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ASYMPTOTIC NORMALITY OF A VARIANCE ESTIMATOR OF A LINEAR COMBIN--ETC(U)  
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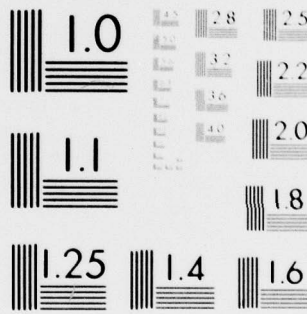
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ASYMPTOTIC NORMALITY OF A VARIANCE ESTIMATOR OF A LINEAR COMBINATION  
OF A FUNCTION OF ORDER STATISTICS

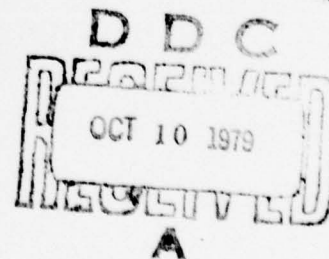
By

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Abstract

An estimator of the asymptotic variance of (a randomly stopped) linear combination of a function of order statistics is considered and its asymptotic normality is studied under appropriate regularity conditions. A comparative study of the regularity conditions pertaining to the asymptotic normality and strong convergence of linear combinations of functions of order statistics and their estimated asymptotic variances is also made.

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Key Words and Phrases: Asymptotic normality; asymptotic variance; almost sure convergence; empirical distribution; linear combination of order statistics; quantile process; stopping time; Wiener process.

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## 1. Introduction.

Let  $\{X_i, i \geq 1\}$  be a sequence of independent and identically distributed random variables (i.i.d.r.v.) with a continuous distribution function (df)  $F$ , defined on the real line  $R = (-\infty, \infty)$ . For every  $n (\geq 1)$ , let  $X_{n,1}, \dots, X_{n,n}$  be the order statistics corresponding to  $X_1, \dots, X_n$  and consider the statistics

$$T_{n,k} = n^{-1} \sum_{i=1}^k c_{n,i} h(X_{n,i}), \quad 1 \leq k \leq n, \quad (1.1)$$

where  $\{c_{n,i}, 1 \leq i \leq n; n \geq 1\}$  is a triangular array of (known) real constants and  $h$  is a specified function. Actually, if we let  $g = h \circ F^{-1}$  and  $\xi_{n,i} = F(X_{n,i}), 1 \leq i \leq n$  (so that  $\xi_{n,1}, \dots, \xi_{n,n}$  are the ordered r.v. of a sample of size  $n$  from the uniform  $(0,1)$  df), we may rewrite (1.1) as

$$T_{n,k} = n^{-1} \sum_{i=1}^k c_{n,i} g(\xi_{n,i}), \quad 1 \leq k \leq n. \quad (1.2)$$

Under suitable regularity conditions (on  $g$  and the  $c_{n,i}$ ), for  $k/n \rightarrow \alpha$  ( $0 < \alpha \leq 1$ ),

$$n^{1/2}(T_{n,k} - \mu(\alpha))/\sigma(\alpha) \xrightarrow{L} N(0,1), \quad (1.3)$$

where for each  $\alpha \in (0,1]$ ,  $\mu(\alpha)$  (asymptotic mean) and  $\sigma^2(\alpha)$  (asymptotic variance) are functionals of  $g$  and the score function  $J$  (which generates the  $c_{n,i}$ ). (1.3) has been proved under diverse regularity conditions by a host of research workers (viz. [1, 3, 4, 5, 6, 7, 8]). Stigler (1969) has also shown that under suitable regularity conditions,

$$n \text{Var}(T_{n,k})/\sigma^2(\alpha) \rightarrow 1 \text{ as } n \rightarrow \infty. \quad (1.4)$$

Let  $\{\tau_n, n \geq 1\}$  be a class of stopping times, where, for each  $n (\geq 1)$ ,  $\tau_n$  is defined in terms of  $X_{n,1}, \dots, X_{n,n}$  and it assumes values in

$\{1, \dots, n\}$ . Gardiner and Sen (1978) have shown that if  $n^{-1}\tau_n \xrightarrow{P} \alpha \in (0, 1]$  and the regularity conditions pertaining to (1.3) hold, then

$$n^{1/2}(T_{n, \tau_n} - \mu(n^{-1}\tau_n))/\sigma(\alpha) \xrightarrow{L} N(0, 1), \quad (1.5)$$

while if  $n^{-1/2}(\tau_n - n\alpha) \xrightarrow{P} 0$ , then in (1.5),  $\mu(n^{-1}\tau_n)$  may also be replaced by  $\mu(\alpha)$ .

In a variety of practical applications,  $\mu(\alpha)$  can be related to the basic (viz., location or scale) parameters of  $F$ , and thereby, confidence intervals or tests of significance for  $\mu(\alpha)$  can be transmitted to yield parallel conclusions for these parameters. In this context, one confronts the problem of estimating  $\sigma^2(\alpha)$  and natural estimators of  $\sigma^2(\alpha)$  can be derived from the sample. The object of the present investigation is to consider such an estimator of  $\sigma^2(\alpha)$  and to study its asymptotic normality. Along with the preliminary notions, the main theorems are presented in Section 2 and their proofs are considered in Section 3. Section 4 is devoted to some general remarks including a comparative study of the regularity conditions pertaining to the almost sure (a.s.) convergence and asymptotic normality of  $T_{n, k}$  and the estimator of  $\sigma^2(\alpha)$ . For the convenience of presentation, some of the technicalities are postponed to the Appendix.

## 2. Preliminary notions and the main theorems.

Define  $g$  as in after (1.1)

and assume that for every  $\theta \in (0, 1/2)$ ,  $g$  is of bounded variation in  $(0, 1-\theta)$ . For each  $n (\geq 1)$ , define  $J_n$  on  $[0, 1]$  by letting  $J_n(t) = c_{n, i}$  for  $(i-1)/n < t \leq i/n$ ,  $1 \leq i \leq n$  and  $J_n(0) = c_{n, 1}$ . Also, let  $\Gamma_n(t) = n^{-1} \sum_{i=1}^n I(\xi_{n, i} \leq t)$ ,  $t \in [0, 1]$  be the empirical df. Then  $T_{n, k}$  in (1.2) can be expressed as

$$T_{n,k} = \int_0^{\Gamma_n^{-1}(\frac{k}{n})} J_n(\Gamma_n(t))g(t)d\Gamma_n(t). \quad (2.1)$$

We define a bounding function

$$B(\cdot, \underline{a}) = \{B(t, \underline{a}) = M t^{-a_1}(1-t)^{-a_2}, t \in (0,1)\} \quad (2.2)$$

where  $M(0 < M < \infty)$ ,  $\underline{a} = (a_1, a_2)$  and  $a_1, a_2$  are real numbers. Also, for fixed  $\beta(> 0)$  and  $\delta(> 0)$ , we define

$$q_\beta = \{q_\beta(t) = [t(1-t)]^{\beta-\delta/2}, t \in (0,1)\}. \quad (2.3)$$

Then, we make the following assumptions:

[A1]:  $|g| \leq B(\cdot, \underline{a})$  for some  $\underline{a} = (a_1, a_2)$ .

