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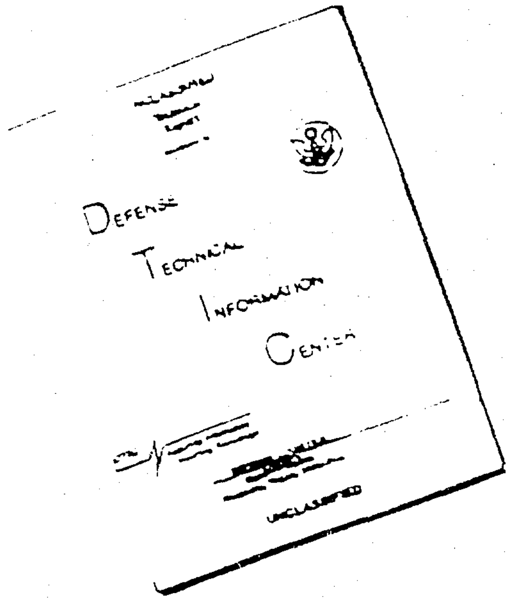

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In order to estimate the rate of vertical transport due to turbulence in the stratosphere, it is necessary to know the degree of mixing that takes place in "Kelvin-Helmholtz billow events." This is estimated by means of a discrete model and the results are compared with published experimental observations. From these considerations it is concluded that a very large degree of mixing probably takes place in such events, and that, therefore, one must use the relation between the bulk vertical eddy diffusivity and layer diffusivity originally proposed by Rosenberg and Dewan instead of the simple one usually employed.

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Preface

I wish to thank Dr. Earl Good and Dr. Antonio Quesada for their many helpful comments on the original draft of this manuscript.

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Mixing in Billow Turbulence and Stratospheric Eddy Diffusion

1. INTRODUCTION

The value of the vertical eddy diffusion coefficient, K_v, which describes the transport effects of small scale turbulence, is at present controversial for the case of the stratosphere. Two relationships have been proposed that relate bulk K_e to the fraction of fluid that is turbulent, P^*, as measured along a given representative length of the stratosphere. One of these, which was first reported by Rosenberg and Dewan,[1] is

$$K_e = \frac{P^* \overline{L^2}}{2 \Delta t_f}, \tag{1}$$

where $\overline{L^2}$ is of the average square of L, and L is defined as the vertical distance upwards from a given altitude (assumed located somewhere within a turbulent layer) to the top of the layer. As shown in Rosenberg and Dewan[1] and Dewan,[2] the manner in which L is calculated yields a value of L that would be somewhat less than one half the thickness of the turbulent layers on average. Rosenberg and Dewan[1] gives

(Received for publication 9 April 1979)

1. Rosenberg, N.W., and Dewan, E.M. (1975) Stratospheric Turbulence and Vertical Effective Diffusion Coefficients, AFCRL-TR-75-0519, AD AO19708.
2. Dewan, E.M. (1979) Estimate of Vertical Eddy Diffusion Due to Turbulent Layers in the Stratosphere, AFGL-TR-79-0042.

the general background and derivation of Eq. (1) while Dewan[2] discusses many details and subtle points omitted in earlier papers. The term Δt_f is the duration of the time frame of the model developed in the works cited above and its value is estimated to be approximately 1500 sec. The other relation, which is used by Woods and Wiley[3] and Lilly[4] is given by

$$K_e = K_L P^* \tag{2}$$

where K_L designates the average eddy diffusivity within the turbulent layers.

It was shown in Dewan[2] that Eqs. (1) and (2) represent two extreme cases that result from the "vertical stack" model for vertical transport in stratified fluids with intermittent layers. It was shown there that Eq. (1) is valid when

$$K_L \Delta t_M >> \frac{4}{\pi^2} \overline{L^2}, \tag{3}$$

where Δt_M is the mixing time which is of order Δt_f (as demonstrated in Dewan)[2]. Eq. (2) is valid when

$$K_L \Delta t_m << \frac{4}{\pi^2} \overline{L^2}. \tag{4}$$

In Dewan,[2] experimental evidence is cited in support of the condition Eq. (3). The purpose of this report is to examine this question theoretically. It will be concluded that, for the case of the stratosphere, the model for mixing developed here supports condition Eq. (3).

2. SIMPLE MODEL FOR MIXING IN A TURBULENT LAYER

The turbulence to be found in the stratosphere is primarily due to the Kelvin-Helmholtz (K-H) shear instability in the context of stable stratification. The purpose of this section is to derive a model for the mixing due to this instability in order to estimate the value of $K_L \Delta t_M$ or its equivalent, so that it can be compared to $[(4\overline{L^2})/\pi^2)]$.

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3. Woods, J.D., and Wiley, R.L. (1972) Billow turbulence and ocean microstructure, Deep Sea Research and Oceanic Abst. 19:87-121.

4. Lilly, D.K., Waco, D.E., and Adelfang, S.I. (1975) Stratospheric mixing estimated from high-altitude turbulence measurements by using energy budget techniques, The Natural Stratosphere of 1974, CIAP Monograph I, Final Report, DOT-TST-75-51, pp. 6-81 to 6-90.

2.1 Definition of K_L

K_L is of course not constant in time in the case of billow turbulence. In contrast to boundary layer turbulence, a billow cannot be described by "steady-state" conditions. Instead it consists of a turbulent breakdown followed by a decay. The term $K_L \Delta t_M$ is, therefore, somewhat vague, and for purposes of calculation we sharpen its meaning by defining

$$K_L^o \Delta t_M = \int_o^\infty K_L(t)\, dt^\dagger , \tag{5}$$

where K_L^o is the initial eddy diffusivity subsequent to breakdown, and K_L (t)is the time dependent value of K_L. Henceforth Eq. (5) or it's "finite difference equivalent" will be used in place of "$K_L^o \Delta t_M$."

K_L (t) is defined here as

$$K_L(t) = u\ell , \tag{6}$$

where u is the turbulent fluctuation velocity, that is, the root mean square deviation from the mean velocity, and ℓ is a representative length scale for the energy containing eddys (compare Tennekes and Lumley[5], pp 44-47, and Pasquill[6]). We shall assume that both ℓ and u can depend upon time. The decay of u is due to viscous dissipation from small scale motion at the small end of the wave number spectrum cascade.

