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EQUIVALENCES AND CONTINUITY IN MULTIVALENT PREFERENCE STRUCTURES

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Abstract

In assessing multiattribute utility functions, the valence approach partitions the elements of each attribute into equivalence classes on the basis of conditional preferences. Attribute interactions are reflected by these equivalence classes, so the functional forms of the utility representations are kept simple. This paper establishes equivalence relations for multivalent forms of additive independence, utility independence, and fractional independence, which lead to several new utility representation theorems. We show also that several simple partitions are not possible in multivalent preference structures if the utility function is continuous. These results should simplify the testing and assessment of utility functions when the attributes are interdependent.

Decision problems are usually complicated if there are numerous alternatives to consider, large uncertainties about the eventual outcomes of different alternatives, or several attributes on which to evaluate the consequences of each alternative. In many cases, multiattribute utility analysis is an appropriate methodology for handling these complexities. This methodology provides systematic procedures for examining preferences for outcomes, attitudes towards risk, and value trade-offs among attributes. The principal goal of a utility analysis is to produce a mathematical representation of preferences that will aid in the evaluation of risky decision alternatives.

Most previous research in multiattribute utility theory has focused on the *decomposition approach*. This approach prescribes how to divide the assessment of a decision maker's utility function into several steps, if certain "independence axioms" hold among the attributes in the decision problem. Each step requires either the determination of scaling constants or the estimation of conditional utility functions involving one or more attributes. The decomposition approach often simplifies the evaluation process in decision making, because less effort is required in completing these steps to assess a multiattribute utility function than might be expended in other methods of evaluation.

One drawback to the decomposition approach in practice, however, is that no independence axioms will be verified in some decision problems. This situation may occur if for some reason it is undesirable to conduct independence tests or if certain axioms are tested and subsequently rejected. Each set of axioms corresponds to a particular utility decomposition, but in the absence of any empirical verification one must guess at the form of the utility function. Depending on the nature of attribute interrelationships, simple approximations to the utility function may prove highly unsatisfactory. Since tests of independence axioms require moderate effort from both the analyst and the decision maker, it is advantageous to have other approaches available.

This paper discusses a *valence approach* for assessing multiattribute utility functions. This approach partitions the elements of each attribute into equivalence classes, called *orbitals*, such that preference orders conditioned on the elements within each orbital are identical. Unlike decomposition methods that use independence axioms over whole attributes, the valence approach considers *multivalent independence* axioms for which particular independence relations hold on the restriction of each attribute to any of its orbitals. This approach makes axiom tests much easier for the decision maker. Since preference interdependencies among attributes are reflected primarily in the orbitals, attribute interactions are readily interpreted and the functional forms of the utility representations are kept simple. Farquhar [6] illustrates the valence approach with single-element conditional preference orders and derives several utility representations with multivalent utility independence axioms.

The purpose of this paper is (1) to establish equivalence relations for generating the partitions in various multivalent preference structures, and (2) to examine the implications of continuity on possible partitions. We define equivalence relations that produce multivalent forms of additive independence, utility independence, and fractional independence, which lead to several new representation theorems. On the other hand, we show that some simple multivalent preference structures cannot occur if the utility function is continuous. These results should simplify the testing and assessment of utility functions when the attributes are interdependent.

1. INTRODUCTION TO MULTIVALENT PREFERENCE STRUCTURES

Let X denote the *outcome space* in a decision problem, and let P denote the space of all simple probability distributions (*lotteries*) over X. We assume that the *preference relation* $\succ$ on P is a strict weak order satisfying the von Neumann-Morgenstern axioms [8, 18]. Thus there exists a real-valued function u on X, called a *utility function* for $\succ$ on P, such that for all $p, q \in P$, $p \succ q$ iff (if and only if) $\sum_{x\in X} p(x)u(x) > \sum_{x \in X} q(x)u(x)$.

Let $X = Y \times Z$ initially, where Y and Z are *attribute* sets each containing at least two elements. Let P_Y denote the set of all lotteries on Y. Then the (single-element) *conditional preference order* $\succ_z$ induced on P_Y by the preference order $\succ$ on P and a fixed element $z \in Z$ is defined by

$$p_Y \succ_z q_Y \quad \text{iff} \quad (p_Y, z) \succ (q_Y, z), \tag{1}$$

where $p_Y, q_Y \in P_Y$.