[A2]: There exists a  $J$ , defined on  $(0,1)$ , such that

$$|J| \leq B(\cdot, \underline{b}) \quad \text{and} \quad |J_n| \leq B(\cdot, \underline{b}), \quad \forall n, \quad (2.4)$$

where  $\underline{b} = (b_1, b_2)$  with real  $b_1, b_2$  and except on a set of  $t$ 's of  $|g|$ -measure zero, both  $J$  is continuous at  $t$  and  $J_n \rightarrow J$  uniformly in some neighborhood of  $t$  as  $n \rightarrow \infty$ .

For each  $\alpha \in (0,1]$ , let us then define

$$\mu_n(\alpha) = \int_0^\alpha J_n(t)g(t)dt, \quad (2.5)$$

$$\sigma^2(\alpha) = \int_0^\alpha \int_0^\alpha (s \wedge t - st)J(s)J(t)dg(s)dg(t); \quad a \wedge b = \min(a,b). \quad (2.6)$$

Note that if

$$a_1 + b_1 = a_2 + b_2 = 1/2 - \delta \quad (2.7)$$

then  $\int_0^1 B(\cdot, \underline{b})q_\beta d|g| < \infty$  and it follows from assumptions A1, A2 that both  $\mu_n(\alpha)$  and  $\sigma^2(\alpha)$  are finite and then (1.3) holds [cf. Shorack (1972)]. If, in addition  $n^{-1} \tau_n \xrightarrow{P} \alpha \in (0,1)$  and  $g$  admits a derivative

at  $\alpha$  or  $n^{-1/2}(\tau_n - n\alpha) = O_p(1)$  and  $g$  is continuous at  $\alpha$  then (1.5) obtains [cf. Gardiner & Sen (1978)].

In the current paper, we consider the following estimator of  $\sigma^2(\alpha)$ :

$$\hat{\sigma}_n^2(\alpha) = \int_0^\alpha \int_0^\alpha \{ \Gamma_n(s \wedge t) - \Gamma_n(s)\Gamma_n(t) \} J_n(\Gamma_n(s)) J_n(\Gamma_n(t)) dg(s) dg(t) \quad (2.8)$$

which can also be written as

$$\hat{\sigma}_n^2(\alpha) = n^{-2} \sum_{i=1}^{n^*-1} \sum_{j=1}^{n^*-1} c_{n,i} c_{n,j} [n(i \wedge j) - ij] \{ h(X_{n,i+1}) - h(X_{n,i}) \} \{ h(X_{n,j+1}) - h(X_{n,j}) \} + r_n, \quad (2.9)$$

where  $n^* = \max\{k: \xi_{n,k} \leq \alpha\}$  and  $r_n = O_p(n^{-1})$ . Also, as in Sen (1978),  $\hat{\sigma}_n^2(\alpha)$  can be interpreted as the conditional variance of  $nT_{n,n^*}$  given  $\{X_{n+k,j}, 1 \leq j \leq n+k \text{ and } k \geq 1\}$ . Our main concern is to study regularity conditions pertaining to the asymptotic normality of  $n^{1/2}(\hat{\sigma}_n^2(\alpha) - \sigma^2(\alpha))$ .

For this purpose we need some additional regularity conditions:

$$[A3]: \quad n^{1/2} \int_0^1 |J_n(\Gamma_n(t)) - J(\Gamma_n(t))| d|g(t)| \xrightarrow{P} 0, \text{ as } n \rightarrow \infty,$$

$$[A4]: \quad \text{except on a set of } t\text{'s of } |g| \text{ measure zero, } J'(t) = (d/dt)J(t) \text{ exists and is continuous at } t, \text{ and for some } \underline{c} = (c_1, c_2),$$

$$|J'| \leq B(\cdot, \underline{c}) \text{ where } 0 \leq c_1 - b_1, c_2 - b_2 \leq 1, \quad (2.10)$$

with  $\underline{b}$  defined in [A2].

Let us now write  $I$  for the identity function on  $(0,1)$  and let

$$J_{(1)} = IJ, \quad J_{(2)} = (1 - I)J; \quad (2.11)$$

$$L_1(t) = 2 \int_t^1 J_{(2)} dg, \quad L_2(t) = 2 \int_0^t J_{(1)} dg, \quad 0 < t < 1; \quad (2.12)$$

$$L_0 = L_1 J'_1 + L_2 J'_2. \quad (2.13)$$

Define

$$\gamma^2 = \int_0^1 \int_0^1 (s \wedge t - st) L_0(s) L_0(t) dg(s) dg(t). \quad (2.14)$$

Then, we have the following.

Theorem 1. Suppose that A1, A2, A3 and A4 hold and

$$\int_0^1 B(\cdot, b) q_{\frac{1}{2}} d|g| < \infty. \quad (2.15)$$

Then, both  $\sigma^2(1)$  and  $\gamma^2$  are finite and

$$n^{\frac{1}{2}}(\hat{\sigma}_n^2(1) - \sigma^2(1)) / \gamma \xrightarrow{L} N(0,1). \quad (2.16)$$

The proof is considered in the next section. We may remark here that in (2.11) through (2.14), if we let  $J(t) = 0$  for  $t \geq \alpha$  (when  $0 < \alpha < 1$ ) and denote the resulting expression in (2.14) by  $\gamma_\alpha^2$ , then (2.16) holds for  $n^{\frac{1}{2}}(\hat{\sigma}_n^2(\alpha) - \sigma^2(\alpha)) / \gamma_\alpha$ . Hence, for the sake of simplicity, we consider the case of  $\alpha = 1$  and, for notational convenience, write  $\hat{\sigma}_n^2(1) = \hat{\sigma}_n^2$ ,  $\sigma^2(1) = \sigma^2$ . We may also remark that whenever  $L_0$  in (2.13) is integrable with respect to the signed measure  $g$  on  $(0,1)$ , a more convenient form of (2.14) can be obtained. Define  $G_0$  on  $(0,1)$  by

$$G_0(t) = \int_0^t L_0(s) dg(s), \quad 0 < t < 1. \quad (2.17)$$

Then, a pedestrian calculation leads us to

$$\gamma^2 = \int_0^1 G_0^2(t) dt - \left( \int_0^1 G_0(t) dt \right)^2. \quad (2.18)$$

Now let us suppose  $0 < \alpha < 1$  and set  $\sigma_n^{*2} = \hat{\sigma}_n^2(n^{-1} \tau_n)$ . In the statement of A4 we assume additionally that  $J$  is continuous at  $\alpha$  and  $J_n \rightarrow J$  uniformly in some neighborhood of  $\alpha$  as  $n \rightarrow \infty$ . For  $t \in (0, \alpha)$  we define  $L_1^+(t) = 2 \int_t^\alpha J(2) dg$  and  $L_0^+ = L_1^+ J'(1) + L_2 J'(2)$ . Let  $(\gamma^+(\alpha))^2 = \int_0^\alpha \int_0^\alpha (s \wedge t - st) L_0^+(s) L_0^+(t) dg(s) dg(t)$ .