To define ℓ, we consider two plausible possibilities: The first is (see Tennekes and Lumley[5], pp. 48 and 62)

$$\ell = \frac{u}{S} , \tag{7}$$

where S is the vertical shear of the horizontal mean velocity. The other is

$$\ell = \frac{u}{N} , \tag{8}$$

where N is the buoyancy frequency. Eq. (8) follows from energy considerations to be described below. In the development of our model below we will choose Eq. (7) for ℓ. This will be done in order to get a lower limit estimate of the degree of mixing.

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†This integral converges when due account is taken of viscosity.

5. Tennekes, H., and Lumley, J. L. (1972) A First Course in Turbulence, MIT Press, Cambridge, Massachusetts.

6. Pasquill, F. (1962) Atmospheric Diffusion, Van Nostrand Co., Ltd.

Next we consider the mixing effects of $K_L(t)$ upon itself through erosion of the mean profiles of temperature and velocity within the mixing layer as the turbulence mixes material within the layers; this mixing will decrease the slope of mean quantities with respect to the vertical coordinate Z. This would, by definition, decrease the values of S and N. This in turn would (from Eqs. (7) and (8) cause ℓ to become larger and in turn, from Eq. (6), tend to increase K_L (t). The maximum value of ℓ (vertically) is the layer thickness; therefore, if at the start ℓ has this value, then erosion effects will not lengthen it.

The value of u decreases monotonically as turbulence decays. This will cause K_L (t) to tend to decrease in time, as will be shown below; however, if ℓ increases or remains constant, then $\int_0^\infty K_L(t)\,dt$ can assume much larger values than $(4L^2)/\pi^2$ (unless, as will be shown, L is quite small). It will be shown that, in this way, stratospheric mixing due to billow events is probably very large within the layers and that therefore Eq. (3) holds.

2.2 Initial Fluctuation Velocity

The initial fluctuation or rms velocity, u_1, is defined as the velocity of turbulent motion immediately after the K-H instability breaks down. It will depend upon the Richardson number, which is defined as

$$R_i = \frac{g}{\Theta}\left(\frac{d\theta}{dZ}\right)\left(\frac{\Delta U}{\Delta Z}\right)^{-2}, \tag{9}$$

where g is gravitational acceleration, Θ the average potential temperature over a scale as large as the entire layer, and θ is the mean potential temperature as a function of height, Z. ΔU is the difference in mean horizontal velocity across the layer, and ΔZ is the layer thickness. This quantity, as defined by Eq. (9), is sometimes called the "layer Richardson number."

To estimate u_1 we employ a simple energy budget approach that is based upon the work of Businger[7] and Ludlam.[8] Figure 1 schematically depicts the situation The layer of thickness, ΔZ, has a shear, S, across it. Since $S = \Delta U/\Delta Z$, $\Delta U = S\Delta Z$. We now estimate availability of kinetic energy of this configuration. For convenience we define $\delta U = \Delta U/2$ and $\delta Z = \Delta Z/2$. Let $\overline{U}$ be the mean horizontal velocity of the fluid in the center of the layer just prior to breakdown. We now calculate the kinetic energy made available by the exchange of the four parcels as indicated in

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7. Businger, J.A. (1969) On the energy supply of clear air turbulence, in Clear Air Turbulence and Its Detection, edited by Y.H. Pao and A. Goldberg, Plenum Press, New York, pp. 100-108.

8. Ludlam, F.H. (1967) Characteristics of billow clouds and their relation to clear air turbulence, Quart. J. Roy. Soc. 93:419-435.

Figure 1 (two parcels being located at the center, and the other two on opposite boundaries of the layer). The resulting available fluctuation kinetic energy (per unit mass), KE, would thus be

$$KE = \frac{1}{2}\left[(\overline{U} + \delta U)^2 + (\overline{U} - \delta U)^2 - 2\overline{U}^2\right]$$

$$= 2\,\delta\,U^2 = \frac{S^2\,\Delta Z^2}{2}\,. \qquad (10)$$

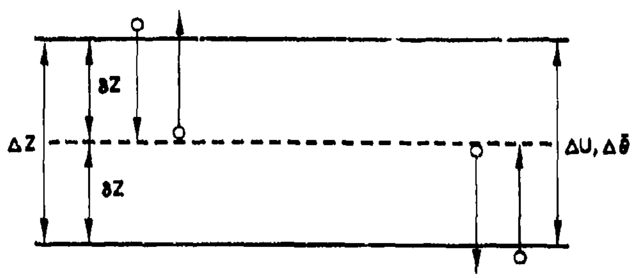

Figure 1. Parcel Exchange Diagram for Billow Energy Budget

To calculate the potential energy change (per unit mass), PE, caused by the process of displacing the four parcels in Figure 1, we assume that the stable temperature profile is not altered significantly by this exchange. We calculate the work per unit mass needed to move one of these parcels a distance δZ against the buoyance force. This is given by

$$\frac{g}{\Theta}\,\frac{d\overline{\theta}}{dZ}\int_0^{\delta Z} Z'\,dZ' = N^2\,\frac{\delta Z^2}{2} \qquad (11)$$

where

$$N^2 = \frac{g}{\Theta}\,\frac{d\overline{\theta}}{dZ}\,, \qquad (12)$$

which is the square of the buoyancy frequency, N. Four parcels thus require four times the work in Eq. (11); therefore,

$$PE = \frac{N^2\,(\Delta Z)^2}{2}\,. \qquad (13)$$

These results allow an estimate of u_1. Subtracting PE from KE we arrive at the total remaining available kinetic energy that can give rise to turbulent velocity fluctuations. The latter will be $3\,(u_1)^2/2$ if we assume isotropy. Thus

$$\frac{3}{2} u_1{}^2 = \frac{(\Delta Z)^2}{2} (S^2 - N^2) , \tag{14}$$

hence

$$u_1 = S \, \Delta \, Z \sqrt{\frac{1 - R_i}{3}} , \tag{15}$$

where

$$R_i = N^2/S^2 .$$

2.3 Estimation of Dissipation Rate and Velocity Decay Rate

Having an estimate for u_1 we now estimate u as a function of time. This is needed, of course, in order to estimate K_L as a function of time from Eq. (6). As shown by Eqs. (7) and (8), there is an interrelation between ℓ and u. To simplify this complication we shall divide the time into finite epochs during which ℓ can be considered as constant. During one of these epochs we shall also assume that S and N are constant. The duration of a particular epoch will be taken as the characteristic time for the fluctuation motion given (from dimensional analysis) by

$$\Delta t = \ell/u , \tag{16}$$

where ℓ and u are the values at the start of the epoch under consideration.

