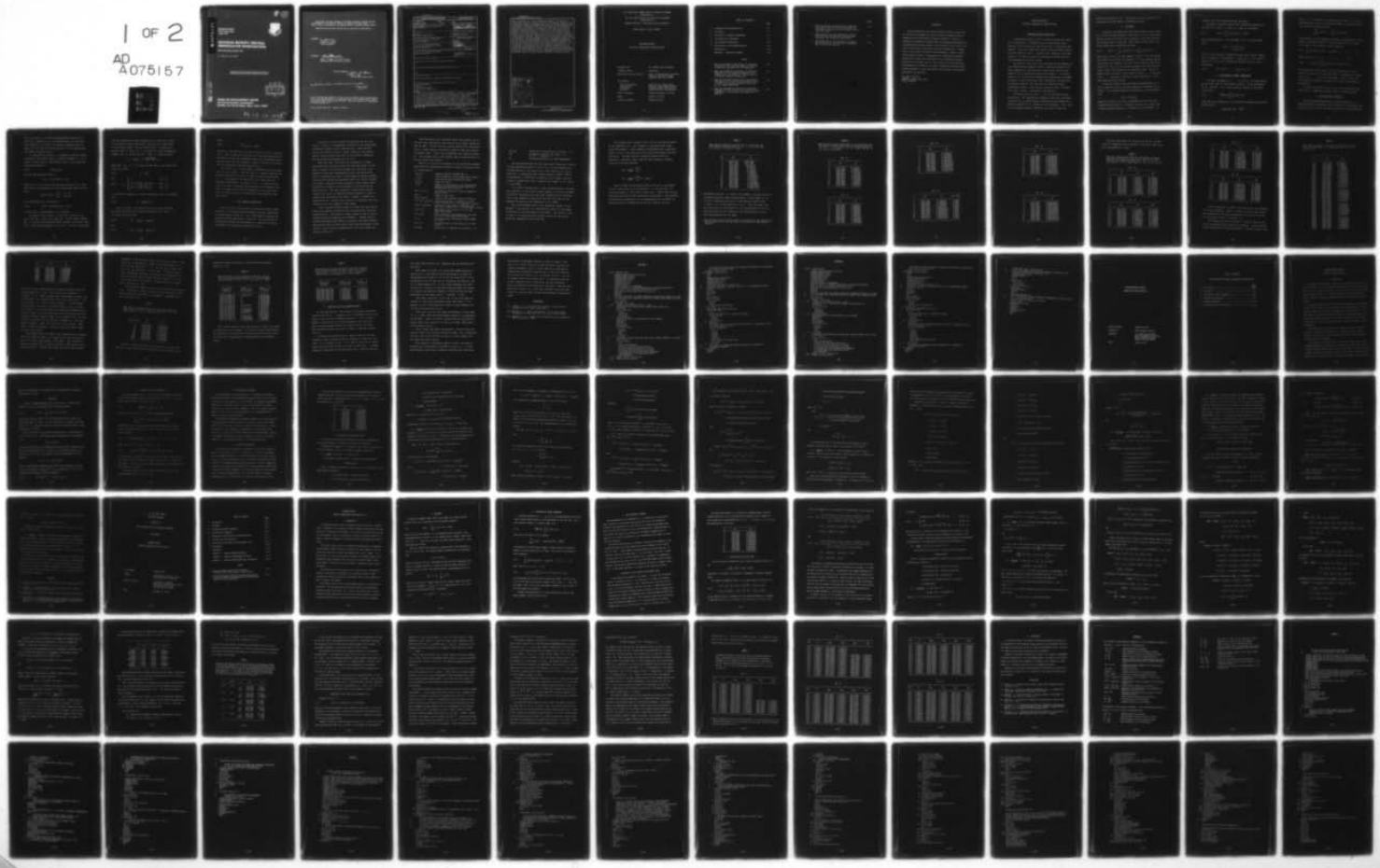
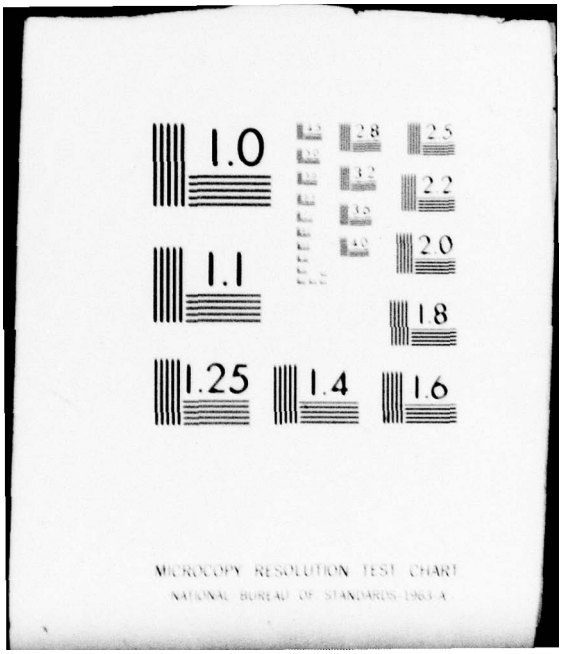
AD-A075 157 STATE UNIV OF NEW YORK COLL AT ONEONTA DEPT OF MATHEM--ETC F/G 17/4
MAXIMUM ENTROPY SPECTRAL DEMODULATOR INVESTIGATION.(U)
AUG 79 R G VAN METER F30602-75-C-0122

UNCLASSIFIED RADC-TR-79-209 NL

1 OF 2
AD A075157




1.0
1.1
1.25
1.4
1.6
2.8
3.2
3.6
4.0
2.5
2.2
2.0
1.8
MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS 1963 A

AD A075157

LEVEL II

12

B.S.

RADC-TR-79-209
Final Technical Report
August 1979

MAXIMUM ENTROPY SPECTRAL DEMODULATOR INVESTIGATION

State University of New York

Dr. Robert Guy Van Meter

APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED

DDC FILE COPY

DDC
RECEIVED
OCT 17 1979
E

ROME AIR DEVELOPMENT CENTER
Air Force Systems Command
Griffiss Air Force Base, New York 13441

79 10 16 077

This report has been reviewed by the RADC Information Office (OI) and is releasable to the National Technical Information Service (NTIS). At NTIS it will be releasable to the general public, including foreign nations.

RADC-TR-79-209 has been reviewed and is approved for publication.

APPROVED: Kenneth E. Wilson

KENNETH E. WILSON
Project Engineer

APPROVED: Fred I Diamond

FRED I. DIAMOND, Technical Director
Communications and Control Division

FOR THE COMMANDER: John P. Huss

JOHN P. HUSS
Acting Chief, Plans Office

Contract Comment on 1473 per Mr. Zimmerman, RADC
MCSumbacher
18 Oct 79

If your address has changed or if you wish to be removed from the RADC mailing list, or if the addressee is no longer employed by your organization, please notify RADC (DCCL), Griffiss AFB NY 13441. This will assist us in maintaining a current mailing list.

Do not return this copy. Retain or destroy.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: RADC-TR-79-209	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): MAXIMUM ENTROPY SPECTRAL DEMODULATOR INVESTIGATION	5. TYPE OF REPORT & PERIOD COVERED: Final Technical Report, June 1977 — September 1978
	6. PERFORMING ORG. REPORT NUMBER: N/A
7. AUTHOR(s): Dr. Robert Guy Van Meter	8. CONTRACT OR GRANT NUMBER(s): F30602-75-C-0122 (Source To be Add To Contract Data)
9. PERFORMING ORGANIZATION NAME AND ADDRESS: The State University of New York at Oneonta NY 13820, Dept. of Math., Sci.	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 62702F 45196IP0
11. CONTROLLING OFFICE NAME AND ADDRESS: Rome Air Development Center (DCCL) Griffiss AFB NY 13441	12. REPORT DATE: August 1979
	13. NUMBER OF PAGES: 98
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): Rome Air Development Center (DCCL) Griffiss AFB NY 13441	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE: N/A

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

Same

18. SUPPLEMENTARY NOTES

RADC Project Engineer: Kenneth E. Wilson, Capt, USAF (DCCL)

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Spectral Estimation
Modulation
Frequency Shift Keyed
Low Frequency Communication
ECCM

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The work presented in this report examined the feasibility of using maximum entropy spectral estimation techniques for demodulating frequency shift keyed signals in the presence of interfering signals and noise. Conventional frequency shift keyed demodulators derive an output representative of the instantaneous frequency, an output which can take only a single value at any instant. Spectral estimation allows multiple values of discrete frequencies to be resolved simultaneously. Maximum entropy spectral estimation is able to resolve such frequencies more accurately than fast Fourier transform techniques,

→ next page

DD FORM 1 JAN 73 1473

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

411 410

given a short data set (a 20 millisecond baud interval). The ability to resolve several values of discrete frequencies simultaneously opens up the way to simultaneous detection and demodulation of both the desired signal and the interference. Communication ~~(Mark - Space)~~ decisions can then be made for the desired signal on the basis of frequency, and interference can be detected, recognized, and ignored based upon frequency. This report examines this demodulator capable of obraining 15 to 20 decibels of interference rejection appears to be feasible given that the state of the art in analog to digital converters is 16 binary digits. Part One of the report studied the effect of analog to digital converter (quantization) noise on the accuracy of estimating a single discrete frequency. Both fortuitous and pathological frequency cases are determined. Part Two developed the theory of the spectral estimator in the no noise case, proving some important non-obvious fundamental relationships, thus placing the procedure being used on a firm theoretical foundation. The analysis exposed the nature of the random variable transformation which would be required to allow a statistically based prediction of performance to be accomplished, and demonstrated that such performance prediction can only be accomplished using numerical techniques. Part Three extends this theoretical work to the limits of mathematical tractability, and then develops and presents the results of a simulation of a digital communication system of the frequency shift keyed variety. All three parts are contained in this single report.

Accession For
NTIS GRA&I [X]
DDC TAB [ ]
Unannounced [ ]
Justification
By
Distribution/
Availability Codes
Dist. | Avail and/or special
A

1977 USAF/ASEE SUMMER FACULTY RESEARCH PROGRAM
sponsored by
THE AIR FORCE OFFICE OF SCIENTIFIC RESEARCH
conducted by
AUBURN UNIVERSITY AND OHIO STATE UNIVERSITY

PARTICIPANT'S FINAL REPORT

MAXIMUM ENTROPY SPECTRAL DEMODULATOR INVESTIGATION

Prepared by:	Dr. Robert Guy Van Meter
Academic Rank:	Professor
Department and University:	Dept. of Mathematical Sciences State University College Oneonta, New York 13820
Assignment:	
(Air Force Base)	Griffiss Air Force Base
(Laboratory)	Rome Air Development Center
(Division)	Communications and Control
(Branch)	Communications Transmission
USAF Colleague:	Captain Kenneth E. Wilson
Date:	August 19, 1977
Contract Number:	F44620-75-0031

TABLE OF CONTENTS

		PAGE
1.	Introduction and Objectives	I-5
2.	The Model	I-6
3.	Estimation of Model Parameters	I-7
4.	The Analytical Approach	I-8
5.	The Computer Simulation	I-11
6.	Conclusions and Recommendations	I-26
	References	I-28
	Appendix - Computer Program	I-29

TABLES

1.	Mean and Standard Deviations of predicted frequencies for varying numbers of analog to digital converter bits of precision.	I-16
2.	Mean and Standard Deviations of predicted frequencies for varying minimized error sample sizes and number of bits of analog to digital converter precision.	I-17
3.	Mean and Standard Deviations of predicted frequencies as a function of number of bits of analog to digital converter precision and of signal amplitude.	I-20
4.	Mean and Standard Deviation of predicted frequencies for 14 bits of precision as the frequency is varied over the range 125 to 5875 Hz.	I-22

PAGE

5. Mean and Standard Deviation of predicted frequencies for 14 bits of precision as the number of minimized error terms varies from 8 to 2048. I-24

6. Approximation of the probability density function and distribution function over frequency for 14 bits of precision. I-25

7. Approximation of the probability density and distribution function over frequency for 27 bits of precision. I-26

EVALUATION

The work presented in this report represents an effort to press upon the frontiers of knowledge in the area of spectral estimation, spectral estimators, and the performance of spectral estimators. To date, no researcher has developed a tractable means for computing the performance of a maximum entropy spectral estimator in the presence of noise, even for the type of noise which is usually most convenient mathematically: Gaussian noise. We have joined the ranks of these researchers. However, we have pressed beyond this point of disappointment by simulating the estimator against noise, signals, and interference, and by composing statistical analysis of the estimator outputs. These results have been sufficiently encouraging that an attempt will be made to implement the maximum entropy spectral estimator in a real time system, and to validate the performance predicted by the simulation.

Kenneth E. Wilson

KENNETH E. WILSON, Capt, USAF
Project Engineer

MAXIMUM ENTROPY
SPECTRAL DEMODULATOR INVESTIGATION

1. INTRODUCTION AND OBJECTIVES

A transmitted signal, which is masked by noise and, possibly, interference, is assumed to be a sinusoid with Hertz-frequency which varies over some finite set F of positive real numbers. Let "s(t)" denote the value of the resulting continuous-time signal at time t. Of particular interest is the case where #F = 2; that is, where the frequency switches back and forth between two distinct values.

The object of this study is to evaluate the performance of the maximum entropy method (= MEM) of spectral estimation for short segments of a discrete-time signal which results from sampling s(t) at uniformly spaced time instants. To be more specific, it is desired to estimate the uncertainty of the MEM estimates of the transmitted signal frequencies, by obtaining confidence intervals, for the cases (a) received signal = transmitted signal + noise and (b) received signal = transmitted signal + noise + (purposeful) interference.

The simple case (a) with $F = \{f_1\}$ was considered almost exclusively. The (analytical) problem of obtaining confidence intervals for f_1, which requires determining the probability density function for f_1, appears to be intractable. Thus, late in the program, computer simulation was used to study the

sampling variability of f_1. The results of this simulation are encouraging; the MEM seems to perform quite well.

2. THE MODEL

In general, the MEM models functional values as the output of a linear discrete system (= LDS); that is, as a linear combination of past functional values (outputs of the LDS) and past and present inputs to the LDS. This leads to the linear, constant coefficient, difference equation

$$(2.1) \qquad f(kT) = -\sum_{j=1}^{p} a_j f((k-j)T) + G\sum_{i=0}^{q} b_i u((k-i)T),$$

where the a_j and b_i are real numbers, $b_0 \underline{\underline{df}}\ 1$, G is the system gain factor (a positive real number), T is the sampling period (a positive real number), u((k-i)T) is the input to the LDS at time (k-i)T, and k is any positive integer such that the functions f and u are defined at the indicated times. As (2.1) enables one to "predict" f(mT) from f((m-1)T), ... , f((m-p)T), u(mT), ... , u((m-q)T), the name "linear prediction" is also associated with this method.

There are several other equivalent representations of a LDS in addition to the difference equation formulation [1; pp. 85-86]. For frequency-domain considerations, the representation

$$S(z) = H(z)U(z),$$

where S(z) and U(z) are the z-transforms of s(kT) and u(kT) respectively and H(z) is a rational function of z called the "system transfer function," is useful [1; pp. 220-282]. In

general, H(z) will have both zeros and poles.

We model a received signal s by a difference equation of the form (2.1) with q = 0 and G = 1; that is, we assume

$$s(kT) = -\sum_{j=1}^{p} a_j s((k-j)T) + u(kT) \tag{2.2}$$

for all appropriate k. For the model (2.2), it can be shown that

$$H(z) = \frac{1}{1 + \sum_{j=1}^{p} a_j z^{-j}}.$$

Thus H is an "all-pole" transfer function with p poles, namely, the p solutions of the equation $1 + \sum_{j=1}^{p} a_j z^{-j} = 0$ or, equivalently (if $z \neq 0$, as is required by the definition of the z-transform),

$$z^p + a_1 z^{p-1} + \ldots + a_{p-1} z + a_p = 0. \tag{2.3}$$

3. ESTIMATION OF MODEL PARAMETERS

The model parameters $a_1, \ldots, a_p$ in (2.2) are approximated by the usual type of least-squares analysis in the time-domain [2; pp. 563-567]. For a given positive integer m, we predict s(mT) to be

$$\widetilde{s(mT)} \underline{\underline{df}} -\sum_{j=1}^{p} \tilde{a}_j s((m-j)T), \tag{3.1}$$

where the $\tilde{a}_j$ are chosen so as to minimize an appropriate function of the errors

$$e(kT) \underline{\underline{df}} s(kT) - \widetilde{s(kT)}$$

for $k < m$. In the case of a deterministic signal, $\sum e^2(kT)$ is minimized over some set of previous samples. In case we minimize the sum of squared errors

$$\sum_{k=m-N}^{m-1} e^2(kT) \quad (= \sum_{k=0}^{N-1} e^2((m-N+k)T)$$

corresponding to the preceding N samples, the technique is called the "covariance method" of linear prediction [2; p. 564]. This leads to the system of linear equations

$$(3.2) \quad \sum_{j=1}^{p} \tilde{a}_j(m,N)\phi_{ij}(m,N) = -\phi_{0i}(m,N) \quad (i = 1, 2, \ldots, p),$$

where, for all $(i,j) \in \{1, \ldots, p\} \times \{1, \ldots, p\}$,

$$(3.3) \quad \phi_{ij}(m,N) \underline{\underline{df}} \sum_{k=0}^{N-1} s((m-N+k-i)T)s((m-N+k-j)T),$$

for determining the $\tilde{a}_j$ which yield the prediction of $s(mT)$. From (3.2) and (3.3), we see that the $N + p$ consecutive samples $s((m-N-p)T), \ldots, s((m-1)T)$ are needed. Thus if we do not allow negative arguments, $s((N+p+1)T)$ is the first sample we can predict.

In making the prediction (3.1), we are assuming that the input $u(mT)$ is completely unknown, which is often the case.

4. THE ANALYTICAL APPROACH

The analytical determination of the frequencies $f_i \in F$ exhibited by the transmitted signal involves (a) calculating the $\phi_{ij}(m,N)$ from (3.3), (b) solving the system of linear equations

(3.2) for the $\tilde{a}_j$, (c) solving the polynomial equation (2.3) with "$\tilde{a}_j$" in place of "a_j", (d) determining θ_i in the polar form $r_i \exp(\pm J\theta_i)$ of each of the complex conjugate pairs of roots of (2.3), and (e) multiplying the θ_i by an appropriate real number to obtain the f_i.

As indicated in Section 1, an attempt was made to handle analytically the simple case of a single frequency ($F = \{f_1\}$) signal in noise with no interference; that is, we assume the transmitted signal is

$$A\sin(2\pi f_1 t) \tag{4.1}$$

and that the received signal is

$$s(t) = A\sin(2\pi f_1 t) + n(t),$$

where n(t) is a zero-mean Gaussian noise process [3; pp. 219-222] which is uncorrelated with the transmitted signal and for which

$$E[n(t)n(t+k)] = \begin{cases} \sigma^2 & (k = 0), \\ 0 & (k \neq 0). \end{cases}$$

Thus the samples of s are given by

$$s(kT) = A\sin(2\pi f_1 kT) + n(kT). \tag{4.2}$$

In this case, a 2-pole model ($p = 2$) suffices.

The roots of (2.3) with $p = 2$ and "$\tilde{a}_j$" in place of "a_j" are $z_1 \underline{\underline{df}} (-\tilde{a}_1 + \sqrt{d})/2$ and $z_1 \underline{\underline{df}} (-\tilde{a}_1 - \sqrt{d})/2$, where $d \underline{\underline{df}} \tilde{a}_1^2 - 4\tilde{a}_2$. It can be shown that the radian argument in the polar form of z_1 is the radian-frequency f_1 in (4.2). (In fact, this poten-

tial for ferreting out the frequencies of a sinusoidal signal is the reason linear prediction is useful to us, rather than its predictive powers. We can always "wait" T seconds and measure s(mT).) Thus, as $f > 0$, z_1 must be a non-real(complex) number; that is, we must have $\tilde{a}_1^2 - 4\tilde{a}_2 < 0$. Hence we have

$$z_1 = -(\tilde{a}_1/2) + J(\sqrt{4\tilde{a}_2 - \tilde{a}_1^2})/2,$$

where $4\tilde{a}_2 - \tilde{a}_1^2 > 0$. It is easy to show that z_1 has polar form $r_1 \exp(J\theta_1)$, where

(4.3) $$r_1 = \sqrt{\tilde{a}_2}$$

and

(4.4) $$\theta_1 = \begin{cases} \pi/2 & (\tilde{a}_1 = 0), \\ \tan^{-1}(-\sqrt{4\tilde{a}_2 - \tilde{a}_1^2}/\tilde{a}_1) + \pi & (\tilde{a}_1 > 0), \\ \tan^{-1}(-\sqrt{4\tilde{a}_2 - \tilde{a}_1^2}/\tilde{a}_1) & (\tilde{a}_1 < 0). \end{cases}$$

If we measure frequencies in Hertz and require that $F \subset (0,6000)$, then

(4.5) $$f_1 = (6000/\pi)\theta_1.$$

Now, $\tilde{a}_1$ and $\tilde{a}_2$ in the expression for f_1 can be obtained explicitly by solving the system of equations (3.2) with $p = 2$. By Cramer's Rule, we have

(4.6) $$\tilde{a}_1 = (\phi_{12}\phi_{02} - \phi_{22}\phi_{01})/\Delta$$

and

(4.7) $$\tilde{a}_2 = (\phi_{21}\phi_{01} - \phi_{02}\phi_{11})/\Delta,$$

where

$$\Delta \stackrel{df}{=} \phi_{11}\phi_{22} - \phi_{12}\phi_{21} \tag{4.8}$$

and the ϕ_{ij} are given by (3.3) and involve the s(kT) as given by (4.2). Even in this very special case, the complicated chain of operations linking the assumptions about the transmitted signal and the noise with the frequency f_1 makes it difficult to determine the probability density function for f_1.