Theorem 2. With the remarks noted above suppose that A1 through A4 hold together with (2.15). Then both  $\sigma^2(\alpha)$  and  $\gamma^+(\alpha)$  are finite and if, in addition,  $n^{-1}\tau_n \xrightarrow{P} \alpha$  then

$$n^{1/2}(\sigma_n^{*2} - \sigma^2(n^{-1}\tau_n))/\gamma^+(\alpha) \xrightarrow{L} N(0,1) \quad (2.19)$$

while if  $n^{-1/2}(\tau_n - n\alpha) \xrightarrow{P} 0$  and  $g$  admits a derivative at  $\alpha$  then in (2.19)  $\sigma^2(n^{-1}\tau_n)$  may also be replaced by  $\sigma^2(\alpha)$ .

### 3. Proofs of Theorems.

Note that by (2.4) and (2.6),

$$\begin{aligned} 0 \leq \sigma^2(\alpha) &= 2 \int_0^\alpha \int_0^s (1-t)J(s)J(t)dg(s)dg(t) \\ &\leq 2 \left( \int_0^\alpha \{t(1-t)\}^{1/2} |J(t)| |d|g(t)| \right)^2 \\ &\leq 2 \left( \int_0^1 \{t(1-t)\}^{1/2} |J(t)| |d|g| \right)^2, \quad \forall \alpha \in (0,1) \\ &\leq 2M^2 \left( \int_0^1 Bq_{1/2} d|g| \right)^2, \quad \forall \alpha \in (0,1). \end{aligned} \quad (3.1)$$

Now (2.15) ensures the less restrictive condition  $\int_0^1 Bq_{1/2} d|g| < \infty$  and so  $\sigma^2(\alpha) < \infty$  for every  $\alpha \in (0,1)$ . Similarly, on noting that under (2.15), by (2.11), (2.12) and (2.13)

$$|L_0(t)| \leq M^* \{t(1-t)\}^{-1/2} B(t, \underline{b}), \quad \forall t \in (0,1), \text{ for some } M^* < \infty, \quad (3.2)$$

we have by (2.14) and (3.2),

$$\begin{aligned} \gamma^2 &\leq 2 \left( \int_0^1 \{t(1-t)\}^{1/2} |L_0(t)| |d|g(t)| \right)^2 \\ &\leq 2(M^*)^2 \left( \int_0^1 \{t(1-t)\}^{1/2} B(t, \underline{b}) d|g(t)| \right)^2 \\ &< 2M^{*2} \left( \int B(\cdot, \underline{b}) q_{1/2} d|g| \right)^2 < \infty, \text{ by (2.15)} \end{aligned} \quad (3.3)$$

Note that by (2.6), (2.11) and (2.12),

$$\sigma^2 = 2 \int_0^1 \int_0^1 s(1-t)J(s)J(t)dg(s)dg(t) = \frac{1}{2} \int_0^1 L_1(t)dL_2(t). \quad (3.4)$$

Again, if we define for each  $n > 1$  and  $t \in [\xi_{n,1}, \xi_{n,n})$ ,

$$L_{n,1}(t) = 2 \int_t^{\xi_{n,n}} (1-\Gamma_n)J_n(\Gamma_n)dg \quad \text{and} \quad L_{n,2}(t) = 2 \int_{\xi_{n,1}}^t \Gamma_n J_n(\Gamma_n)dg, \quad (3.5)$$

with both  $L_{n,1}$  and  $L_{n,2}$  set equal to zero otherwise, we may write

$$\hat{\sigma}_n^2 = (1/2) \int_0^1 L_{n,1}(t)dL_{n,2}(t). \quad (3.6)$$

From (3.4) and (3.6), we have

$$n^{1/2}(\hat{\sigma}_n^2 - \sigma^2) = \frac{1}{2} \{S_{n,1} + S_{n,2} + R_n\}, \quad (3.7)$$

where

$$S_{n,1} = \int_0^1 n^{1/2}(L_{n,1}(t) - L_1(t))dL_2(t), \quad (3.8)$$

$$S_{n,2} = \int_0^1 L_1(t)d\{n^{1/2}(L_{n,2}(t) - L_2(t))\}, \quad (3.9)$$

$$R_n = \int_0^1 n^{1/2}(L_{n,1}(t) - L_1(t))d(L_{n,2}(t) - L_2(t)). \quad (3.10)$$

Let  $(\Omega, \mathcal{B}, P)$  be the underlying probability space and let  $U_n = n^{1/2}(\Gamma_n - 1)$  be the uniform empirical process on  $[0,1]$ . Suppose  $U$  denotes a standard Brownian bridge process on  $[0,1]$  defined on the same probability space.  $(\Omega, \mathcal{B}, P)$  may not be rich enough to support  $U$ . However, by one of the usual techniques of embedding [cf. Shorack (1972)], we may construct another probability space  $(\Omega^*, \mathcal{B}^*, P^*)$  where the distributions of our original variables are preserved and which is rich enough to support  $U$ .] Let  $Q$  be the class of all nonnegative, continuous  $q$  on  $[0,1]$  which are bounded below by functions  $\bar{q}$  nondecreasing (nonincreasing) on  $[0, 1/2]$  ( $[1/2, 1]$ ) and

satisfy  $\int_0^1 \frac{1}{q} dL < \infty$ . Let  $\rho_q(f, g) = \sup\{|f(t) - g(t)|/q(t) : 0 < t < 1\}$  be the usual sup-norm metric. Then, it is known that for each  $q \in Q$

$$\rho_q(U_n, U) = o_p(1) \quad \text{and} \quad \rho_q(U_n, 0) = o_p(1) = \rho_q(U, 0). \quad (3.11)$$

Note that by our definitions,

$$\begin{aligned} \frac{1}{2} n^{\frac{1}{2}} (L_{n,1} - L_1) &= - \int_t^{\xi_{n,n}} U_n J_n(\Gamma_n) dg + \int_t^{\xi_{n,n}} (1-I) n^{\frac{1}{2}} (J_n(\Gamma_n) - J(\Gamma_n)) dg \\ &\quad + \int_t^{\xi_{n,n}} (1-I) n^{\frac{1}{2}} (J(\Gamma_n) - J) dg - n^{\frac{1}{2}} \int_{\xi_{n,n}}^1 (1-I) J dg, \\ &\quad \text{for } t \in [\xi_{n,1}, \xi_{n,n}) \\ &= -n^{\frac{1}{2}} \int_t^1 (1-I) J dg, \text{ otherwise} \end{aligned} \quad (3.12)$$

$$\begin{aligned} \frac{1}{2} n^{\frac{1}{2}} (L_{n,2} - L_2) &= \int_{\xi_{n,1}}^t U_n J_n(\Gamma_n) dg + \int_{\xi_{n,1}}^t I n^{\frac{1}{2}} (J_n(\Gamma_n) - J(\Gamma_n)) dg \\ &\quad + \int_{\xi_{n,1}}^t I n^{\frac{1}{2}} (J(\Gamma_n) - J) dg - n^{\frac{1}{2}} \int_0^{\xi_{n,1}} I J dg, \\ &\quad \text{for } t \in [\xi_{n,1}, \xi_{n,n}), \\ &= -n^{\frac{1}{2}} \int_0^t I J dg, \text{ otherwise.} \end{aligned} \quad (3.13)$$