If preferences for lotteries on Y depend on the elements in Z, then one can partition Z according to the distinct conditional preference orders on P_Y (see Farquhar [1, 5, 6]).

Definition 1: The *multivalent preference structure of* Y *given* Z is defined by $(Y, \Omega_Z, [Z])$ where for some nonempty index set $\mathcal{J}$,

(i) $\Omega_Z \equiv \{\succ_j : j \in \mathcal{J}\}$ denotes a collection of distinct preference orders, called *base orders*, on P_Y; and

(ii) $[Z] \equiv \{Z^j : j \in \mathcal{J}\}$ denotes a partition of Z into nonempty classes, called *orbitals*, such that $\succ_z = \succ_j$ for all $z \in Z^j$ and $j \in \mathcal{J}$.

Thus two elements z' and z" in Z belong to the same orbital iff $\succ_{z'} = \succ_{z''}$ on P_Y.

In the preference structure $(Y, \Omega_Z, [Z])$, *valence* refers to the cardinality of [Z]. At one extreme, the preference structure is *univalent* if $[Z] = \{Z\}$; at the other extreme, complete dependence of Y on Z occurs when [Z] consists of all single-element subsets of Z. Multivalent preference structures, therefore, cover an entire spectrum of interdependencies between attributes.

Instead of using conditional preference orders to determine [Z], we can obtain the same partition by using equivalent transformations of conditional utility functions. For example,

Definition 2: The relation *utility equivalence* (UE) on Z is defined by

z'(UE)z" iff there exist constants a and b with $b > 0$ such that

$$u(y, z'') = a + bu(y, z') \quad \text{for all } y \in Y. \tag{2}$$

Note that z'(UE)z" iff $\succ_{z'} = \succ_{z''}$, so the equivalence classes generated by (UE) are precisely the orbitals in [Z] above.

Although multiple-element conditional preference orders [1-4, 9-12] can partition Z into orbitals, it is simpler to use the corresponding equivalence relations. With single-element conditional preference orders, there is no advantage in either approach over the other.

Equivalence relations lead directly to multivalent independence axioms. Utility equivalence, for instance, leads to (cf., Lemma 1 in [6]),

Definition 3: Y is *multivalent utility independent* of [Z], denoted by Y(UI)[Z], iff [Z] is a partition of Z such that for all $\hat{Z} \in [Z]$,

$$z \in \hat{Z} \text{ iff } u(y, z) = \alpha_2(z) + \beta_2(z)u(y, \hat{z}) \quad \text{for all } y \in Y, \tag{3}$$

where $\hat{z}$ is fixed arbitrarily in $\hat{Z}$, and α_2 and β_2 are real functions on Z with $\beta_2 > 0$.

Analogously, $Z(UI)[Y]$ iff $[Y]$ is a partition of Y such that for all $\hat{Y} \in [Y]$,

$$y \in \hat{Y} \text{ iff } u(y, z) = \alpha_1(y) + \beta_1(y)u(\hat{y}, z) \quad \text{for all } z \in Z, \tag{4}$$

where $\hat{y}$ is fixed arbitrarily in $\hat{Y}$, and α_1 and β_1 are real functions on Y with $\beta_1 > 0$.

Univalent utility independence is the same as ordinary utility independence [13, 15, 17]. However, multivalent independence axioms yield representations of multiattribute utility functions when ordinary independence axioms fail. For example, Farquhar [1, 6] proves (see Meyer [16] also),

THEOREM 1: Let u be a von Neumann-Morgenstern utility function on the outcome space $Y \times Z$. Suppose $Y(UI)[Z]$ and $Z(UI)[Y]$. Then there exist real functions α_1 and β_1 on Y with $\beta_1 > 0$, real functions α_2 and β_2 on Z with $\beta_2 > 0$, and constants $\hat{k}$ depending on only the sets $\hat{Y} \times \hat{Z}$, where $\hat{Y} \in [Y]$ and $\hat{Z} \in [Z]$, such that u has one of the following *additive-multiplicative representations* for all $y \in \hat{Y}$ and $z \in \hat{Z}$:

$$u(y, z) = \alpha_1(y) + \alpha_2(z) + u(\hat{y}, \hat{z}), \tag{5a}$$

$$u(y, z) = \alpha_1(y) + \beta_1(y)u(\hat{y}, \hat{z}), \tag{5b}$$

$$u(y, z) = \alpha_2(z) + \beta_2(z)u(\hat{y}, \hat{z}), \tag{5c}$$

$$u(y, z) = \hat{k} + \beta_1(y)\beta_2(z)[u(\hat{y}, \hat{z}) - \hat{k}]. \tag{5d}$$

These additive-multiplicative representations are readily determined from the assessment of several constants and conditional utility functions [6].