In case #F = 2, a 4-pole model is appropriate and the equation (2.3) is quartic. The four roots of a quartic equation can be given explicitly in terms of radicals and the coefficients $a_1, \ldots, a_4$; however, the expressions for these roots are extremely complicated. Of course, if #F > 2, a model with at least six poles is required and no analog of the quadratic and quartic formulas exists for the equation (2.3) in such cases. Thus, for these cases, we can not mimic the treatment of the 2-pole case.

5. THE COMPUTER SIMULATION

The first part of the program involved an extensive investigation of linear prediction techniques and their application to communications theory [1]. This investigation included the analytical effort described in Section 4. The second part of the program involved the development of a computer simulation to determine the sampling variability of the f_1.

In Sections 1 and 4, we did not elaborate on the term "noise." Noise is a fundamental limitation on the performance of physical systems such as radar devices. Some possible sources of noise are clutter, cosmic radiation, and thermal motion of the electrons and ions in the receiver components and the antenna surroundings [3; pp. 3-5].

There are other limitations on performance in signal-processing that are based on the fact that the values of variables in models of real-world systems are, typically, real numbers with decimal representations which are non-terminating or terminate only after a large number of digits, whereas, digital signal-processing equipment (which for various reasons has largely supplanted analog equipment) seldom allows representations of numbers (or other symbols) with more than 64 precision bits. One such limitation, "quantization noise," results from use of an analog-to-digital converter (= ADC) in sampling the continuous-time signal. Another, "roundoff (or chopping) noise," results from rounding (or chopping) sums and products to fit the computer's word length.

In the simulating done to date, an effort has been made to assess the sampling variability due to quantization noise and roundoff noise. Such noise is always present as we can not do infinite-precision sampling and arithmetic. Further work is needed to determine the effect of purposeful man-made interference and the type of noise cited in the second paragraph of this section. Also, we have considered only the 2-pole model discussed in Section 4.

Some features of the simulation which look peculair owe to the fact that, previously, some processing of actual signals was done at RADC. An 8-bit ADC was used to produce 8-bit approximations of signal values and blocks of 2048 of these were stored on magnetic tape. Later, this information was computer-processed. The programs used in the present study include portions of the previously-used program.

Below is a list of key variables in the simulation programs along with the corresponding variable (if any) in Section 4 and its interpretation:

TSA (T)	sampling period (1/12000 sec.);
F1 (f_1)	Hertz-frequency of the transmitted signal;
A (A)	amplitude of the transmitted signal
NP (p)	number of poles;
N (N)	number of error terms in the minimization process for determining the prediction coefficients $\tilde{a}_j$ in (3.1);
NN (= N + p)	the number of previous samples needed to predict s(mT)
NB ( )	number of ADC bits;
S(J) (s(jT))	value of the received signal at time jT;
P(I,J) ($\phi_{ij}(m,N)$)	coefficient of $\tilde{a}_j$ in equation i of the linear system (3.2), ϕ is defined in (3.3);
PO(I) ($\phi_{0i}(m,N)$)	constant on the right in equation i of the linear system (3.2), ϕ is defined in (3.3);
R(K) (r_1)	magnitude of the root of (2.3) in the upper half-plane;
F(K) (f_1)	Hertz-frequency corresponding to the root of (2.3) in the upper half-plane;
FC ( )	the radian-Hertz conversion factor $6000/\pi$ in (4.5);
DEL (Δ)	the determinant of the system (3.3) (see (4.8));
A1 ($\tilde{a}_1$)	coefficient in equation (2.3) with p = 2;

A2 (a_2)	coefficient in equation (2.3) with p = 2;
AM	arithmetic mean of F $\underline{\underline{df}}$ {f(mT): m ε {6, 7, ... , 2048}};
SD	standard deviation of F, the frequencies

We assume henceforth that the amplitude A (measured in volts) of the transmitted signal is in [-5, 5]. In the simulation of the ADCs behavior, the signal S(J) in the interval [-5, 5], of length 10, is mapped into the integer interval $[-2^{NB-1}, 2^{NB-1}]$ by following $x \longmapsto (2^{NB}/10)x$ by chopping to an integer, and the integer is mapped back into a computer real number in [-5, 5] by $x \longmapsto (10/2^{NB})x$.

In a certain average sense, binary representations of numbers have $\log_2(10)$ (= 3.32) times as many symbols as do the corresponding decimal representations when both representations terminate. Thus it was apparent to the writer that the 8-bit ADC was not adequate for dealing with frequencies in (0, 6000).

One of the programs used in the simulation appears in the Appendix. Its purpose is to determine the effect of the number of ADC bits on the performance of the model. The results, given in Table 1, of executing this program confirm the conclusion about the inadequacy of the 8-bit ADC.

This program uses 5 samples (N=3, p=2) to form each estimate of the frequency (f_1). The frequency is estimated 2043 times, using 2048 samples. These samples are generated with a specified number of bits of ADC precision (NB) using the procedure outlined previously. The mean (AM) and standard deviation (SD) are computed in subroutine STATS from the 2043 frequency estimates using the following formulas:

$$AM = \frac{1}{2043} \sum_{i=1}^{2043} f_i$$

$$SD = \frac{1}{2043} \sum_{i=1}^{2043} (f_i - 3250)^2$$

Table 2 shows the variation of AM and SD with N, the number of error terms used in the minimization. In the case NB = 14, central processor (= CPU) time is given also. These times include the execution time for the statistical calculations. Thus absolute differences are meaningful to the determination of the effect on execution time of changing the value of N, whereas relative

TABLE 1

Mean AM and standard deviation SD of predictions for the number of ADC bits NB = 2, 3, ... , 16, 27* with A = 1.5, N = 3, and F1 = 3250.

NB	AM	SD
2	6000.0000	2749.9994
3	3497.7974	1677.3801
4	3252.4888	226.23133
5	3248.9677	78.865247
6	3249.5983	33.414235
7	3249.6557	21.808373
8	3249.9485	11.663445
9	3249.9887	4.8569507
10	3249.9991	2.4628091
11	3250.0001	0.77203118
12	3250.0004	0.22295413
13	3249.9985	0.18255417
14	3249.9960	0.15308648
15	3250.0002	0.075419844
16	3250.0008	0.034125918
27	3250.0001	0.0066758151

differences are not. It appears that adding 1 to the value of N increases execution time by approximately (1/3)(0.0001) hr. or 0.12 sec. Note that execution of the program involves 2043 (= 2048 - (3 + 2)) calculations of an f_1. Thus the added time for calculation of each value of f_1 is approximately $5 \cdot 10^{-5}$ sec. Execution times are of interest as the ultimate goal is real-time implementation of the model.

*Here we have used 27 precision-bit floating-point representations and arithmetic with the ADC simulation portion of the program deleted.

TABLE 2

Mean AM and standard deviation SD of predictions for the number of error terms used in the minimization N = 2, 3, ... , 12 with A = 1.5, F1 = 3250, and NB = 8, 10, 12, 14, 16, and 27.

NB = 8

N	AM	SD
2	3250.0737	15.632812
3	3249.9485	11.663445
4	3249.9371	9.3071191
5	3249.8821	7.1128755
6	3249.8813	6.0597079
7	3249.8710	5.2067635
8	3249.8677	4.4239790
9	3249.8625	3.3021908
10	3249.8594	2.8550067
11	3249.8582	2.2166742
12	3249.8565	2.0223015

NB = 10

N	AM	SD
2	3250.0018	2.9927573
3	3249.9991	2.4628091
4	3249.9970	2.0885355
5	3249.9975	1.7026238
6	3249.9979	1.6084557
7	3249.9978	1.2938950
8	3249.9922	1.0680774
9	3249.9925	0.73630817
10	3249.9945	0.56215978
11	3249.9921	0.39859506
12	3249.9906	0.35790581

NB = 12

N	AM	SD
2	3249.9978	0.28106676
3	3250.0004	0.22295413
4	3250.0015	0.14402156
5	3250.0015	0.13972569
6	3250.0001	0.081098709
7	3250.0009	0.067171544
8	3249.9994	0.059041822
9	3250.0011	0.044649824
10	3250.0009	0.047537731
11	3249.9999	0.052555363
12	3250.0009	0.046175324

NB = 14

N	AM	SD	CPU TIME
2	3249.9988	0.21686259	0.0009
3	3249.9960	0.15309648	0.0009
4	3249.9997	0.11826708	0.0010
5	3249.9983	0.10109097	0.0010
6	3249.9958	0.066045111	0.0010
7	3249.9998	0.059400090	0.0011
8	3249.9980	0.052958412	0.0011
9	3249.9972	0.032415771	0.0011
10	3250.0021	0.037008012	0.0012
11	3249.9973	0.038732876	0.0012
12	3250.0007	0.033104122	0.0012

NB = 16

N	AM	SD
2	3249.9999	0.045065880
3	3250.0008	0.034125918
4	3249.9995	0.024391433
5	3249.9991	0.021149200
6	3250.0010	0.019371934
7	3249.9993	0.015833400
8	3249.9985	0.013296252
9	3249.9994	0.011714947
10	3249.9999	0.010651756
11	3249.9995	0.0083492643
12	3250.0000	0.0071696392

NB = 27

N	AM	SD
2	3250.0003	0.0087628514
3	3250.0001	0.0066758151
4	3250.0000	0.0042304078
5	3250.0000	0.0027758344
6	3250.0000	0.0017828080
7	3250.0000	0.0020461476
8	3250.0000	0.0021599298
9	3250.0000	0.0019517387
10	3250.0000	0.0015333511
11	3250.0000	0.00091071260
12	3250.0000	0.0010223139

The next table shows the variation of AM, SD, and the signal-to-(quantization and roundoff) noise ratio SNR, given by $20 \cdot \log_{10}(2^{NB}A/10)$, with A.

TABLE 3

Mean AM, standard deviation SD, and signal-to-(quantization and roundoff) noise ratio SNR for the amplitude of the transmitted signal A = 0.5, 1.0, 1.5, ... 5.0 with N = 3, F1 = 3250, and NB = 10, 12, 14, and 16.

NB = 10

A	AM	SD	SNR
0.5	3249.9668	6.9402777	34.185400
1.0	3249.9982	2.0219451	40.205999
1.5	3249.9991	2.4628091	43.727824
2.0	3249.9991	1.5623139	46.226599
2.5	3249.9962	1.5531899	48.164799
3.0	3250.0001	0.77203118	49.748425
3.5	3250.0002	1.3622908	51.087360
4.0	3250.0012	0.72204416	52.247199
4.5	3250.0019	0.42808743	53.270249
5.0	3249.9961	0.98476960	54.185400

NB = 12

A	AM	SD	SNR
0.5	3249.9991	1.5623139	46.226599
1.0	3250.0012	0.72204416	52.247199
1.5	3250.0004	0.22295413	55.769024
2.0	3249.9994	0.53745331	58.267799
2.5	3249.9979	0.44305741	60.205999
3.0	3249.9985	0.18255417	61.789624
3.5	3250.0007	0.20943003	63.128560
4.0	3249.9981	0.27760254	64.288399
4.5	3250.0016	0.16504455	65.311449
5.0	3250.0001	0.16447618	66.226500

NB = 14

A	AM	SD	SNR
0.5	3249.9994	0.53745331	58.267799
1.0	3249.9981	0.27760254	64.288399
1.5	3249.9960	0.15309648	67.810224
2.0	3250.0001	0.12638791	70.308999
2.5	3250.0036	0.086964880	72.247199
3.0	3250.0002	0.075419844	73.830824
3.5	3249.9987	0.056363066	75.169760
4.0	3250.0002	0.056470926	76.329598
4.5	3250.0023	0.025813821	77.352649
5.0	3249.9998	0.049159881	78.267799

NB = 16

A	AM	SD	SNR
0.5	3250.0001	0.12638791	70.308999
1.0	3250.0002	0.056470926	76.329598
1.5	3250.0008	0.034125918	79.851424
2.0	3249.9998	0.033097830	82.350199
2.5	3250.0016	0.024226625	84.288399
3.0	3250.0005	0.017549278	85.872024
3.5	3249.9992	0.021185211	87.210959
4.0	3249.9992	0.018053080	88.370798
4.5	3250.0004	0.015683560	89.393849
5.0	3249.9998	0.015091892	90.308999

Table 4 shows the variation of AM and SD with the frequency F1 of the transmitted signal. As there are 14-bit ADCs available commercially and model performance is quite good with NB = 14 according to Table 1, we let NB = 14.

The strange variation of SD with F1 results from the sampling process and finite precision arithmetic. As the possible signal frequencies are less than 6000 Hz., we must sample at the

TABLE 4

Mean AM and standard deviation SD of predictions for F1 = 125, 250, 375, ... , 5875 with NB = 14, N = 3, and A = 1.5.

F1	AM	SD
125	119.24084	36.594588
250	249.51668	13.343529
375	374.99050	2.1931926
500	499.99242	2.2354304
625	624.99677	1.6018316
750	750.00167	0.41421278
875	874.99947	0.93162946
1000	999.99837	0.73667024
1125	1124.9997	0.35508527
1250	1250.0013	0.36510814
1375	1374.9998	0.33071683
1500	1499.9985	0.24333926
1625	1624.9985	0.24277011
1750	1750.0004	0.10973179
1875	1875.0006	0.10363140
2000	2000.0000	0.
2125	2124.9995	0.18358279
2250	2250.0019	0.073796298
2375	2375.0001	0.21530845
2500	2499.9964	0.090456136
2625	2625.0006	0.12147366
2750	2750.0028	0.15310039
2875	2875.0002	0.18313020
3000	3000.0000	0.
3125	3124.9978	0.18266767
3250	3249.9960	0.15309648
3375	3374.9995	0.12147751
3500	3500.0056	0.090457460
3625	3625.0008	0.21415340
3750	3749.9958	0.073787051
3875	3874.9984	0.18436528
4000	4000.0000	0.
4125	4124.9997	0.10363143
4250	4249.9976	0.10972426
4375	4374.9999	0.24238029
4500	4499.9988	0.24332672
4625	4624.9993	0.33069280
4750	4750.0037	0.36511440
4875	4875.0005	0.35509441

F1	AM	SD
5000	4999.9958	0.73666593
5125	5124.9982	0.90562031
5250	5250.0022	0.41419801
5375	5375.0008	1.5328385
5500	5500.0052	2.2354247
5625	5625.0086	2.1931942
5750	5750.4873	13.343526
5875	5880.7836	36.559757

rate of at least 12000 samples per second according to the Sampling Theorem [1; p. 291]. Thus we are led to letting T = 1/12000. For F1 = 2000 (respectively, 3000, 4000), we have α df $2k\pi F1/12000 = k\pi/3$ (respectively, $k\pi/2$, $2k\pi/3$) and $\sin(\alpha) = .50$ (respectively, 1.0, .50) in exact arithmetic. As the library SIN function is accurate to 8 decimal digits, $\sin(\alpha)$ is exact and $1.5\sin(\alpha)$ is also exact with two significant decimal digits. In these cases, it can be shown that the first step in the simulation of the ADCs behavior leads to an integer if NB = 14; thus the ADC simulation leads to the exact frequency F1 as the predicted frequency. Thus, the value 0 for SD in case F1 = 2000, 3000, or 4000 is explained. Similar reasons for other irregularities can be given.

The program leading to Table 4 was executed with F1 ranging from 125 to 5875 by steps of 25; however, for obvious reasons not all of these values appear in the table. The original attempt at execution for F1 ranging from 25 to 5975 lead to execution-time errors. Further work showed that the trouble begins

somewhere in the intervals [100, 125) and (5875, 6000). Obviously, as F1 approaches 0^+ (6000^-), the radian measure of θ_1 in (4.4) must approach 0^+ (π^-). Thus the discriminant d $\underline{df}$ $4\tilde{a}_2 - \tilde{a}_1^2$ must approach 0 in both cases. Printing values of some intermediate variables for the bad cases showed that d took on negative values with increasing frequency as F1 got closer and closer to 0^+ (6000^-). Quantization and roundoff noise had taken its toll.

The next table shows the variation of AM and SD with block size. Ultimately, a block size smaller than 2048 may be used. We note that the performance of the model is reasonably uniform over block size.

TABLE 5

Mean AM and standard deviation SD of predictions for block size = 8, 16, 32, 64, 128, 256, 512, 1024, and 2048 with NB = 14, F1 = 3250, and A = 1.5.

BLOCK SIZE	AM	SD
8	3249.9163	0.10430552
16	3249.9741	0.19046609
32	3249.9907	0.14852447
64	3249.9952	0.16077075
128	3249.9980	0.15215203
256	3249.9989	0.15498016
512	3250.0002	0.15291365
1024	3249.9998	0.15360796
2048	3249.9960	0.15309648

Finally, relative frequencies and cumulative relative frequencies of the predictions were obtained in the absence of the

probability density function (= PDF) and the distribution function (= DF).

TABLE 6

Approximations of the probability density function PDF and the distribution function DF of the predictions with NB = 14, N = 3, A = 1.5, and F1 = 3250.

PREDICTION INTERVAL	APPROX. PDF ON INTERVAL	APPROX. DF AT RIGHT
3249.45-3249.50	0.041605482	0.041605482
3249.50-3249.55	0.	0.041605482
3249.55-3249.60	0.	0.041605482
3249.60-3249.65	0.	0.041605482
3249.65-3249.70	0.	0.041605482
3249.70-3249.75	0.	0.041605482
3249.75-3249.80	0.16691140	0.20851689
3249.80-3249.85	0.	0.20851689
3249.85-3249.90	0.083700441	0.29221733
3249.90-3249.95	0.16691140	0.45912873
3249.95-3250.00	0.20802741	0.66715614
3250.00-3250.05	0.	0.66715614
3250.05-3250.10	0.16642193	0.83357808
3250.10-3250.15	0.083210965	0.91078904
3250.15-3250.20	0.083210965	1.00000000

The "grainy" nature of the above results is again the effect of quantization and roundoff. To partially negate these effects the program was run again with the quantization portion missing so that 27-bit floating-point representations and arithmetic prevail.

TABLE 7

Approximations of the probability density function PDF and the distribution function DF of the predictions with N = 3, A = 1.5, F1 = 3250, and 27-bit floating-point representations and arithmetic.

PREDICTION INTERVAL	APPROX. PDF ON INTERVAL	APPROX. DF AT RIGHT
3249.985-3249.990	0.12922173	0.12922173
3249.990-3249.995	0.081742535	0.21096427
3249.995-3250.000	0.28095937	0.49192364
3250.000-3250.005	0.36514929	0.85707294
3250.005-3250.010	0.064121390	0.92119432
3250.010-3250.015	0.042094959	0.95790504
3250.015-3250.020	0.042094959	1.00000000

6. CONCLUSIONS AND RECOMMENDATIONS

As indicated earlier, the problem of obtaining confidence intervals for the $f_i \in F$ appears to be intractable, even for the simple case of a single-frequency signal ($F = \{f_1\}$) in noise. However, simulation of the single-frequency case, with the only noise being quantization and roundoff noise, has shown that the MEM performs quite well in terms of sampling variability of f_1.

If results similar to those in Table 1 hold for the two-frequency signal simulation and the frequencies differ by, say, 50 Hz., it is obvious (as SD = 11.7) that an 8-bit ADC is inadequate and that an ADC giving 12-16 bits is desirable. A catalog search has shown that 12-bit ADCs with 2μsec. sampling time and

and 14-bit ADCs with 50 μsec. sampling time are available commercially.