Substituting (3.12) in (3.8), we write

$$S_{n,1} = -S_{n,1}^{(1)} + S_{n,1}^{(2)} + S_{n,1}^{(3)} - S_{n,1}^{(4)}. \quad (3.14)$$

Define  $\xi_1 = \int_0^1 \int_t^1 U J dg dL_2$  and let  $X_{n,1}, X_{n,2}$  denote the indicators of  $[\xi_{n,1}, \xi_{n,n})$  and  $[t, \xi_{n,n})$ ,  $t \in (0, 1)$  respectively. Then

$$\begin{aligned}
|S_{n,1}^{(1)} - \zeta_1| &\leq \int_0^1 IB(\cdot, \underline{b}) \chi_{n,1} d|g| \left\{ \int_t^1 |\chi_{n,2} U_n J_n(\Gamma_n) - UJ| d|g| \right\} \\
&\quad + \int_0^1 IB(\cdot, \underline{b}) \bar{\chi}_{n,1} d|g| \left\{ \int_t^1 |UJ| d|g| \right\} \\
&= S_{n,11}^{(1)} + S_{n,12}^{(1)}, \text{ say,}
\end{aligned} \tag{3.15}$$

where  $\bar{\chi}_{n,1}$  is the indicator of the complement of  $[\xi_{n,1}, \xi_{n,n})$  relative to  $(0,1)$ . To handle  $S_{n,11}^{(1)}$  note that for  $\chi_{n,1} = 1$  and  $\chi_{n,2} = 1$  we have

$$|\chi_{n,2} U_n J_n(\Gamma_n) - UJ| \leq |U_n - U| B(\cdot, \underline{b}) + |J_n(\Gamma_n) - J| |U_n|. \tag{3.16}$$

Furthermore,  $|J_n(\Gamma_n) - J| \leq 2B(\Gamma_n, \underline{b}) \vee B(I, \underline{b})$ , and since  $0 < \Gamma_n < 1$ , in the range under consideration, we obtain by Theorem 2 of Wellner (1977) that there exists a set  $A \subset \Omega$  such that  $P(A) = 1$  and for each  $\omega \in A$  there exists an integer  $n_\omega$  for which  $n \geq n_\omega$  implies

$$|J_n(\Gamma_n) - J| \leq M^0 B(\cdot, \underline{b}) q_{\frac{1}{2}} / \tilde{q}, \tag{3.17}$$

where  $M^0 (< \infty)$  is a constant and  $\tilde{q} = \{I(1-I)\}^{1-\delta/4}$ . For such  $\omega$  and  $n$ , therefore, from (3.11) and (3.17), the right hand side of (3.16) is bounded by

$$\rho_{q_{\frac{1}{2}}} (U_n, U) B(\cdot, \underline{b}) q_{\frac{1}{2}} + M^0 \rho_{\tilde{q}} (U_n, 0) B(\cdot, \underline{b}) q_{\frac{1}{2}} = O_P(1) B(\cdot, \underline{b}) q_{\frac{1}{2}} \tag{3.18}$$

whenever  $\chi_{n,1} = 1$  and  $\chi_{n,2} = 1$ . When  $\chi_{n,1} = 1$  and  $\chi_{n,2} = 0$ , however the left hand side of (3.16) is again dominated by  $O_P(1) B(\cdot, \underline{b}) q_{\frac{1}{2}}$ . We note that  $\Gamma_n \rightarrow I$  uniformly on  $[0,1]$  and thus by [A4],  $J_n(\Gamma_n) \rightarrow J$  (a.s.), pointwise a.e.  $|g|$ . Since  $\xi_{n,n} \rightarrow 1$  a.s. and from (3.11) we have for each  $t \in (0,1)$ ,  $\chi_{n,2} U_n J_n(\Gamma_n) \rightarrow UJ$  (a.s.), pointwise a.e.  $|g|$ . Hence the dominated convergence theorem applies and for each  $t \in (0,1)$ , we obtain

$$\chi_{n,1}(t) \int_t^1 |\chi_{n,2}^{U_n J_n}(\Gamma_n) - UJ|d|g|| \stackrel{P}{\rightarrow} 0. \quad (3.19)$$

Again for each  $t \in (0,1)$ , we have using the upper bound in (3.18)

$$\begin{aligned} IB(\cdot, \underline{b}) \left\{ \int_t^1 |\chi_{n,2}^{U_n J_n}(\Gamma_n) - UJ|d|g| \right\} \\ \leq \{I(1-I)\}^{\delta/2} B(\cdot, \underline{b}) q_{\frac{1}{4}} \left\{ \int_0^1 B(\cdot, \underline{b}) q_{\frac{1}{4}} d|g| \right\} o_p(1), \end{aligned} \quad (3.20)$$

where the right hand side is a  $|g|$ -integrable function. It then follows

from (3.19) and the dominated convergence theorem that  $S_{n,11}^{(1)} \stackrel{P}{\rightarrow} 0$  as

$n \rightarrow \infty$ .

To handle  $S_{n,12}^{(1)}$  we write

$$\begin{aligned} S_{n,12}^{(1)} \leq \int_0^{\xi_{n,1}} IB(\cdot, \underline{b}) d|g| \left\{ \int_t^1 |U|B(\cdot, \underline{b}) d|g| \right\} \\ + \int_{\xi_{n,n}}^1 IB(\cdot, \underline{b}) d|g| \left\{ \int_t^1 |U|B(\cdot, \underline{b}) d|g| \right\}. \end{aligned} \quad (3.21)$$

The first term on the right hand side may be bounded by

$$\begin{aligned} \int_0^{\xi_{n,1}} IB(\cdot, \underline{b}) d|g| \left\{ \int_t^1 \{I(1-I)\}^{\frac{1}{4}} B(\cdot, \underline{b}) q_{\frac{1}{4}} d|g| \right\} \rho_{q_{\frac{1}{2}}}(U, 0) \\ \leq \left( \int_0^{\xi_{n,1}} \{I(1-I)\}^{\delta/2} B(\cdot, \underline{b}) q_{\frac{1}{4}} d|g| \right) \left( \int_0^1 B(\cdot, \underline{b}) q_{\frac{1}{4}} d|g| \right) o_p(1) \end{aligned}$$

$= o_p(1)$ , since  $\xi_{n,1} \stackrel{P}{\rightarrow} 0$  and the integral converges.