2. MULTIVALENT ADDITIVE REPRESENTATIONS

Additive representations have received considerable attention in utility theory because they are so simple and easy to use. Fishburn [7-11], Pollak [17], and others describe independence axioms that produce additivity. When these axioms do not hold, however, it is often possible to obtain a multivalent additive representation. This approach offers distinct advantages over nonadditive utility decompositions which rely on more complicated sets of axioms.

Definition 4: The relation *additive equivalence* (AE) on Z is defined by

$$z'(AE)z'' \quad \text{iff} \quad u(y, z'') - u(y, z') = u(y^0, z'') - u(y^0, z'), \tag{6}$$

for all $y \in Y$, where y^0 is fixed arbitrarily in Y.

Additive equivalence induces a partition on Z which characterizes the following multivalent independence axiom.

Definition 5: Y is *multivalent additive independent* of [Z], denoted Y(AI)[Z], iff [Z] is a partition of Z such that for all $\hat{Z} \in [Z]$,

$$z \in \hat{Z} \quad \text{iff} \quad u(y, z) - u(y, \hat{z}) = u(y^0, z) - u(y^0, \hat{z}) \quad \text{for all } y \in Y, \tag{7}$$

where y^0 and $\hat{z}$ are fixed arbitrarily in Y and $\hat{Z}$, respectively.

Analogously, Z(AI)[Y] iff [Y] is a partition of Y such that for all $\hat{Y} \in [Y]$,

$$y \in \hat{Y} \text{ iff } u(y, z) - u(\hat{y}, z) = u(y, z^0) - u(\hat{y}, z^0) \quad \text{for all } z \in Z, \tag{8}$$

where $\hat{y}$ and z^0 are fixed arbitrarily in $\hat{Y}$ and Z, respectively.

We note that one univalent additive independence axiom (that is, either $[Z] = \{Z\}$ in (7) or $[Y] = \{Y\}$ in (8)) implies the other. Thus if we let $\hat{z} = z^0$ when $[Z] = \{Z\}$ or $\hat{y} = y^0$ when $[Y] = \{Y\}$, we obtain the usual additive representation,

$$u(y, z) = u(y, z^0) + u(y^0, z) - u(y^0, z^0), \tag{9}$$

for all $y \in Y$ and $z \in Z$. The utility function is often scaled so that $u(y^0, z^0)$ equals zero in (9).

The (univalent) additive representation in (9) is a special case of

THEOREM 2: Let u be a von Neumann-Morgenstern utility function on the outcome space $Y \times Z$. Suppose that $Y(AI)[Z]$ and $Z(AI)[Y]$. Then u has a *multivalent additive representation* on $\hat{Y} \times \hat{Z}$ such that for all $y \in \hat{Y}$ and $z \in \hat{Z}$ where $\hat{Y} \in [Y]$ and $\hat{Z} \in [Z]$,

$$u(y, z) = u(y, z^0) + u(y^0, z) - u(\hat{y}, z^0) - u(y^0, \hat{z}) + u(\hat{y}, \hat{z}), \tag{10}$$

where y^0, $\hat{y}$, z^0, and $\hat{z}$ are fixed arbitrarily in Y, $\hat{Y}$, Z, and $\hat{Z}$, respectively.

Proof: Since $Y(AI)[Z]$ and $Z(AI)[Y]$, equations (7) and (8) imply

$$u(y, z) = u(y^0, z) + u(y, \hat{z}) - u(y^0, \hat{z}) \quad \text{for all } (y, z) \in Y \times \hat{Z}, \tag{11}$$

$$u(y, z) = u(y, z^0) + u(\hat{y}, z) - u(\hat{y}, z^0) \quad \text{for all } (y, z) \in \hat{Y} \times Z, \tag{12}$$

where (y^0, z^0) is fixed arbitrarily in $Y \times Z$. Replacing z by $\hat{z}$ in (12) and substituting this result into (11) yields equation (10). ∎

Theorem 2 yields a spectrum of multivalent additive representations. We noted earlier that if either $[Y] = \{Y\}$ or $[Z] = \{Z\}$ above, then (10) reduces to the usual additive representation, $u(y, z) = u(y, z^0) + u(y^0, z) - u(y^0, z^0)$. At the other extreme, if each orbital in $[Y]$ and $[Z]$ contains exactly one element, then (10) trivially becomes $u(y, z) = u(y, z)$. Thus multivalent additivity ranges from full additivity to none at all.