From Table 2 with NB = 14, we see that model behavior is good with N = 3 but that SD can be decreased by a factor of approximately 4/5 (that is, to 1/5 of its value for N = 3) by increasing N to 12 at the cost of increasing CPU time by a factor of approximately 1/3. If the single-frequency case were of practical interest, it would not seem desirable to increase N in view of the desire to operate in real-time and the good performance of the model with N = 3 (and NB = 12-14).

From Table 3 with NB = 12 (or 14), we see that model performance is good with transmitted signal amplitude A = 1.5. However, SD can be decreased by a factor of approximately 3/4 (or 5/6) by increasing A to 4.5.

From Table 4, we see that model performance is quite good for $f_1 \in (1000, 5000)$ but deteriorates rapidly as f_1 approaches 0^+ and 6000^-. Again, if similar results hold for the two-frequency signal case, values of f_1 and f_2 in [1500, 4500] would give excellent results.

Table 5 shows that model performance is quite uniform and good for block sizes ranging from 128 to 2048. Thus a reduction of block size to the vicinity of 128, as is anticipated, will not adversely effect results.

There remains a substantial amount of work to be done in assessing the performance of the MEM in case $\#F \geq 2$. In the two-frequency signal case, operation counting shows that Gauss

elimination is decidedly superior to use of Cramer's Rule. Also, it is likely that use of some iterative algorithm for finding polynomial roots is better than use of the quartic formula (which generally requires use of the cubic formula) for solving (2.3) with p = 4. Of course, for #F $\geq$ 3 (p $\geq$ 6) no formula exists for solving (2.3). A substantial portion of the computing time is required by the COV subroutine, which is probably as efficient as is possible. With the real-time goal in mind, it is vital that the most efficient algorithms for solving linear systems and polynomial equations be found.

REFERENCES

[1] Cadzow, J. A., Discrete-time Systems, Prentice-Hall, Englewood Cliffs, New Jersey,1973.

[2] Makhoul, J., "Linear Prediction: A Tutorial Review," Proceedings of the IEEE, Vol. 63, No. 4, April 1975.

[3] Robinson, E. A., Statistical Communication and Detection, Hafner, New York, 1967.

APPENDIX

```
CMETA  PROGRAM META
       DOUBLE PRECISION TSA,TPI,TC,W1
       COMMON/SAMP/S(2048)
       COMMON/FREQ/F(2048)
       COMMON/RMAG/R(2048)
       COMMON/PHI/P(2,2)
       COMMON/PHO/PO(2)
       COMMON/TFRQ/F1
C      ASSIGN VALUES TO VARIABLES
       DATA TSA,TPI/83.3333333333333D-6,6.28318530717958647/
       DATA PI,FC,F1/3.14159265,1909.85931,3250./
       DATA KU,A,NP,N,AM,SD/2048,1.5,2,3,0.,0./
       NN=NP+N
       KL=NN+1
C      LOOP TO CALCULATE AND PRINT PREDICTED FREQUENCIES FROM (4.4) AND
C      (4.5)AND MEAN AND STANDARD DEVIATION THEREOF FOR NUMBER OF BITS =
C      2,3,...,16
       DO 5 NB=2,16
         WRITE (6,200) NB
  200    FORMAT (18H NUMBER OF BITS = ,I3//)
C        CALCULATE EXACT TRANSMITTED SIGNAL VALUE FROM (4.1)
         W1=F1*TPI
         DO 10 J=1,2048
           FJ=J-1
           TC=FJ*TSA
           TS=W1*TC
   10    S(J)=A*SIN(TS)
C        SIMULATION OF ADC QUANTIZATION OF THE SIGNAL
         NBP=2**NB
         PNB=NBP
         DV=10./PNB
         DVI=PNB/10.
         DO 15 J=1,2048
           X=S(J)
           X=X*DVI
           KP=X
           X=KP
   15    S(J)=X*DV
C        CALCULATE COEFFICIENTS FOR AND SOLVE LINEAR SYSTEM (3.2) USING
C        (4.6)-(4.8)
         DO 20 K=KL,KU
           KK=K
           CALL COV(NP,NN,KK)
           DEL=P(1,1)*P(2,2)-P(1,2)*P(2,1)
           A1=(P(1,2)*PO(2)-PO(1)*P(2,2))/DEL
           A2=(P(2,1)*PO(1)-PO(2)*P(1,1))/DEL
C          CALCULATE AND PRINT PREDICTED FREQUENCY
           F(K)=FC*ARCTA(-A1,SQRT(4*A2-A1*A1))
           R(K)=SQRT(A2)
   20    WRITE (6,300) K,R(K),F(K)
  300    FORMAT (1X,I8,2G20.8)
```

```
C       CALCULATE AND PRINT MEAN AND STANDARD DEVIATION OF PREDICTIONS
        CALL STATS (AM,SD,KL,KU)
    5 WRITE (6,400) AM,SD
  400 FORMAT (1X,2G20.8///)
      STOP
      END
      SUBROUTINE COV(NP,NN,LP)
      COMMON/SAMP/S(2048)
      COMMON/PHI/P(2,2)
      COMMON/PHO/PO(2)
C     CALCULATE THE PHI(I,J) OF (3.3)
C     CALCULATE DIAGONAL ELEMENTS, PHI(J,J), OF COVARIANCE MATRIX -
C     ASSIGN TO P(J,J)
      L=LP-1
      NI=NN-NP
      NL=LP-NI
      B=0.
      DO 5 J=NL,L
    5 B=B+S(J)*S(J)
      DO 10 J=1,NP
        K=LP-J
        I=NL-J
        B=B+S(I)*S(I)-S(K)*S(K)
   10 P(J,J)=B
C     CALCULATE REMAINDING PHI(I,J)
      DO 15 KK=1,NP
        B=0.
C       CALCULATE PHI(0,KK) - ASSIGN TO PO(KK)
        DO 20 J=1,NI
          N=LP-J
          M=N-KK
   20   B=B+S(N)*S(M)
        PO(KK)=B
C       CALCULATE PHI(I,J),J-I=KK,1<I<NP-1,2<J<NP - ASSIGN TO P(I,J)
        IF (KK.EQ.NP) GO TO 15
        DO 25 K=1,NP-KK
          I=K
          J=KK+K
          N=LP-K
          M=N-KK
          N1=NL-K
          M1=N1-KK
          B=B+S(N1)*S(M1)-S(N)*S(M)
          P(I,J)=B
   25   P(J,I)=B
C       THE PREVIOUS STATEMENT TAKES ADVANTAGE OF SYMMETRY OF
C       COVARIANCE MATRIX
   15 CONTINUE
      RETURN
      END
```

APPENDIX

```
CMETA  PROGRAM META
       DOUBLE PRECISION TSA,TPI,TC,W1
       COMMON/SAMP/S(2048)
       COMMON/FREQ/F(2048)
       COMMON/RMAG/R(2048)
       COMMON/PHI/P(2,2)
       COMMON/PHQ/PO(2)
       COMMON/TFRQ/F1
C      ASSIGN VALUES TO VARIABLES
       DATA TSA,TPI/83.3333333333333D-6,6.28318530717958647/
       DATA PI,FC,F1/3.14159265,1909.85931,3250./
       DATA KU,A,NP,N,AM,SD/2048,1.5,2,3,0.,0./
       NN=NP+N
       KL=NN+1
C      LOOP TO CALCULATE AND PRINT PREDICTED FREQUENCIES FROM (4.4) AND
C      (4.5)AND MEAN AND STANDARD DEVIATION THEREOF FOR NUMBER OF BITS =
C      2,3,...,16
       DO 5 NB=2,16
         WRITE (6,200) NB
  200    FORMAT (18H NUMBER OF BITS = ,I3//)
C        CALCULATE EXACT TRANSMITTED SIGNAL VALUE FROM (4.1)
         W1=F1*TPI
         DO 10 J=1,2048
           FJ=J-1
           TC=FJ*TSA
           TS=W1*TC
   10    S(J)=A*SIN(TS)
C        SIMULATION OF ADC QUANTIZATION OF THE SIGNAL
         NBP=2**NB
         PNB=NBP
         DV=10./PNB
         DVI=PNB/10.
         DO 15 J=1,2048
           X=S(J)
           X=X*DVI
           KP=X
           X=KP
   15    S(J)=X*DV
C        CALCULATE COEFFICIENTS FOR AND SOLVE LINEAR SYSTEM (3.2) USING
C        (4.6)-(4.8)
         DO 20 K=KL,KU
           KK=K
           CALL COV(NP,NN,KK)
           DEL=P(1,1)*P(2,2)-P(1,2)*P(2,1)
           A1=(P(1,2)*PO(2)-PO(1)*P(2,2))/DEL
           A2=(P(2,1)*PO(1)-PO(2)*P(1,1))/DEL
C          CALCULATE AND PRINT PREDICTED FREQUENCY
           F(K)=FC*ARCTA(-A1,SQRT(4*A2-A1*A1))
           R(K)=SQRT(A2)
   20    WRITE (6,300) K,R(K),F(K)
  300    FORMAT (1X,I8,2G20.8)
```

```
C       CALCULATE AND PRINT MEAN AND STANDARD DEVIATION OF PREDICTIONS
        CALL STATS (AM,SD,KL,KU)
    5 WRITE (6,400) AM,SD
  400 FORMAT (1X,2G20.8///)
      STOP
      END
      SUBROUTINE COV(NP,NN,LP)
      COMMON/SAMP/S(2048)
      COMMON/PHI/P(2,2)
      COMMON/PHO/PO(2)
C     CALCULATE THE PHI(I,J) OF (3.3)
C     CALCULATE DIAGONAL ELEMENTS, PHI(J,J), OF COVARIANCE MATRIX -
C     ASSIGN TO P(J,J)
      L=LP-1
      NI=NN-NP
      NL=LP-NI
      B=0.
      DO 5 J=NL,L
    5 B=B+S(J)*S(J)
      DO 10 J=1,NP
        K=LP-J
        I=NL-J
        B=B+S(I)*S(I)-S(K)*S(K)
   10 P(J,J)=B
C     CALCULATE REMAINDING PHI(I,J)
      DO 15 KK=1,NP
        B=0.
C       CALCULATE PHI(0,KK) - ASSIGN TO PO(KK)
        DO 20 J=1,NI
          N=LP-J
          M=N-KK
   20   B=B+S(N)*S(M)
        PO(KK)=B
C       CALCULATE PHI(I,J),J-I=KK,1<I<NP-1,2<J<NP - ASSIGN TO P(I,J)
        IF (KK.EQ.NP) GO TO 15
        DO 25 K=1,NP-KK
          I=K
          J=KK+K
          N=LP-K
          M=N-KK
          N1=NL-K
          M1=N1-KK
          B=B+S(N1)*S(M1)-S(N)*S(M)
          P(I,J)=B
   25   P(J,I)=B
C       THE PREVIOUS STATEMENT TAKES ADVANTAGE OF SYMMETRY OF
C       COVARIANCE MATRIX
   15 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE STATS (AM,SD,KL,KU)
C     STATS CALCULATES THE MEAN AND STANDARD DEVIATION OF THE
C     KU-(NP+N) PREDICTED FREQUENCIES F(J)
      COMMON/FREQ/F(2048)
      COMMON/TFRQ/F1
      S1=0.
      S2=0.
      XN=KU-KL+1
      DO 5 J=KL,KU
        S1=S1+F(J)
    5 S2=S2+(F(J)-F1)**2
      AM=S1/XN
      SD=SQRT(S2/XN)
      RETURN
      END
      FUNCTION ARCTA(X,Y)
C     ARCTA CALCULATES RADIAN FREQUENCY DETERMINED BY ROOT X+JY OF
C     (2.3) BY USE OF (4.4)
      DATA PI,HPI/3.14159265,1.57079632/
      IF (X) 1,2,3
    1 ARCTA=ATAN(Y/X)+PI
      RETURN
    2 ARCTA=HPI
      RETURN
    3 ARCTA=ATAN(Y/X)
      RETURN
      END
```

MAXIMUM ENTROPY SPECTRAL

DEMODULATOR INVESTIGATION II

Contract Number:	F30602-75-C-0122
Contractor:	SUNY, Buffalo, New York
Performer:	Dr. Robert Guy Van Meter Dept of Mathematical Sciences State University College Oneonta, New York 13820
Date:	June 14, 1978

TABLE OF CONTENTS

MAXIMUM ENTROPY SPECTRAL DEMODULATOR INVESTIGATION II

		Page
1.	Introduction	II-3
2.	The Model	II-4
3.	Estimation of Model Parameters	II-5
4.	The Analytical Approach	II-6
5.	Computer Simulation	II-6
6.	The Two-pole Case without Noise	II-7

MAXIMUM ENTROPY SPECTRAL DEMODULATOR INVESTIGATION II

1. INTRODUCTION

A transmitted signal, which is masked by noise and, possibly, interference, is assumed to be a sinusoid with Hertz-frequency which varies over some finite set F of positive real numbers. Let s(t) be the value of this continuous-time signal at time t. Of particular interest is the case where #F = 2; that is, where the frequency is switched back and forth between two distinct values.

The object of this study is to evaluate the performance of the Maximum Entropy Method (= MEM) of spectral estimation for short segments of the discrete-time signal which results from sampling s(t) every T seconds. To be more specific, we wish to determine the accuracy of the MEM estimates of the transmitted signal frequencies for the cases (a) received signal = transmitted signal + noise and (b) received signal = transmitted signal + noise + interference.

The ultimate goal of this project is the real-time use of the MEM (if feasible) to identify the frequencies of a transmitted signal, thereby countering the effects of noise and interference (such as that produced by a keyed, slewing, or CW jammer).

In this paper, we obtain a result (Corollary 6.5) which explains the excellent performance of the MEM in a simulation [4] for the simple case of a single-frequency sinusoidal signal with no noise ($s(kT) = A\sin(2\pi fkT + P)$). An effort to resolve the problem in case (a) above ($t(kT) = s(kT) + n(kT)$, where s(kT) is as above and n(kT) is independent, zero-mean, Gaussian noise)

has not been successful. This effort will be continued with the support of Grant # AFOSR 78 - 3614.

2. THE MODEL

We model the sampled signal s(kT) as the output of a linear discrete system (= LDS) [1]; in particular, by the difference equation

$$s(kT) = -\sum_{j=1}^{p} a_j\, s((k-j)T) + u(kT), \tag{2.1}$$

where the a_j are real numbers, T is the sampling period, and u(kT) is the transmitted signal at time kT. As (2.1) enables one to "predict" s(mT) from s((m - 1)T), ... , s((m - p)T), and u(mT), the name "linear prediction" is also associated with the MEM.

There are several other equivalent representations of the LDS given by (2.1) [1; pp. 85 - 86]. For frequency-domain considerations, the representation

$$S(z) = H(z)\, U(z),$$

where S(z) and U(z) are the z-transforms of s(kT) and u(kT) respectively and H(z) is a rational function of z called the "system transfer function," is useful [1; pp. 220 - 282]. The transfer function corresponding to (2.1) is given by

$$H(z) = 1 / (1 + \textstyle\sum_{j=1}^{p} a_j\, z^{-j}).$$

Thus H is an "all-pole" transfer function with p poles, namely, the p solutions of the equation $1 + \sum_{j=1}^{p} a_j\, z^{-j} = 0$ or, equivalently (if $z \neq 0$, as is required by the definition of the z-transform),

$$z^p + a_1 z^{p-1} + \ldots + a_{p-1} z + a_p = 0. \tag{2.2}$$

3. ESTIMATION OF MODEL PARAMETERS

The model parameters $a_1, \ldots, a_p$ in (2.1) are approximated by the usual type of least-squares analysis in the time-domain [2; pp. 563 - 567]. For a given positive integer m, we predict s(mT) to be

$$\overline{s(mT)} \underline{\underline{\text{df}}} - \sum_{j=1}^{p} \overline{a_j}\, s((m-j)T), \tag{3.1}$$

where the $\overline{a_j}$ are chosen so as to minimize the mean square error

$$\sum_{k=m-N}^{m-1} e^2(kT) \qquad (e(kT) \underline{\underline{\text{df}}} s(kT) - \overline{s(kT)})$$

corresponding to the preceding N samples (thereby using the "covariance method" of linear prediction [2; p. 564]). This leads to the system of linear equations

$$\sum_{j=1}^{p} \overline{a_j}(m,N)\, \phi_{ij}(m,N) = -\phi_{0i}(m,N) \qquad (i = 1, 2, \ldots, p), \tag{3.2}$$

where, for all $(i,j) \in \{0, \ldots, p\} \times \{1, \ldots, p\}$

$$\phi_{ij}(m,N) \underline{\underline{\text{df}}} \sum_{k=0}^{N-1} s((m-N+k-i)T)\, s((m-N+k-j)T), \tag{3.3}$$

for determining the $\overline{a_j}$ which yield the prediction of s(mT). From (3.2) and (3.3), we see that N + p consecutive samples $s((m-N-p)T), \ldots, s((m-1)T)$ are needed. Thus if we do not allow negative arguments, s((N + p + 1)T) is the first sample we can predict.

In making the prediction (3.1), we are assuming that u(mT) is completely unknown, which is often the case.

4. THE ANALYTICAL APPROACH

The determination of the frequencies $f_i \in F$ exhibited by the transmitted signal involves (a) calculating the $\phi_{ij}(m, N)$ from (3.3), (b) solving the system of linear equations (3.2) for the $\overline{a_j}$, (c) solving the polynomial equation (2.2) with "$\overline{a_j}$" in place of "a_j", (d) determining the radian-frequency θ_j in the polar form $r_j \exp(\pm i\theta_j)$ of each of the complex conjugate pairs of roots of (2.2), and (e) multiplying θ_j by an appropriate real number to obtain the Hertz-frequency.

The problem of determining a confidence interval for f_i, which requires finding a probability density function, is difficult, even for the simple case $F = \{f_1\}$ with noise. This case, in which a two-pole model ($p = 2$ and the equation (2.2) is quadratic) is appropriate, was considered by the writer in 1977 under the USAF/ASEE Summer Faculty Research Program. Late in this program, this analytical effort was abandoned in favor of a computer simulation. In this paper, we return to the theoretical effort.

5. COMPUTER SIMULATION

In the above-cited simulation, we assumed $F = \{f_1\}$ and the transmitted signal is $A\sin(2\pi f_1 T)$, with $f_1 \in (0, 6000)$ and $T = 1/12000$ (in accordance with The Sampling Theorem [1; p. 291]), ignoring noise of the type cited in [3; pp. 3 - 5] and interference; thus the only noise is "quantization noise" (from use of a simulated analog-to-digital converter (= ADC) in sampling the continuous-time signal) and "roundoff noise" (from performing the MEM calculations with a digital computer). This simulation showed excellent performance of the MEM for sampling with 12 - 16 bit ADCs (which are available commercially) if f_1 is not too close to either 0 or 6000 [4; pp. 10 - 26].

The table below from [4; p. 17] gives the arithmetic mean (= AM) and standard deviation (= SD) of 2043 predicted frequencies for the number of error terms used in the minimization N = 2, 3, ... , 12 with A = 1.5, f_1 = 3250, and simulation of a 16-bit ADC.

N	AM	SD
2	3249.9999	0.045065880
3	3250.0008	0.034125918
4	3249.9995	0.024391433
5	3249.9991	0.021149200
6	3250.0010	0.019371934
7	3249.9993	0.015833400
8	3249.9985	0.013296252
9	3249.9994	0.011714947
10	3249.9999	0.010651756
11	3249.9995	0.0083492643
12	3250.0000	0.0071696392

6. THE TWO-POLE CASE WITHOUT NOISE

In this section, we give some theoretical results which explain the excellent performance of the MEM in the above-cited simulation

Throughout the remainder of this paper, we use R and Z^+ to denote the set of real numbers and the set of positive integers respectively.

6.1. Lemma. For all $B, P \in R$, $k, m, n \in Z^+$,

$$\sin((k+m)B+P)\sin((k+n)B+P) - \sin((k+m+n)B+P)\sin(kB+P)$$

$$= \sin(mB)\sin(nB).$$

Proof. By familiar trigonometric identities, the left-hand side of the above equality is equal to

$$(1/2)(\cos((m-n)B) - \cos((2k+m+n)B+2P))$$

$$-(1/2)(\cos((m+n)B) - \cos((2k+m+n)B+2P))$$

$$= (1/2)(\cos((m-n)B) - \cos((m+n)B)$$

$$= \sin(((m+n+m-n)/2)B)\sin(((m+n-(m-n))/2)B)$$

$$= \sin(mB)\sin(nB).$$

6.2. Corollary. If, for all $k \in Z^+$,

$$(s_k \overset{df}{=}) \quad s(kT) = A\sin(2\pi fkT + P),$$

where $f, P, T \in R$, then, for all $k, m, n \in Z^+$

$$s_{k+m}s_{k+n} - s_{k+m+n}s_k = A^2 \sin(2\pi fmT)\sin(2\pi fnT).$$

In particular, for $m = n = 1$, we have $s_{k+1}^2 - s_k s_{k+2} = A^2 \sin^2(2\pi fT)$.