The same argument will also show that the second term on the right hand side of (3.21) is  $o_p(1)$ . Hence, finally it follows from (3.15) that

$$S_{n,1}^{(1)} \stackrel{P}{\rightarrow} \tau_1 = \int_0^1 \left[ \int_t^1 UJ dg \right] dL_2, \quad \text{as } n \rightarrow \infty. \quad (3.22)$$

Next, we note that by [A3] and the definition of  $S_{n,1}^{(2)}$ ,

$$\begin{aligned}
|S_{n,1}^{(2)}| &\leq \int_0^1 IB(\cdot, \underline{b}) d|g| \left\{ \int_t^1 (1-I)n^{\frac{1}{2}} |J_n(\Gamma_n) - J(\Gamma_n)| d|g| \right\} \\
&\leq \left( \int_0^1 \{I(1-I)\}^{\delta/2} B(\cdot, \underline{b}) q_{\frac{1}{2}} d|g| \right) \left( \int_0^1 n^{\frac{1}{2}} |J_n(\Gamma_n) - J(\Gamma_n)| d|g| \right) \\
&= o(1) o_p(1). \tag{3.23}
\end{aligned}$$

To handle  $S_{n,1}^{(3)}$  we note that it may be written in the form

$$S_{n,1}^{(3)} = \int_0^1 dL_2(t) \chi_{n,1}(t) \left\{ \int_t^1 \chi_{n,2}(1-I) U_n((J(\Gamma_n) - J)/(\Gamma_n - I)) dg \right\} \tag{3.24}$$

where the indicators  $\chi_{n,1}, \chi_{n,2}$  were defined preceding (3.15). Define

$$\zeta_2 = \int_0^1 \left\{ \int_t^1 (1-I) UJ' dg \right\} dL_2. \text{ Then}$$

$$|S_{n,1}^{(3)} - \zeta_2| \leq S_{n,13}^{(1)} + S_{n,14}^{(1)}, \tag{3.25}$$

where

$$S_{n,13}^{(1)} = \int_0^1 IB(\cdot, \underline{b}) \chi_{n,1} d|g| \left\{ \int_t^1 (1-I) |\chi_{n,2} U_n(J(\Gamma_n) - J)/(\Gamma_n - I) - UJ'| d|g| \right\}, \tag{3.26}$$

$$S_{n,14}^{(1)} = \int_0^1 IB(\cdot, \underline{b}) \bar{\chi}_{n,1} d|g| \left\{ \int_t^1 (1-I) |UJ'| d|g| \right\}. \tag{3.27}$$

The analysis of  $S_{n,13}^{(1)}$  is very similar to that of  $S_{n,11}^{(1)}$ . Note that

$$|J(\Gamma_n) - J|/|\Gamma_n - I| \leq B(\Gamma_n, \underline{c}) \vee B(I, \underline{c}) \text{ by [A4]}. \text{ Once again since } 0 < \Gamma_n < 1$$

in the range under consideration in (3.26) we may invoke Theorem 2 of

Wellner (1972): for some  $A^* \subset \Omega$  with  $P(A^*) = 1$ , there exists for each

$\omega^* \in A^*$ , an integer  $n_{\omega^*}$  such that for  $n \geq n_{\omega^*}$

$$|J(\Gamma_n) - J|/|\Gamma_n - I| \leq M_0 B(\cdot, \underline{c}) q_{\frac{1}{2}} / \tilde{q}, \tag{3.28}$$

where  $M_0 (< \infty)$  is a constant and  $\tilde{q}$  is defined as in (3.17). By steps

similar to (3.16) through (3.19) for  $S_{n,11}^{(1)}$  and the continuity of  $J'$ , we

obtain, for each  $t \in (0,1)$

$$\chi_{n,1}(t) \int_t^1 (1-I) |\chi_{n,2} U_n (J(\Gamma_n) - J) / (\Gamma_n - I) - UJ' |d|g| \rho_0 \quad (3.29)$$

Furthermore in view of [A4],

$$\begin{aligned} IB(\cdot, b) \left\{ \int_t^1 (1-I) |\chi_{n,2} U_n (J(\Gamma_n) - J) / (\Gamma_n - I) - UJ' |d|g| \right\} \\ \leq \{I(1-I)\}^{\delta/2} B(\cdot, b) q_{1/4} \left\{ \int_0^1 I(1-I) B(\cdot, \xi) q_{1/4} d|g| \right\} o_p(1) \end{aligned} \quad (3.30)$$

and the right hand side is a  $|g|$ -integrable function. Hence from (3.29)

and the dominated convergence theorem we obtain  $S_{n,13}^{(1)} \xrightarrow{p} 0$  as  $n \rightarrow \infty$ .

Finally from (3.27) and [A4]

$$\begin{aligned} S_{n,14}^{(1)} \leq \int_0^{\xi_{n,1}} IB(\cdot, b) d|g| \left\{ \int_t^1 (1-I) |U| B(\cdot, \xi) d|g| \right\} \\ + \int_{\xi_{n,n}}^1 IB(\cdot, b) d|g| \left\{ \int_t^1 (1-I) |U| B(\cdot, \xi) d|g| \right\}, \end{aligned} \quad (3.31)$$

and as in the treatment of (3.21) the first term on the right hand side of

(3.31) may be bounded by

$$\begin{aligned} \int_0^{\xi_{n,1}} IB(\cdot, b) d|g| \left\{ \int_t^1 I^{1/4} (1-I)^{5/4} B(\cdot, \xi) q_{1/4} d|g| \right\} \rho_{q_{1/2}}(U, 0) \\ \leq \left( \int_0^{\xi_{n,1}} \{I(1-I)\}^{\delta/2} B(\cdot, b) q_{1/4} d|g| \right) \left( \int_0^1 I(1-I) B(\cdot, \xi) q_{1/4} d|g| \right) o_p(1) \\ = o_p(1), \text{ since the integral converges and } \xi_{n,1} \xrightarrow{p} 0. \end{aligned}$$

The same argument applies to the second term on the right hand side of

(3.31) and so we have  $S_{n,14}^{(1)} \xrightarrow{p} 0$  and hence finally from (3.25)

$$S_{n,1}^{(3)} \xrightarrow{p} \zeta_2 = \int_0^1 \left\{ \int_t^1 (1-I) UJ' d|g| \right\} dL_2 \quad (3.32)$$

In the Appendix, Lemma 1, we show  $S_{n,1}^{(4)} \xrightarrow{p} 0$  as  $n \rightarrow \infty$ . Thus from (3.14),

(3.22), (3.23), (3.32) and the above, it follows that

$$S_{n,1} \stackrel{P}{\rightarrow} \zeta_2 - \zeta_1 \quad \text{as } n \rightarrow \infty. \quad (3.33)$$

