=\pi/2, h/R=3/256, \bar{\nu}=0.3$) with a Free Edge, versus Circumferential Wave Number (m)

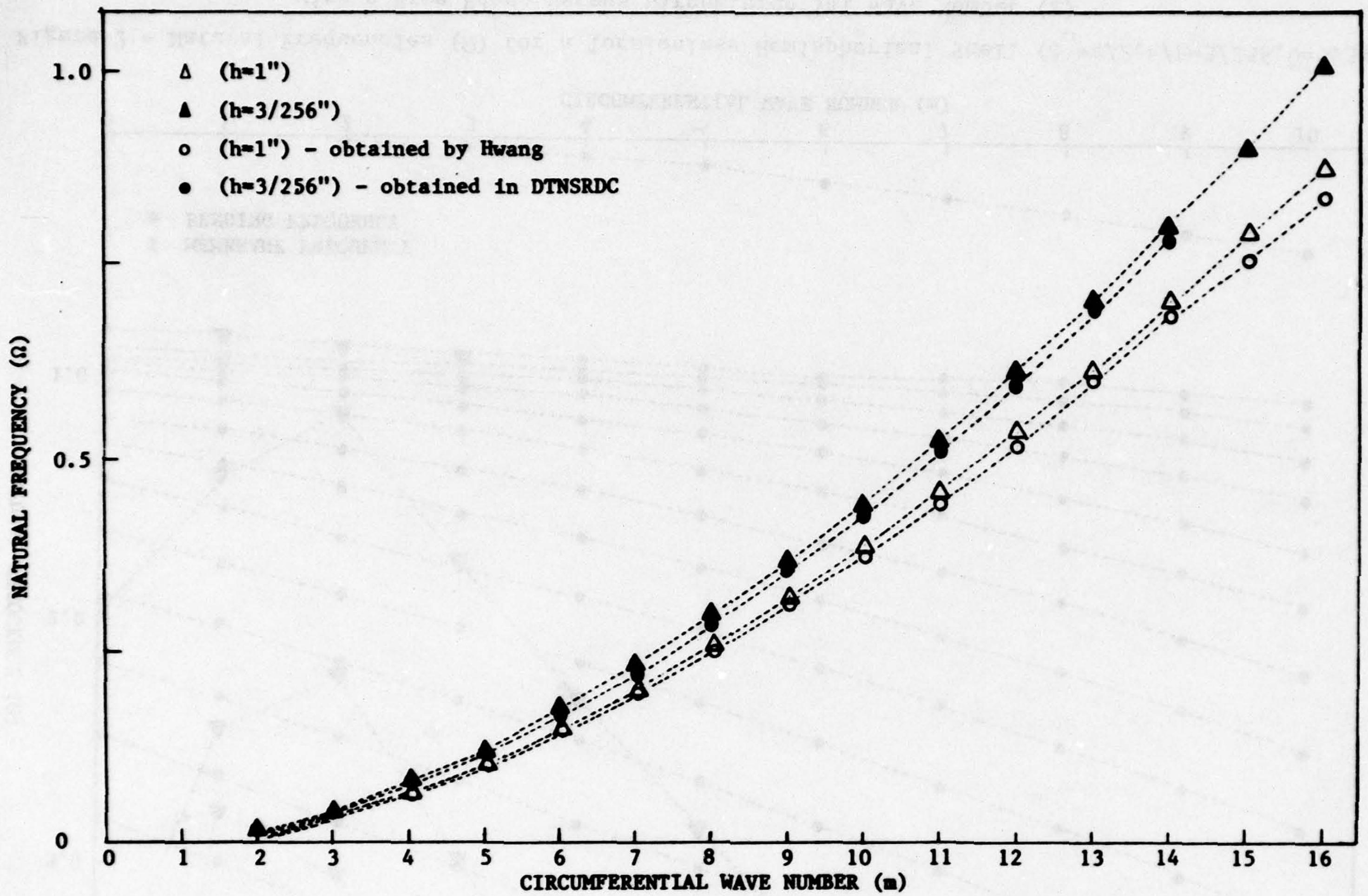

Figure 3 - Natural Frequencies (Ω) of Free-Edged Hemispherical Shells ($\phi_o=\pi/2, h/R=3256, \nu=0.3$) versus Circumferential Wave Number (m) in the Low (Rayleigh Bending) Frequency Region (Comparisons between Theoretical Predictions Δ and ▲ by Present Solutions and Experimental Results o and ●)

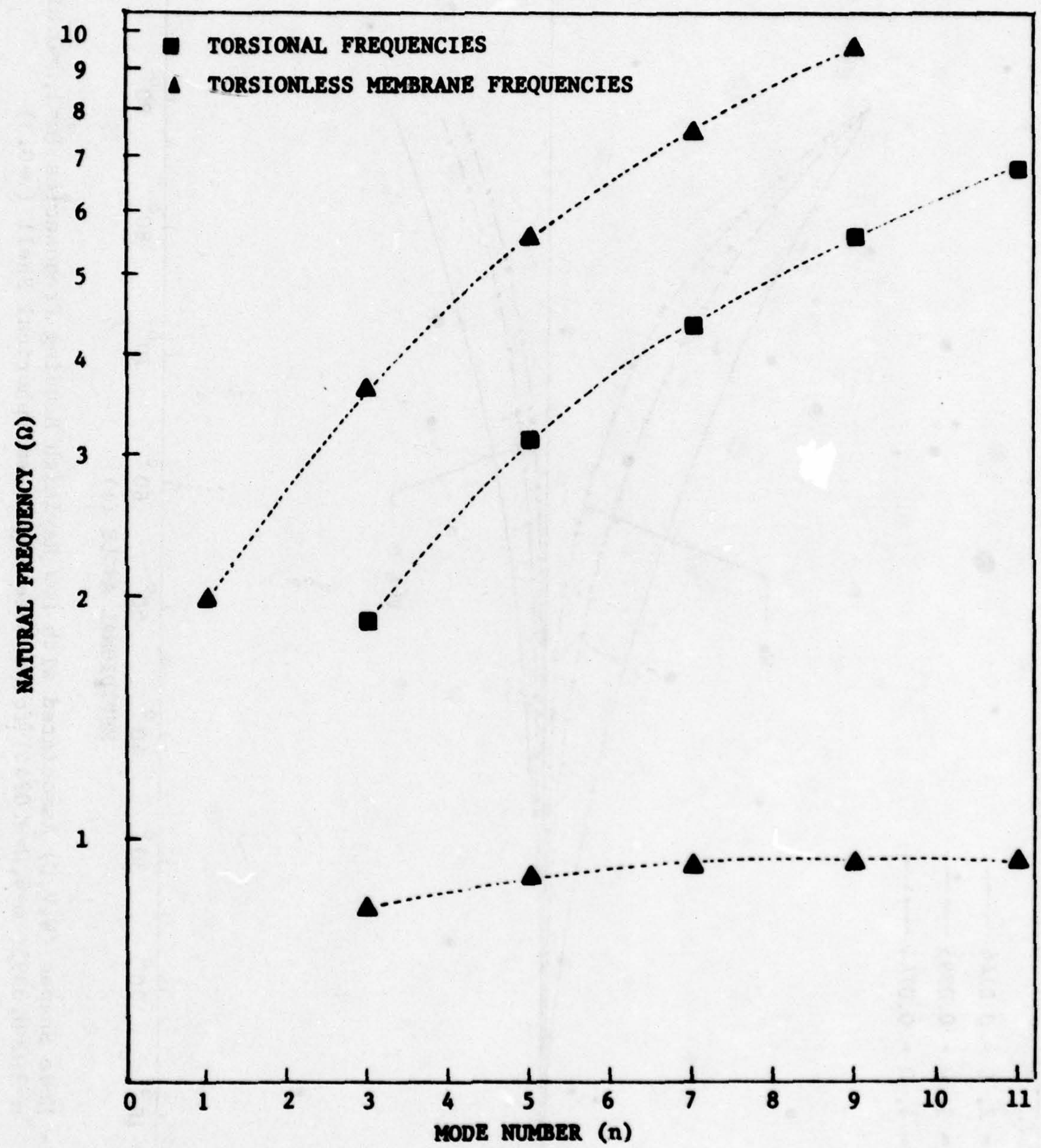

Figure 4 - Torsional and Torsionless Natural Frequencies (Ω) of a Free-Edged, Axisymmetric Hemispherical Membrane Shell versus Mode Number (n) ($m=0, \bar{\nu}=0.3$)