Next we employ the well known relation between the dissipation rate, ϵ, and the velocity and length scales given by

$$\epsilon = \frac{u^3}{\ell} . \tag{17}$$

From this, one can calculate decay from

$$\frac{d\left(\frac{3}{2} u^2\right)}{dt} = - \frac{u^3}{\ell} . \tag{18}$$

Thus, in order to calculate the decrease in u in the time interval Δt, we integrate the separable Eq. (18) and obtain

$$\frac{1}{u_f} - \frac{1}{u_i} = \frac{\Delta t}{3\ell}, \tag{19}$$

where u_i and u_f are the initial and final fluctuation velocities of the epoch Δt (compare Dewan)[9]. Using Eq. (18) for Δt (u_i = u in that equation), we have

$$u_f = \frac{3}{4} u_i . \tag{20}$$

We shall use this relation below for an iterated sequence of epochs.

2.4 Estimation of the Erosion of S and N and the Time Dependence of ℓ

We now examine the erosion of mean temperature and mean horizontal velocity profiles caused by turbulent mixing. For generality we shall employ the symbol $\Psi(Z)$ to stand for either the mean temperature profile $\overline{\theta}(Z)$ or the mean horizontal velocity profile U(Z). Figure 2 shows our assumed linear initial profile $\Psi(Z)$ within the layer. Denoting the layer thickness by Λ,† letting Z = 0 at the bottom of the layer and letting Ψ_o be the value of Ψ at the top of the layer with Ψ = 0 at the bottom for convenience, we have

$$\Psi = \frac{\Psi_o Z}{\Lambda} . \tag{21}$$

Both $\overline{\theta}(Z)$ and U(Z) within the layer obey the diffusion equation during mixing (assuming K_L constant within the layer). Thus,

$$K_L \frac{\partial^2 \Psi}{\partial Z^2} = \frac{\partial \Psi}{\partial t} . \tag{22}$$

While this is clearly reasonable for $\overline{\theta}(Z)$, one may wish to consult Batchelor[10], pp. 186-189, for the case of U(Z). The initial conditions for Eq. (22) in the region $o < Z < \Lambda$ is given by Eq. (21). The boundary condition will be taken as

$$\left.\frac{\partial \Psi}{\partial Z}\right|_{Z=0,\,\Lambda} = 0 . \tag{23}$$

† Formerly denoted by ΔZ.

9. Dewan, E.M. (1976) Theoretical Explanation of Spectral Slopes in Stratospheric Turbulence Data and Implications for Vertical Transport, AFGL-TR-76-0247, AD AO36307.

10. Batchelor, G.K. (1967) An Introduction to Fluid Dynamics, Cambridge Univ. Press.

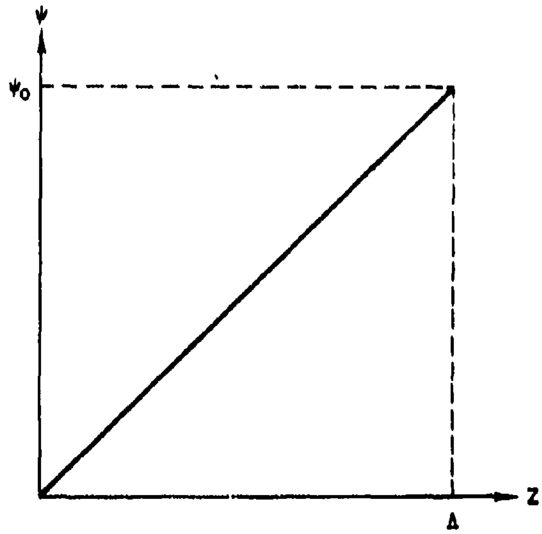

Figure 2. Initial Density, Potential Temperature, or Velocity Profile

This condition represents the assumption that the boundaries of the layer are to be regarded as perfectly insulated. The observations of Woods and Wiley[3] for the ocean lend some support to this assumption, and it would presumably hold to an even greater degree for the atmosphere (see Section 4.2 below).

The standard methods for solving this problem, using Fourier series, lead to

$$\Psi(Z,t) = \Psi_o \left\{ \frac{1}{2} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{g(n)}{n^2} \cos\left(\frac{n\pi z}{\Lambda}\right) e^{-[K_L (n\pi/\Lambda)^2 t]} \right\} . \tag{24}$$

where g (n) = -1 for odd values of n and zero for even values. Due to the rapid decay of the higher harmonics and rapid convergence of the series, we need consider only the dominant, first term. Thus

$$\Psi(Z,t) \sim \Psi_o \left\{ \frac{1}{2} - \frac{4}{\pi^2} \cos\left(\frac{\pi z}{\Lambda}\right) e^{-[K_L (\pi/\Lambda)^2 t]} \right\} . \tag{25}$$

Figure 3 shows $\Psi(Z,O)$ as well as $\Psi(Z,t)$ during diffusion. Figure 3 is only a schematic, not a mathematical graph. Next we wish to use Eq. (25) to determine S or N^2 as a function of time. Since the latter quantities are proportional to $\partial\Psi/\partial Z$, we estimate a representative value of $\partial\Psi/\partial Z$ by joining the end points of $\Psi(Z,t)$ at $Z = O$ and $Z = \Lambda$ as indicated in Figure 3. Thus,

$$\frac{\partial\Psi(t)}{\partial Z} \sim \frac{\Psi(\Lambda,t) - \Psi(o,t)}{\Lambda} . \tag{26}$$

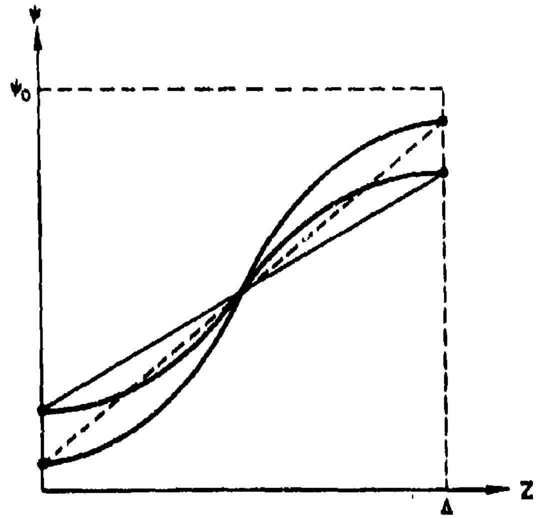

Figure 3. Profile of Mean Quantities as a Function of Time

From Eq. (21):

$$\frac{\partial \Psi(t)}{\partial Z} \sim \left(\frac{8}{\Pi^2}\right)\frac{\Psi_o}{\Lambda}\, e^{-K_L(\Pi/\Lambda)^2 t} \tag{27}$$