The analysis of  $S_{n,2}$  is entirely analogous, and hence, in the interest of brevity, we omit the details and present only the final result:

$$S_{n,2} \stackrel{P}{\rightarrow} \zeta_3 + \zeta_4 \quad \text{as } n \rightarrow \infty, \quad (3.34)$$

where

$$\zeta_3 = \int_0^1 L_1 U J dg \quad \text{and} \quad \zeta_4 = \int_0^1 1 L_1 U J' dg. \quad (3.35)$$

Finally, a very similar analysis leads to the conclusion that

$$R_n \stackrel{P}{\rightarrow} 0 \quad \text{as } n \rightarrow \infty. \quad (3.36)$$

Hence from (3.7), (3.33), (3.34) and (3.36) we obtain that

$$n^{1/2}(\hat{\sigma}_n^2 - \sigma^2) \stackrel{P}{\rightarrow} \zeta_2 - \zeta_1 + \zeta_3 + \zeta_4 \quad \text{as } n \rightarrow \infty. \quad (3.37)$$

In Lemma 2 of the Appendix we show that

$$\zeta_2 - \zeta_1 + \zeta_3 + \zeta_4 = \int_0^1 U L_0 dg = S, \quad \text{say,} \quad (3.38)$$

where  $L_0$  is defined by (2.13). Therefore with  $\gamma^2$  defined by (2.14),  $S$  has the normal distribution with mean 0 and variance  $\gamma^2$ . Q.E.D.

The proof of Theorem 2 proceeds very much along the same lines and so we omit some details here. Corresponding to (2.12) and (3.5) we define for each  $n > 1$ ,

$$L_{n,1}^*(t) = 2 \int_t^{n^{-1}\tau_n} (1 - \Gamma_n) J_n(\Gamma_n) dg, \quad \text{for } t \in [\xi_{n,1}, n^{-1}\tau_n] \quad (3.39)$$

with  $L_{n,1}^*$  set equal to zero otherwise and

$$L_{1,n}^*(t) = 2 \int_t^{n^{-1}\tau_n} J_{(2)} dg, \quad \text{for } t \in (0, n^{-1}\tau_n], \quad (3.40)$$

with  $L_{1,n}^*$  set equal to zero otherwise.

For simplicity we shall write  $L_1^*$  for  $L_{1,n}^*$  in the sequel. Now

$$\sigma_n^{*2} = \frac{1}{2} \int_0^1 L_{n,1}^* dL_{n,2} \quad \text{and} \quad \sigma^2(n^{-1}\tau_n) = 1/2 \int_0^1 L_1^* dL_2. \quad (3.41)$$

Therefore corresponding to (3.7) through (3.10) we have

$$n^{1/2}(\sigma_n^{*2} - \sigma^2(n^{-1}\tau_n)) = \frac{1}{2} (S_{n,1}^* + S_{n,2}^* + R_n^*) \quad (3.42)$$

where

$$S_{n,1}^* = \int_0^{n^{-1}\tau_n} n^{1/2} (L_{n,1}^* - L_1^*) dL_2, \quad (3.43)$$

$$S_{n,2}^* = \int_0^{n^{-1}\tau_n} L_1^* d\{n^{1/2}(L_{n,2} - L_2)\}, \quad (3.44)$$

$$R_n^* = \int_0^{n^{-1}\tau_n} n^{1/2} (L_{n,1}^* - L_1^*) d(L_{n,2} - L_2). \quad (3.45)$$

The decomposition corresponding to (3.12) reads

$$\begin{aligned} \frac{1}{2}(L_{n,1}^* - L_1^*) &= - \int_t^{n^{-1}\tau_n} U_n J_n(\Gamma_n) dg + \int_t^{n^{-1}\tau_n} (1 - I) n^{1/2} (J_n(\Gamma_n) - J(\Gamma_n)) dg \\ &\quad + \int_t^{n^{-1}\tau_n} (1 - I) n^{1/2} (J(\Gamma_n) - J) dg, \quad t \in [\xi_{n,1}, n^{-1}\tau_n) \\ &= -n^{1/2} \int_t^{n^{-1}\tau_n} (1 - I) J dg, \quad t \in (0, \xi_{n,1}]. \end{aligned} \quad (3.46)$$

Since  $n^{-1}\tau_n \stackrel{P}{\rightarrow} \alpha (< 1)$  by assumption and  $\xi_{n,n} \stackrel{P}{\rightarrow} 1$ , the set on which  $n^{-1}\tau_n < \xi_{n,n}$  has probability which tends to one as  $n \rightarrow \infty$ . The argument used to examine (3.14) now applies with only minor modifications. For instance

$$\int_{\xi_{n,1}}^{n^{-1}\tau_n} dL_2 \left\{ \int_t^{n^{-1}\tau_n} U_n J_n(\Gamma_n) dg - \int_t^{n^{-1}\tau_n} U J dg \right\} \stackrel{P}{\rightarrow} 0, \quad (3.47)$$

and the usual argument shows that provided  $n^{-1} \tau_n \xrightarrow{P} \alpha$

$$\int_{\xi_{n,1}}^{n^{-1} \tau_n} dL_2 \left\{ \int_t^{n^{-1} \tau_n} UJ dg \right\} \xrightarrow{P} \zeta_1^* = \int_0^\alpha dL_2 \left\{ \int_t^\alpha UJ dg \right\}. \quad (3.48)$$

Likewise  $\int_{\xi_{n,1}}^{n^{-1} \tau_n} dL_2 \left\{ \int_t^{n^{-1} \tau_n} (1 - I) n^{1/2} (J_n(\Gamma_n) - J(\Gamma_n)) dg \right\} \xrightarrow{P} 0$  and

$$\int_{\xi_{n,1}}^{n^{-1} \tau_n} dL_2 \left\{ \int_t^{n^{-1} \tau_n} (1 - I) n^{1/2} (J(\Gamma_n) - J) dg \right\} \xrightarrow{P} \zeta_2^* \quad (3.49)$$

where

$$\zeta_2^* = \int_0^\alpha dL_2 \left\{ \int_t^\alpha (1 - I) UJ' dg \right\}. \quad (3.50)$$

Finally  $n^{1/2} \int_0^{\xi_{n,1}} dL_2 \left\{ \int_t^{n^{-1} \tau_n} J^{(2)} dg \right\} \xrightarrow{P} 0$ , by the argument in Lemma 1 of the Appendix. From (3.46) through (3.49) we obtain

$$S_{n,1}^* \xrightarrow{P} \zeta_2^* - \zeta_1^*. \quad (3.51)$$

Finally for  $S_{n,2}^*$  and  $R_n^*$  the results are

$$S_{n,2}^* \xrightarrow{P} \zeta_3^* + \zeta_4^* \quad \text{and} \quad R_n^* \xrightarrow{P} 0 \quad (3.52)$$

where

$$\zeta_3^* = \int_0^\alpha L_1^+ UJ dg \quad \text{and} \quad \zeta_4^* = \int_0^\alpha IL_1^+ UJ dg. \quad (3.53)$$