The assessment of (10) requires one conditional utility function on each attribute, $u(y, z^0)$ and $u(y^0, z)$, and the utilities of the representative outcomes $(\hat{y}, \hat{z})$ for each $\hat{Y} \times \hat{Z}$. Thus the assessment effort is comparable to that required of the usual additive representation, though the number of constants needed increases with the number of orbitals in $[Y]$ and $[Z]$.

One can also obtain a multivalent additive representation from Theorem 1 when $\beta_1(y) \equiv 1$ on each $\hat{Y} \in [Y]$ in (4) and $\beta_2(z) \equiv 1$ on each $\hat{Z} \in [Z]$ in (3). These assumptions lead to the multivalent form in (5a), which is equivalent to (10). Further details are in [6].

Extension to more than two attributes

Suppose the outcome space is $X = X_1 \times \ldots \times X_n$. The multivalent additive representation is easily extended to n attributes by letting $Y \equiv X_{\bar{i}}$ and $Z \equiv X_i$ for $i = 1, \ldots, n$ in Theorem 2.

Let $x^0 = (x_1^0, \ldots, x_n^0)$ be fixed arbitrarily in X, and define $x_{(\bar{i})}^0 \equiv (x_1^0, \ldots, x_{i-1}^0, x_{i+1}^0, \ldots, x_n^0)$ for $i = 1, \ldots, n$. Then from Definition 5, $X_{\bar{i}}$ is multivalent additive independent of $[X_i]$, denoted $X_{\bar{i}}(AI)[X_i]$, if and only if $[X_i]$ is a partition of X_i for $i = 1, \ldots, n$, such that for all $\hat{X}_i \in [X_i]$,

$$x_i \in \hat{X}_i \quad \text{iff} \quad u(x_i, x_{\bar{i}}) - u(\hat{x}_i, x_{\bar{i}}) = u(x_i, x_{(\bar{i})}^0) - u(\hat{x}_i, x_{(\bar{i})}^0), \tag{13}$$

for all $x_{\bar{i}} \in X_{\bar{i}}$, where $\hat{x}_i$ is fixed arbitrarily in $\hat{X}_i$.

THEOREM 3: Let u be a von Neumann-Morgenstern utility function on the outcome space $X = X_1 \times \ldots \times X_n$. Suppose that $X_{\bar{i}}(AI)[X_i]$ for all i = 1, ..., n. For $\hat{X}_i \in [X_i]$, u has the following *multivalent additive representation* on each $\hat{X}_1 \times \ldots \times \hat{X}_n$,

$$u(x_1, \ldots, x_n) = \alpha_1(x_1) + \ldots + \alpha_n(x_n) + u(\hat{x}_1, \ldots, \hat{x}_n), \tag{14}$$

where $\alpha_i(x_i) \equiv u(x_i, x_{(\bar{i})}^0) - u(\hat{x}_i, x_{(\bar{i})}^0)$, and where x_i^0 and $\hat{x}_i$ are fixed arbitrarily in X_i and $\hat{X}_i$, respectively, for i = 1, ..., n.

Proof: Since $X_{\bar{i}}(AI)[X_i]$ for i = 1, ..., n, equation (13) yields

$$u(x_1, \ldots, x_n) = \alpha_i(x_i) + u(\hat{x}_i, x_{\bar{i}}) \quad \text{for all } x_{\bar{i}} \in X_{\bar{i}}. \tag{15}$$

When i = 1, the above equation becomes $u(x_1, \ldots, x_n) = \alpha_1(x_1) + u(\hat{x}_1, x_2, \ldots, x_n)$. With i = 2 and $x_1 = \hat{x}_1$ in (15), we obtain an expression for $u(\hat{x}_1, x_2, \ldots, x_n)$ and substitute it into the last equation to get $u(x_1, \ldots, x_n) = \alpha_1(x_1) + \alpha_2(x_2) + u(\hat{x}_1, \hat{x}_2, x_3, \ldots, x_n)$. Repeated substitution in (15) for i = 1, ..., n thus yields the result in (14). ■

In Theorem 3, the *joint multivalent additive independence* assumption $X_{\bar{i}}(AI)[X_i]$ for i = 1, ..., n produces a representation requiring the assessment of one conditional utility function on each of n attributes, and the utilities of the representative outcomes $(\hat{x}_1, \ldots, \hat{x}_n)$. This assessment requirement is far less than that required by most other nonadditive utility representations.