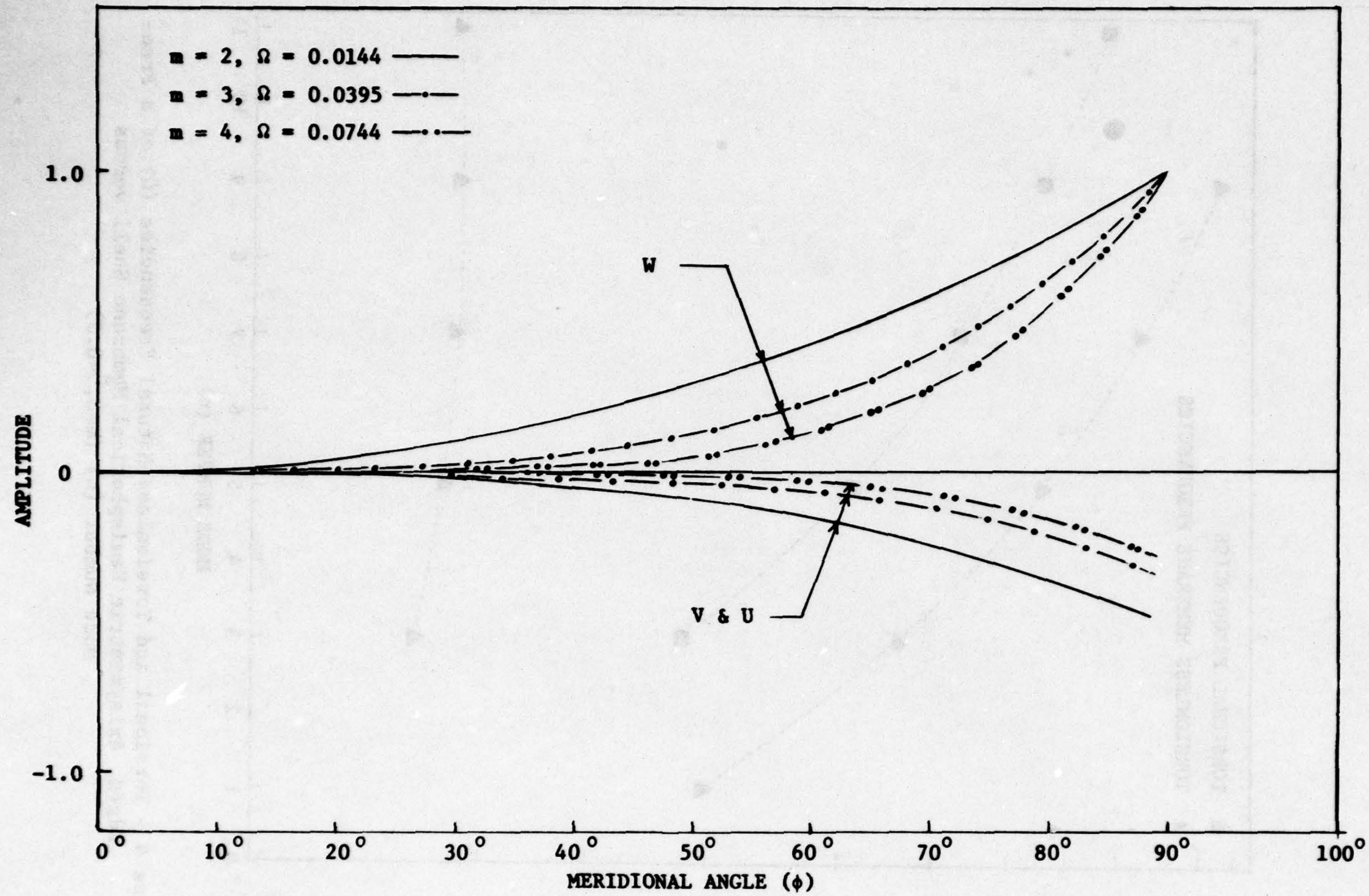

Figure 5 - Mode Shapes (W,V,U) Associated with Low Rayleigh Bending Frequencies ($m=2,\Omega=0.0144$; $m=3,\Omega=0.0395$; $m=4,\Omega=0.0744$) for a Free-Edged Hemispherical Shell ($\bar{\nu}=0.3$)

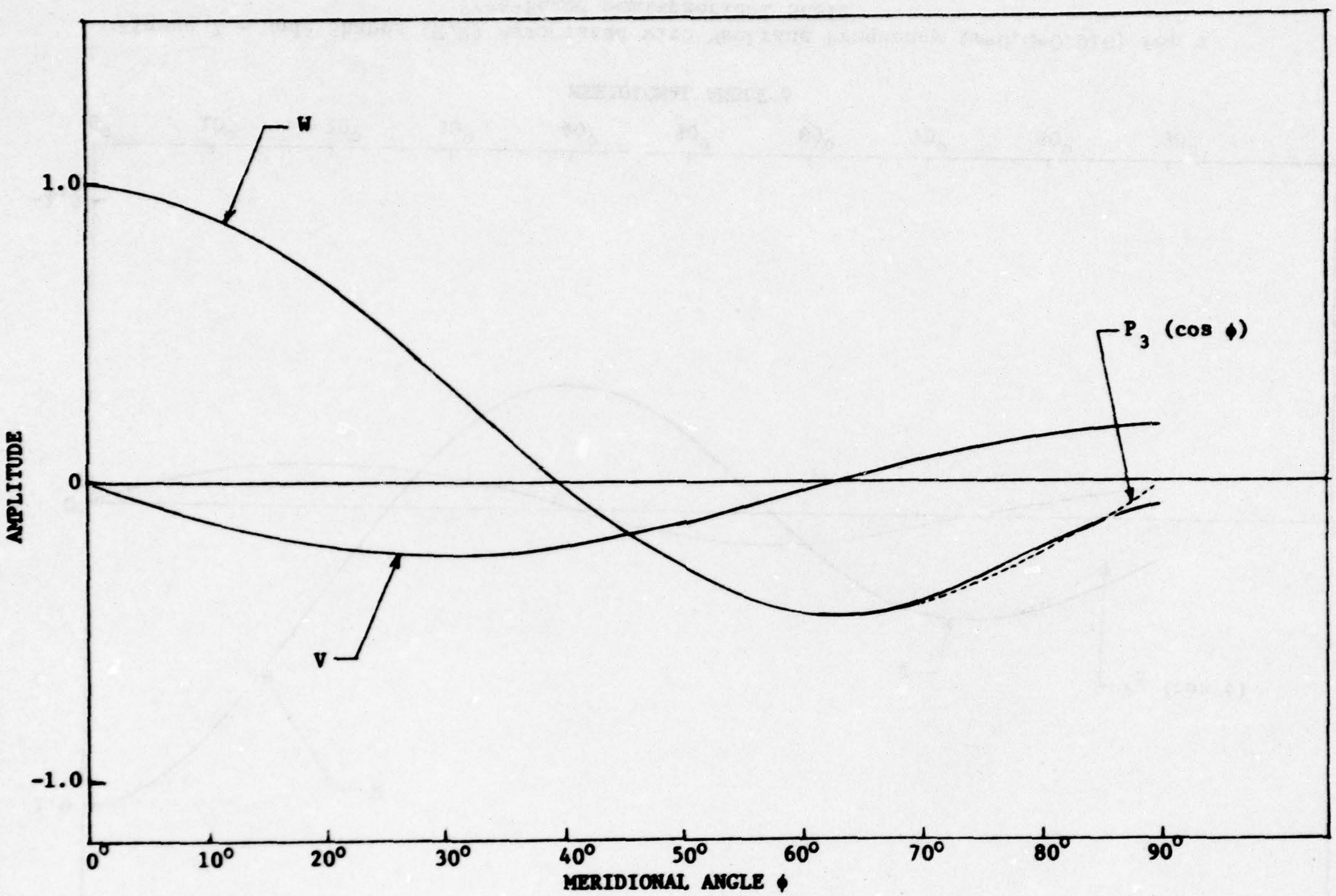

Figure 6 - Mode Shapes (W,V) Associated with Membrane Frequency (m=0,Ω=0.831) for a Free-Edged Hemispherical Shell