6.3. Theorem. If $p = 2$, $N \in Z^+ - \{1\}$, $m = N+p+1$, s_k is defined as in Corollary 6.2, $0 \leq P \leq 2\pi$, and $0 \leq f \leq 1/(2T)$, then (a) the system (3.2) is singular if and only if $f = 0$ or $f = 1/(2T)$ and (b) in the non-singular case, its solution is the pair $(\overline{a_1}, \overline{a_2})$ given by $\overline{a_1} = -2\cos(2\pi fT)$ and $\overline{a_2} = 1$.

Proof. (a) If $p = 2$ and $m = N+p+1$, then (from (3.3))

$$\phi_{ij}(m,N) = \sum_{k=0}^{N-1} s_{3+k-i}\, s_{3+k-j}$$

and the system (3.2) is as follows:

$$(s_2^2 + s_3^2 + \ldots + s_{N+1}^2)\overline{a_1} + (s_1 s_2 + s_2 s_3 + \ldots + s_N s_{N+1})\overline{a_2}$$

$$= -(s_2 s_3 + s_3 s_4 + \ldots + s_{N+1} s_{N+2}) \tag{6.1}$$

$$(s_1 s_2 + s_2 s_3 + \ldots + s_N s_{N+1})\overline{a_1} + (s_1^2 + s_2^2 + \ldots + s_N^2)\overline{a_2}$$

$$= -(s_1 s_3 + s_2 s_4 + \ldots + s_N s_{N+2}).$$

Let D be the determinant of the matrix of coefficients in (6.1). Then

$$D = (s_1^2 + \ldots + s_N^2)(s_2^2 + \ldots + s_{N+1}^2) - (s_1 s_2 + s_2 s_3 + \ldots + s_N s_{N+1})^2.$$

It is convenient to arrange the terms of D in a 2N by N array

$$\begin{pmatrix} U \\ \hline V \end{pmatrix}$$

where both U and V are N by N arrays, $u_{ij} \overset{df}{=\!=} s_i^2 s_{j+1}^2$ and $v_{ij} \overset{df}{=\!=} -s_i s_{i+1} s_j s_{j+1}$ for all $(i,j) \in \{1, 2, \ldots, N\}^2$. The main diagonal elements of U cancel with the corresponding elements of V; that is, $u_{ii} + v_{ii} = 0$ for all $i \in \{1, 2, \ldots, N\}$. The remaining terms of D can be grouped as follows:

$$(T_{ij} \overset{df}{=\!=}) \quad u_{ij} + v_{ij} + u_{ji} + v_{ji} \qquad (i = 1, \ldots, N-1;\ j = i+1, \ldots, N).$$

Thus

$$D = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} T_{ij}.$$

It is helpful to replace this sum by one in which we sum along "lines" $j = i + k$, $k = 1, 2, \ldots, N-1$; that is

$$D = \sum_{k=1}^{N-1} \sum_{i=1}^{N-k} T_{i,\,i+k}.$$

Obviously,

$$T_{ij} = s_i^2 s_{j+1}^2 - s_i s_{i+1} s_j s_{j+1} + s_j^2 s_{i+1}^2 - s_j s_{j+1} s_i s_{i+1}$$

$$= (s_j s_{i+1} - s_{j+1} s_i)^2.$$

Hence, by use of Corollary 6.2 (with $k = i$, $m = k$, and $n = 1$), we have

$$T_{i,i+k} = (s_{i+k}s_{i+1} - s_{i+k+1}s_i)^2$$

$$= (A^2 \sin(2\pi fkT) \sin(2\pi fT))^2$$

$$= A^4 \sin^2(2\pi fT) \sin^2(2\pi fkT).$$

Therefore,

$$D = \sum_{k=1}^{N-1} (N-k) A^4 \sin^2(2\pi fT) \sin^2(2\pi fkT)$$

$$= A^4 \sin^2(2\pi fT) \sum_{k=1}^{N-1} (N-k) \sin^2(2\pi fkT).$$

Hence, $D = 0$ if and only if $\sin(2\pi fT) = 0$ or $\sin(2\pi fkT) = 0$ for all $k \in \{1, 2, \ldots, N-1\}$, which is equivalent to $\sin(2\pi fT) = 0$. As $0 \leq f \leq 1/(2T)$, $0 \leq 2\pi fT \leq \pi$ and, therefore, $\sin(2\pi fT) = 0$ if and only if $f = 0$ or $f = 1/(2T)$.

(b) In case $D \neq 0$, Cramer's rule gives the solution $(\overline{a_1}, \overline{a_2})$, where $\overline{a_1} = D^{(1)}/D$, $\overline{a_2} = D^{(2)}/D$,

$$D^{(1)} = -(s_1^2 + s_2^2 + \ldots + s_N^2)(s_2 s_3 + s_3 s_4 + \ldots + s_{N+1} s_{N+2})$$

$$+ (s_1 s_3 + s_2 s_4 + \ldots + s_N s_{N+2})(s_1 s_2 + s_2 s_3 + \ldots + s_N s_{N+1}),$$

and

$$D^{(2)} = -(s_2^2 + s_3^2 + \ldots + s_{N+1}^2)(s_1 s_3 + s_2 s_4 + \ldots + s_N s_{N+2})$$

$$+ (s_2 s_3 + s_3 s_4 + \ldots + s_{N+1} s_{N+2})(s_1 s_2 + s_2 s_3 + \ldots + s_N s_{N+1}).$$

To calculate $D^{(1)}$, we proceed as with D. Let $u_{ij}^{(1)} \overset{df}{=} -s_i^2 s_{j+1} s_{j+2}$, $v_{ij}^{(1)} \overset{df}{=} s_i s_{i+2} s_j s_{j+1}$, and

$$T_{ij}^{(1)} \stackrel{df}{=} u_{ij}^{(1)} + v_{ij}^{(1)} + u_{ji}^{(1)} + v_{ji}^{(1)} \quad (i=1,\ldots,N-1;\ j=i+1,\ldots,N).$$

It is easy to show that

$$T_{ij}^{(1)} = (s_i s_{j+1} - s_j s_{i+1})(s_{i+2} s_j - s_i s_{j+2}).$$

Now, by two uses of Corollary 6.2, we have

$$T_{i,i+k}^{(1)} = (s_i s_{i+k+1} - s_{i+k} s_{i+1})(s_{i+2} s_{i+k} - s_i s_{i+k+2})$$

$$= (-A^2 \sin(2\pi fkT) \sin(2\pi fT))(A^2 \sin(2\pi fkT) \sin(2\pi f2T))$$

$$= -A^4 \sin(2\pi fT) \sin(4\pi fT) \sin^2(2\pi fkT).$$

Thus

$$D^{(1)} = \sum_{k=1}^{N-1} \sum_{i=1}^{N-k} T_{i,i+k}^{(1)}$$

$$= -A^4 \sin(2\pi fT) \sin(4\pi fT) \sum_{k=1}^{N-1} (N-k) \sin^2(2\pi fkT).$$

Finally, let $u_{ij}^{(2)} \stackrel{df}{=} -s_{i+1}^2 s_j s_{j+2}$, $v_{ij}^{(2)} \stackrel{df}{=} s_{i+1} s_{i+2} s_j s_{j+1}$, and

$$T_{ij}^{(2)} \stackrel{df}{=} u_{ij}^{(2)} + v_{ij}^{(2)} + u_{ji}^{(2)} + v_{ji}^{(2)}$$

$$= (s_{i+1} s_j - s_i s_{j+1})(s_{i+2} s_{j+1} - s_{i+1} s_{j+2}).$$

By Corollary 6.2,

$$T_{i,i+k}^{(2)} = (s_{i+1} s_{i+k} - s_i s_{i+k+1})(s_{i+2} s_{i+k+1} - s_{i+1} s_{i+k+2})$$

$$= (s_{i+1} s_{i+k} - s_i s_{i+k+1})(s_{(i+1)+1} s_{(i+1)+k} - s_{i+1} s_{(i+1)+k+1})$$

$$= (A^2 \sin(2\pi fT) \sin(2\pi fkT))\,(A^2 \sin(2\pi fT) \sin(2\pi fkT))$$

$$= A^4 \sin^2(2\pi fT) \sin^2(2\pi fkT)$$

$$= T_{i,\,i+k}.$$

Hence, $D^{(2)} = D$.

Thus,

$$\overline{a_1} = \frac{D^{(1)}}{D} = \frac{-A^4 \sin(2\pi fT) \sin(4\pi fT) \sum_{k=1}^{N-1} (N-k) \sin^2(2\pi fkT)}{A^4 \sin^2(2\pi fT) \sum_{k=1}^{N-1} (N-k) \sin^2(2\pi fkT)}$$

$$= -2\cos(2\pi fT)$$

and

$$\overline{a_2} = \frac{D^{(2)}}{D} = \frac{D}{D} = 1.$$

We should remark that D could have been evaluated easily by use of Lagrange's identity; however, this is not the case for $D^{(1)}$ and $D^{(2)}$.

6.4. Remarks. In case $p = 2$, the minimization of Section 3 must involve at least two error terms; hence, we have assumed $N \neq 1$ in Theorem 6.3. In case $N = 1$, the system (6.1) takes the form

$$s_2^2 \overline{a_1} + s_1 s_2 \overline{a_2} = -s_2 s_3$$

$$s_1 s_2 \overline{a_1} + s_1^2 \overline{a_2} = -s_1 s_3$$

and $D = D^{(1)} = D^{(2)} = 0$. Thus, (6.1) does not have a unique solution.

There is no loss of generality in assuming $m = N + p + 1$ in Theorem 6.3. This is an obvious consequence of Corollary 6.2. For example, if $m = N + p + 2$,

then each of the subscripts in the coefficients of the system (3.2) and in the expressions for D, $D^{(1)}$, and $D^{(2)}$ is one larger than in the case of $m = N + p + 1$; thus, by Corollary 6.2, D, $D^{(1)}$, and $D^{(2)}$ have the same values.

Examination of the cases N = 2 and N = 3 suggested the proof of Theorem 6.3. It may be helpful to give the proof of this theorem in case N = 2. First,

$$D = (s_1^2 + s_2^2)(s_2^2 + s_3^2) - (s_1 s_2 + s_2 s_3)^2$$

$$= + s_1^2 s_2^2 + s_1^2 s_3^2$$

$$+ s_2^2 s_2^2 + s_2^2 s_3^2$$

$$- s_1 s_2 s_1 s_2 - s_1 s_2 s_2 s_3$$

$$- s_2 s_3 s_1 s_2 - s_2 s_3 s_2 s_3$$

$$= s_1^2 s_3^2 + s_2^4 - s_1 s_2^2 s_3 - s_1 s_2^2 s_3$$

$$= (s_2^2 - s_1 s_3)^2$$

$$= (A^2 \sin^2(2\pi fT))^2$$

$$= A^4 \sin^4(2\pi fT).$$

Obviously, D = 0 if and only if $\sin(2\pi fT) = 0$ or if and only if f = 0 or $f = 1/(2T)$. Also,

$$D^{(1)} = -(s_1^2 + s_2^2)(s_2 s_3 + s_3 s_4) + (s_1 s_3 + s_2 s_4)(s_1 s_2 + s_2 s_3)$$

$$= - s_1^2 s_2 s_3 - s_1^2 s_3 s_4$$

$$- s_2^2 s_2 s_3 - s_2^2 s_3 s_4$$

$$+ s_1 s_3 s_1 s_2 + s_1 s_3 s_2 s_3$$

$$+ s_2 s_4 s_1 s_2 + s_2 s_4 s_2 s_3$$

$$= - s_1^2 s_3 s_4 - s_2^3 s_3 + s_1 s_2 s_3^2 + s_1 s_2^2 s_4$$

$$= - (s_2^2 - s_1 s_3)(s_2 s_3 - s_1 s_4)$$

$$= - (A^2 \sin^2(2\pi fT))(A^2 \sin(2\pi fT) \sin(2\pi f2T))$$

$$= - A^4 \sin^3(2\pi fT) \sin(4\pi fT)$$

and

$$D^{(2)} = - (s_2^2 + s_3^2)(s_1 s_3 + s_2 s_4) + (s_1 s_2 + s_2 s_3)(s_2 s_3 + s_3 s_4)$$

$$= - s_2^2 s_1 s_3 - s_2^2 s_2 s_4$$

$$- s_3^2 s_1 s_3 - s_3^2 s_2 s_4$$

$$+ s_1 s_2 s_2 s_3 + s_1 s_2 s_3 s_4$$

$$+ s_2 s_3 s_2 s_3 + s_2 s_3 s_3 s_4$$

$$= - s_2^3 s_4 - s_1 s_3^3 + s_1 s_2 s_3 s_4 + s_2^2 s_3^2$$

$$= (s_2^2 - s_1 s_3)(s_3^2 - s_2 s_4)$$

$$= (A^2 \sin^2(2\pi fT))\,(A^2 \sin^2(2\pi fT))$$

$$= A^4 \sin^4(2\pi fT)$$

$$= D.$$

Hence, if $D \neq 0$,

$$\overline{a_1} = \frac{D^{(1)}}{D} = \frac{-A^4 \sin^3(2\pi fT) \sin(4\pi fT)}{A^4 \sin^4(2\pi fT)} = -2\cos(2\pi fT)$$

and

$$\overline{a_2} = \frac{D^{(2)}}{D} = \frac{D}{D} = 1.$$

6.5. Corollary. If the hypotheses of Theorem 6.3 hold, then

$$\overline{s((N+p+1)T)} = s((N+p+1)T).$$

Proof. From (3.1), Theorem 6.3 (b), and familiar trigonometric identities, we have

$$\overline{s((N+p+1)T)} = -\overline{a_1}\, s((N+p)T) - \overline{a_2}\, s((N+p-1)T)$$

$$= -(-2\cos(2\pi fT))\,(A\sin(2\pi f(N+p)T+P))$$

$$-(1)\,(A\sin(2\pi f(N+p-1)T+P))$$

$$= A(\sin(2\pi fT + 2\pi f(N+p)T + P) - \sin(2\pi fT - 2\pi f(N+p)T - P))$$

$$- A\sin(2\pi f(N+p-1)T+P)$$

$$= A\sin(2\pi f(N+p+1)T+P) + A\sin(2\pi f(N+p-1)T+P)$$

$$- A\sin(2\pi f(N+p-1)T+P)$$

$$= A\sin(2\pi f(N+p+1)T+P) = s((N+p+1)T).$$

6.6. Remark. Corollary 6.5 explains the remarkably good simulation results cited in Section 5 for the case of no noise. We assumed in Theorem 6.3 and Corollary 6.5 that the first $N+p$ $(= N+2)$ samples of the received signal are actually the first $N+p$ samples of the transmitted signal and showed that the minimization process of linear prediction by the covariance method leads to values of $\overline{a_1}$ and $\overline{a_2}$ such that the predicted signal sample $\overline{s((N+p+1)T)}$ is equal to the transmitted signal sample $s((N+p+1)T)$. Thus there is no model error, according to this corollary; all the error is due to the quantization and the computational process.

6.7. Theorem. Suppose the hypotheses of Theorem 6.3 hold with $0 < f < 1/(2T)$ (so that the system (3.2) is non-singular).

(a) The zeros of $z^2 + \overline{a_1} z + \overline{a_2}$ (that is, the poles of $H(z)$), with $\overline{a_1} = -2\cos(2\pi fT)$ and $\overline{a_2} = 1$ as in Theorem 3.2 (b), are

$$z_1 \stackrel{df}{=} \cos(2\pi fT) - \underline{i}\sin(2\pi fT) = \exp(-2\pi fT\underline{i})$$

and

$$z_2 \stackrel{df}{=} \cos(2\pi fT) + \underline{i}\sin(2\pi fT) = \exp(2\pi fT\underline{i}).$$

(b) These zeros, z_1 and z_2, have magnitude $r = 1$ (that is, can be represented by points on the unit circle) or, in terms of $\overline{a_1}$ and $\overline{a_2}$,

$$r = \sqrt{a_2}.$$

(c) If $T = 1/12000$ (so that $0 < f < 6000$), then

$$(6.2) \qquad f = \begin{cases} (6000/\pi)\tan^{-1}(\mathrm{Im}(z_2)/\mathrm{Re}(z_2)) & (\text{if } 0 < 2\pi fT < \pi/2), \\ 3000 & (\text{if } 2\pi fT = \pi/2), \\ (6000/\pi)(\tan^{-1}(\mathrm{Im}(z_2)/\mathrm{Re}(z_2)) + \pi) & (\text{if } \pi/2 < 2\pi fT < \pi) \end{cases}$$

or, in terms of $\overline{a_1}$ and $\overline{a_2}$,

$$(6.3) \qquad f = \begin{cases} (6000/\pi)\tan^{-1}(-\sqrt{4\overline{a_2} - \overline{a_1}^2}\,/\,\overline{a_1}) & (\text{if } \overline{a_1} < 0), \\ 3000 & (\text{if } \overline{a_1} = 0), \\ (6000/\pi)(\tan^{-1}(-\sqrt{4\overline{a_2} - \overline{a_1}^2}\,/\,\overline{a_1}) + \pi) & (\text{if } \overline{a_1} > 0). \end{cases}$$

Proof. (a) From Theorem 6.3 (b) and the quadratic formula, the zeros of $z^2 + \overline{a_1}\,z + \overline{a_2}$ are

$$(-\overline{a_1}\,/\,2) \pm (\sqrt{\overline{a_1}^2 - 4\overline{a_2}}\,/\,2)$$

or, as $\overline{a_1}^2 - 4\overline{a_2} = 4\cos^2(2\pi fT) - 4 \leq 0$,

$$(-\overline{a_1}\,/\,2) \pm \underline{i}(\sqrt{4\overline{a_2} - \overline{a_1}^2}\,/\,2)$$

$$= \cos(2\pi fT) \pm \underline{i}\sin(2\pi fT).$$

(b) Obviously,

$$|z_1| = |z_2| = \sqrt{(-\overline{a_1}\,/\,2)^2 + (\sqrt{4\overline{a_2} - \overline{a_1}^2}\,/\,2)^2} = \sqrt{a_2} = \sqrt{1} = 1.$$

(c) Let $X \underline{\text{df}} \text{Re}(z_2) = \cos(2\pi fT)$ and $Y \underline{\text{df}} \text{Im}(z_2) = \sin(2\pi fT)$.

Case 1 ($0 < 2\pi fT < \pi/2$). In this case, $X > 0$, $Y > 0$, and $Y/X = \tan(2\pi fT) > 0$; thus, $0 < \tan^{-1}(Y/X) < \pi/2$ and, hence, $2\pi fT = \tan^{-1}(Y/X)$. As $T = 1/12000$,

$$f = (6000/\pi)\tan^{-1}(Y/X).$$

Case 2 ($2\pi fT = \pi/2$). As $T = 1/12000$, $f = 3000$ follows immediately from the case assumption.

Case 3 ($\pi/2 < 2\pi fT < \pi$). In this case, $X < 0$, $Y > 0$, and $Y/X =$

$\tan(2\pi fT) < 0$; thus, $-\pi/2 < \tan^{-1}(Y/X) < 0$ and, hence, $2\pi fT = \tan^{-1}(Y/X) + \pi$. As $T = 1/12000$,

$$f = (6000/\pi)(\tan^{-1}(Y/X) + \pi).$$

Finally, (6.3) results immediately from (6.2), as $\mathrm{Re}(z_2) = -\overline{a_1}/2$ and $\mathrm{Im}(z_2) = \sqrt{4\overline{a_2} - \overline{a_1}^2}/2$.

6.8. Remarks. Theorem 6.7 shows that the frequency of a sinusoidal signal without noise of any kind (including quantization noise and roundoff noise) can be determined (by use of (6.2) or (6.3)) from a pole of the all-pole transfer function associated with the LDS (2.1) with $p = 2$. In subsequent work, we will use (6.3) to approximate f in the case of a sinusoidal signal with Gaussian noise.

The simulation results given in Section 5 show that the standard deviation of the predicted values of f decreases as N increases. This might seem strange in view of the fact that $\overline{a_1}$ ($= -2\cos(2\pi fT)$) and $\overline{a_2}$ ($= 1$) are independent of N. However, the computer program used calls for the computation of D, $D^{(1)}$, and $D^{(2)}$ which do depend on N; that is, no cancellation is done in the computer computation.

REFERENCES

[1] Cadzow, J. A., Discrete-time Systems, Prentice-Hall, Englewood Cliffs, New Jersey, 1973.

[2] Makhoul, J., "Linear Prediction: A Tutorial Review," Proceedings of the IEEE, Vol. 63, No. 4, April 1975.

[3] Robinson, E. A., Statistical Communication and Detection, Hafner, New York, 1967.