The constant $(8/\Pi^2) = 0.811$; however, we shall approximate it by unity in order to obtain the correct slope for $t = O$ at the beginning of the first epoch, and for $t = \Delta t$:

$$\left.\frac{\partial \Psi}{\partial Z}\right|_{t=\Delta t} \sim \frac{\Psi_o}{\Lambda}\, e^{-[K_L(\Pi/\Lambda)^2 \Delta t]} . \tag{28}$$

Thus

$$N^2(\Delta t) = N^2(o)\, e^{-[K_L(\Pi/\Lambda)^2 \Delta t]} \tag{29a}$$

$$S(\Delta t) = S(o)\, e^{-[K_L(\Pi/\Lambda)^2 \Delta t]} . \tag{29b}$$

We could now estimate the decay of N^2 or S from epoch to epoch on the basis of Eq. (29). Before doing so we first derive Eq. (8) from energy considerations. From this point on, u will refer to the isotropic velocity, but for the present purpose we will consider it to be in the vertical direction. Let W represent the work done by a vertically moving parcel as it goes from its initial equilibrium position to a point located vertically a distance ℓ away. Then

$$W = \int_0^{\ell} \rho\, N^2\, Z dZ = \frac{\rho N^2 \ell^2}{2}, \tag{30}$$

where ρ is the density of the fluid. Equating W to the kinetic energy of the parcel,

$$\frac{\rho u^2}{2} = \frac{\rho N^2 \ell^2}{2}. \tag{31}$$

Hence,

$$\ell = \frac{u}{N}. \tag{32}$$

The important point to notice about this relation is that as u decays, N could decrease at such a rate that ℓ remains constant or even increases in time. This consideration applies to ℓ defined by Eq. (7) as well.

2.5 A Simple Model for Mixing

It is now possible to set up a flow chart for a discrete time or "finite difference" model for mixing that employs all of the above observations, so as to give K_L as a function of time.

Let ℓ_1, u_1, N_1, S_1, and K_1 be the initial values of the respective variables. We have here suppressed the subscript L in K_L. From previous considerations we have:

$$\ell_1 = u_1/N_1 \tag{33}$$

from Eq. (32), where N_1 could be replaced by S_1;

$$u_1 = \sqrt{\frac{1 - R_1}{3}}\; S_1 \Lambda \tag{34}$$

from Eq. (15);

$$K_1 = \ell_1\, u_1 \tag{35}$$

from Eq. (6), and

$$\Delta t_1 = \ell_1/u_1 \tag{36}$$

from Eq. (16).

To calculate the effect of erosion of N_n or S_n in one time step, Eq. (35) gives

$$\frac{N_{n+1}}{N_n} = \exp\left(-\frac{1}{2} K_n \Delta t_n \frac{\pi^2}{\Lambda^2}\right), \tag{37}$$

where the value n = 1 designates the initial step and the factor $\frac{1}{2}$ in the exponent would be absent in the case of S_{n+1}/S_n. Subsequent epochs would be indicated by successive integral values of n. From Eqs. (20), (8), (6), and (16) we obtain the following relations between contiguous epochs:

$$\begin{aligned} u_{n+1} &= \frac{3}{4} u_n \\ \ell_{n+1} &= u_{n+1}/N_{n+1} \\ K_{n+1} &= u_{n+1}\, \ell_{n+1} \\ \Delta t_{n+1} &= \ell_{n+1}/u_{n+1} \, . \end{aligned} \tag{38}$$

Equation (37) completes this set. Before setting up the flow chart for this model, it is useful to put Eq. (38) into dimensionless form. We use N_1 and Λ as the representative scales and make the following definitions:

$$\begin{aligned} u_n' &= u_n/(N_1 \Lambda) \\ N_n' &= N_n/N_1 \\ \ell_n' &= \ell_n/\Lambda \\ K_n' &= K_n/(N_1 \Lambda^2) \\ \Delta t_n' &= N_1/N_n \, , \end{aligned} \tag{39}$$

where, in the last equation of this set, we replaced ℓ by its value given in Eq. (38) (compare with Eq. (36)).

3. STABILITY AND MIXING

Our objective is to use the above model to ascertain when the criteria, Eqs. (3) and (4), are valid. When this is done, the degree of molecular mixing will be considered.

3.1 Stability

We now consider the stability of the model given by Eqs. (38) and (37). In order to arrive at a conservative estimate of what layer thickness and R_i gives rise to effectively total mixing, we reconsider the model in view of the ambiguity of using either S or N in the definition of ℓ, and redesign it to contain the most conservative aspects of both approaches.

For example, Eq. (29) gives us

$$N_2 = N_1 \exp\left[-\frac{1}{2} K_1 \left(\frac{\Pi}{\Lambda}\right)^2 \Delta t\right] \tag{40}$$

and

$$S_2 = S_1 \exp\left[-K_1\left(\frac{\Pi}{\Lambda}\right)^2 \Delta t\right]. \tag{41}$$

In Eq. (41) the erosion could be significantly faster than in Eq. (40) due to the fact that the absolute value of the negative exponent is twice as large. To be conservative we therefore choose the milder erosion rate given by Eq. (40) (N is an inverse time scale). On the other hand, consider the case where $R_i = 0.25$ (the threshold for turbulence in most cases), and compare the values of ℓ_1, the initial scale length, for the two definitions of ℓ:

$$\ell_1 = \frac{u_1}{N_1} = \frac{\Lambda S_1}{N_1}\sqrt{\frac{1 - R_i}{3}} = \Lambda \sqrt{\frac{1 - R_i}{3 R_i}} = \Lambda \tag{42}$$

$$\ell_1 = \frac{u_1}{S_1} = \Lambda \sqrt{\frac{1 - R_i}{3}} = \frac{\Lambda}{2}. \tag{43}$$