Hence from (3.42), (3.51) and (3.52) we get

$$n^{1/2} (\sigma_n^{*2} - \sigma^2(n^{-1} \tau_n)) \xrightarrow{P} \frac{1}{2} (\zeta_2^* - \zeta_1^* + \zeta_3^* + \zeta_4^*).$$

A minor modification of Lemma 2 of the Appendix will show that

$$\frac{1}{2} (\zeta_2^* - \zeta_1^* + \zeta_3^* + \zeta_4^*) = \int_0^\alpha UL_0^+ dg,$$

and thus the first part of Theorem 2 is proven. For the second part we need only recognize that for  $n^{-1}\tau_n < \alpha$ ,

$$n^{1/2}(\sigma^2(n^{-1}\tau_n) - \sigma^2(\alpha)) = -2n^{1/2} \left( \int_0^{n^{-1}\tau_n} J_{(1)} dg \right) \left( \int_{n^{-1}\tau_n}^{\alpha} J_{(2)} dg \right) - 1/2 n^{1/2} \int_{n^{-1}\tau_n}^{\alpha} L_1^+ dL_2$$

with a similar expression if  $n^{-1}\tau_n > \alpha$ . Now, for the first term we may use the argument of Lemma 1 to show that it converges to zero in probability as  $n \rightarrow \infty$ . For the second term the additional assumptions on  $g$  and  $\tau_n$  gives

$$\begin{aligned} \left| n^{1/2} \int_{n^{-1}\tau_n}^{\alpha} L_1^+ dL_2 \right| &\leq O(1)n^{1/2} |g(n^{-1}\tau_n) - g(\alpha)| \\ &\leq O(1)n^{1/2} |n^{-1}\tau_n - \alpha| |g'(\alpha) + o_p(1)| \\ &= o_p(1). \end{aligned}$$

Hence Theorem 2 is proven.

#### 4. Some general remarks.

Consider the following example. Let  $X_1, \dots, X_n$  be iid rv's with df  $F$  and  $E|X_1|^r < \infty$  for some  $r > 4$ . We shall consider the sample variance

$$(4.1) \quad \hat{\sigma}_n^2 = n^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2$$

where  $\bar{X}_n = n^{-1} \sum_{i=1}^n X_i$  is the sample mean. Then in the notation of Theorem 1 we have  $c_{n,i} = 1$  for all  $i$ ,  $1 \leq i \leq n$  and  $g = F^{-1}$ . Note that  $E|X_1|^r < \infty$  implies  $\lim_{t \rightarrow 0+, 1-} t(1-t)|F^{-1}(t)|^r = 0$ . Thus  $|g| \leq D$  on  $(0,1)$  with  $a_1 = a_2 = 1/r$ . Also  $J = J_n = 1$  so that  $b_1 = b_2 = 0$  and  $A3, A4$  hold trivially. Therefore if we take  $\delta$  such that  $1/r = 1/4 - \delta$  we have  $\delta > 0$  provided  $r > 4$ . By an integration by parts (2.15) holds.

Now  $L_0$  of (2.13) reduces to

$$L_0(t) = -2F^{-1}(t) + 2 \int_0^1 F^{-1}(s) ds.$$

For simplicity let us take  $EX_1 = 0$  so that  $\int_0^1 F^{-1}(s) ds = 0$ . Then in (2.17) we may take  $G_0(t) = -\{F^{-1}(t)\}^2$ . So we obtain  $\gamma^2 = \mu_4 - \sigma^4$  where  $\mu_4 = EX_1^4$  and  $\sigma^2 = EX_1^2 = \sigma^2(1)$ . Hence Theorem 1 yields

$$n^{1/2}(\hat{\sigma}_n^2 - \sigma^2) \xrightarrow{L} N(0, \mu_4 - \sigma^4), \quad (4.2)$$

a result which is obtainable under the assumption  $r = 4$ . In this context Theorem 1 "just fails" to yield the slightly stronger conclusion.

The above example presents a very interesting observation pertaining to the different sets of conditions that suffice to yield the almost sure (a.s.) convergence of the statistics  $T_{n,n}$ , their asymptotic normality and the asymptotic normality of the estimator  $\hat{\sigma}_n^2$  of the asymptotic variance. We have noted that if A1, A2 obtain and (2.7) holds then

$$\int_0^1 B(\cdot, \underline{b}) q_{1/2} d|g| < \infty \quad (4.3)$$

and both  $\mu_n(1)$  of (2.5) and  $\sigma^2(1)$  of (2.6) are finite and

$$n^{1/2}(T_{n,n} - \mu_n(1)) \xrightarrow{L} N(0, \sigma^2(1)). \quad (4.4)$$

Under A1, A2 (2.7) ensures the finiteness of  $\mu_n(1)$  and (4.3) that of  $\sigma^2(1)$ . To obtain the asymptotic normality of the variance estimator of  $\delta_n^2(1)$  of (2.8) we impose the additional assumptions A3, A4 and replace (4.3) by the stronger condition (2.15). The a.s. convergence of  $T_{n,n}$  has been studied by Wellner (1977). If A1, A2 obtain and  $a_1 + b_1 = a_2 + b_2 = 1 - \delta$  then  $\mu_n(1)$  is finite and

$$(T_{n,n} - \mu_n) \xrightarrow{\text{a.s.}} 0. \quad (4.5)$$

Sen (1978) has obtained the a.s. convergence of  $\hat{\sigma}_n^2(1)$  following a different technique.

For the "stopped statistics"  $T_{n,\tau_n}$  their asymptotic normality is derived in [2]. Again if A1, A2 obtain and (2.7) holds

$$n^{1/2}(T_{n,\tau_n} - \mu_n(n^{-1}\tau_n)) \xrightarrow{L} N(0, \sigma^2(\alpha)) \quad (4.6)$$

provided  $n^{-1}\tau_n \xrightarrow{P} \alpha \in (0,1)$  and  $g$  admits a derivative at  $\alpha$  or  $n^{1/2}(n^{-1}\tau_n - \alpha) = o_p(1)$  and  $g$  is continuous at  $\alpha$ . In the latter case if we further assume the stronger condition  $n^{1/2}(n^{-1}\tau_n - \alpha) \xrightarrow{P} 0$  then  $\mu_n(n^{-1}\tau_n)$  in (4.6) can be also replaced by  $\mu_n(\alpha)$ .

The a.s. convergence of  $T_{n,\tau_n}$  can be discussed along the lines of Wellner (1977) assuming  $n^{-1}\tau_n \xrightarrow{\text{a.s.}} \alpha$ .

## 5. Appendix

Lemma 1: Under the hypothesis of Theorem 1 and  $S_{n,1}^{(4)}$  defined by (3.12)

and (3.14), we have  $S_{n,1}^{(4)} \xrightarrow{P} 0$  as  $n \rightarrow \infty$ .