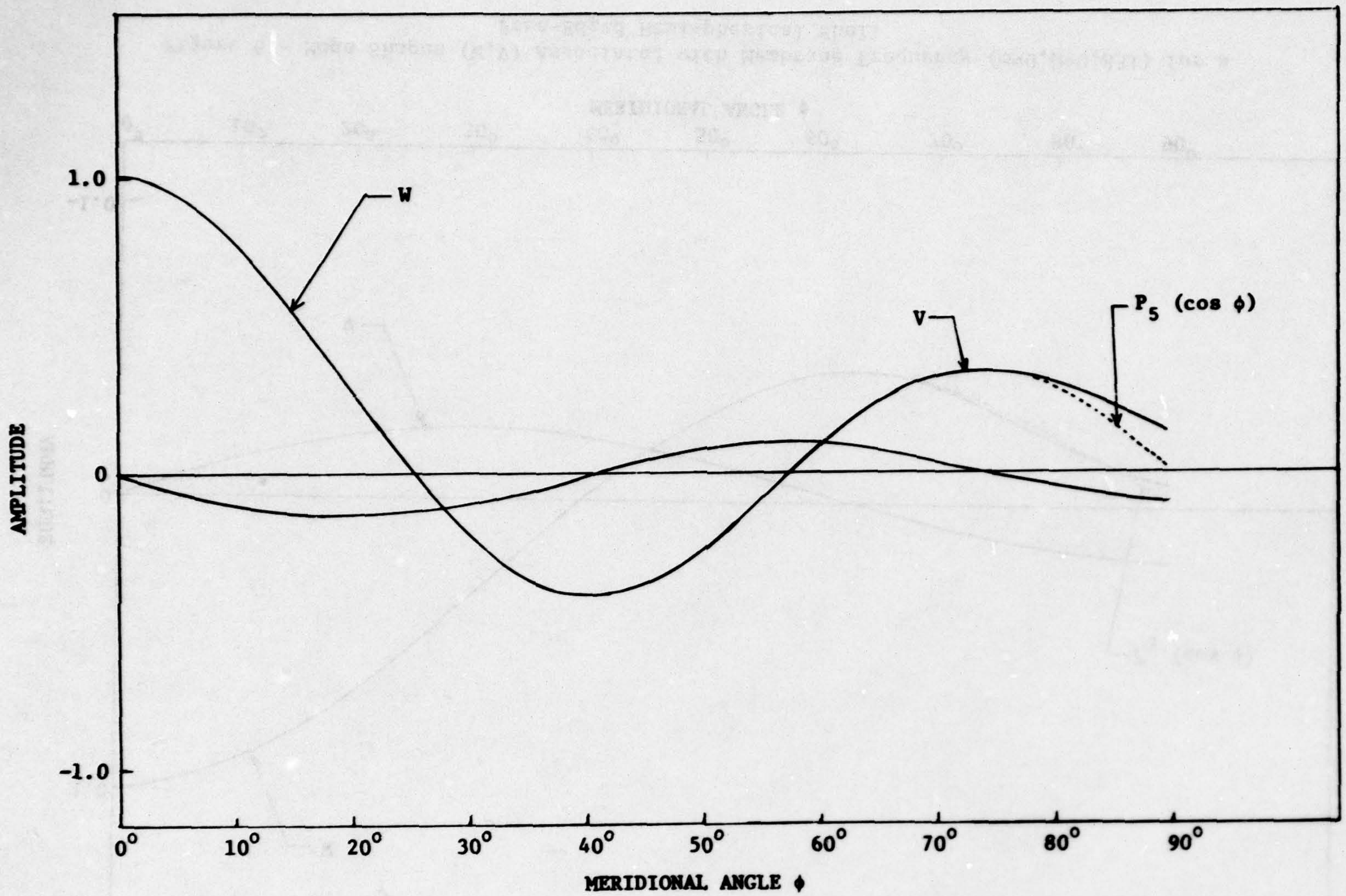

Figure 7 - Mode Shapes (W,V) Associated with Membrane Frequency (m=0,Ω=0.910) for a Free-Edged Hemispherical Shell

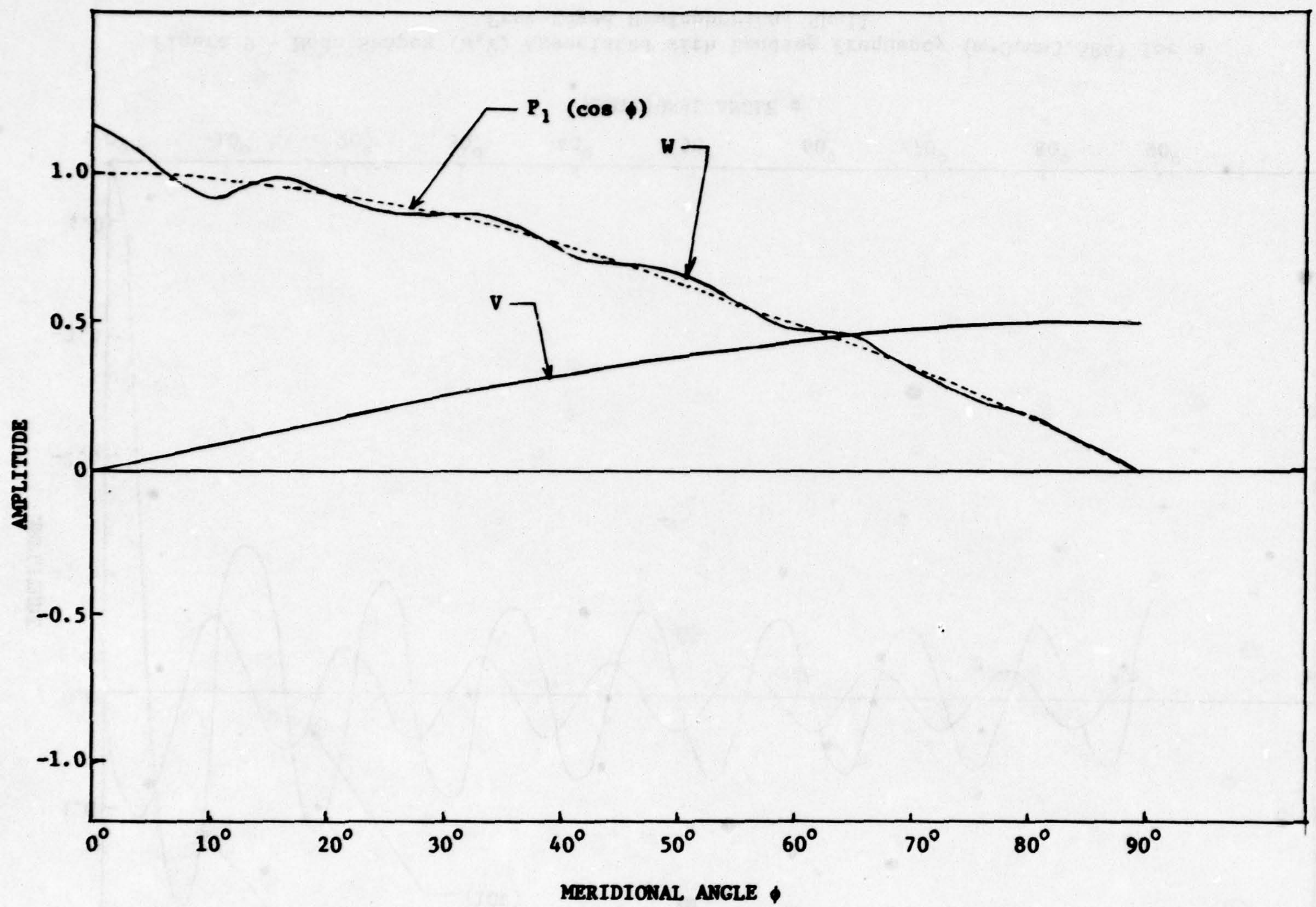

Figure 8 - Mode Shapes (W,V) Associated with Membrane Frequency (m=0,Ω=1.975) for a Free-Edged Hemispherical Shell