On the other hand, Theorem 2 can be extended to n attributes with individual independence assumptions where $Y \equiv X_i$ and $Z \equiv X_{\bar{i}}$ for $i = 1, \ldots, n$. Thus *individual multivalent additive independence* holds on X whenever $X_i(AI)[X_{\bar{i}}]$ for all $i = 1, \ldots, n$. A special type of individual independence requires partitions of the attributes $[X_1], \ldots, [X_n]$, such that $[X_{\bar{i}}]$ is expressed by $[X_{(\bar{i})}] \equiv [X_1] \times \ldots \times [X_{i-1}] \times [X_{i+1}] \times \ldots \times [X_n]$ for all $i = 1, \ldots, n$. *Correlative multivalent additive independence* holds on X iff $X_i(AI)[X_{(\bar{i})}]$ for all $i = 1, \ldots, n$. Correlative independence assures a stable orbital structure on individual attributes that is not necessarily found in arbitrary partitions of $X_{\bar{i}}$.

We use correlative independence in establishing another type of multivalent additive representation. Let $(x_i, \hat{x}_{(\bar{i})})$ denote the outcome $(\hat{x}_1, \ldots, \hat{x}_{i-1}, x_i, \hat{x}_{i+1}, \ldots, \hat{x}_n)$ where $\hat{x}_j \in \hat{X}_j$, $\hat{X}_j \in [X_j]$, for all $j \neq i$ and $i = 1, \ldots, n$. From Definition 5, $X_i(AI)[X_{(\bar{i})}]$ for all $i = 1, \ldots, n$, if and only if there exist partitions of the attributes $[X_1], \ldots, [X_n]$, such that for all $\hat{X}_{(\bar{i})} \in [X_{(\bar{i})}]$,

$$x_{\bar{i}} \in \hat{X}_{(\bar{i})} \quad \text{iff} \quad u(x_i, x_{\bar{i}}) - u(x_i, \hat{x}_{(\bar{i})}) = u(x_i^0, x_{\bar{i}}) - u(x_i^0, \hat{x}_{(\bar{i})}), \tag{16}$$

for all $x_i \in X_i$.

<u>THEOREM 4</u>: Let u be a von Neumann-Morgenstern utility function on the outcome space $X = X_1 \times \ldots \times X_n$. Suppose that $X_i(AI)[X_{(\bar{i})}]$ for all $i = 1, \ldots, n$. For $\hat{X}_i \in [X_i]$, u has the following *multivalent additive representation* on each $\hat{X}_1 \times \ldots \times \hat{X}_n$,

$$u(x_1, \ldots, x_n) = \hat{\alpha}_1(x_1) + \ldots + \hat{\alpha}_n(x_n) + u(x_1^0, \ldots, x_n^0), \tag{17}$$

where $\hat{\alpha}_i(x_i) \equiv u(x_i, \hat{x}_{(\bar{i})}) - u(x_i^0, \hat{x}_{(\bar{i})})$, for x_i^0 and $\hat{x}_i$ fixed arbitrarily in X_i and $\hat{X}_i$, respectively, for all $i = 1, \ldots, n$.

Proof: Since $X_i(AI)[X_{(\bar{i})}]$ for all $i = 1, \ldots, n$, (16) gives

$$u(x_1, \ldots, x_n) = \hat{\alpha}_i(x_i) + u(x_i^0, x_{\bar{i}}) \quad \text{for all } x_i \in X_i. \tag{18}$$

If $i = 1$, (18) gives $u(x_1, \ldots, x_n) = \hat{\alpha}_1(x_1) + u(x_1^0, x_2, \ldots, x_n)$. With $i = 2$ and $x_1 = x_1^0$ in (18), we substitute for $u(x_1^0, x_2, \ldots, x_n)$ in this last equation and obtain $u(x_1, \ldots, x_n) = \hat{\alpha}_1(x_1) + \hat{\alpha}_2(x_2) + u(x_1^0, x_2^0, x_3, \ldots, x_n)$. Successive substitution in (18) for $i = 1, \ldots, n$ yields the result in (17). ■