Thus, in the case of Eq. (43) the initial mixing would be less than that for the case of Eq. (42) in view of the fact that K_1 is the product of u_1 and ℓ_1 in Eq. (35). Equation (43) is, therefore, the more conservative for our purposes. Finally, we must choose between two definitions of Δt_n: Δt_n could be given by either S_n^{-1} of N_n^{-1}. We seek the most conservative value that by definition, would give the smallest value for $(K_n \Delta t_n)$ in the exponent of Eq. (37). Since $S > N$ for $R_i < 1$, and since we shall only be interested in the range $0 < R_i < 1$ (stratified turbulence is limited to this range), it follows that the most appropriate value of Δt_n is

$$\Delta t_n = 1/S_n . \tag{44}$$

With these changes we now define

$$S_n' = \frac{S_n}{N_1} \tag{45}$$

and

$$\Delta t_n' = \frac{N_1}{S_n} . \tag{46}$$

The flow chart that makes use of the conservative choice above and the non-dimensional notation is given in Figure 4.[†] Appendix A gives a hand calculator program based on Figure 4 and designed for the HP25.

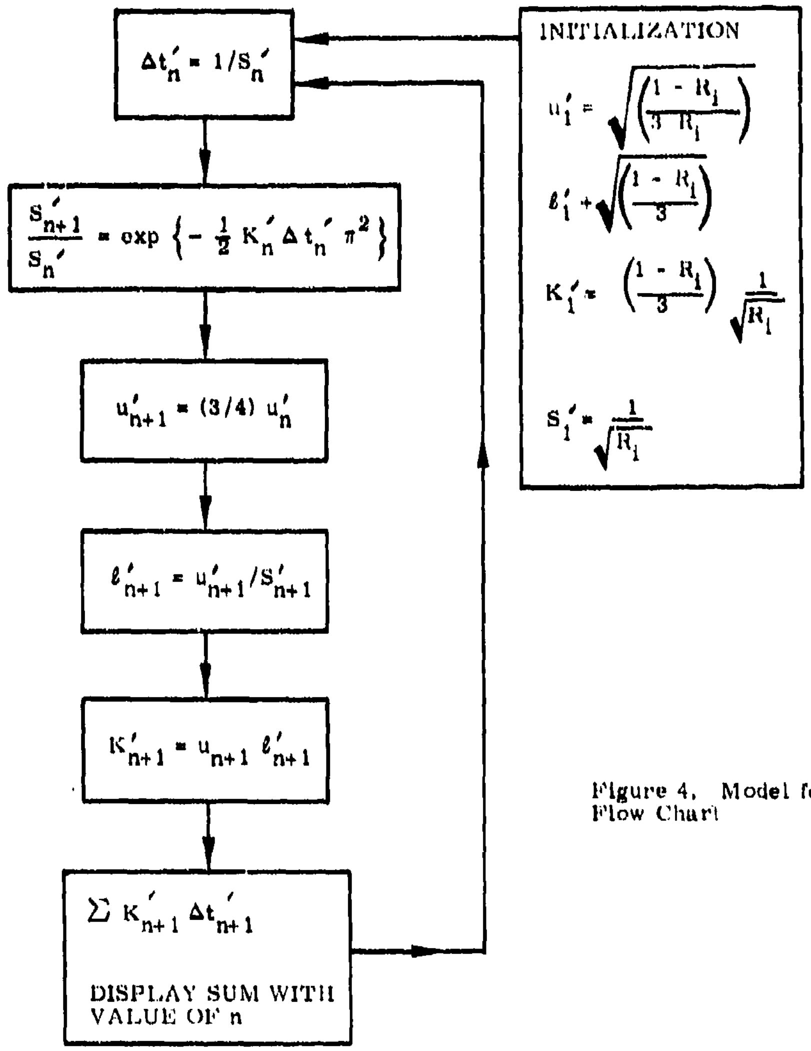

Figure 4. Model for Mixing: Flow Chart

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† This program displays the value of $\left(\sum_{n=1}^{n} K_n' t_n'\right)$ for each value of n (that is, each iteration). This quantity should be multiplied by Λ^2 to obtain the dimensional value.

Experiments with the hand calculator program showed that if initial R_i were small enough to cause ℓ to grow from epoch to epoch, then inevitably $\left(\sum_{n=1}^{n} K_n' \Delta t_n'\right)$ would grow indefinitely. In the program, ℓ was replaced by the upper bound Λ when the calculated value of ℓ was equal to or larger than Λ (that is, when $\ell' \geq (1)$ then $\ell' \equiv 1$). It was found experimentally that this instability, which caused ℓ to grow, occurred when $R_i < 0.825$. Of course, it is also useful to derive this result from analysis.

To obtain this result algebraically, we first prove the statement that if ℓ remains a constant, then

$$\underset{n \to \infty}{\text{Limit}} \sum_{n=1}^{n} K_n' t_n' \to \infty \, . \tag{47}$$

Using Eq. (44) for Δt, Eq. (38) for ℓ_n and K_n, Eq. (39) for the non-dimensional values, Eq. (47) can be written

$$\underset{n \to \infty}{\text{Lim}} \left[\sum_{n=1}^{n} \frac{u_n' \ell_n'}{S_n} \right] = \underset{n \to \infty}{\text{Lim}} \left[\sum_{n=1}^{n} \left(\frac{u_n'}{S_n'} \right)^2 \right] = \underset{n \to \infty}{\text{Lim}} \sum_{n=1}^{n} (\ell_n')^2 \, , \tag{48}$$

and for ℓ_n = constant, the sum becomes equal to a constant $\times \left(\sum_{n=1}^{n} 1 \right)$ and, hence, is unbounded as n becomes arbitrarily large. To find the crucial value of R_i we therefore examine successive values of ℓ_n. In particular, we consider the ratio of (ℓ_{n+1}/ℓ_n).