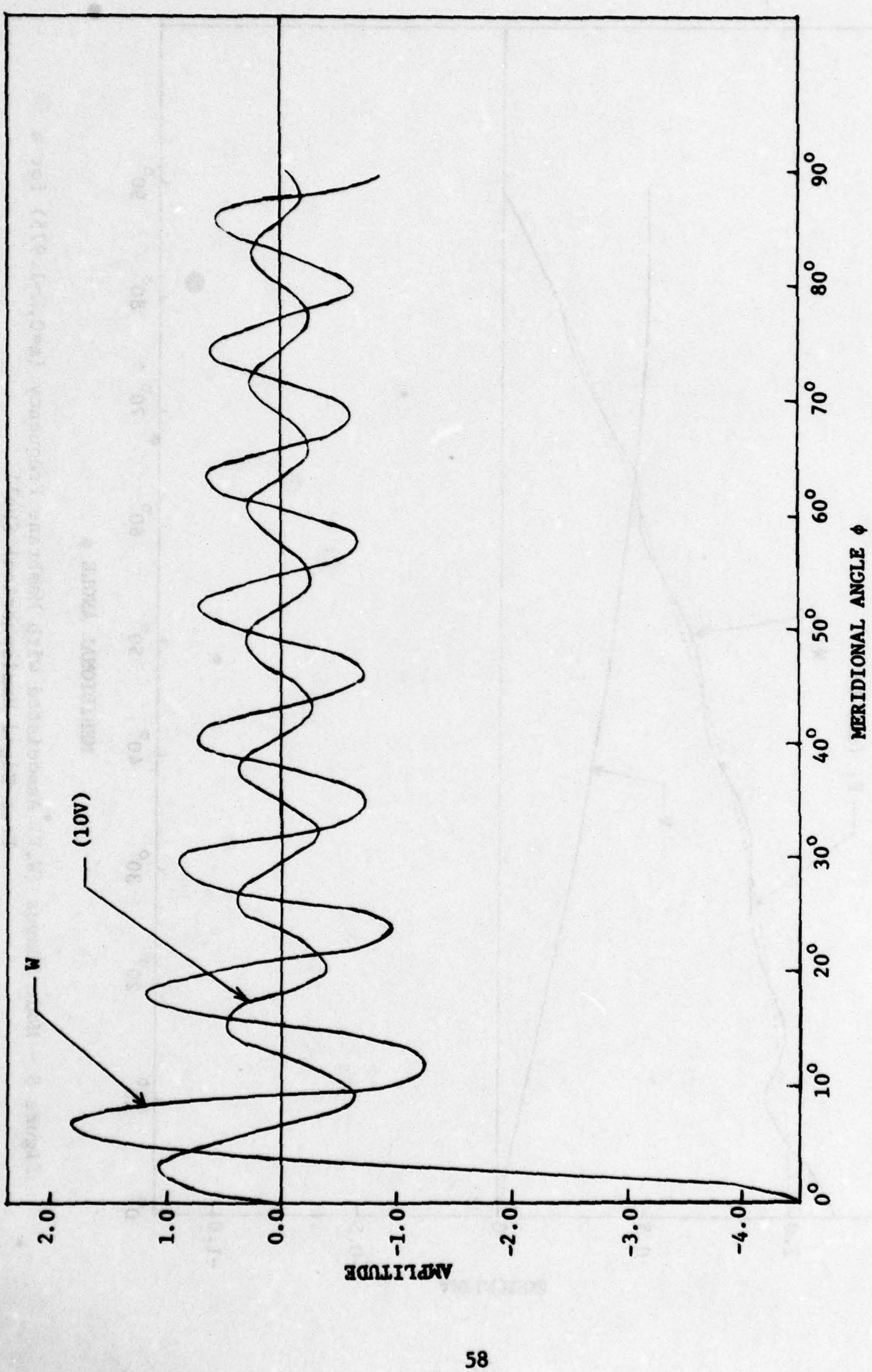

Figure 9 - Mode Shapes (W,V) Associated with Bending Frequency (m=0,Ω=3.584) for a Free-Edged Hemispherical Shell

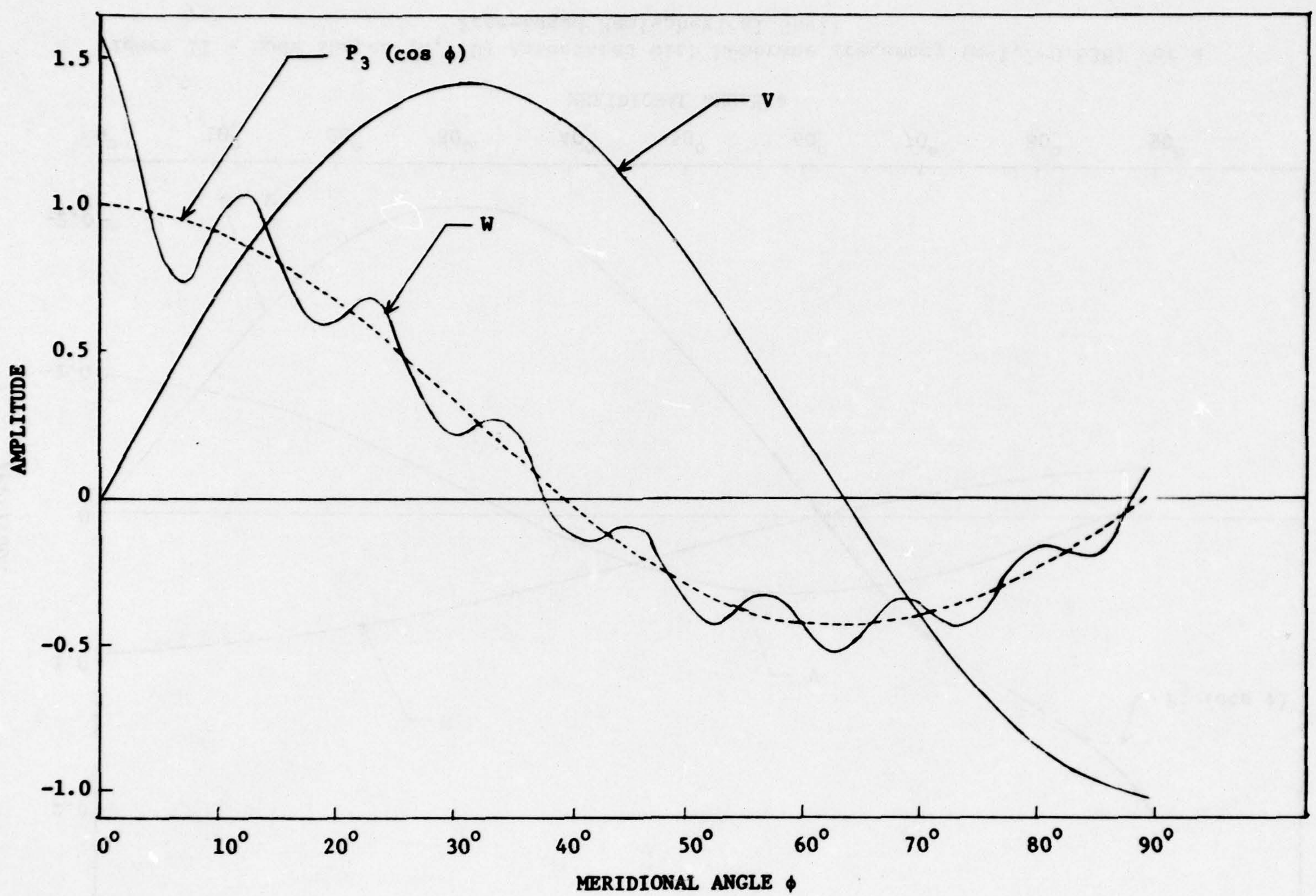

Figure 10 - Mode Shapes (W,V) Associated with Membrane Frequency ($m=0, \Omega=3.635$) for a Free-Edged Hemispherical Shell