[4] Van Meter, R. G., "Maximum Entropy Spectral Demodulator Investigation," Report No. 31, 1977 USAF/ASEE Summer Faculty Research Program Research Reports, Vol. 2, AFOSR-TR-78-0349, September 1977.

U. S. AIR FORCE GRANTS

FOR BASIC RESEARCH

sponsored by

THE AIR FORCE OFFICE OF SCIENTIFIC RESEARCH

FINAL REPORT

MAXIMUM ENTROPY

SPECTRAL DEMODULATOR INVESTIGATION III

Grant Number: AFOSR-78-3614

Grantee: The Research Foundation of the State University of New York

Project Director: Dr. Robert G. Van Meter
Professor of Mathematical Sciences
State University College
Oneonta, New York 13820

Date: September 29, 1978

Table of Contents

		Page
1.	Introduction	III-3
2.	The Model	III-4
3.	Estimation of Model Parameters	III-5
4.	The Analytical Approach	III-6
5.	Simulation in the Two Pole Case Without Noise	III-6
6.	The Two Pole Case With Noise	III-7
7.	Simulation In the Signal and Interference Case	III-17
8.	Conclusions	III-24
	References	III-24
	Appendix 1 Computer Program Variables	III-25
	Appendix 2 Computer Program MESD With Noise	III-27
	Appendix 3 Computer Program MESD With Interference	III-31

Tables

1.	Mean and Standard Deviation of Predicted Frequencies for varying signal to noise ratios.	III-16
2.	Mean and Standard Deviation of Predicted Frequencies for varying signal to interference ratios and varying numbers of bits of precision.	III-21

MAXIMUM ENTROPY

SPECTRAL DEMODULATOR INVESTIGATION III

1. INTRODUCTION

A transmitted signal, which is masked by noise and, possibly, interference, is assumed to be a sinusoid with Hertz-frequency which varies over some finite set F of positive real numbers. Let s(t) be the value of this continuous-time signal at time t. Of particular interest is the case where #F = 2; that is, where the frequency is switched back and forth between two distinct values.

The object of this study is to evaluate the performance of the Maximum Entropy Method (= MEM) of spectral estimation for short segments of the discrete-time signal which results from sampling s(t) every T seconds. To be more specific, we wish to determine the accuracy of the MEM estimates of the transmitted signal frequencies for the cases (a) received signal = transmitted signal + noise and (b) received signal = transmitted signal + noise + interference.

The ultimate goal of this project is the real-time use of the MEM (if feasible) to identify the frequencies of a transmitted signal, thereby countering the effects of noise and interference (such as that produced by a keyed, slewing, or CW jammer).

The problems stated in the second paragraph appear to be theoretically intractable. We give some partial theoretical results and discouraging simulation results for the case (a). For case (b) without noise, we give simulation results which point to the need for analog-to-digital conversion of greater precision than is possible with currently available equipment.

2. THE MODEL

We model the sampled signal s(kT) as the output of a linear discrete system (= LDS) [1]; in particular, by the difference equation

$$s(kT) = -\sum_{j=1}^{p} a_j\, s((k-j)T) + u(kT), \tag{2.1}$$

where the a_j are real numbers, T is the sampling period, and u(kT) is the transmitted signal at time kT. As (2.1) enables one to "predict" s(mT) from s((m - 1)T), ... , s((m - p)T), and u(mt), the name "linear prediction" is also associated with the MEM.

There are several other equivalent representations of the LDS given by (2.1) [1; pp. 85 - 86]. For frequency-domain considerations, the representation

$$S(z) = H(z)\, U(z),$$

where S(z) and U(z) are the z-transforms of s(kT) and u(kT) respectively and H(z) is a rational function of z called the "system transfer function," is useful [1; pp. 220 - 282]. The transfer function corresponding to (2.1) is given by

$$H(z) = 1 / (1 + \sum_{j=1}^{p} a_j\, z^{-j}).$$

Thus H is an "all-pole" transfer function with p poles, namely, the p solutions of the equation $1 + \sum_{j=1}^{p} a_j\, z^{-j} = 0$ or, equivalently (if $z \neq 0$, as is required by the definition of the z-transform),

$$z^p + a_1\, z^{p-1} + \ldots + a_{p-1}\, z + a_p = 0. \tag{2.2}$$

3. ESTIMATION OF MODEL PARAMETERS

The model parameters $a_1, \ldots, a_p$ in (2.1) are approximated by the usual type of least-squares analysis in the time-domain [2; pp. 563 - 567]. For a given positive integer m, we predict s(mT) to be

$$\overline{s(mT)} \stackrel{df}{=} -\sum_{j=1}^{p} \overline{a_j}\, s((m-j)T), \tag{3.1}$$

where the $\overline{a_j}$ are chosen so as to minimize

$$\sum_{k=m-N}^{m-1} e^2(kT) \qquad (e(kT) \stackrel{df}{=} s(kT) - \overline{s(kT)})$$

corresponding to the preceding N samples (thereby using the "covariance method" of linear prediction [2; p. 564]). This leads to the system of linear equations

$$\sum_{j=1}^{p} \overline{a_j}(m,N)\, \phi_{ij}(m,N) = -\phi_{0i}(m,N) \qquad (i = 1, 2, \ldots, p), \tag{3.2}$$

where, for all $(i,j) \in \{1, \ldots, p\}^2$,

$$\phi_{ij}(m,N) \stackrel{df}{=} \sum_{k=0}^{N-1} s((m-N+k-i)T)\, s((m-N+k-j)T), \tag{3.3}$$

for determining the $\overline{a_j}$ which yield the prediction $\overline{s(mT)}$. From (3.2) and (3.3), we see that N + p consecutive samples $s((m-N-p)T), \ldots, s((m-1)T)$ are needed. Thus if we do not allow negative arguments, $s((N+p+1)T)$ is the first sample we can predict.

In making the prediction (3.1), we are assuming that u(mT) is completely unknown, which is often the case.

4. THE ANALYTICAL APPROACH

The determination of the frequencies $f_i \in F$ exhibited by the transmitted signal involves (a) calculating the $\phi_{ij}(m, N)$ from (3.3), (b) solving the system of linear equations (3.2) for the $\overline{a_j}$, (c) solving the polynomial equation (2.2) with "$\overline{a_j}$" in place of "a_j", (d) determining the radian-frequency θ_j in the polar form $r_j \exp(\pm i\theta_j)$ of each of the complex conjugate pairs of roots of (2.2), and (e) multiplying θ_j by an appropriate real number to obtain the Hertz-frequency.

The problem of determining a confidence interval for the frequency of the transmitted signal at a given instant and for the frequency of the interference (if any), which requires finding probability density functions, is difficult, even for the simple case $F = \{f_1\}$ with noise. This case, in which a two-pole model ($p = 2$ and the equation (2.2) is quadratic) is appropriate, was considered by the writer in 1977 in the USAF/ASEE Summer Faculty Research Program. The analytical approach was abandoned in favor of a computer simulation. In this paper (Section 6), we return to this theoretical effort.

5. SIMULATION IN THE TWO-POLE CASE WITHOUT NOISE

In the above-cited simulation, we assumed $F = \{f_1\}$ and the transmitted signal is $A\sin(2\pi f_1 T)$, with $f_1 \in (0, 6000)$ and $T = 1/12000$ (in accordance with The Sampling Theorem [1; p. 291]), ignoring noise of the type cited in [4; pp. 3 - 5] and interference; thus the only noise is "quantization noise" (from use of a simulated analog-to-digital converter (= ADC) in sampling the continuous-time signal) and "roundoff noise" (from performing the MEM calculations with a digital computer). This simulation showed excellent performance of the MEM for sampling with 12 - 16 bit ADCs (which are available commercially) if f_1 is not too close to either 0 or 6000 [5; pp. 10 - 26].

The table below from [5; p. 17] gives the arithmetic mean (= AM) and standard deviation (= SD) of 2043 predicted frequencies for the number of error terms used in the minimization $N = 2, 3, \ldots, 12$ with $A = 1.5$, $f_1 = 3250$, and simulation of a 16-bit ADC.

N	AM	SD
2	3249.9999	0.045065880
3	3250.0008	0.034125918
4	3249.9995	0.024391433
5	3249.9991	0.021149200
6	3250.0010	0.019371934
7	3249.9993	0.015833400
8	3249.9985	0.013296252
9	3249.9994	0.011714947
10	3249.9999	0.010651756
11	3249.9995	0.008349264
12	3250.0000	0.007169639

6. THE TWO-POLE CASE WITH NOISE

In this section, we consider the case in which the number of poles $p = 2$ and

$$(s_k \underline{\underline{df}}) \quad s(kT) = t(kT) + n(kT),$$

where $t(kT) = A\sin(2\pi fT + P)$ and $n(kT)$ is independent, zero-mean, Gaussian noise.

We assume for simplicity that $N = 2$, in which case the system (3.2) is

$$\begin{aligned} (s_2^2 + s_3^2)\ \overline{a_1} + (s_1 s_2 + s_2 s_3)\,\overline{a_2} &= -(s_2 s_3 + s_3 s_4), \\ (s_1 s_2 + s_2 s_3)\,\overline{a_1} + (s_1^2 + s_2^2)\ \overline{a_2} &= -(s_1 s_3 + s_2 s_4). \end{aligned} \tag{6.1}$$

As was shown in [6; p. 6, Theorem 6.3], this system of equations is singular if and only if $f = 0$ or $f = 1/(2T)$, in case we restrict f to $[0, 1/(2T)]$.

In the non-singular case, its solution is the ordered pair $(\overline{a_1}, \overline{a_2})$ given by

$$(6.2) \qquad \overline{a_1} = \frac{-(s_1^2 + s_2^2)(s_2 s_3 + s_3 s_4) + (s_1 s_3 + s_2 s_4)(s_1 s_2 + s_2 s_3)}{(s_1^2 + s_2^2)(s_2^2 + s_3^2) - (s_1 s_2 + s_2 s_3)^2}$$

$$= (s_2^2 - s_1 s_3)(s_1 s_4 - s_2 s_3) / (s_2^2 - s_1 s_3)^2$$

$$= (s_1 s_4 - s_2 s_3) / (s_2^2 - s_1 s_3)$$

and

$$(6.3) \qquad \overline{a_2} = \frac{-(s_2^2 + s_3^2)(s_1 s_3 + s_2 s_4) + (s_1 s_2 + s_2 s_3)(s_2 s_3 + s_3 s_4)}{(s_1^2 + s_2^2)(s_2^2 + s_3^2) - (s_1 s_2 + s_2 s_3)^2}$$

$$= (s_2^2 - s_1 s_3)(s_3^2 - s_2 s_4) / (s_2^2 - s_1 s_3)^2$$

$$= (s_3^2 - s_2 s_4) / (s_2^2 - s_1 s_3).$$

The problem of determining the probability density functions of the random variables $\overline{a_1}$ and $\overline{a_2}$ is intractable according to my colleague T. S. Bolis, an expert in probability theory. The numerators and (common) denominator in the expressions for $\overline{a_1}$ and $\overline{a_2}$ have distributions which look "something like" non-central chi-square distributions. However, Bolis has shown (by use of the characteristic function - a complex variable analog of the moment generating function) that the numerator and denominator are dependent for both $\overline{a_1}$ and $\overline{a_2}$; thus we cannot easily calculate bounds on $\overline{a_1}$ and $\overline{a_2}$ (through bounding the numerators and denominator - which requires independence).

As shown in [6; pp. 14 - 16, Theorem 6.7], for the two-pole case without noise ($s(kT) = A\sin(2\pi fT + P)$), the frequency, f, of the transmitted signal

is given by

$$(6.4)\qquad f = \begin{cases} (6000/\pi)\tan^{-1}(-\sqrt{4\bar{a}_2 - \bar{a}_1^2}\,/\,\bar{a}_1) & (\text{if } \bar{a}_1 < 0), \\ 3000 & (\text{if } \bar{a}_1 = 0), \\ (6000/\pi)\,(\tan^{-1}(-\sqrt{4\bar{a}_2 - \bar{a}_1^2}\,/\,\bar{a}_1) + \pi) & (\text{if } \bar{a}_1 > 0). \end{cases}$$

We also use (6.4) to "predict" f in the two-pole case with noise. If we could bound $\bar{a}_1$ and $\bar{a}_2$, we could try to get bounds on f as given by (6.4).

We can, however, obtain the expected value and variance of the numerator and denominator in the expressions for both $\bar{a}_1$ and $\bar{a}_2$ (Theorem 6.4).

6.1. Lemma. For all $B, P \in R$, $k, m, n \in Z^+$,

$$\sin((k+m)B+P)\sin((k+n)B+P) - \sin((k+m+n)B+P)\sin(kB+P)$$
$$= \sin(mB)\sin(nB).$$

Proof. By familiar trigonometric identities, the left-hand side of the above equality is equal to

$$(1/2)\,(\cos((m-n)B) - \cos((2k+m+n)B+2P))$$
$$-\,(1/2)\,(\cos((m+n)B) - \cos((2k+m+n)B+2P))$$
$$= (1/2)\,(\cos((m-n)B) - \cos((m+n)B)$$
$$= \sin(((m+n+m-n)/2)B)\sin(((m+n-(m-n))/2)B)$$
$$= \sin(mB)\sin(nB).$$

6.2. Corollary. If, for all $k \in Z^+$,

$$(s_k \overset{df}{=})\quad s(kT) = A\sin(2\pi fkT + P),$$

where $f, P, T \in R$, then, for all $k, m, n \in Z^+$,

$$s_{k+m}s_{k+n} - s_{k+m+n}s_k = A^2 \sin(2\pi fmT)\sin(2\pi fnT).$$

In particular, for $m = n = 1$, we have $s_{k+1}^2 - s_k s_{k+2} = A^2 \sin^2(2\pi fT)$.

6.3. Lemma. If X is a normally distributed random variable, $E(X) = 0$, and $\mathrm{Var}(X) = \sigma^2$, then, for all $n \in Z^+$,

(a) $E(X^{2n+1}) = 0$

and

(b) $E(X^{2n}) = \sigma^{2n} \prod_{i=1}^{n} (2i - 1)$.

Proof. This is a well-known result; (a) follows immediately from the fact that if f is an odd function, then $\int_{-\infty}^{+\infty} f(x)\,dx = 0$ and (b) follows readily from

$$\int_{-\infty}^{+\infty} x^{2n} e^{-\alpha x^2}\,dx = (2\alpha)^{-n}(\pi/\alpha)^{1/2} \prod_{i=1}^{n} (2i - 1).$$

6.4. Theorem. If, for all $k \in Z^+$, $s_k = t_k + n_k$, where

$$t_k \overset{\text{df}}{=\!=} t(kT) = A\sin(2\pi fkT + P),$$

$A, P, T \in R$, $0 \leq P \leq 2\pi$, $0 < f < 1/(2T)$, and $n_k \overset{\text{df}}{=\!=} n(kT)$ is independent, zero-mean, Gaussian noise (n_k has normal distribution with $E(n_k) = 0$, $\mathrm{Var}(n_k) = \sigma^2$, and $E(n_i n_j) = E(n_i)E(n_j) = 0$ if $i \neq j$), then the following hold:

(a) for all $k \in Z^+$, s_k has normal distribution with $E(s_k) = t_k$ and $\mathrm{Var}(s_k) = \sigma^2$;

(b) if $p = N = 2$, then the solution (a_1, a_2) of the (6.1) is given by (6.2) and (6.3) and

$$E(s_1 s_4 - s_2 s_3) = -A^2 \sin(2\pi fT)\sin(4\pi fT),$$

$$\mathrm{Var}(s_1 s_4 - s_2 s_3) = 2\sigma^4 + A^2\sigma^2 \sum_{k=1}^{4} \sin^2(2\pi fkT + P),$$

$$E(s_3^2 - s_2 s_4) = \sigma^2 + A^2 \sin^2(2\pi fT),$$

$$\mathrm{Var}(s_3^2 - s_2 s_4) = 3\sigma^4 + A^2\sigma^2 [\sin^2(4\pi fT + P) + 4\sin^2(6\pi fT + P) + \sin^2(8\pi fT + P)],$$

$$E(s_2^2 - s_1 s_3) = \sigma^2 + A^2 \sin^2(2\pi fT),$$

and

$$\mathrm{Var}(s_2^2 - s_1 s_3) = 3\sigma^4 + A^2\sigma^2 [\sin^2(2\pi fT + P) + 4\sin^2(4\pi fT + P) + \sin^2(6\pi fT + P)].$$

Proof. The proof of (a) is trivial. We have already shown, at the beginning of this section that (6.2) and (6.3) give the solution of (6.1). We now complete the proof of (b).

Let NUM1 $\underline{\underline{\mathrm{df}}}$ $s_1 s_4 - s_2 s_3$, NUM2 $\underline{\underline{\mathrm{df}}}$ $s_3^2 - s_2 s_4$, and DEN $\underline{\underline{\mathrm{df}}}$ $s_2^2 - s_1 s_3$. Now,

$$E(\mathrm{NUM1}) = E((t_1 + n_1)(t_4 + n_4) - (t_2 + n_2)(t_3 + n_3))$$

$$= E(t_1 t_4 - t_2 t_3 + t_1 n_4 + t_4 n_1 + n_1 n_4 - t_2 n_3 - t_3 n_2 - n_2 n_3)$$

$$= t_1 t_4 - t_2 t_3 + t_1 E(n_4) + t_4 E(n_1) + E(n_1 n_4) - t_2 E(n_3)$$

$$- t_3 E(n_2) - E(n_2 n_3).$$

By hypothesis, the expectations involving the n_i are zero; thus

$$E(\mathrm{NUM1}) = t_1 t_4 - t_2 t_3.$$

Hence, by use of Corollary 6.2 (with k = m = 1 and n = 2), we have

$$E(\mathrm{NUM1}) = -A^2 \sin(2\pi fT) \sin(4\pi fT).$$

We can view

$$\mathrm{NUM1} - E(\mathrm{NUM1}) \quad (= s_1 s_4 - s_2 s_3 - (t_1 t_4 - t_2 t_3))$$

as a function of (s_1, s_2, s_3, s_4) and expand in a Taylor series about (t_1, t_2, t_3, t_4) to obtain

$$\begin{aligned}\text{NUM1} - E(\text{NUM1}) &= t_4 (s_1 - t_1) - t_3 (s_2 - t_2) - t_2 (s_3 - t_3) \\ &\quad + t_1 (s_4 - t_4) + (s_1 - t_1)(s_4 - t_4) \\ &\quad - (s_2 - t_2)(s_3 - t_3) \\ &= t_4 n_1 - t_3 n_2 - t_2 n_3 + t_1 n_4 + n_1 n_4 - n_2 n_3.\end{aligned}$$

Hence,

$$\begin{aligned}\text{Var}(\text{NUM1}) &= E((\text{NUM1} - E(\text{NUM1}))^2) \\ &= E(t_4^2 n_1^2 + t_3^2 n_2^2 + t_2^2 n_3^2 + t_1^2 n_4^2 + n_1^2 n_4^2 + n_2^2 n_3^2 \\ &\quad - 2 t_3 t_4 n_1 n_2 - 2 t_2 t_4 n_1 n_3 + 2 t_1 t_4 n_1 n_4 + 2 t_4 n_1^2 n_4 \\ &\quad - 2 t_4 n_1 n_2 n_3 + 2 t_2 t_3 n_2 n_3 - 2 t_1 t_3 n_2 n_4 - 2 t_3 n_1 n_2 n_4 \\ &\quad + 2 t_3 n_2^2 n_3 - 2 t_1 t_2 n_3 n_4 - 2 t_2 n_1 n_3 n_4 + 2 t_2 n_2 n_3^2 \\ &\quad + 2 t_1 n_1 n_4^2 - 2 t_1 n_2 n_3 n_4 - 2 n_1 n_2 n_3 n_4).\end{aligned}$$

By use of properties of expectation, Lemma 6.3, and hypotheses, we have

$$\begin{aligned}\text{Var}(\text{NUM1}) &= t_4^2 E(n_1^2) + t_3^2 E(n_2^2) + t_2^2 E(n_3^2) \\ &\quad + t_1^2 E(n_4^2) + E(n_1^2 n_4^2) + E(n_2^2 n_3^2) \\ &= \sigma^2 (t_1^2 + t_2^2 + t_3^2 + t_4^2) + 2\sigma^4 \\ &= 2\sigma^4 + A^2 \sigma^2 \sum_{k=1}^{4} \sin^2(2\pi f k T + P).\end{aligned}$$

Next,

$$E(NUM2) = E((t_3 + n_3)^2 - (t_2 + n_2)(t_4 + n_4))$$

$$= E(t_3^2 - t_2 t_4 + 2 t_3 n_3 - t_2 n_4 - t_4 n_2 - n_2 n_4 + n_3^2)$$

$$= t_3^2 - t_2 t_4 + 2 t_3 E(n_3) - t_2 E(n_4) - t_4 E(n_2) - E(n_2 n_4) + E(n_3^2)$$

$$= t_3^2 - t_2 t_4 + \sigma^2.$$

Thus, by Corollary 6.2,

$$E(NUM2) = \sigma^2 + A^2 \sin^2(2\pi fT).$$

We can view

$$NUM2 - E(NUM2) \quad (= s_3^2 - s_2 s_4 - (t_3^2 - t_2 t_4 + \sigma^2))$$

as a function of (s_2, s_3, s_4) and expand in a Taylor series about (t_2, t_3, t_4) to obtain

$$NUM2 - E(NUM2) = -\sigma^2 - t_4 (s_2 - t_2) + 2 t_3 (s_3 - t_3) - t_2 (s_4 - t_4)$$

$$- (s_2 - t_2)(s_4 - t_4) + (s_3 - t_3)^2$$

$$= -\sigma^2 - t_4 n_2 + 2 t_3 n_3 - t_2 n_4 - n_2 n_4 + n_3^2.$$

Proceeding as in the calculation of Var(NUM1), we can show that

$$Var(NUM2) = \sigma^4 + t_4^2 E(n_2^2) + 4 t_3^2 E(n_3^2) + t_2^2 E(n_4^2) + E(n_2^2 n_4^2)$$

$$+ E(n_3^4) - 2\sigma^2 E(n_3^2)$$

$$= \sigma^4 + \sigma^2 (t_4^2 + 4 t_3^2 + t_2^2) + \sigma^4 + 3\sigma^4 - 2\sigma^4$$

$$= 3\sigma^4 + \sigma^2 (t_4^2 + 4 t_3^2 + t_2^2)$$

$$= 3\sigma^4 + A^2\sigma^2 [\sin^2(4\pi fT + P) + 4\sin^2(6\pi fT + P) + \sin^2(8\pi fT + P)].$$

Similarly, we can obtain expressions for E(DEN) and Var(DEN) which are identical and almost identical, respectively, to those for E(NUM2) and Var(NUM2). (Note that, by Corollary 6.2, $t_3^2 - t_2 t_4 = t_2^2 - t_1 t_3$.)