Using $\ell_n' = u_n'/S_n'$ and Eq. (40) for S_n (where N_n is replaced by S_n) so that (nondimensionally)

$$S_{n+1}' = S_n' \exp \left(- \frac{\Pi^2}{2} K_n' \Delta t_n' \right) , \tag{49}$$

we find

$$\left(\frac{\ell_{n+1}'}{\ell_n'} \right)^2 = \left(\frac{u_{n+1}'/S_{n+1}'}{u_n'/S_n'} \right)^2 = \left(\frac{3}{4} \right)^2 \exp \left(\Pi^2 \ell_n^{2} \right) , \tag{50}$$

(Figure 4 also defines these terms) where we used $u_{n+1} = (3/4\ u_n)$ to get Eq. (50). Initial values are

$$\ell_n' = \sqrt{\frac{1 - R_i}{3}} \tag{51}$$

$$S_1' = (R_i)^{-1} . \tag{52}$$

To explore critical behavior we set $(\ell_2/\ell_1)^2 = (1+\gamma)$ where γ is a small number. Then, from Eq. (50), for n = 1

$$(1+\gamma) = \left(\frac{3}{4}\right)^2 \exp(\pi^2 \ell_1^{\,2}) \tag{53}$$

thus

$$\ell_1^{\,2} = \frac{\log_e \left\{\left(\frac{4}{3}\right)^2 (1+\gamma)\right\}}{\pi^2} \tag{54}$$

Next we shall show that if γ is positive then (ℓ_3/ℓ_2) is also greater than unity. Using Eq. (50) with n = 2,

$$\left(\frac{\ell_3}{\ell_2}\right)^2 = \left(\frac{3}{4}\right)^2 \exp(\pi^2 \ell_2^{\,2}) , \tag{55}$$

but, since $\ell_2^{\,2} = \ell_1^{\,2}(1+\gamma)$ and using Eq. (54) for $\ell_1^{\,2}$, we have

$$\left(\frac{\ell_3}{\ell_2}\right)^2 = \left(\frac{3}{4}\right)^2 \exp\left\{(1+\gamma) \log e \left[\left(\frac{4}{3}\right)^2 (1+\gamma)\right]\right\} . \tag{56}$$

If $\gamma = 0$, $(\ell_3/\ell_1) = 1$. If γ is positive it is easy to prove (by simple expansions that $\ell_3/\ell_2 > 1$. Furthermore, in a similar manner it can be easily shown that if $[\ell_n/\ell_{n-1}] > 1$ it follows that $[\ell_{n+1}/\ell_n] > 1$. By mathematical induction, then, we have instability if $(\ell_2/\ell_1) > 1$. It can also be shown that if $(\ell_2/\ell_1) < 1$, the value of ℓ_n continues to decrease and one has stability. Thus, setting $\gamma = 0$ in Eq. (54) and determining the value of R_i (critical) from this result (that is, substituting into Eq. (51) we find that $R_{i_{CRIT}} = 0.825$* QED. Since $R_i \leq 0.25$ is assumed here to be necessary for the initiation of turbulence, it follows that there will always be an instability of ℓ and, hence, total mixing according to this model.

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*Thorpe[11] found vertical velocity fluctuations of order $(\frac{\Delta U}{3})$ at the start of billow turbulence in his laboratory. This is smaller than our estimate by a factor of $(\sqrt{3})^{-1}$. If we used $u_1 = (\frac{\Delta U}{3})\sqrt{\frac{1-R_i}{R_i}}$ instead of Eq. (15), then $R_{i_{CRIT}} = 0.475$ instead of 0.825 and our conclusion concerning mixing remains unchanged.

11. Thorpe, S.A. (1973) Turbulence in stably stratified fluids: A review of laboratory experiments. Boundary Layer Met. 5:94-119.

Since the above model does not take into account the fact that at late times during decay Eq. (18) is not valid, we now consider this effect. We shall assume that $\ell = \Lambda$ throughout the process and estimate $\int^{\infty} K(t)\,dt$ analytically. To do so we shall divide the time into two regimes (cf. Ref. 2).[6] In the first we shall use

$$-\epsilon = (-u_I^3/\Lambda) = \frac{d}{dt}\left[(3/2)\,u_I^2\right]. \tag{57}$$

The second regime starts when the Reynolds number, R_e, reaches the value of about 10 (Tennekes and Lumley,[5] p. 25). At that point, we shall use

$$\epsilon = \frac{C\,\nu\,(u_{II})^2}{\Lambda^2}. \tag{58}$$

For consistency between Eqs. (57) and (58) when $R_e = 10$, C must be set = 10.

Next we estimate u as a function of time. Integrating Eq. (57) as before, we obtain

$$u_I(t) = \frac{3\,\Lambda\,u_i}{3\,\Lambda + u_i\,t}, \tag{59}$$

where u_I refers to the first phase. In the second phase we obtain

$$\frac{d\left[\left(\frac{3}{2}\right)u_{II}^2\right]}{dt} = -\frac{10\,\nu u_{II}^2}{\Lambda^2}. \tag{60}$$

Now, defining $\tilde{u}$ and $\tilde{t}$ at the moment when $R_e = 10$, we obtain, by integrating Eq. (60),

$$u_{II}(t) = \tilde{u}\exp\left[-\frac{10\,\nu}{3\,\Lambda^2}\,(t-\tilde{t})\right]. \tag{61}$$

The values of $\tilde{u}$ and $\tilde{t}$ are obtained from $R_e\,[(u\,\Lambda/\nu)] = 10$ and Eq. (59) with u_I taken as $\tilde{u}$. Thus

$$\tilde{u} = \frac{10\,\nu}{\Lambda} \tag{62}$$

$$\tilde{t} = \left[\frac{1}{\tilde{u}} - \frac{1}{u_i}\right](3\,\Lambda). \tag{63}$$

Hence

$$\int_0^{\infty} K(t)\,dt = \int_0^{\infty} u(t)\,\Lambda\,dt = \Lambda \int_0^{\tilde{t}} u_I\,dt + \Lambda \int_{\tilde{t}}^{\infty} u_{II}\,dt\,. \tag{64}$$

Thus

$$\int_0^{\infty} K(t)\,dt = 3\,\Lambda^2 \left\{ \log_e \left(\frac{R_e^{\,i}}{10} \right) + 1 \right\}, \tag{65}$$

where $R_e^{\,i}$ is the initial Reynolds number $[(u_i\,\Lambda)/\nu]$. Thus, Eq. (65) is always finite when the final manner of decay is taken into account [see Eq. (60)].