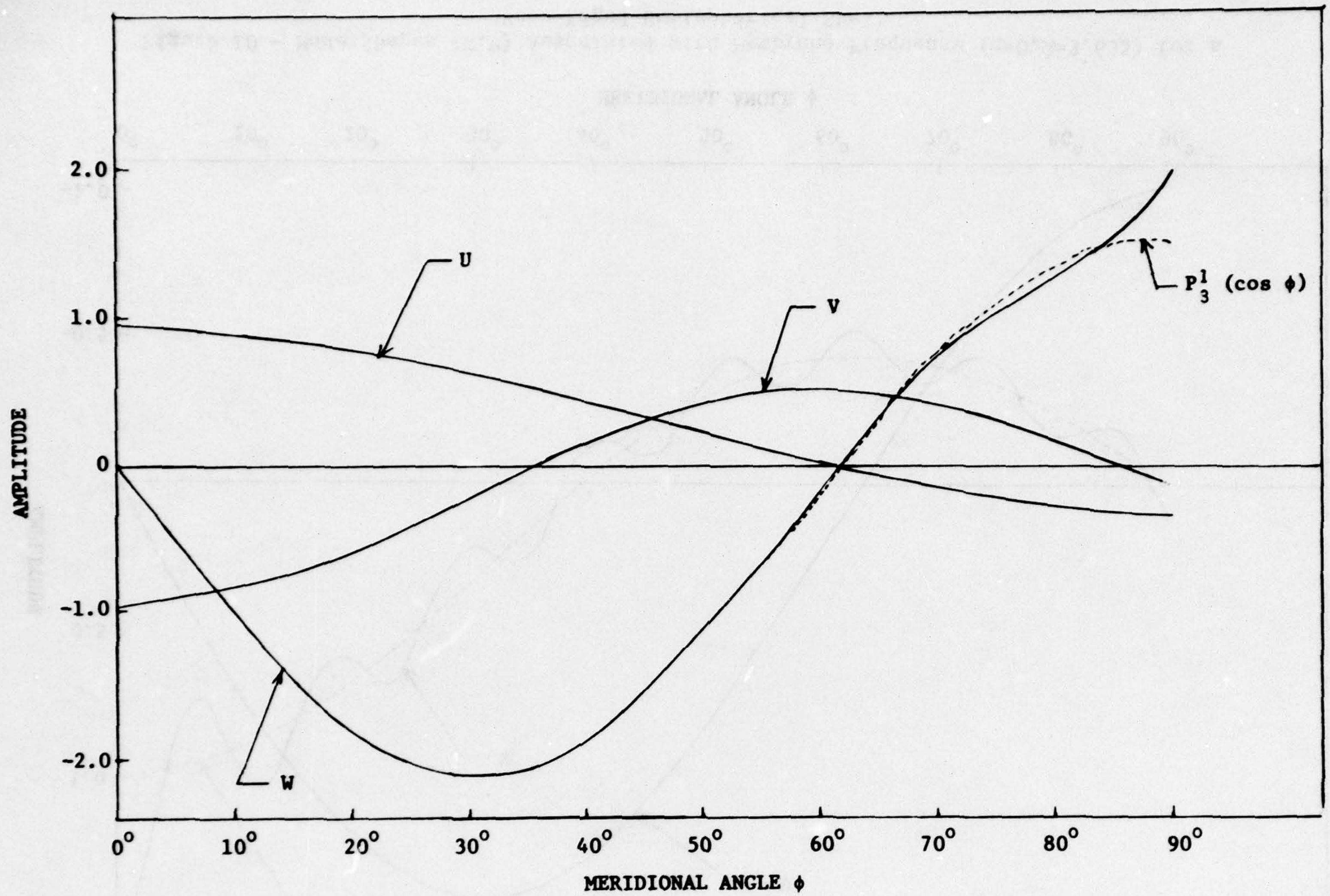

Figure 11 - Mode Shapes (W,V,U) Associated with Membrane Frequency (m=1,Ω=0.836) for a Free-Edged Hemispherical Shell

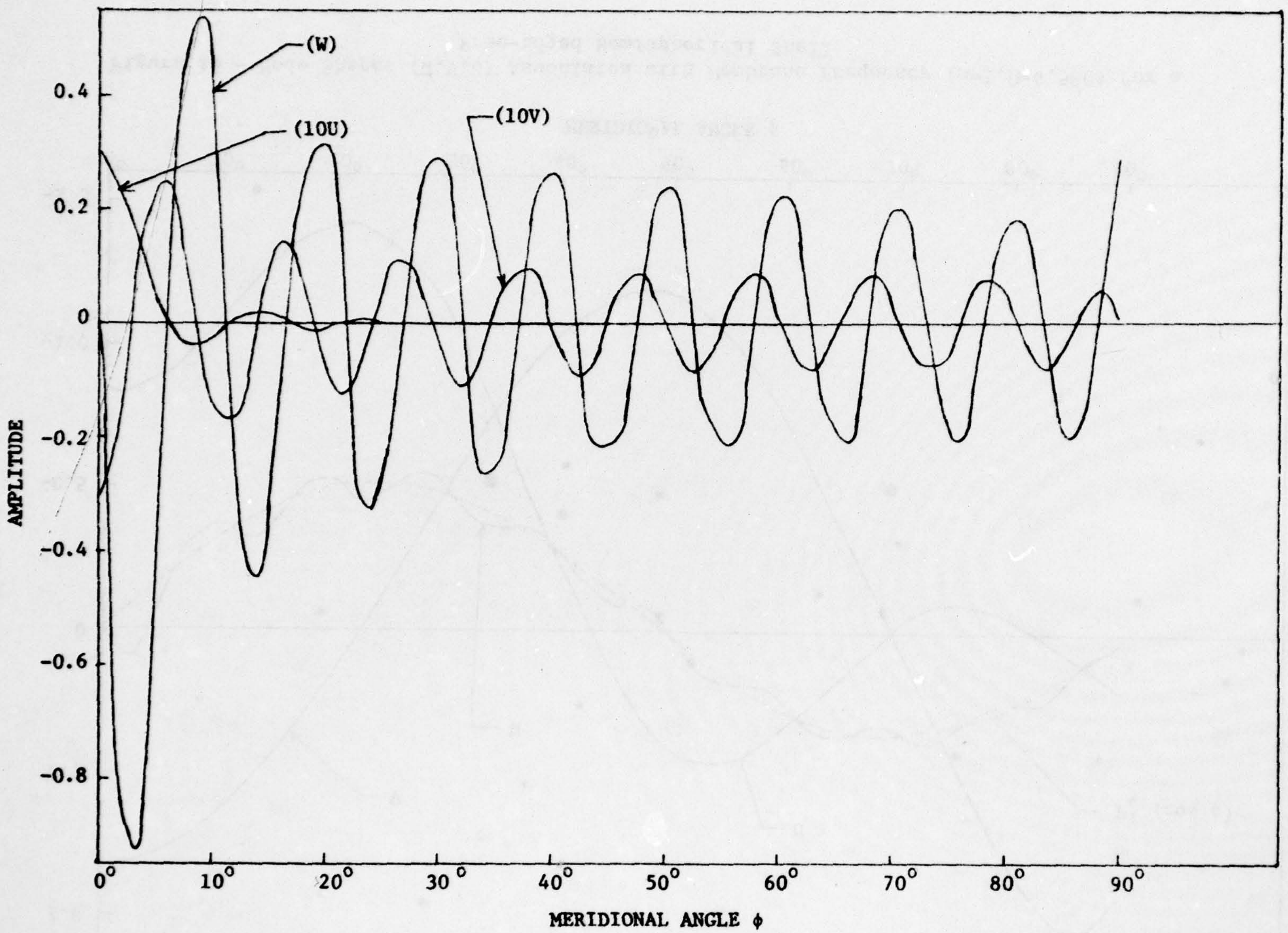

Figure 12 - Mode Shapes (W,V,U) Associated with Bending Frequency ($m=1, \Omega=4.243$) for a Free-Edged Hemispherical Shell