As a check on the reasonableness of the expectations in Theorem 6.4 (b), let us consider some consequences of <u>assuming</u> that NUM1, NUM2, and DEN actually take their expectations as values so that

$$\overline{a_1}(\sigma) = (-A^2 \sin(2\pi fT) \sin(4\pi fT)) / (\sigma^2 + A^2 \sin^2(2\pi fT))$$

and

$$\overline{a_2}(\sigma) = (\sigma^2 + A^2 \sin^2(2\pi fT)) / (\sigma^2 + A^2 \sin^2(2\pi fT)) = 1.$$

(Note: We are not claiming that $\overline{a_1}$ = E(NUM1) / E(DEN), nor that $E(\overline{a_1})$ = E(NUM1) / E(DEN).) One can readily show that

$$\overline{a_1}(0) = -2\cos(2\pi fT) \quad \text{and} \quad \overline{a_2}(0) = 1,$$

which we have shown [5; p. 6, Theorem 6.3 (b)] gives the solution of the system (6.1). Thus, from (6.4), we get $\overline{f} = f$. Also,

$$\lim_{\sigma \to +\infty} \overline{a_1}(\sigma) = 0 \quad \text{and} \quad \lim_{\sigma \to +\infty} \overline{a_2}(\sigma) = 1.$$

Thus, in this limiting case, we get $\overline{f} = 3000$ from (6.4). (In this case, the equation (2.2) is $z^2 + 1 = 0$ which has solutions $\pm \underline{i}$ yielding $\overline{\theta} = \pi / 2$ and $\overline{f} = (6000 / \pi)(\pi / 2) = 3000$.) Therefore, under the above assumptions, we get the same result, $\overline{f} = f$, as in the (deterministic) all signal - no noise case if $\sigma = 0$ and we get $\overline{f} = 3000$, which is midway along the frequency band (0, 6000) (and obviously minimizes the error $f - \overline{f}$), in the all noise - no signal case σ "=" $+\infty$.

The following table gives the expectations, variances and standard deviations from Theorem 6.4 (b) for f = 3250, A = 1, and σ = 0.01, 0.1, and 1.0.

	σ		
	1.0	0.1	0.01
E(NUM1)	0.2566	0.2566	0.2566
Var(NUM1)	4.1535	0.02174	0.0002154
Std(NUM1)	2.0380	0.1473	0.01468
E(NUM2)	1.9830	0.9930	0.9831
Var(NUM2)	6.7312	0.03761	0.0003732
Std(NUM2)	2.5945	0.1939	0.01932
E(DEN)	1.9830	0.9930	0.9831
Var(den)	3.1045	0.02114	0.0002105
Std(DEN)	1.7619	0.1451	0.01451

Some experimenting with reasonable departures of NUM1, NUM2, and DEN from their means in case σ = 1 will show that the predicted value of f, $\bar{f}$, given by (6.4), can depart substantially from 3250.

Table 1 gives the results of a simulation of the signal and Gaussian noise case. The variables AM and SD respectively are the arithmetic mean and standard deviation of 100 predicted values, $\bar{f}$, of f. The simulation program is given as Appendix 2.

We use a procedure given by Knuth [2; p. 104] for generating random numbers with distribution N(0, 1); then, numbers with any normal distribution can be generated by a simple linear transformation (if X is N(0, 1), then Y $\underline{\underline{df}}$ $\mu + \sigma X$ is $N(\mu,\sigma^2)$). This procedure is given below.

Until satisfied, do:

(1) Generate two uniformly distributed random numbers r_1 and r_2.

(2) Set $V_1 = 2r_1 - 1$ and set $V_2 = 2r_2 - 1$.

(3) Set $S = V_1^2 + V_2^2$.

(4) If $S < 1$, set $x_1 = V_1 (-2 \ln (S) / S)^{\frac{1}{2}}$ and set $x_2 = V_2 (-2 \ln (S) / S)^{\frac{1}{2}}$; otherwise, go to (1).

As not all pairs of uniformly distributed random numbers lead to a pair of normally distributed random numbers, an excess of such pairs must be generated. The uniformly distributed random numbers were obtained from use of the Honeywell library function RANDT.

TABLE 1

Arithmetic mean AM and standard deviation SD of 100 predicted values f of the Hertz-frequency f (= 3250) of the transmitted signal in the signal and Gaussian noise case for the standard deviation of the noise σ = 0.001, 0.01, 0.05, 0.1, and 0.2 with the amplitude of the transmitted signal A = 1, the number of error terms in the minimization (of Section 3) N = 2, 501, and 1000, and the number of ADC bits NB = 28. SNR is the signal-to-noise ratio $20 \log_{10} (A / \sigma)$.

σ	SNR	N	AM	SD
0.001	60.00	2	3249.9703	1.4424
		501	3249.9993	0.5466E-2
		1000	3249.9995	0.2802E-2
0.01	40.0	2	3249.7444	14.5075
		501	3249.9546	0.5509E-1
		1000	3249.9389	0.2798E-1
0.05	26.02	2	3249.7308	72.3807
		501	3248.9504	0.2803
		1000	3248.4615	0.1399
0.1	20.00	2	3252.1990	145.6529
		501	3245.9330	0.5945
		1000	3243.9413	0.2873
0.2	13.98	2	3268.8310	307.4860
		501	3234.9594	1.4030
		1000	3227.3130	0.6397

In the no noise case studied in [5], we showed that increasing N, the number of error terms in the minimization of Section 3, significantly decreases the variability of the $\bar{f}$ values (see the table on page 5 of this paper). In the present simulation, we give N the values 2, 501, and 1000.

The program calls for giving the standard deviation of the zero-mean Gaussian noise, σ, the sequence of values 0.001, 0.01, 0.05, 0.1, 0.2, 0.3, 0.5. With $\sigma = 0.3$, the variable DISCR took on some negative values which led to an attempt to find the square root of a negative number. DISCR has value $4.0\,A2 - A1^2$, which is the negative of the discriminant of the quadratic $Z^2 + A1\,Z + A2$. Thus, with $\sigma = 0.3$, zeros of some of the 100 quadratics are real and (6.4) does not apply.

We note from Table 1 that the accuracy of the predicted values, $\bar{f}$, of f decreases (3250 - AM increases and SD increases) as σ increases. Also, we note that the sequence of AM values (with N = 1000) 3249.9995, 3249.9389, 3248.4615, 3243.9413, 3227.3130 respectively corresponding to $\sigma = 0.001$, 0.01, 0.05, 0.1, 0.2 are receding from 3250 toward 3000 (see the comments on page 12).

7. SIMULATION IN THE SIGNAL AND INTERFERENCE CASE

Assume, for all $k \in Z^+$,

$$s(kT) = A_t \sin(2\pi f_t kT) + A_i \sin(2\pi f_i kT), \tag{7.1}$$

where f_t is the Hertz-frequency of the transmitted signal and f_i is the Hertz-frequency of the interference. We assess, by simulation, the sampling variability of the predicted values of f_t and of f_i due to a combination of quantization noise and roundoff noise.

Suppose we wish f_t to take two distinct values (# F = 2) which are 50 Hertz apart, say 9975 and 10025, and $f_i = 10000$. In accordance with the Sampling

Theorem [1, p. 291], we must sample at a rate of at least 20050 (= 2 · 10025) samples per second. However, in view of the poor results (reported in [5]) obtained in case $s(kT) = A\sin(2\pi fkT)$ and f is "close to" an end-point of the frequency band under consideration, we sampled at 25000 samples per second (T = 1 / 25000).

We give simulation results for f_t = 10025; similar results were obtained for f_t = 9975. We did not study model behavior when f_t is switched instantaneously from 10025 to 9975 or vice versa. The simulation program is given as Appendix 3.

In case s(kT) is given by (7.1), a four-pole model (2.1) (p = 4) is appropriate and the equation (2.2) is quartic. Rather than use the tedious quartic formula to solve (2.2), we use the Honeywell library subprogram ZORP2, which uses a modified Downhill-Newton method. For solving the linear system (3.2), we use the Honeywell library subroutine LINSS, which uses Gauss elimination with pivoting.

Both ZORP2 and LINSS were modified as early runs led to no results ("DEGENERATE MATRIX ..." error messages from LINSS) or poor results. Somewhat better results were obtained by replacing the suggested value, 10^{-6} or 10^{-7}, of the variable EPS by 10^{-9}. However, satisfactory results were obtained only upon going to double-precision representations of data and double-precision arithmetic throughout the main program and all the subprograms and decreasing the value of EPS to 10^{-18}. The Honeywell version of ZORP2 consists of single-precision subroutines DOWNH, GRAD, MTALGD, DIV, and POLY. In DOWNH, a number of logical IF statements involve constants 10^{-6} and 10^{-7}. Upon going to double-precision, these constants were replaced by 10^{-12} and 10^{-14}, respectively; however, this resulted in "EXP UFL" error messages. These constants were replaced

ultimately by 10^{-7} and 10^{-8}, respectively.

The program in Appendix 3 calculates $\overline{f_t}$ and $\overline{f_i}$ values from FC * ARCTA(X, Y), where X and Y respectively are the real part and imaginary part of a zero of the quartic and $Y \geq 0$. (For a given complex conjugate pair of solutions of (2.2), we use the solution with positive imaginary part to get a value of $\overline{f_t}$ or $\overline{f_i}$ in (0, 12500).) In case $Y \geq 0$, ARCTA returns a number in $[0, \pi]$ so that FC * ARCTA(X, Y) is in [0, 12500]. In case the quartic has real zeros (Y = 0), FC * ARCTA(X, Y) has value either 0 or 12500. The program in Appendix 2 calculates $\overline{f}$ values from FC * ARCTA(- A1, DSQRT(DISCR)), where DISCR has as value the negative of the discriminant of the quadratic (2.2) with p = 2. Thus, as noted in Section 6, execution breaks down in the case of real zeros of the quadratic (for which DISCR has negative values).

We assume that the amplitudes (measured in volts), A_t and B_t, of the transmitted signal and the interference, respectively, are in [-5, 5]. In the simulation of the behavior of an NB-bit analog-to-digital converter (= ADC), the signal S(J) in the interval [-5, 5], of length 10, is mapped into the integer interval $[-2^{NB-1}, 2^{NB-1}]$ by following $x \longmapsto (2^{NB} / 10)\, x$ with chopping to an integer and this integer is then mapped back into a floating-point real number in [-5, 5] by $x \longmapsto (10 / 2^{NB})\, x$.

The simulation results appear in Table 2 below. The number of ADC bits, NB, takes values 16, 20, 24, 27, and 30. These values were selected because 16-bit ADCs are the most accurate ADCs presently available and 20, 24, 27, and 30 bits respectively correspond to approximately 6, 7, 8, and 9 decimal digits of precision (due to the fact that, in a certain average sense, binary representations of floating-point real numbers have $\log_2(10)$ (= 3.32) times as many symbols as do the corresponding decimal representations). The signal-to-

interference ratio, SIR, is given by

$$\text{SIR} \underline{\underline{\text{df}}} \; 10 \log_{10}(A_t^2 / A_i^2) = 20 \log_{10}(A_t / A_i)$$

For economy, Table 2 does not give the simulation results for SIR < 0, which are similar to those for SIR > 0 except that the approximation of f_t (respectively, f_i) is better than that of f_i (respectively, f_t) for SIR > 0 (respectively, < 0). More specifically, for large positive (respectively, negative) values of SIR, the arithmetic mean of the $\overline{f_t}$ (respectively, $\overline{f_i}$) values was much closer to f_t (respectively, f_i) than was the arithmetic mean of the $\overline{f_i}$ (respectively, $\overline{f_t}$) values to f_i (respectively, f_t). As SIR approaches 0 this disparity diminishes. This is to be expected, as SIR > 0 (respectively, < 0) implies $A_t > A_i$ (respectively, $A_i < A_t$). With a 27 (or 30) bit ADC, the predicted frequencies, $\overline{f_t}$ and $\overline{f_i}$, are quite accurate (error in the arithmetic mean of the f_t (respectively, f_i) < 0.00001 (respectively, 0.4)) for SIR ε [0, 70]; for SIR ε [-70, 0], this statement also holds if we interchange "f_t" and "f_i". This is surprising in view of the fact that for SIR = 70 (respectively, -70), $A_t > 3200\, A_i$ (respectively, $A_i > 3200\, A_t$).

The program in Appendix 3 calls for calculating and printing the arithmetic means of the predicted magnitudes of the complex solutions of (2.2) in addition to the arithmetic means of the predicted frequencies. The arithmetic means of the magnitudes of the complex solutions which yield the predicted values of f_t (= 10025) are all 1.00000 to five decimal places except for the cases NB = 16 and SIR = 0, 5, 10, and 15 respectively in which the arithmetic means are 0.99978, 0.99974, 0.99985, and 0.99993. The arithmetic means of the magnitudes of the complex solutions which yield the predicted values of f_i (= 10000) are all 1.00 to two decimal places except for the cases NB = 16

and SIR = 15, 20, ... , 80, NB = 20 and SIR = 40, 45, ... , 80, and NB = 24 and SIR = 65, 70, 75, 80. These results are expected as the poles of the transfer function should be on the unit circle.

TABLE 2

Arithmetic mean AM_{FT} (respectively, AM_{FI}) and standard deviation SD_{FT} (respectively, SD_{FI}) of 100 predicted values of the Hertz-frequency of the transmitted signal (respectively, interference) for the signal-to-interference ratio SIR = 0.0, 5.0, 10.0, ... , 80.0 with AT = 5.0, N = 496, and the number of ADC bits NB = 16, 20, 24, 27, and 30.

NB = 16

SIR	AI	AM_{FT}	SD_{FT}	AM_{FI}	SD_{FI}
80.0	0.5000E-3	10025.00001	0.14875E-4	*	
75.0	0.8891E-3	10025.00000	0.55818E-4	*	
70.0	0.1581E-2	10025.00001	0.62049E-4	*	
65.0	0.2812E-2	10024.99997	0.62494E-4	*	
60.0	0.5000E-2	10025.00000	0.13758E-3	*	
55.0	0.8891E-2	10024.99997	0.21750E-3	*	
50.0	0.1581E-1	10024.99984	0.59548E-3	*	
45.0	0.2812E-1	10024.99931	0.11306E-2	*	
40.0	0.5000E-1	10024.99734	0.20244E-2	*	
35.0	0.8891E-1	10024.99198	0.45764E-2	9904.55117	0.12742E+1
30.0	0.1581E+0	10024.97501	0.13919E-1	9782.96934	0.51214E+1
25.0	0.2812E+0	10024.92350	0.60974E-1	9891.21543	0.20241E+1
20.0	0.5000E+0	10024.79049	0.28133E+0	9962.73950	0.34268E+0
15.0	0.8891E+0	10024.59744	0.93372E+0	9987.73936	0.70680E+0
10.0	0.1581E+1	10024.53250	0.14395E+1	9996.65213	0.12077E+1
5.0	0.2812E+1	10024.46491	0.12766E+1	9998.76546	0.11762E+1
0.0	0.5000E+1	10024.44931	0.70598E+0	9999.60803	0.56777E+0

* AM_{FI} is grossly in error due to the occurrence, in calculating some or all of the predicted values of FI, of real pairs (0 and 12500), rather than complex pairs, of solutions of (2.2).

NB = 20

SIR	AI	AM_{FT}	SD_{FT}	AM_{FI}	SD_{FI}
80.0	0.5000E-3	10024.99999	0.31531E-5	*	
75.0	0.8891E-3	10025.00000	0.24865E-4	*	
70.0	0.1581E-2	10024.99999	0.60063E-4	*	
65.0	0.2812E-2	10025.00000	0.11830E-3	*	
60.0	0.5000E-2	10025.00000	0.24188E-3	10299.64857	0.90224E+1
55.0	0.8891E-2	10024.99994	0.75772E-3	9775.32456	0.10371E+1
50.0	0.1581E-1	10024.99960	0.30767E-2	9873.41665	0.10457E+1
45.0	0.2812E-1	10024.99904	0.14200E-1	9955.79372	0.18207E+1
40.0	0.5000E-1	10024.99895	0.51282E-1	9985.21355	0.87810E+0
35.0	0.8891E-1	10025.00235	0.88389E-1	9995.09369	0.48957E+0
30.0	0.1581E+0	10025.00380	0.73353E-1	9998.37917	0.24974E+0
25.0	0.2812E+0	10024.99760	0.42521E-1	9999.50158	0.74370E-3
20.0	0.5000E+0	10025.00373	0.24424E-1	9999.85185	0.78603E-1
15.0	0.8891E+0	10025.00365	0.15603E-1	9999.94350	0.40965E-1
10.0	0.1581E+1	10024.99374	0.63482E-2	9999.98351	0.88233E-2
5.0	0.2812E+1	10025.00370	0.30933E-2	9999.99406	0.13460E-1
0.0	0.5000E+1	10025.00169	0.11661E-1	9999.99412	0.69500E-2

NB = 24

SIR	AI	AM_{FT}	SD_{FT}	AM_{FI}	SD_{FI}
80.0	0.5000E-3	10025.00000	0.31543E-4	9813.03648	0.39144E+1
75.0	0.8891E-3	10025.00000	0.14769E-3	9853.07626	0.14610E+1
70.0	0.1581E-2	10025.00003	0.63502E-3	9943.78711	0.83820E+0
65.0	0.2812E-2	10024.99997	0.26975E-2	9981.51822	0.83104E+0
60.0	0.5000E-2	10025.00016	0.53971E-2	9993.55141	0.11223E+0
55.0	0.8891E-2	10025.00026	0.60723E-2	9997.80849	0.74932E-1
50.0	0.1581E-1	10025.00035	0.31791E-2	9999.42349	0.34453E-1
45.0	0.2812E-1	10025.00033	0.20466E-2	9999.79271	0.20292E-1
40.0	0.5000E-1	10024.99983	0.19149E-2	9999.90712	0.10978E-2
35.0	0.8891E-1	10025.00032	0.41177E-3	9999.97580	0.67684E-2
30.0	0.1581E+0	10025.00032	0.27273E-3	9999.99337	0.38625E-2
25.0	0.2812E+0	10025.00032	0.17491E-3	9999.99774	0.21642E-2
20.0	0.5000E+0	10025.00012	0.11124E-3	9999.99949	0.46476E-2
15.0	0.8891E+0	10024.99993	0.28941E-3	10000.00030	0.36852E-3
10.0	0.1581E+1	10025.00031	0.17481E-3	10000.00010	0.39369E-3
5.0	0.2812E+1	10025.00032	0.51933E-4	9999.99995	0.22110E-3
0.0	0.5000E+1	10025.00001	0.91795E-3	9999.99985	0.31839E-3