The next question† is, what value of Λ will cause

$$\int_0^{\infty} K(t)\,dt << \frac{4\,\overline{L^2}}{\pi^2} \tag{66}$$

which is the criterion for small mixing [Eq. (4)]. In Eq. (66) we shall use the equality for definiteness. In that case, using Eq. (65) we obtain (using $\overline{L^2} \sim \frac{\Lambda^2}{4}$)

$$3 \left[\log_e \left(\frac{R_e^{\,i}}{10} \right) + 1 \right] = \frac{1}{\pi^2} \tag{67}$$

Using $N = 0.0225\,s^{-1}$, $\nu = 1.64 \times 10^{-4}\ m^2/s$ (standard atmosphere values at 20 km), $R_i = 0.25$ (hence $S = 0.045\,s^{-1}$), and using $u_i = u_1$ in Eq. (34) we obtain

$$\Lambda = 16.7\ \mathrm{cm}^{*}\,. \tag{68}$$

This is about three orders of magnitude smaller than the typical values of Λ reported by Rosenberg and Dewan[1] where values of L should be multiplied by 2 or more in order to be to equal Λ), and Anderson[12], Cadet[13,14], and Barat.[15,16] Therefore, if all mixing were essentially irreversible, then, from the above results, one could conclude that Eq. (1) and not Eq. (2) is the correct one to use in the stratosphere. We must therefore examine the postulate that a significant amount of molecular mixing takes place. Unless this latter condition is met, the parcels of fluid would migrate back to their stable altitudes.[17]

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†Note that if one were to arbitrarily set $\tilde{t} = \infty$, then, before, $\int_0^{\infty} K(t)\,dt$ would be unbounded. $\tilde{t}$ finite makes Eq. (65) finite.

*This would be 22 cm if $u_i = \frac{\Delta U}{3}\sqrt{\frac{1 - R_i}{3\,R_i}}$ were used (that is, using Thorpe's value for initial vertical velocity).

(Because of the number of references mentioned in the above text, they will not be listed here. Refer to Reference Page 29, for References 12 through 17.)

3.2 Degree of Molecular Mixing

We now wish to establish whether or not molecular mixing is sufficient to ensure that irreversible mixing takes place. To make this estimate we make use of a suggestion of Koop[17] to compare the time scale of the K-H, that is, the "turn over time," τ_T, given by

$$\tau_T = N^{-1}, \tag{69}$$

to the molecular mixing time. τ_T is roughly the time available for the turbulence to perform a significant amount of mixing action.

The molecular mixing time (or diffusion time), τ_D, is estimated by

$$\tau_D = \eta_D^2/(D), \tag{70}$$

where D is the molecular diffusion constant and η_D is the Kolmogorov microscale for the cascade (see Tennekes and Lumley[5]). Irreversible mixing will occur when $\tau_D << \tau_T$. To obtain η_D we employ the viscous microscale given by

$$\eta = \left(\frac{\nu^3}{\epsilon}\right)^{1/4}, \tag{71}$$

where ν here must be replaced by D in order to obtain η_D. D is obtained from the Prandtl number P_r, and ν by

$$D = \frac{\nu}{P_r}. \tag{72}$$

Thus

$$\eta_D = \left(\frac{\nu^3}{\epsilon}\right)^{1/4} P_r^{-3/4}. \tag{73}$$

$P_r = 0.73$ for air (Tennekes and Lumley,[5] and for water $P_r \sim 10$ (Hill[19]); $\nu = 1.64 \times 10^{-4}\ m^2/s$ for the lower stratosphere (20 km—U.S. Standard Atmosphere)[18], and $\nu = 10^{-6}\ m^2/s$ for water (Hill[19]). From Eq. (71) we obtain η_D for the atmosphere, η_{DA}, and water, η_{DW} respectively:

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18. U.S. Standard Atmosphere, 1976, NASA, USAF, NOAA - S/T 76-1562.
19. Hill, M.N. (1962) The Sea, Interscience Pub.

$$\eta_{DA} = \frac{1.84 \times 10^{-3}}{\epsilon^{1/4}} \tag{74}$$

$$\eta_{DW} = \frac{5.6 \times 10^{-6}}{\epsilon^{1/4}} \; . \tag{75}$$

Making use of Table 1 in Thorpe[11] we tabulate here, in our Table 1, the values of ϵ that are typical for atmosphere, ocean, and laboratory. For large mixing, $\tau_D << \tau_T$. Table 1 gives the values of all the relevant quantities, and it can be seen that in all three cases $\tau_D/\tau_T << 1$. This condition holds most strongly for the case of the atmosphere. Thus, it seems that irreversible mixing is assured in the case of the atmosphere. Thus, Eq. (3) is indeed the valid criterion.

Table 1. Estimation of Irreversible Mixing

	$\epsilon\,(m^2/s^3)$	Chosen for Calculation of τ's $\epsilon\,(m^2/s^3)$	η_D (m)	τ_D (s)	$N (s^{-1})$	τ_T sec	τ_D/τ_T
Laboratory	3.8×10^{-4}	3.8×10^{-4}	4×10^{-5} m	2×10^{-2}	9	1.1×10^{-1}	1.4×10^{-1}
Ocean	10^{-6} to 10^{-8}	10^{-7}	3×10^{-4} m	9×10^{-1}	7.8×10^{-2}	13	7×10^{-2}
Atmosphere	$(2 \text{ to } 35) \times 10^{-3}$	10^{-2}	5.82×10^{-3} m	1.5×10^{-1}	1.4×10^{-2}	71	2.1×10^{-3}

4. EXPERIMENTAL MEASUREMENTS IN THE LABORATORY, OCEAN, AND ATMOSPHERE

4.1 Laboratory and Oceanic Observations

In Woods[20] there is reported a typical profile of temperature in the thermocline of the upper ocean. The step-like structures seen there have been attributed to mixing events that are K-H billows of the type discussed here. Businger[7] describes the mixing and spreading process of K-H events and, again, total mixing or nearly total mixing is assumed to occur within the layer. In view of the analogy between the turbulence in the upper ocean and stratosphere, it follows that the step-like structures seen in the former case argue for the existence of similar structures in the potential temperature profile of the stratosphere.

In the case of laboratory measurements there have been no observations of zero gradient layers in density. Thorpe[11] has consistently found an approximately linear gradient across the layer subsequent to breakdown and mixing. The same effect

20. Woods, J.D. (1968) Wave-induced shear instability in the summer thermocline; J. Fluid. Mech. 32:791-800.

occurs in the observation of Koop.[17] Since, in fact, there seems to be no laboratory observations of the "steplike" structures (which in turn would indicate total mixing), it would be desirable to explain the discrepancy.