Proof. We first write  $S_{n,1}^{(4)}$  in the form

$$S_{n,1}^{(4)} = -(S_{n,11}^{(4)} + S_{n,12}^{(4)} + S_{n,13}^{(4)}) \quad (5.1)$$

where

$$S_{n,11}^{(4)} = (n^{1/2}) \int_{\xi_{n,1}}^{\xi_{n,n}} IJdg \left( \int_{\xi_{n,n}}^1 (1-I)Jdg \right),$$

$$S_{n,12}^{(4)} = n^{1/2} \int_0^{\xi_{n,1}} IJdg \left( \int_t^1 (1-I)Jdg \right),$$

$$S_{n,13}^{(4)} = n^{1/2} \int_{\xi_{n,n}}^1 IJdg \left( \int_t^1 (1-I)Jdg \right).$$

Therefore,

$$|S_{n,11}^{(4)}| \leq (n^{\frac{1}{2}} \int_{\xi_{n,1}}^{\xi_{n,n}} |B(\cdot, \mathfrak{h}) d|g|) \left( \int_{\xi_{n,n}}^1 (1 - I) B(\cdot, \mathfrak{h}) d|g| \right). \quad (5.2)$$

Now

$$\int_{\xi_{n,1}}^{\xi_{n,n}} |B(\cdot, \mathfrak{h}) d|g| \leq \xi_{n,n}^{3/4} \int_{\xi_{n,1}}^{\xi_{n,n}} I^{\frac{1}{4}} B(\cdot, \mathfrak{h}) d|g|, \quad (5.3)$$

and

$$\int_{\xi_{n,n}}^1 (1 - I) B(\cdot, \mathfrak{h}) d|g| \leq (1 - \xi_{n,n})^{\frac{1}{2}} \int_{\xi_{n,n}}^1 I^{\delta/2 - \frac{1}{2}} (1 - I)^{\delta/2 + \frac{1}{2}} B(\cdot, \mathfrak{h}) q_{\frac{1}{2}} d|g|, \quad (5.4)$$

and therefore from (5.2),

$$\begin{aligned} |S_{n,11}^{(4)}| &\leq n^{\frac{1}{2}} \int_{\xi_{n,1}}^{\xi_{n,n}} \{I(1 - I)\}^{\frac{1}{2}} B(\cdot, \mathfrak{h}) d|g| \int_{\xi_{n,n}}^1 \{I(1 - I)\}^{\delta/2 + \frac{1}{2}} B(\cdot, \mathfrak{h}) q_{\frac{1}{2}} d|g| \\ &\leq \left( \int_{\xi_{n,1}}^{\xi_{n,n}} \{I(1 - I)\}^{\delta/2} B(\cdot, \mathfrak{h}) q_{\frac{1}{2}} d|g| \right) \left( \int_{\xi_{n,n}}^1 B(\cdot, \mathfrak{h}) q_{\frac{1}{2}} d|g| \right) n^{\frac{1}{2}} (1 - \xi_{n,n})^{\frac{1}{2}} \\ &= o_p(1) o_p(1) o_p(1) = o_p(1). \end{aligned}$$

The argument for  $S_{n,13}^{(4)}$  is entirely analogous while for  $S_{n,12}^{(4)}$  the steps are similar except that we use  $n\xi_{n,1} = o_p(1)$ . Hence  $S_{n,1i}^{(4)} \rightarrow 0$  for  $i = 1, 2, 3$  and the lemma follows from (5.1).

Lemma 2. With  $\zeta_1, \zeta_2, \zeta_3, \zeta_4$  defined as in (3.22), (3.32) and (3.35) equation (3.38) holds.

Proof. From (2.11) and (3.35) we have

$$\zeta_3 + \zeta_4 = \int_0^1 L_1 UJ'_1(1) dg. \quad (5.5)$$

Also from (3.22) and (3.32)

$$\zeta_2 - \zeta_1 = \int_0^1 dL_2 \left\{ \int_t^1 UJ'_2 dg \right\}. \quad (5.6)$$

Integrating by parts, we obtain

$$\zeta_2 - \zeta_1 = \int_0^1 L_2 UJ'(2) dg + L_2(t) \left( \int_t^1 UJ'(2) dg \right) \Big|_{t=0}^{t=1}.$$

We shall show

$$\lim L_2(t) \int_t^1 UJ'(2) dg = 0 \quad (5.7)$$

where the limit is taken in each of the two cases  $t \rightarrow 0+$  and  $t \rightarrow 1-$ . In what follows this is to be interpreted whenever the limit is not explicitly stated.

Now for each  $t \in (0,1)$

$$|L_2(t) \int_t^1 UJ dg| \leq 2\rho_{q_{1/2}}(U,0) \left( \int_0^t |B(\cdot, \underline{b}) d|g| \right) \left( \int_t^1 |B(\cdot, \underline{b}) q_{1/2} d|g| \right). \quad (5.8)$$

In view of relations similar to (5.3) and (5.4) the function on the right hand side of (5.8) is dominated by

$$\left( \int_0^t \{I(1-I)\}^{\delta/2} |B(\cdot, \underline{b}) q_{1/2} d|g| \right) \left( \int_t^1 |B(\cdot, \underline{b}) q_{1/2} d|g| \right), \quad (5.9)$$

and so by (2.15), (5.9) must vanish in the limit as  $t \rightarrow 0+$  and  $t \rightarrow 1-$ .

Hence

$$\lim L_2(t) \int_t^1 UJ dg = 0. \quad (5.10)$$

Again for each  $t \in (0,1)$

$$|L_2(t) \int_t^1 (1-I)UJ' dg| \leq 2\rho_{q_{1/2}}(U,0) \left( \int_0^t |B(\cdot, \underline{b}) d|g| \right) \left( \int_t^1 (1-I)|B(\cdot, \underline{g}) q_{1/2} d|g| \right). \quad (5.11)$$

and the function on the right hand side of (5.11) is dominated by

$$\left( \int_0^t \{I(1-I)\}^{\delta/2} |B(\cdot, \underline{b}) q_{1/2} d|g| \right) \left( \int_t^1 I(1-I)|B(\cdot, \underline{g}) q_{1/2} d|g| \right). \quad (5.12)$$

It follows from (2.10) and (2.15) that (5.12) vanishes in the limit as  $t \rightarrow 0+$  and  $t \rightarrow 1-$ . So

$$\lim L_2(t) \int_t^1 (1 - t) UJ' dg = 0. \quad (5.13)$$

But (5.10) and (5.13) imply (5.7) and therefore  $\zeta_2 - \zeta_1 = \int_0^1 L_2 UJ'_{(2)} dg$ .

Hence using (5.5) and (2.13),

$$\frac{1}{2}(\zeta_2 - \zeta_1 + \zeta_3 + \zeta_4) = \int_0^1 UL_0 dg,$$

which is (3.38).

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