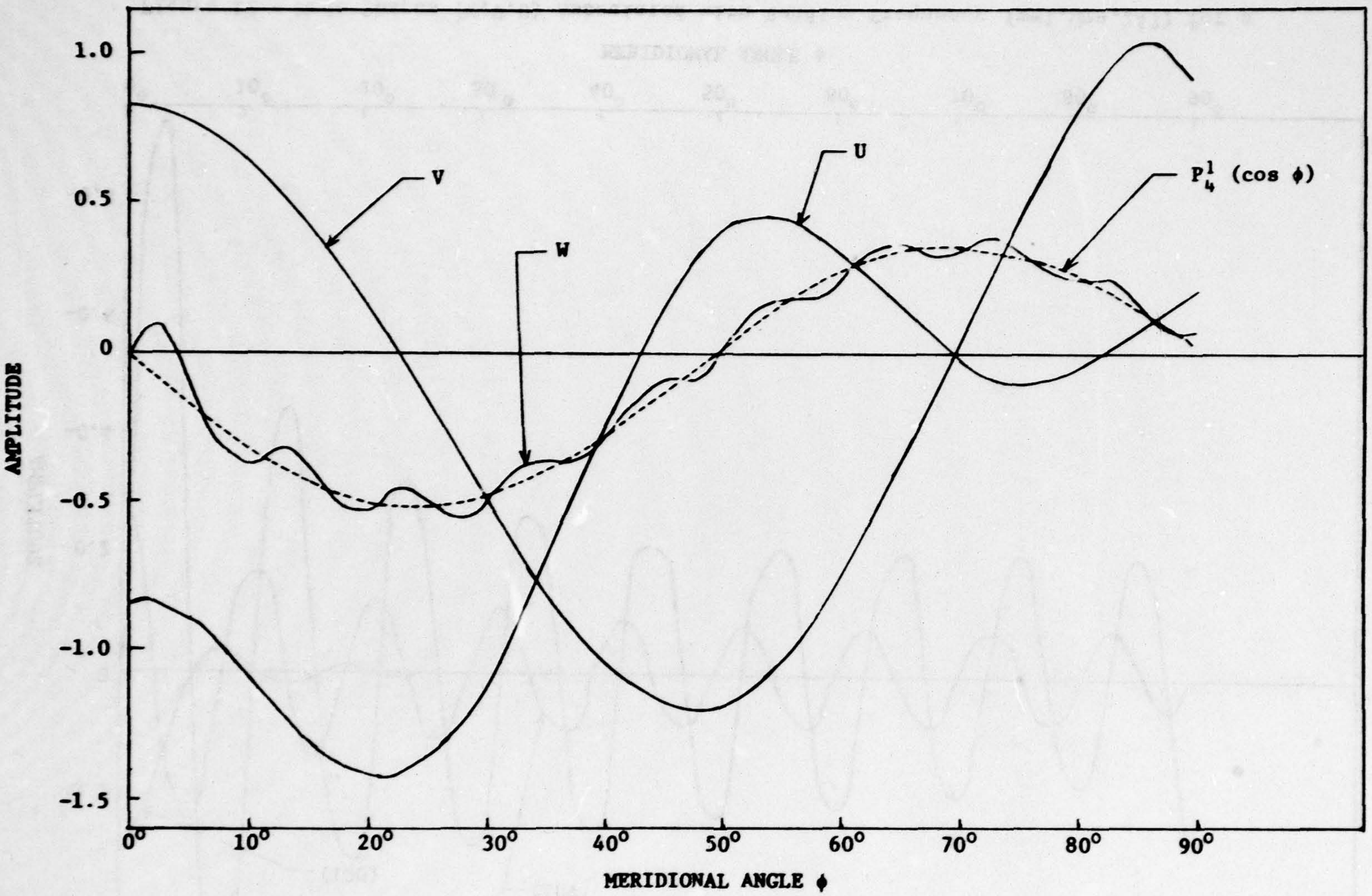

Figure 13 - Mode Shapes (W,V,U) Associated with Membrane Frequency (m=1,Ω=4.580) for a Free-Edged Hemispherical Shell

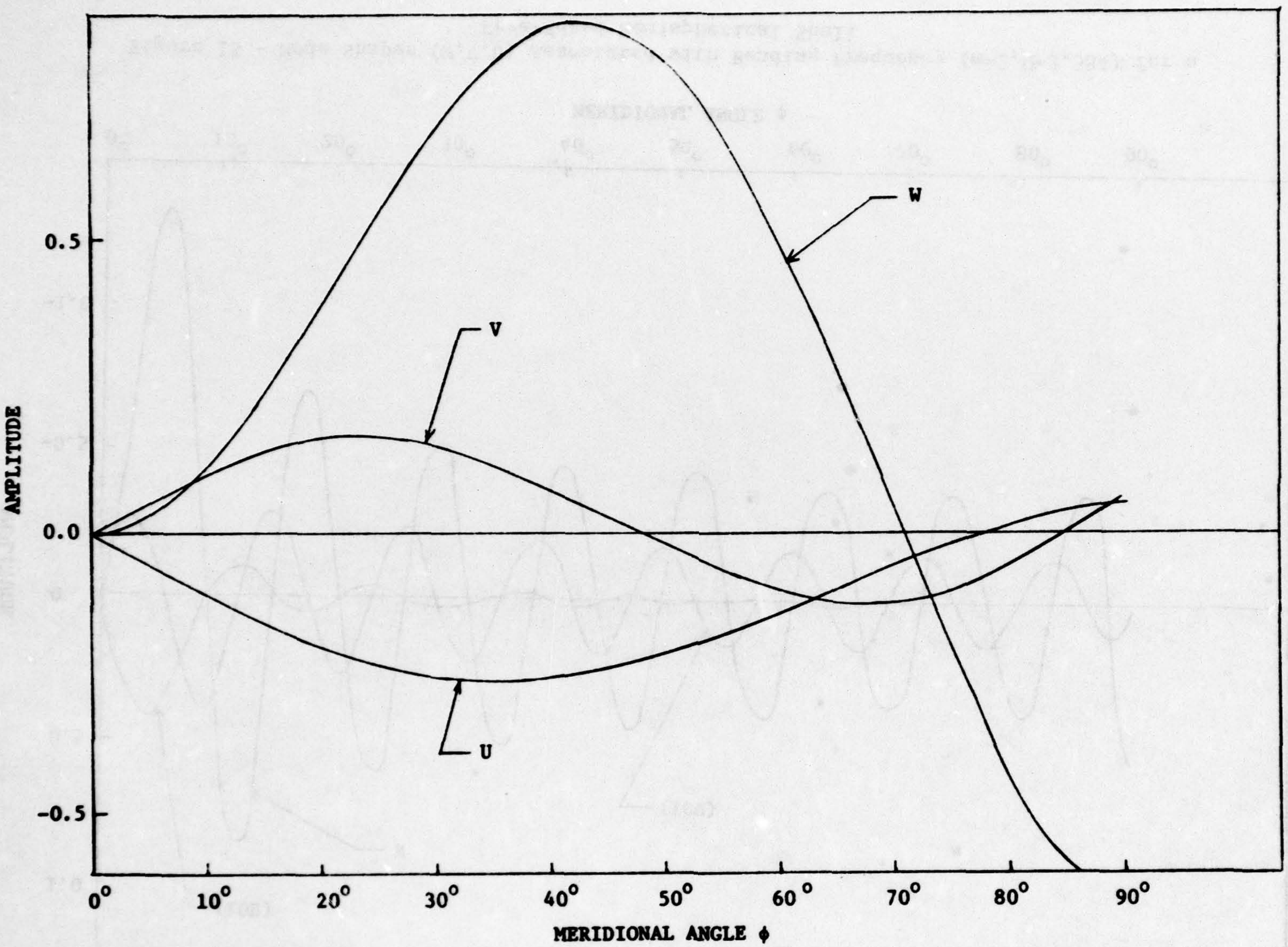

Figure 14 - Mode Shapes (W,V,U) Associated with Membrane Frequency (m=2,Ω=0.896) for a Free-Edged Hemispherical Shell

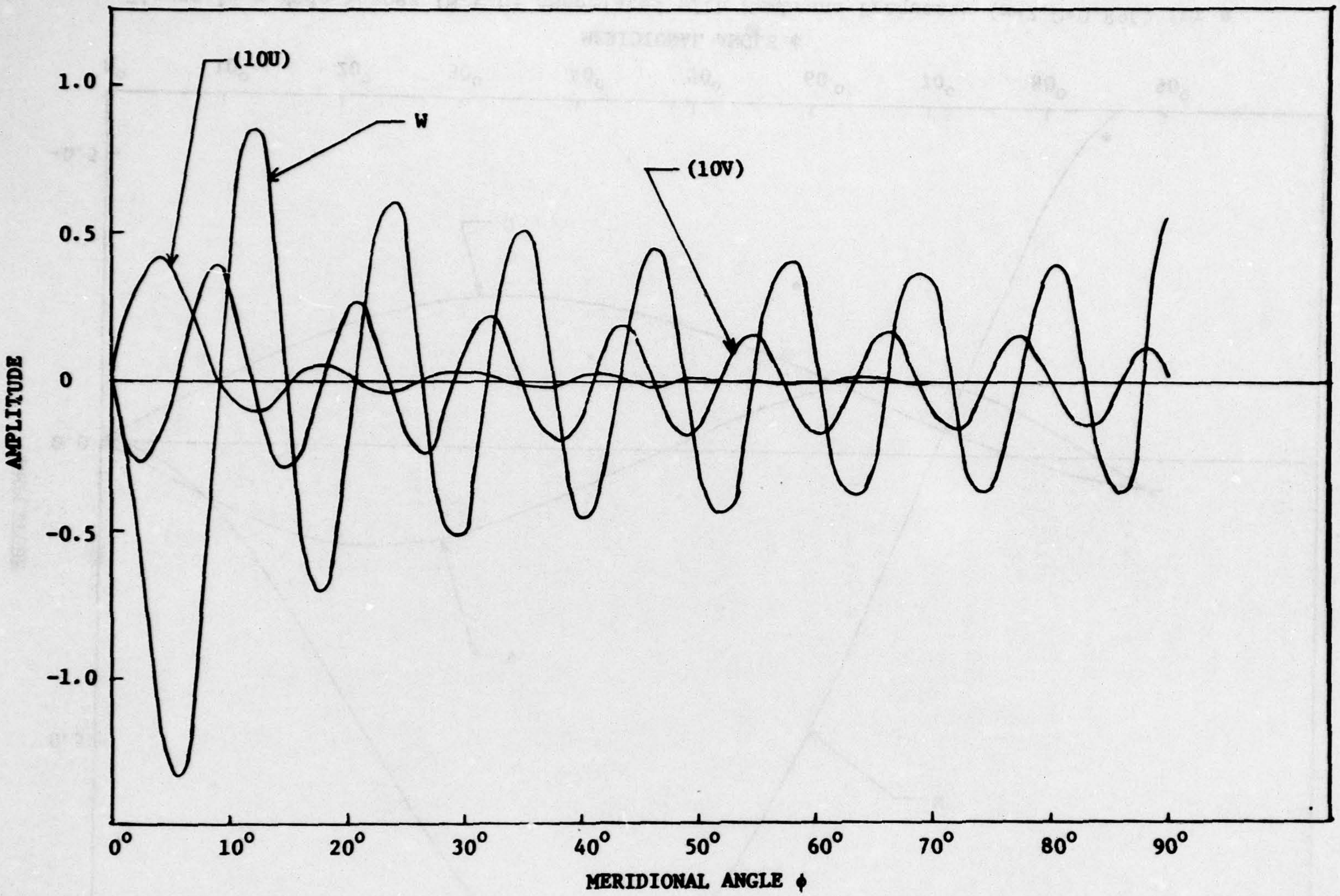

Figure 15 - Mode Shapes (W,V,U) Associated with Bending Frequency (m=2,Ω=3.584) for a Free-Edged Hemispherical Shell