NB = 27

SIR	AI	AM_{FT}	SD_{FT}	AM_{FI}	SD_{FI}
80.0	0.5000E-3	10025.00001	0.54492E-3	9993.09929	0.15019E+0
75.0	0.8891E-3	10025.00004	0.54903E-3	9997.66742	0.38508E+0
70.0	0.1581E-2	10025.00003	0.24169E-3	9999.39654	0.32871E+0
65.0	0.2812E-2	10025.00000	0.28053E-3	9999.71835	0.21289E-1
60.0	0.5000E-2	10025.00003	0.21474E-3	9999.88056	0.63278E-1
55.0	0.8891E-2	10025.00004	0.12446E-3	9999.97034	0.19248E-1
50.0	0.1581E-1	10025.00000	0.15211E-3	9999.97841	0.33529E-2
45.0	0.2812E-1	10024.99999	0.41514E-4	9999.99853	0.16168E-1
40.0	0.5000E-1	10025.00000	0.11558E-3	9999.99448	0.10382E-2
35.0	0.8891E-1	10025.00000	0.10051E-3	10000.00254	0.57312E-3
30.0	0.1581E+0	10025.00003	0.12463E-3	9999.99850	0.20064E-2
25.0	0.2812E+0	10024.99999	0.28505E-4	9999.99985	0.16137E-2
20.0	0.5000E+0	10025.00000	0.10251E-3	9999.99950	0.10498E-3
15.0	0.8891E+0	10025.00000	0.95455E-4	10000.00030	0.62359E-4
10.0	0.1581E+1	10025.00000	0.11685E-3	10000.00015	0.34832E-4
5.0	0.2812E+1	10025.00002	0.62149E-4	9999.99999	0.18437E-3
0.0	0.5000E+1	10024.99998	0.24879E-4	9999.99996	0.13870E-4

NB = 30

SIR	AI	AM_{FT}	SD_{FT}	AM_{FI}	SD_{FI}
80.0	0.5000E-3	10025.00000	0.11038E-2	10006.70448	0.49575E-1
75.0	0.8891E-3	10025.00000	0.77445E-3	10002.07125	0.86676E-2
70.0	0.1581E-2	10025.00000	0.41554E-3	10000.66534	0.13332E-1
65.0	0.2812E-2	10025.00000	0.22524E-3	10000.20150	0.55550E-2
60.0	0.5000E-2	10025.00000	0.12143E-3	10000.06493	0.11716E-2
55.0	0.8891E-2	10025.00000	0.68928E-4	10000.02024	0.37293E-3
50.0	0.1581E-1	10025.00000	0.49600E-4	10000.00622	0.18744E-3
45.0	0.2812E-1	10025.00000	0.31128E-4	10000.00295	0.22967E-2
40.0	0.5000E-1	10024.99999	0.14440E-4	10000.00057	0.43003E-3
35.0	0.8891E-1	10024.99999	0.27836E-4	10000.00057	0.32625E-3
30.0	0.1581E+0	10025.00000	0.22309E-4	10000.00023	0.40442E-3
25.0	0.2812E+0	10024.99999	0.15418E-4	10000.00004	0.80662E-4
20.0	0.5000E+0	10025.00000	0.17013E-4	10000.00001	0.29009E-4
15.0	0.8891E+0	10025.00000	0.10302E-4	9999.99998	0.62105E-4
10.0	0.1581E+1	10024.99999	0.13689E-4	10000.00001	0.22865E-4
5.0	0.2812E+1	10025.00000	0.25597E-4	10000.00002	0.22428E-4
0.0	0.5000E+1	10025.00000	0.21318E-4	10000.00001	0.19165E-4

8. CONCLUSIONS

As indicated earlier, the problem of obtaining confidence intervals for the frequencies of the transmitted signal appears to be intractable, both for the signal and noise case and for the signal and interference case. The case of a signal in noise and interference was not considered.

Simulation (and partial theoretical) results in the signal in independent, zero-mean, Gaussian noise case were disappointing. The writer is not certain whether the Gaussian noise assumption, so common in the literature, is made because it is realistic or because of its mathematical niceties.

Simulation in the signal and interference case indicates the need for an ADC of accuracy at least 27 bits. At present, 16-bit ADCs are the most accurate available.

REFERENCES

[1] Cadzow, J. A., Discrete-time Systems, Prentice-Hall, Englewood Cliffs, New Jersey, 1973.

[2] Knuth, D. E., The Art of Computer Programming, Vol. 2: Seminumerical Algorithms, Addison-Wesley, Reading, Mass., 1969.

[3] Makhoul, J., "Linear Prediction: A Tutorial Review," Proceedings of the IEEE, Vol. 63, No. 4, April 1975.

[4] Robinson, E. A., Statistical Communication and Detection, Hafner, New York, N, Y., 1967.

[5] Van Meter, R. G., "Maximum Entropy Spectral Demodulator Investigation," Report No. 31, 1977 USAF/ASEE Summer Faculty Research Program Research Reports, Vol. 2, AFOSR-TR-78-0349, September 1977.

[6] Van Meter, R. G., "Maximum Entropy Spectral Demodulator Investigation. II," Research Report, SCEEE Contract F30602-75-C-0122, to appear.

APPENDIX 1

Key variables in the program in Appendix 2, the corresponding variables in Sections 2, 3, and 6, and their interpretations:

TSA	(T)	sampling period (in seconds);
F1	(f)	Hertz-frequency of the transmitted signal;
A	(A)	amplitude (in volts) of the transmitted signal;
NP	(p)	number of poles of the transfer function;
N	(N)	number of error terms in the minimization process for determining the prediction coefficients $\overline{a_j}$ in (3.1);
NN	(N + p)	number of previous samples of the signal needed to predict s(mT);
NB	(NB)	number of ADC bits;
SIG(K)	(σ)	standard deviation of the Gaussian noise;
S(J)	(s(kT))	value of the received signal at time JT;
P(I,J)	($\phi_{ij}(m,N)$)	coefficient of $\overline{a_j}$ in equation i of the linear system (3.3);
PO(I)	($\phi_{0i}(m,N)$)	constant on the right in equation i of the linear system (3.1);
A1, A2	($\overline{a_1}$, $\overline{a_2}$)	coefficients in equation (2.2) with p = 2;
DISCR	($4\overline{a_2} - \overline{a_1}^2$)	negative of the discriminant of the quadratic in equation (2.2) with p = 2;
F(K)	($\overline{f}$)	predicted value of the Hertz-frequency f of the transmitted signal;
FC	( )	radian-to-Hertz conversion factor 6000 / π;
AM	(AM)	arithmetic mean of 100 $\overline{f}$ values;
SD	(SD)	standard deviation of 100 $\overline{f}$ values.

Key variables in the program in Appendix 3, the corresponding variables in Sections 2, 3, and 7, and their interpretations:

TSA	(T)	sampling period (in seconds);
FC	( )	radian-to-Hertz conversion factor 12500 / π;
FT	(f_t)	Hertz-frequency of the transmitted signal;
FI	(f_i)	Hertz-frequency of the interference;

AT	(A_t)	amplitude (in volts) of the transmitted signal;
AI	(A_i)	amplitude (in volts) of the interference;
NP	(p)	number of poles of the transfer function;
N	(N)	number of error terms in the minimization process for determining the prediction coefficients $\overline{a_j}$ in (3.1);
NN	(N + p)	number of previous samples of the signal needed to predict s(mT);
NB	(NB)	number of ADC bits;
SIR	(SIR)	signal-to-interference ratio (in decibels);
FONE	($\overline{f_t}$)	predicted value (in Hertz) of the frequency of the transmitted signal;
FTWO	($\overline{f_i}$)	predicted value (in Hertz) of the frequency of the interference.

APPENDIX 2

```
C         MAXIMUM ENTROPY SPECTRAL DEMODULATION
C           SIGNAL AND (GAUSSIAN) NOISE CASE
C
      DOUBLE PRECISION PI,TSA,TPI,A,AM,SD,F1,FC,R1(1300),R2(1300)
      DOUBLE PRECISION V1,V2,S1,W(1300),V11(1300),V21(1300),SIG(7)
      DOUBLE PRECISION X1(2600),W1,FJ,TC,TS,U,S(1102),PNB,DV,DVI,X
      DOUBLE PRECISION P(2,2),PO(2),DEL,DEL1,DEL2,A1,A2,DISCR,F(1102)
      COMMON/SAMP/S
      COMMON/PHI/P
      COMMON/PHO/PO
      COMMON/FREQ/F
      COMMON/RAD/PI
      DATA PI,TSA/3.14159265358979324D0,83.3333333333333333D-6/
      DATA NP,A,AM,SD,F1/2,1.0D0,0.0D0,0.0D0,3250.0D0/
      DATA SIG(1),SIG(2),SIG(3),SIG(4)/0.001D0,0.01D0,0.05D0,0.1D0/
      DATA SIG(5),SIG(6),SIG(7)/0.2D0,0.3D0,0.5D0/
      TPI=6.283185307179958648
      FC=6000.0D0/PI
C         GENERATE GAUSSIAN NOISE (0160-0360)
      DO 10 J=1,1300
   10 R1(J)=RANDT(1.0)
      DO 20 J=1,1300
   20 R2(J)=RANDT(1.0)
      I=1
      DO 30 J=1,1300
      V1=2.0D0*R1(J)-1.0D0
      V2=2.0D0*R2(J)-1.0D0
      S1=V1*V1+V2*V2
      IF(S1.GE.1.0D0) GO TO 30
      W(I)=-2.0D0*DLOG(S1)/S1
      V11(I)=V1
      V21(I)=V2
      I=I+1
   30 CONTINUE
      L=I-1
C         LOOP TO CALCULATE AND PRINT MEAN AND STANDARD
C         DEVIATION OF 100 PREDICTED FREQUENCIES FOR
C         VARIOUS VALUES OF SIGMA
      DO 40 K1=1,7
      DO 50 J=1,L
```

```
      Z=SIG(K1)*DSQRT(W(J))
      X1(2*J-1)=V11(J)*Z
   50 X1(2*J)=V21(J)*Z
C          CALCULATE TRANSMITTED SIGNAL PLUS NOISE
      W1=F1*TPI
      DO 60 J=1,1102
      FJ=J
      TC=FJ*TSA
      TS=W1*TC
      U=A*DSIN(TS)
   60 S(J)=U+X1(J)
C          SIMULATE ANALOG-TO-DIGITAL CONVERSION OF SIGNAL
      DO 45 NB=16,28,12
      WRITE(6,1) NB
    1 FORMAT(1X,"NUMBER OF BITS = ",I5/)
      NBP=2**NB
      PNB=NBP
      DV=10.0D0/PNB
      DVI=PNB/10.0D0
      DO 70 J=1,1102
      X=S(J)
      X=X*DVI
      KP=X
      X=KP
   70 S(J)=X*DV
C          LOOP TO VARY N (= THE NUMBER OF ERROR TERMS IN
C          THE MINIMIZATION OF SECTION 3)
      DO 55 N=2,1000,499
      NN=NP+N
      KL=NN+1
      KU=KL+99
C          LOOP TO GENERATE 100 (= KU-KL+1) PREDICTED FREQUENCIES
      DO 80 K=KL,KU
      KK=K
C          CALCULATE COEFFICIENTS FOR LINEAR SYSTEM (3.2)
C          USING SUBROUTINE COV (COVARIANCE METHOD)
      CALL COV(NP,NN,KK)
C          SOLVE LINEAR SYSTEM (3.2) BY CRAMER'S RULE
      DEL=P(1,1)*P(2,2)-P(1,2)*P(2,1)
      DEL1=P(1,2)*PO(2)-P(2,2)*PO(1)
      DEL2=P(2,1)*PO(1)-P(1,1)*PO(2)
      A1=DEL1/DEL
      A2=DEL2/DEL
C          SOLVE EQUATION (2.2) BY QUADRATIC FORMULA
      DISCR=4.0D0*A2-A1*A1
      IF(DISCR.GE.0.0D0) GO TO 80
      WRITE(6,2) K
    2 FORMAT(1X,"DISCR NEG AT STEP ",I8)
C          CALCULATE PREDICTED FREQUENCY BY USE OF (6.4)
   80 F(K)=FC*ARCTA(-A1,DSQRT(DISCR))
```

```
C         CALCULATE AND PRINT MEAN AND STANDARD DEVIATION
C         OF PREDICTED FREQUENCIES
      CALL STATS(AM,SD,KL,KU)
      WRITE(6,3) N,AM,SD
    3 FORMAT(1X,I5,2G24.8/)
   55 CONTINUE
   45 CONTINUE
   40 CONTINUE
      STOP
      END
C
      SUBROUTINE COV(NP,NN,LP)
C
      DOUBLE PRECISION S(1102),P(2,2),PO(2),B
      COMMON/SAMP/S
      COMMON/PHI/P
      COMMON/PHQ/PO
      L=LP-1
      NI=NN-NP
      NL=LP-NI
      B=0.0D0
C         LOOPS TO CALCULATE PHI(J,J), 1<=J<=NP
      DO 10 J=NL,L
   10 B=B+S(J)*S(J)
      DO 11 J=1,NP
      K=LP-J
      I=NL-J
      B=B+S(I)*S(I)-S(K)*S(K)
   11 P(J,J)=B
      DO 12 KK=1,NP
      B=0.0D0
C         LOOP TO CALCULATE PHI(0,KK), 1<=KK<=NP / STORE IN PO(KK)
      DO 13 J=1,NI
      N=LP-J
      M=N-KK
   13 B=B+S(N)*S(M)
      PO(KK)=B
C         LOOP TO CALCULATE PHI(I,K) FOR I NOT = K
      IF(KK.EQ.NP) GO TO 12
      DO 14 J=1,NP-KK
      I=J
      K=KK+J
      N=LP-J
      M=N-KK
      N1=NL-J
      M1=N1-KK
      B=B+S(N1)*S(M1)-S(N)*S(M)
      P(I,K)=B
   14 P(K,I)=B
   12 CONTINUE
      RETURN
      END
```

```
C
      SUBROUTINE STATS(AM,SD,KL,KU)
C
C          STATS CALCULATES THE MEAN AND STANDARD DEVIATION
C          OF KU-KL+1 PREDICTED FREQUENCIES F(J)
      DOUBLE PRECISION F(1102),S1,S2,RM,AM,SD
      COMMON/FREQ/F
      S1=0.0D0
      S2=0.0D0
      RM=KU-KL+1
      DO 10 J=KL,KU
   10 S1=S1+F(J)
      AM=S1/RM
      DO 20 J=KL,KU
   20 S2=S2+(F(J)-AM)*(F(J)-AM)
      SD=DSQRT(S2/RM)
      RETURN
      END
C
      FUNCTION ARCTA(X,Y)
C
C          SUBPROGRAM TO CALCULATE ARCTANGENT
C          VALUES IN -PI TO PI
      DOUBLE PRECISION PI,HPI,X,Y,ARCTA
      COMMON/RAD/PI
      HPI=1.57079632679489662D0
      IF(X) 1,2,3
    1 ARCTA=DATAN(Y/X)+PI
      RETURN
    2 ARCTA=HPI
      RETURN
    3 ARCTA=DATAN(Y/X)
      RETURN
      END
```