Certain differences between the laboratory conditions and those found in the natural environment may explain the apparent contradiction. For example, the experiments of Thorpe and Koop involved two stably superposed fluids that were homogeneous but differed in density across a very narrow interface. The homogeneity of the individual fluids would allow a maximum amount of layer spreading to take place after the onset of turbulence. This spreading in turn would cause entrainment, and the latter might effectively prevent the final temperature or density profile from resembling a step. The step-like quality is essential in our model, since, in Section 2 it was assumed that the boundaries of the layer acted as "insulators" [see Eq. (23)]. It is therefore unfortunate that a mathematical explanation for the discrepancy between laboratory observations and the oceanic measurements is not yet available.

Table 2 based on Thorpe[11] shows the values of three other parameters relevant to mixing, namely, R_e, the Reynolds number, R_a, the Rayleigh number, and R_i. These are given for laboratory, ocean, and atmosphere. It can be seen that R_e and R_a are both very large for the case of the atmosphere (in comparison to the other cases). While there seems to be no clear-cut theory relating these parameters to the manner that mixing takes place, there is nevertheless some indication that one would expect thorough mixing in the atmosphere (Thorpe[11] related R_a to mixing).

Table 2. Parameters Associated With Mixing

	R_e	R_a	R_i Initial
Laboratory	1.5×10^3	4.0×10^8	0.05
Ocean	10^3	1.6×10^7	0.15
Atmosphere	6.7×10^7	7.5×10^{15}	0.15

4.2 Atmospheric Observations

Perhaps the most relevant measurements for our purposes are those of Mantis and Peppin[21]. These consisted of radiosonde observations of temperature profiles in the stratosphere and troposphere. Consider the following quotations from their paper:

> "Perhaps more important.....is the observation that both the stable stratosphere and relatively unstable troposphere contain numerous shallow layers with near adiabatic lapse rates....."*

> ".....characteristics of the temperature structure lend support to the hypothesis that there is a change in the turbulent regime over vertical scales of order 100 meters. The deeper unstable layers have these dimensions and in the stratosphere the unstable layers are frequently capped with a very sharp inversion."**

These remarks are very strongly in accord with the model proposed here and with the conclusion that large amounts of mixing take place in K-H events in the atmosphere. The step-like structure discussed by Woods and Wiley[3] has also been seen in the stratosphere.

Still further support to the idea of large mixing in an "insulated layer" will be found in the radar observations of Browning and Watkins[22] and Atlas et al.[23] In both cases a double layer of high reflectivity remained for a long time after the onset of turbulence. This can be considered as evidence for a strong inversion at the top and bottom of a layer that has undergone complete "insulated" mixing in its interior. The layers would then represent the edges of a region of nearly adiabatic lapse rate.

5. CONCLUSION

A model has been proposed for estimating the degree of mixing due to K-H turbulence events in the stratosphere. It was shown that this model indicates that

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*Recall that complete mixing would yield adiabatic lapse rates.

**Complete mixing would give steep gradients or inversions on both boundaries of the mixing layer.

21. Mantis, H.T., and Pepin, T.J. (1971) Vertical temperature structure of the free atmosphere at mesoscale, J. Geophs. Res. 20:8621-8628.

22. Browning, K.A., and Watkins, C.D. (1970) Observations of clear air turbulence by high power radar, Nature 227:260-263.

23. Atlas, D., Metcalf, J.I., Richter, J.H., and Gossard, E.E. (1970) The birth of 'CAT' and microscale turbulence, J. Atmos. Sci. 27:903-913.

large amounts of mixing will take place within the layer and that it is irreversible due to molecular diffusion. This conclusion was compared to laboratory and environmental measurements, and the latter were found to give further support to the theoretical prediction of large mixing.

In conclusion, Eq. (2) for K_e must now be considered invalid and in its place Eq. (1) should be used to estimate K_e in the stratosphere (and probably also in the ocean). K_e is, of course, the overall vertical eddy diffusion coefficient for small scale turbulence.[2] This conclusion casts doubt upon the previous estimate of Lilly et al[4] of $K_e \sim 0.01\ m^2/s$, because he used Eq. (2) to perform his calculations.

Note added in proof

Reference: Bottom p. 21

The value $R_i = 0.825$ was obtained by using the initial values of u_n and ℓ_n of each n^{th} epoch. It could be objected that the resulting K_n for each epoch is over-estimated and that therefore the calculation overestimates mixing. It is thus important to estimate a lower bound for K_n. This can be done by replacing u_n and ℓ_n by the lower bounds $\frac{3}{4}u_n$ and $\frac{3}{4}\ell_n$ for each epoch. When this is carried out, the resulting critical R_i for total mixing is then 0.689. Our conclusion therefore remains unaltered.

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Appendix A

Program for Model (HP-25)

1	RCL 2
2	RCL 1
3	÷
4	STO 0 (store Δt_n)
5	π
6	x^2
7	×
8	RCL 3
9	×
10	2
11	÷
12	CHS
13	e^x
14	RCL 4
15	×
16	STO 4 (store S_n)
17	RCL 1
18	•
19	7
20	5
21	×
22	STO 1 (store U_n)
23	RCL 4
24	÷
25	1
26	$x < y$
27	GTO 47
28	$x \leftrightarrow y$
29	STO 2 (store ℓ_n)
30	RCL 1
31	×
32	STO 3 (store K_n)
33	RCL 0
34	×
35	STO + 5 (store $\sum K_n' \, \Delta t_n'$
36	RCL 2
37	PAUSE (show ℓ_n)
38	RCL 4
39	PAUSE (show S_n)
40	RCL 6
41	PAUSE (show n)
42	1
43	STO + 6 (increment h)
44	GTO 1
45	
46	
47	STO 2 (if $\ell_n = \Lambda$)
48	GTO 30
49	

REGISTERS		INITIALIZATION OF REGISTERS	
0	$\Delta t_n'$	0	O
1	U_n'	1	$u_1' = \sqrt{(1 - R_1)/(3\ R_1)}$
2	ℓ_n'	2	$\ell_1' = \sqrt{(1 - R_1)/(3)}$
3	K_n'	3	$K_1' = (u_1' \ell_1')$
4	S_n'	4	$S_1' = (R_1)^{-1/2}$
5	$\sum K_n' \Delta t_n'$	5	0
6	n	6	2