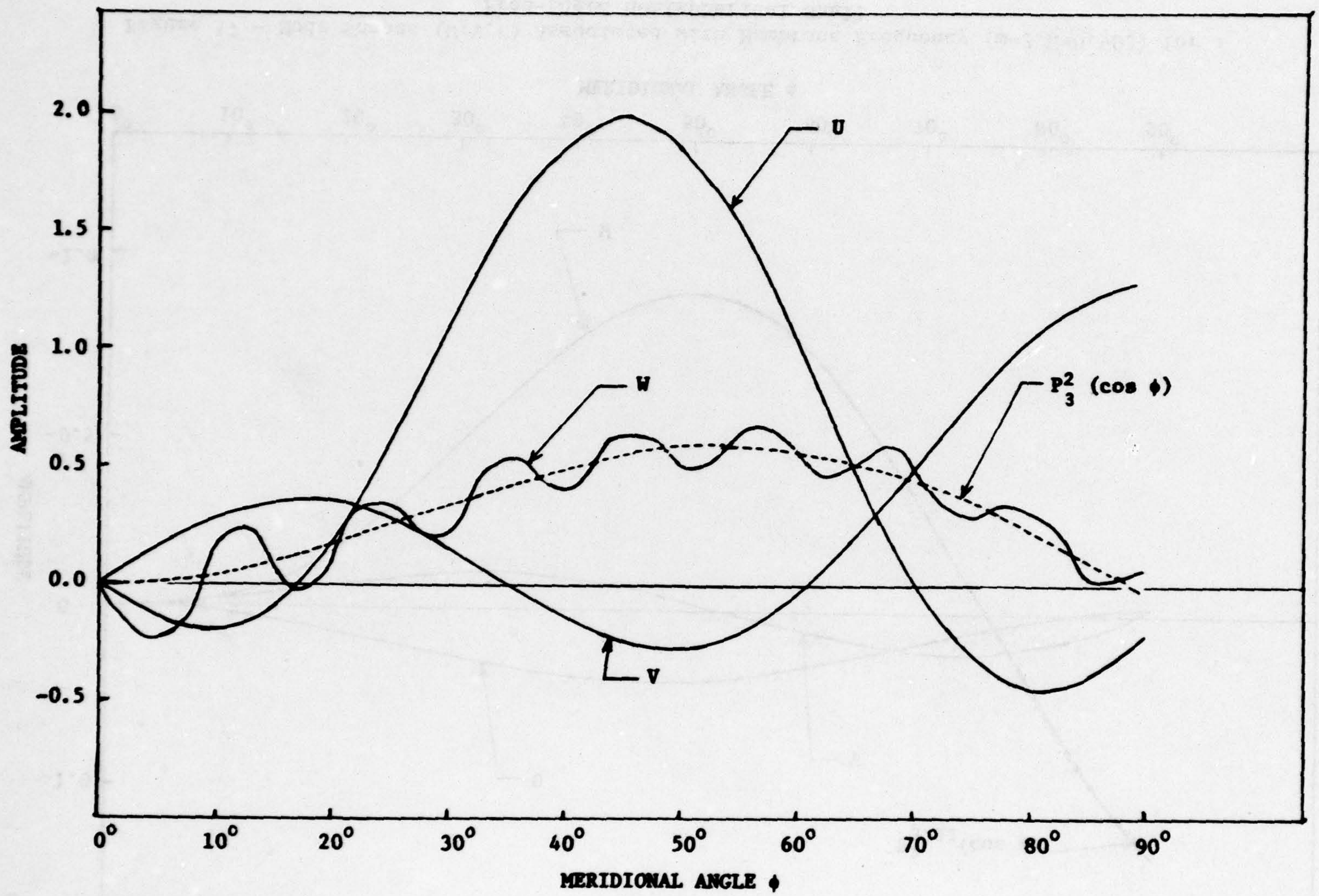

Figure 16 - Mode Shapes (W,V,U) Associated with Membrane Frequency (m=2,Ω=3.667) for a Free-Edged Hemispherical Shell

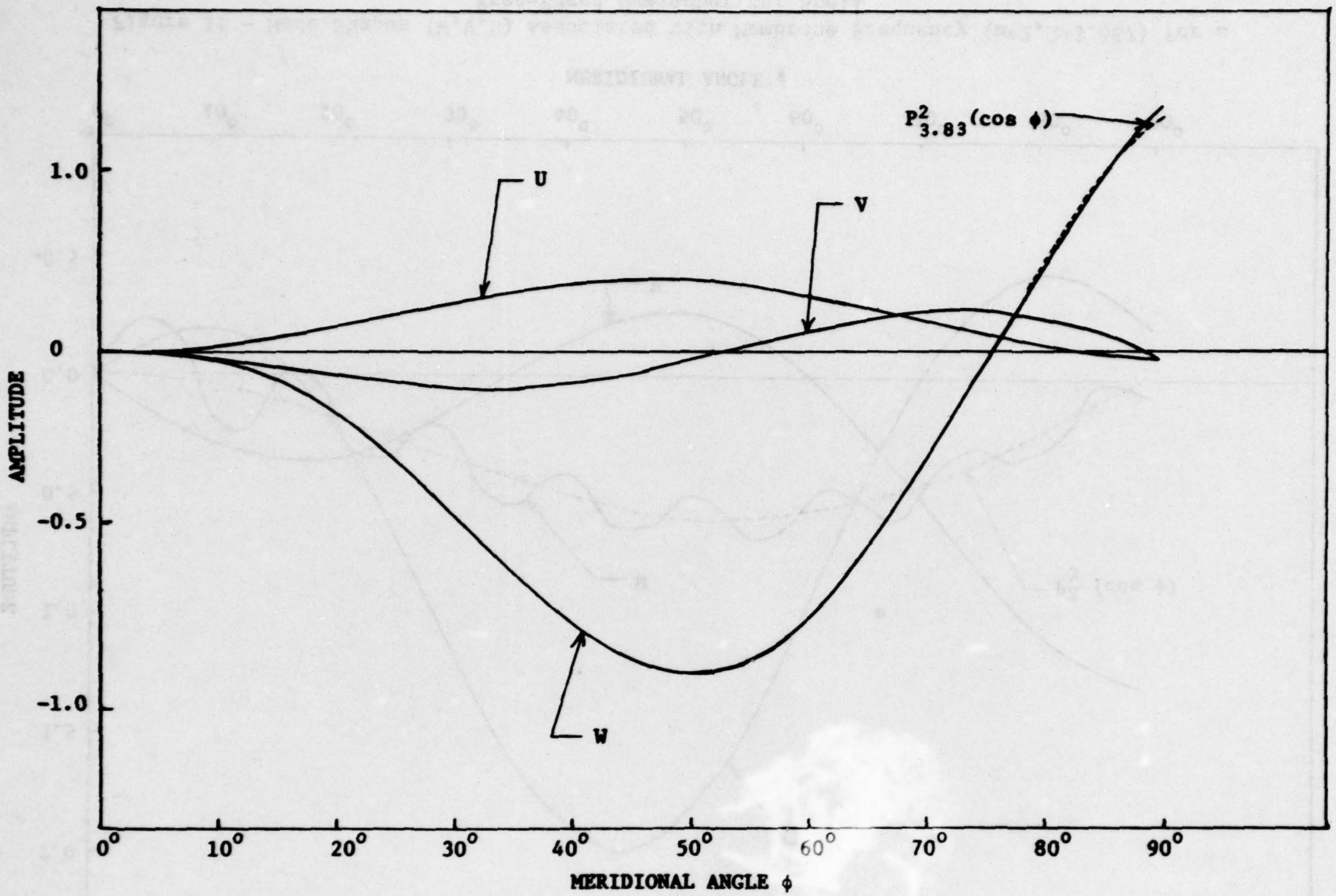

Figure 17 - Mode Shapes (W.V,U) Associated with Membrane Frequency (m=3,Ω=0.903) for a Free-Edged Hemispherical Shell