APPENDIX 3

```
C         MAXIMUM ENTROPY SPECTRAL DEMODULATION
C              SIGNAL AND INTERFERENCE CASE
C
      DOUBLE PRECISION PI,TPI,TSA,FC,AM,SD,AT,AI,FT,FI,SIR,WT,WI
      DOUBLE PRECISION FJ,TC,TST,TSI,S(500),PNB,DV,DVI,X,Y,A(10)
      DOUBLE PRECISION P(4,4),PO(4),RR(5),CR(5),B(5),AUX(4),R(4)
      DOUBLE PRECISION TEMP,RMAG1(100),RMAG2(100),FONE(100)
      DOUBLE PRECISION FTWO(100),F11(2),R11(2)
      COMMON/SAMP/S
      COMMON/PHI/P
      COMMON/PHO/PO
      COMMON/FREQ/FONE,FTWO
      COMMON/MAG/RMAG1,RMAG2
      COMMON/SIME/A,R,AUX
      COMMON/BRC/B,RR,CR
      DATA AM,SD,AT,FT,FI/0.0D0,0.0D0,5.0D0,10025.0D0,10000.0D0/
      DATA NP,N/4,496/
      PI=3.14159265358979324
      TPI=6.28318530717958647
      TSA=1.0D0/25000.0D0
      FC=12500.0D0/PI
      NN=NP+N
      KL=NN+1
C        LOOP TO GIVE SIR VALUES 60,55,50, ... ,5,0
      SIR=60.0D0
   11 WRITE(6,201) SIR
  201 FORMAT(1X,"SIR =",G17.8//)
      AI=AT*10.0D0**(-SIR/20.0D0)
      WRITE(6,111) AI
  111 FORMAT(1X,"AI =",G17.8//)
      WT=FT*TPI
      WI=FI*TPI
C        LOOP TO CALCULATE 100 PREDICTED VALUES OF FT AND OF FI
      DO 33 K=1,100
      DO 34 J=1,500
      FJ=J+500*(K-1)
      TC=FJ*TSA
      TST=WT*TC
      TSI=WI*TC
C        CALCULATE RECEIVED SIGNAL FROM (7.1)
      S(J)=AT*DSIN(TST)+AI*DSIN(TSI)
   34 CONTINUE
```

```
C         SIMULATE 16-BIT ANALOG-TO-DIGITAL CONVERSION OF SIGNAL
      NB=16
      NBP=2**NB
      PNB=NBP
      DV=10.0D0/PNB
      DVI=PNB/10.0D0
      DO 15 J=1,500
      X=S(J)
      X=X*DVI
      KP=X
      X=KP
   15 S(J)=X*DV
C         CALCULATE COEFFICIENTS FOR LINEAR SYSTEM (3.2)
C         USING SUBROUTINE COV (COVARIANCE METHOD)
      CALL COV(NP,NN,KL)
      L=0
      DO 1 J=1,NP
      DO 9 I=1,J
      LL=L+I
      A(LL)=P(I,J)
    9 CONTINUE
      L=L+J
      R(J)=-PO(J)
    1 CONTINUE
      IER=0
C         SOLVE LINEAR SYSTEM (3.2) USING HONEYWELL SUBROUTINE LINSS
      CALL LINSS(NP,IER)
      IF(IER) 2,3,5
    5 NQ=NP-IER
      WRITE(6,202) NQ,IER
  202 FORMAT(1X,"DEGENERATE MATRIX OF DEGENERACY",I8," RANK =",I8)
      GO TO 33
    2 WRITE(6,203)
  203 FORMAT(1X,"MATRIX POSSIBLY SINGULAR")
      GO TO 33
C         ASSIGN ELEMENTS IN SOLUTION 4-TUPLE OF SYSTEM (3.2)
C         (DENOTED BY "R(J)" IN LINSS IN PLACE OF "A(J)" IN (3.2))
C         TO B(J) WITH B(5)=1,B(4)=R(1),B(3)=R(2),B(2)=R(3),
C         B(1)=R(4) AND SOLVE THE EQUATION B(5)*Z**4 + B(4)*Z**3
C         + B(3)*Z**2 + B(2)*Z + B(1) = 0 OF THE FORM (2.2)
C         BY USE OF HONEYWELL SUBPROGRAM ZORP2 WHICH CONSISTS
C         OF SUBROUTINES DOWNH,GRAD,MTALGD,DIV, AND POLY
    3 NAC=NP+1
      B(NAC)=1.0D0
      DO 7 J=1,NP
    7 B(J)=R(NAC-J)
      CALL DOWNH(B,NP,RR,CR)
      I=1
      DO 44 L=1,NP
      IF(CR(L).LT.0.0D0) GO TO 44
      X=RR(L)
      Y=CR(L)
      R11(I)=DSQRT(X*X+Y*Y)
```

```
C         CALCULATE PREDICTED FREQUENCY
          F11(I)=FC*ARCTA(X,Y)
          I=I+1
       44 CONTINUE
          IF(F11(1).LE.F11(2)) GO TO 43
          TEMP=F11(1)
          F11(1)=F11(2)
          F11(2)=TEMP
          TEMP=R11(1)
          R11(1)=R11(2)
          R11(2)=TEMP
       43 RMAG1(K)=R11(1)
          RMAG2(K)=R11(2)
          FONE(K)=F11(1)
          FTWO(K)=F11(2)
       33 CONTINUE
C         CALCULATE AND PRINT MEAN AND STANDARD DEVIATION
C         OF 100 PREDICTED VALUES OF RMAG1,RMAG2,F1, AND F2
          CALL STATS(AM,SD,RMAG1)
          WRITE(6,400) AM,SD
      400 FORMAT(1X,2D25.18)
          CALL STATS(AM,SD,RMAG2)
          WRITE(6,400) AM,SD
          CALL STATS(AM,SD,FONE)
          WRITE(6,400) AM,SD
          CALL STATS(AM,SD,FTWO)
          WRITE(6,400) AM,SD
          WRITE(6,888)
      888 FORMAT(1X,//)
          SIR=SIR-5.0D0
          IF(SIR.LT.0.0D0) STOP
          GO TO 11
          END
C
          SUBROUTINE COV(NP,NN,LP)
C
C         COV WAS WRITTEN BY CAPT. KENNETH WILSON OF RADC TO
C         IMPLEMENT THE COVARIANCE METHOD OF LINEAR PREDICTION
C         (SEE MAKHOUL (2, P. 564))
          DOUBLE PRECISION S(500),P(4,4),PO(4),A
          COMMON/SAMP/S
          COMMON/PHI/P
          COMMON/PHO/PO
          L=LP-1
          NI=NN-NP
          NL=LP-NI
          A=0.0D0
C              LOOPS TO CALCULATE PHI(J,J), 1<=J<=NP
          DO 10 J=NL,L
       10 A=A+S(J)*S(J)
          DO 11 J=1,NP
          K=LP-J
          I=NL-J
          A=A+S(I)*S(I)-S(K)*S(K)
       11 P(J,J)=A
```

```
      DO 12 KK=1,NP
      A=0.0D0
C        LOOP TO CALCULATE PHI(0,KK), 1<KK<NP / STORE IN PO(KK)
      DO 13 J=1,NI
      N=LP-J
      M=N-KK
   13 A=A+S(N)*S(M)
      PO(KK)=A
C           LOOP TO CALCULATE PHI(I,K) FOR I NOT = K
      IF(KK.EQ.NP) GO TO 12
      DO 14 J=1,NP-KK
      I=J
      K=KK+J
      N=LP-J
      M=N-KK
      N1=NL-J
      M1=N1-KK
      A=A+S(N1)*S(M1)-S(N)*S(M)
      P(I,K)=A
   14 P(K,I)=A
   12 CONTINUE
      RETURN
      END
C
      SUBROUTINE LINSS(M,IER)
C
C        LINSS IS A DOUBLE PRECISION VERSION OF THE HONEYWELL
C        SUBROUTINE LINSS FOR SOLVING A LINEAR SYSTEM WITH
C        SYMMETRIC COEFFICIENT MATRIX.  LINSS USES GAUSS ELIMINATION
C        WITH PIVOTING IN THE MAIN DIAGONAL ONLY, TO PRESERVE
C        SYMMETRY.  IER IS AN ERROR RETURN AS FOLLOWS:  IER=0
C        INDICATES NO ERROR; IER=-1 INDICATES NO RESULT AS NP<0
C        OR A PIVOT ELEMENT WAS ZERO DURING ELIMINATION; IER=K
C        IS A WARNING OF POSSIBLE LOSS OF SIGNIFICANCE (OF L
C        SIGNIFICANT DIGITS IF EPS - 10**(-L)) AT ELIMINATION
C        STEP K+1 AND , WITH WELL-CONDITIONED A AND APPROPRIATE
C        EPS, THAT A MAY HAVE A RANK OF K.
      DOUBLE PRECISION A(10),R(4),AUX(4),PIV,TB,TOL,PIVI,EPS
      COMMON/SIME/A,R,AUX
      EPS=1.0D-18
      IF(M.LE.0) GO TO 24
C        SEARCH FOR PIVOT
    1 IER=0
      PIV=0.0D0
      L=0
      DO 3 K=1,M
      L=L+K
      TB=DABS(A(L))
      IF(TB-PIV) 3,3,2
    2 PIV=TB
      I=L
      J=K
    3 CONTINUE
```

```
      TOL=EPS*PIV
      LST=0
      LEND=M-1
C        ELIMINATION LOOP
      DO 18 K=1,M
      IF(PIV) 24,24,4
    4 IF(IER) 7,5,7
    5 IF(PIV-TOL) 6,6,7
    6 IER=K-1
    7 LT=J-K
      LST=LST+K
C        PIVOT ROW REDUCTION AND ROW INTERCHANGE IN RIGHT SIDE R
      PIVI=1.0D0/A(I)
      TB=PIVI*R(J)
      R(J)=R(K)
      R(K)=TB
      IF(K-M) 9,19,19
C        ROW AND COLUMN INTERCHANGE AND PIVOT ROW REDUCTION A
C        PIVOT COLUMN SAVED IN AUX
    9 LR=LST+(LT*(K+J-1))/2
      LL=LR
      L=LST
      DO 14 II=K,LEND
      L=L+II
      LL=LL+1
      IF(L-LR) 12,10,11
   10 A(LL)=A(LST)
      TB=A(L)
      GO TO 13
   11 LL=L+LT
   12 TB=A(LL)
      A(LL)=A(L)
   13 AUX(II)=TB
   14 A(L)=PIVI*TB
      A(LST)=LT
C        ELEMENT REDUCTION AND SEARCH FOR NEXT PIVOT
      PIV=0.0D0
      LLST=LST
      LT=0
      DO 18 II=K,LEND
      PIVI=-AUX(II)
      LL=LLST
      LT=LT+1
      DO 15 LLD=II,LEND
      LL=LL+LLD
      L=LL+LT
   15 A(L)=A(L)+PIVI*A(LL)
      LLST=LLST+II
      LR=LLST+LT
      TB=DABS(A(LR))
      IF(TB-PIV) 17,17,16
   16 PIV=TB
      I=LR
```

```
      J=II+1
   17 LL=II+1
   18 R(LL)=R(LL)+PIVI*R(K)
C          BACK SOLUTION AND INTERCHANGE
   19 IF(LEND) 24,23,20
   20 II=M
      DO 22 I=2,M
      LST=LST-II
      II=II-1
      L=A(LST)+0.5D0
      TB=R(II)
      LL=II
      K=LST
      DO 21 LT=II,LEND
      LL=LL+1
      K=K+LT
   21 TB=TB-A(K)*R(LL)
      K=II+L
      R(II)=R(K)
   22 R(K)=TB
   23 RETURN
   24 IER=-1
      RETURN
      END
C
      SUBROUTINE DOWNH(A,NAR,RR,CR)
C
      DOUBLE PRECISION A(10),RR(5),CR(5),Q(10),B(3)
      DOUBLE PRECISION ANPP,DISC,X,Y,C
      J=0
      N=NAR
      NPL1=N+1
      ANPP=A(NPL1)
      DO 102 I=1,NPL1
      IF(A(I)) 103,102,103
  102 CONTINUE
  103 C=DABS(A(I)/A(NPL1))
      LU=120
      LL=-120
      IF(C-2.0D0**LU) 100,100,101
  100 IF(C-2.0D0**LL) 101,105,105
  101 NAR=-NAR
      GO TO 5001
  105 II=(LU+LL)/2
      IF(C-2.0D0**II) 110,110,109
  109 LL=II
      GO TO 111
  110 LU=II
  111 IF(LU-LL-1) 5001,112,105
  112 IB=II/N
      IF(IB) 114,120,114
  114 DO 115 I=1,NPL1
      II=I-1
  115 A(I)=A(I)*(2.0D0**(II*IB))
```

```
  120 DO 121 J1=1,NPL1
  121 A(J1)=A(J1)/A(NPL1)
  201 IF(N) 2001,2001,206
  206 IF(A(1)) 301,211,301
  211 J=J+1
      RR(J)=0.0D0
      CR(J)=0.0D0
      DO 221 J1=1,N
  221 A(J1)=A(J1+1)
      N=N-1
      GO TO 201
  301 IF(N-2) 601,501,401
  401 CALL GRAD(A,N,X,Y)
  421 IF(DABS(Y)-DABS(X*1.0D-8)) 431,431,441
  431 Y=0.0D0
  441 J=J+1
      RR(J)=X
      CR(J)=Y
      IF(Y) 461,1021,461
  461 J=J+1
      RR(J)=X
      CR(J)=-Y
      GO TO 1011
  501 DISC=A(2)**2-4.0D0*A(1)
      IF(DISC) 521,541,541
  521 Y=DSQRT(-DISC)/2.0D0
      X=-A(2)/2.0D0
      GO TO 421
  541 J=J+1
      RR(J)=(-A(2)+DSQRT(DISC))/2.0D0
      CR(J)=0.0D0
      GO TO 1021
  601 J=J+1
      RR(J)=-A(1)
      CR(J)=0.0D0
      GO TO 2001
 1011 B(1)=X**2+Y**2
      B(2)=-2.0D0*X
      B(3)=1.0D0
      NB=2
      GO TO 1041
 1021 B(1)=-RR(J)
      B(2)=1.0D0
      NB=1
 1041 CALL DIV(A,B,N,NB,Q)
      DO 1061 J1=1,N
 1061 A(J1)=Q(J1)
      IF(CR(J)) 1081,1071,1081
 1071 N=N-1
      GO TO 201
 1081 N=N-2
      GO TO 201
 2001 IF(IB) 2002,2005,2002
```

```
 2002 DO 2000 I=1,NAR
      RR(I)=RR(I)*(2.0D0**(IB))
 2000 CR(I)=CR(I)*(2.0D0**(IB))
 2005 NP1=NAR+1
      DO 2011 I=2,NP1
 2011 A(I)=0.0D0
      A(1)=1.0D0
      NA=0
      J=1
 2021 IF(CR(J)) 2041,2061,2041
 2041 NB=2
      B(3)=1.0D0
      B(2)=-2.0D0*RR(J)
      B(1)=RR(J)**2+CR(J)**2
      J=J+2
      GO TO 2081
 2061 NB=1
      B(2)=1.0D0
      B(1)=-RR(J)
      J=J+1
 2081 CALL MTALGD(A,NA,B,NB,Q)
      NA=NB+NA
      NAPL1=NA+1
      DO 2091 I=1,NAPL1
 2091 A(I)=Q(I)
      IF(NA-NAR) 2021,3001,3001
 3001 DO 3011 J2=1,NPL1
 3011 A(J2)=A(J2)*ANPP
 5001 RETURN
      END
C
      SUBROUTINE GRAD(A,N,XZ,YZ)
C
      DOUBLE PRECISION A(10),X(3),Y(3),RP(3),CP(3),RHO(3),PHI(3)
      DOUBLE PRECISION ABSP(3),PR(3),PC(3),PI,XZ,YZ,RHOZ,PHIZ,SU,U
      DOUBLE PRECISION PSI,TOP,BOT,COSI,SINE,DZ,ABSPZ,PRZ,PCZ,RZ
      DOUBLE PRECISION CZ,THETA,DTHETA,RHOS,PHIS
      PI=3.14159265358979324
      MTST=1
  101 XZ=0.0D0
      YZ=1.0D0
      DZ=2.0D0
      RHOZ=1.0D0
      PHIZ=PI/2.0D0
  201 CALL POLY(N,A,RZ,CZ,PRZ,PCZ,RHOZ,PHIZ)
  221 SU=DSQRT(PRZ**2+PCZ**2)
      ABSPZ=DSQRT(RZ**2+CZ**2)
      U=2.0D0*ABSPZ*SU
      PSI=DATAN(U)
      TOP=RZ*PCZ-CZ*PRZ
      BOT=-(RZ*PRZ+CZ*PCZ)
      THETA=ARCTA(BOT,TOP)
      COSI=DCOS(THETA+PHIZ)
```

```
      SINE=DSIN(THETA+PHIZ)
      IF(ABSPZ) 300,5001,300
  300 IF(SU) 301,501,301
  301 IF(RHOZ) 321,401,321
  321 IF(ABSPZ/(RHOZ*SU)-1.0D-8) 5001,5001,701
  351 IF(ABSPZ/(RHOZ*SU)-10.0D0**(-MTST)) 801,801,401
  401 DZ=DZ/8.0D0
      IM=0
      DO 431 I=1,3
      DZ=2.0D0*DZ
      X(I)=XZ+DZ*COSI
      Y(I)=YZ+DZ*SINE
      RHO(I)=DSQRT(X(I)**2+Y(I)**2)
      PHI(I)=ARCTA(X(I),Y(I))
      CALL POLY(N,A,RP(I),CP(I),PR(I),PC(I),RHO(I),PHI(I))
      ABSP(I)=DSQRT(RP(I)**2+CP(I)**2)
      IF(ABSPZ-ABSP(I)) 431,431,421
  421 ABSPZ=ABSP(I)
      IM=I
  431 CONTINUE
      IF(IM) 441,441,461
  441 DZ=DZ/8.0D0
      IF(RHOZ) 443,445,443
  443 IF(DZ/RHOZ-1.0D-8) 451,451,401
  445 IF(DZ-1.0D-8) 451,451,401
  451 IF(SU-ABSPZ) 501,501,5001
  461 DZ=(2.0D0**(IM-2))*DZ
      XZ=X(IM)
      YZ=Y(IM)
      PHIZ=PHI(IM)
      PRZ=PR(IM)
      PCZ=PC(IM)
      RHOZ=RHO(IM)
      RZ=RP(IM)
      CZ=CP(IM)
      GO TO 221
  501 DZ=1.0D0
      DTHETA=PI/10.0D0
  521 THETA=0.0D0
      DO 561 I=1,20
      THETA=THETA+DTHETA
      XS=XZ+DZ*DCOS(PHIZ+THETA)
      YS=YZ+DZ*DSIN(PHIZ+THETA)
      RHOS=DSQRT(XS**2+YS**2)
      PHIS=ARCTA(XS,YS)
      CALL POLY(N,A,RS,CS,PRS,PCS,RHOS,PHIS)
      ABSP(1)=DSQRT(RS**2+CS**2)
      IF(ABSPZ-ABSP(1)) 561,561,601
  561 CONTINUE
      DZ=DZ/2.0D0
      IF(RHOS) 563,565,563
  563 IF(DZ/RHOS-1.0D-8) 5001,5001,521
  565 IF(DZ-1.0D-8) 5001,5001,521
```

```
  601 XZ=XS
      YZ=YS
      PHIZ=PHIS
      RHOZ=RHOS
      ABSPZ=ABSP(1)
      PRZ=PRS
      PCZ=PCS
      RZ=RS
      CZ=CS
      GO TO 221
  701 IF(PSI-1.0D-7) 711,711,351
  711 IF(SU-ABSPZ) 501,501,351
  801 RHO(1)=RHOZ+BOT/SU**2
      IF(RHO(1)) 901,901,816
  816 PHI(1)=PHIZ+TOP/(RHOZ*SU**2)
  821 CALL POLY(N,A,RZ,CZ,PRZ,PCZ,RHO(1),PHI(1))
      ABSP(1)=DSQRT(RZ**2+CZ**2)
      IF(ABSP(1)-ABSPZ) 851,881,881
  841 XZ=RHOZ*DCOS(PHIZ)
      YZ=RHOZ*DSIN(PHIZ)
      GO TO 5001
  851 RHOZ=RHO(1)
      ABSPZ=ABSP(1)
      PHIZ=PHI(1)
      TOP=RZ*PCZ-CZ*PRZ
      BOT=-(RZ*PRZ+CZ*PCZ)
      SU=DSQRT(PRZ**2+PCZ**2)
      IF(SU) 855,501,855
  855 U=2.0D0*ABSPZ*SU
      PSI=DATAN(U)
      IF(ABSPZ/(RHOZ*SU)-10.0D0**(-MTST)) 861,861,901
  861 IF(ABSPZ/(RHOZ*SU)-1.0D-8) 841,841,871
  871 IF(PSI-1.0D-7) 881,881,801
  881 IF(SU-ABSPZ) 501,501,901
  901 DZ=ABSPZ/SU
      XZ=RHOZ*DCOS(PHIZ)
      YZ=RHOZ*DSIN(PHIZ)
      MTST=MTST+1
      GO TO 201
 5001 RETURN
      END
C
      SUBROUTINE MTALGD(AARG,NA,BARG,NB,C)
C
      DOUBLE PRECISION AARG(10),BARG(10),C(10),A(10),B(10),TEMP
     1 NAPL1=NA+1
      DO 21 J1=1,NAPL1
   21 A(J1)=AARG(J1)
      NBPL1=NB+1
      DO 41 J1=1,NBPL1
   41 B(J1)=BARG(J1)
      NCPL1=NAPL1+NBPL1-1
      DO 91 J1=1,NCPL1
```

```
      TEMP=0.0D0
      DO 81 J2=1,J1
      IF(J2-NAPL1) 61,61,81
   61 N2=J1-J2+1
      IF(N2-NBPL1) 71,71,81
   71 TEMP=TEMP+A(J2)*B(N2)
   81 CONTINUE
      C(J1)=TEMP
   91 CONTINUE
      RETURN
      END
C
      SUBROUTINE DIV(A,B,NA,NB,Q)
C
      DOUBLE PRECISION A(10),B(10),Q(10),TEMP
      I1=NA-NB+1
      DO 61 J1=1,I1
   61 Q(J1)=0.0D0
  101 KKMAX=NA-NB+1
      DO 391 KK=1,KKMAX
      K=KK-1
  201 TEMP=0.0D0
      IF(K-1) 301,211,211
  211 DO 291 JJ=1,K
      J=JJ-1
      I1=NB-K+J
      IF(I1) 291,221,221
  221 I2=NA-NB-J
      TEMP=TEMP+B(I1+1)*Q(I2+1)
  291 CONTINUE
  301 I1=NA-NB-K
      I2=NA-K
  391 Q(I1+1)=A(I2+1)-TEMP
 5001 RETURN
      END
C
      SUBROUTINE POLY(N,A,R,C,PR,PC,RHO,PHI)
C
      DOUBLE PRECISION A(10),R,C,PR,PC,RHO,PHI,V1,V2,W1,W2,T1
      IF(RHO) 10,5,10
    5 R=A(1)
      C=0.0D0
      PR=A(2)
      PC=0.0D0
      RETURN
   10 V1=1.0D0
      V2=0.0D0
      R=A(1)
      C=0.0D0
      PR=0.0D0
      PC=0.0D0
      W1=RHO*DCOS(PHI)
      W2=RHO*DSIN(PHI)
```

AD-A075 157 STATE UNIV OF NEW YORK COLL AT ONEONTA DEPT OF MATHEM--ETC F/G 17/4
MAXIMUM ENTROPY SPECTRAL DEMODULATOR INVESTIGATION.(U)
AUG 79 R G VAN METER F30602-75-C-0122

UNCLASSIFIED RADC-TR-79-209 NL

2 OF 2
AD A075157



END
DATE FILMED
11-79
DDC

```
      NN=N+1
      DO 20 I=2,NN
      T1=W1*V1-W2*V2
      V2=W2*V1+W1*V2
      V1=T1
      R=R+A(I)*V1
      C=C+A(I)*V2
      PR=PR+A(I)*(I-1)*V1
   20 PC=PC+A(I)*(I-1)*V2
      PR=PR/RHO
      PC=PC/RHO
 5001 RETURN
      END
C
      FUNCTION ARCTA(X,Y)
C
C        SUBPROGRAM TO COMPUTE ARCTANGENT
C        VALUES IN INTERVAL -PI TO PI
      DOUBLE PRECISION PI,HPI,X,Y,ARCTA
      PI=3.14159265358979324
      HPI=1.57079632679489662
      IF(X)1,2,3
    1 ARCTA=DATAN(Y/X)+PI*DSIGN(1.0D0,Y)
      RETURN
    3 ARCTA=DATAN(Y/X)
      RETURN
    2 IF(Y) 4,5,6
    4 ARCTA=-HPI
      RETURN
    5 ARCTA=0.0D0
      RETURN
    6 ARCTA=HPI
      RETURN
      END
C
      SUBROUTINE STATS(AM,SD,G)
C
C        STATS CALCULATES MEAN AND STANDARD DEVIATION
C        OF A SET OF 100 NUMBERS
      DOUBLE PRECISION G(100),S1,S2,RM,AM,SD
      COMMON/FREQ/FONE,FTWO
      COMMON/MAG/RMAG1,RMAG2
      S1=0.0D0
      S2=0.0D0
      RM=100.0D0
      DO 10 J=1,100
   10 S1=S1+G(J)
      AM=S1/RM
      DO 20 J=1,100
   20 S2=S2+(G(J)-AM)*(G(J)-AM)
      SD=DSQRT(S2/RM)
      RETURN
      END
```

MISSION
of
Rome Air Development Center

RADC plans and executes research, development, test and selected acquisition programs in support of Command, Control Communications and Intelligence (C^3I) activities. Technical and engineering support within areas of technical competence is provided to ESD Program Offices (POs) and other ESD elements. The principal technical mission areas are communications, electromagnetic guidance and control, surveillance of ground and aerospace objects, intelligence data collection and handling, information system technology, ionospheric propagation, solid state sciences, microwave physics and electronic reliability, maintainability and compatibility.