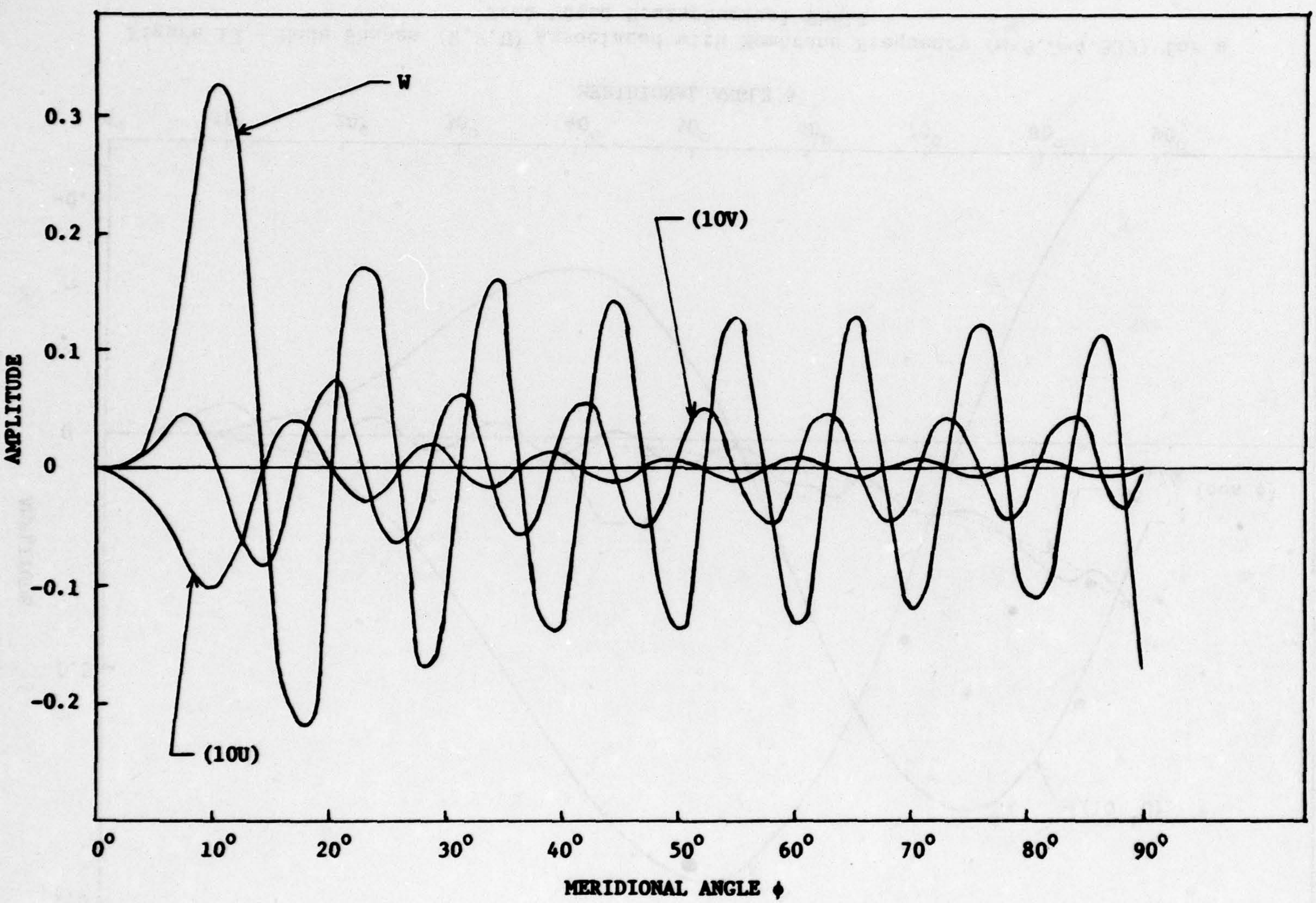

Figure 18 - Mode Shapes (W,V,U) Associated with Bending Frequency (m=5,Ω=4.246) for a Free-Edged Hemispherical Shell

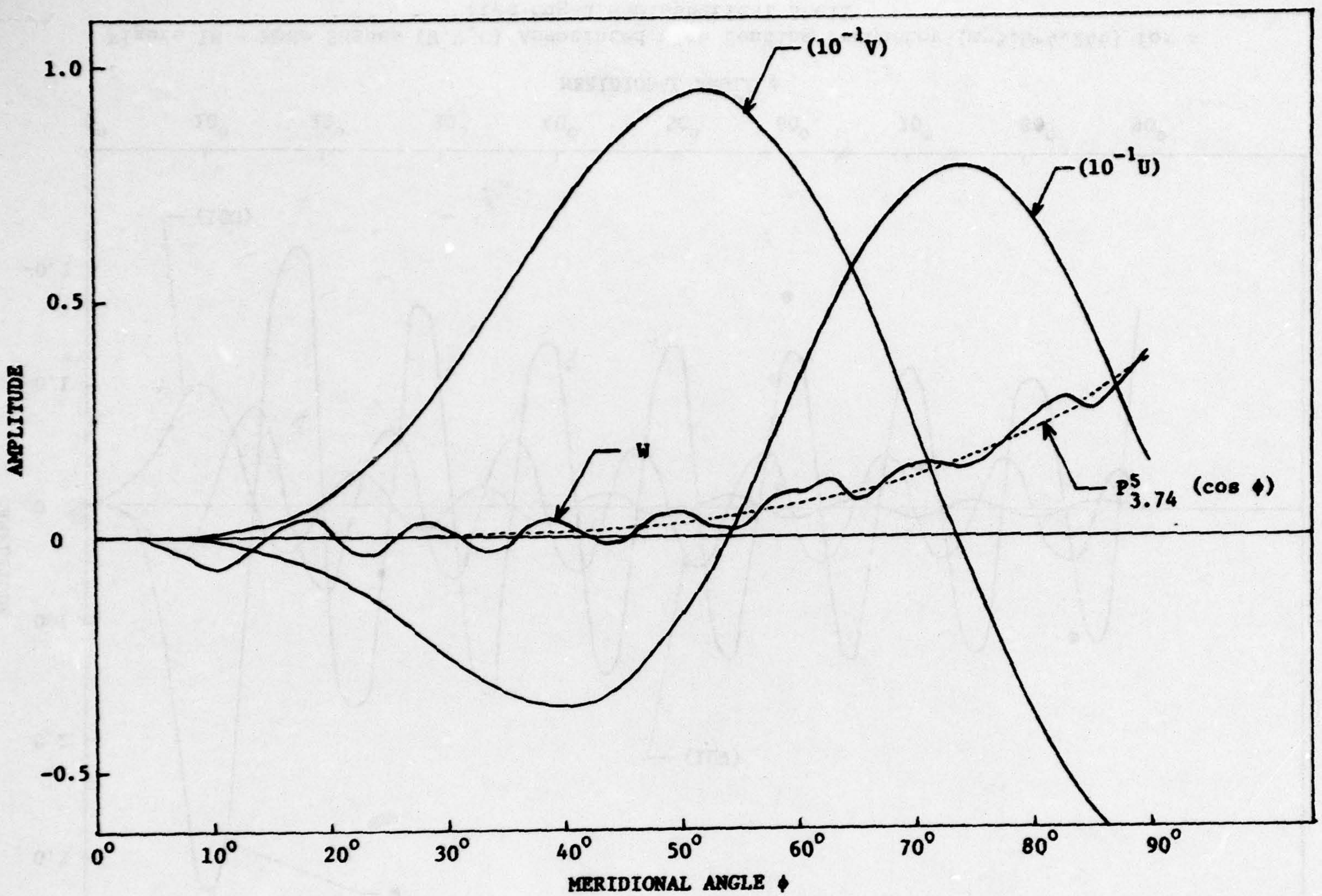

Figure 19 - Mode Shapes (W,V,U) Associated with Membrane Frequency (m=5,Ω=4.339) for a Free-Edged Hemispherical Shell

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23. Bosor 4 results: m=0, Ω=0.70 (232 cps); 0.88 (292 cps); 0.92 (306 cps)

m=1, Ω=0.53 (175 cps)

present analysis: m=0, Ω=0.83; 0.91; 0.94

m=1, no such frequency around Ω=0.53 is found.

where $\Omega = \frac{\omega R}{c_p}$, $c_p = \{E/[(1-\bar{\nu}^2)\rho_s\}^{1/2}$

24. For example, it doesn't matter what the constraint conditions imposed at the apex of the shell, the results are always the same.

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