When $[X_i] = \{X_i\}$ for all $i = 1, \ldots, n$, we let $\hat{x}_i = x_i^0$ in either (14) or (17) and obtain the usual additive representation $u(x_1, \ldots, x_n) = u(x_1, x_{(\bar{1})}^0) + \ldots + u(x_n, x_{(\bar{n})}^0) - (n-1)u(x_1^0, \ldots, x_n^0)$. In general, however, the representation in (17) may require several conditional utility functions on each attribute, while the representation in (14) requires only one conditional utility function on each attribute.

There are other sets of axioms that give multivalent additive representations, but we do not discuss them here. Further details are in [4-7, 9-11].

3. MULTIVALENT FRACTIONAL REPRESENTATIONS

A *hypercube* is a collection of 2^n n-dimensional vertices of the form $(a_1, \ldots, a_n)$ where $a_j \in \{0, 1\}$ for all $j = 1, \ldots, n$. A *fractional hypercube*, or *fraction*, H_i on dimension i is a subset of hypercube vertices satisfying

$$(a_1, \ldots, a_i, \ldots, a_n) \in H_i \quad \text{implies} \quad (a_1, \ldots, \bar{a}_i, \ldots, a_n) \notin H_i, \tag{19}$$

for a particular $i = 1, \ldots, n$, where $\bar{a}_i \equiv 1 - a_i$. A fraction H_i therefore contains at most one vertex on any i-edge of a hypercube.

One can define a class of multiple-element conditional preference orders based on fractional hypercubes [1-4]. These fractional orders yield a variety of independence axioms for modeling attribute interactions. Equivalence relations and multivalent independence axioms for fractional hypercubes are discussed next.

A *primal fraction* F_i is a fractional hypercube containing the apex $(1, \ldots, 1)$. Let $|F_i|$ denote the number of vertices in F_i.

Definition 6: Suppose $x^0 = (x_1^0, \ldots, x_n^0)$ is fixed in X. The *fractional order* $(\succ_i | x_{\bar{i}}, F_i)$ induced on P_{X_i} by the preference order $\succ$ on P for a given $x_{\bar{i}} \in X_{\bar{i}}$ and primal fraction F_i, where $i = 1, \ldots, n$, is defined by

$$p_i^1 \, (\succ_i | x_{\bar{i}}, F_i) \, p_i^0 \quad \text{iff} \quad \mathcal{F}_i \succ \bar{\mathcal{F}}_i \tag{20}$$

for $p_i^0, p_i^1 \in P_{X_i}$. $\mathcal{F}_i$ and $\bar{\mathcal{F}}_i$ are lotteries on P defined by

$$\mathcal{F}_i \equiv \sum \frac{1}{|F_i|} (x_1^{b_1}, \ldots, x_{i-1}^{b_{i-1}}, p_i^{a_i}, x_{i+1}^{b_{i+1}}, \ldots, x_n^{b_n}), \tag{21a}$$

$$\bar{\mathcal{F}}_i \equiv \sum \frac{1}{|F_i|} (x_1^{b_1}, \ldots, x_{i-1}^{b_{i-1}}, p_i^{\bar{a}_i}, x_{i+1}^{b_{i+1}}, \ldots, x_n^{b_n}), \tag{21b}$$

where the summations range over all $(a_1, \ldots, a_n) \in F_i$; when $a_j = 1$, b_j indicates the absence of any superscript, and when $a_j = 0$, $b_j = 0$, for all $j \neq i$ in (21).

Definition 7: Suppose $x^0 = (x_1^0, \ldots, x_n^0)$ is fixed in X. The *generator function* $g_i(p_i | x_{\bar{i}}, F_i)$ on P_{X_i} for a given $x_{\bar{i}} \in X_{\bar{i}}$ and a primal fraction F_i, for $i = 1, \ldots, n$, is defined by

$$g_i(p_i | x_{\bar{i}}, F_i) = \sum (-1)^{\bar{a}_i} u(x_1^{b_1}, \ldots, x_{i-1}^{b_{i-1}}, p_i, x_{i+1}^{b_{i+1}}, \ldots, x_n^{b_n}), \tag{22}$$

where the summation ranges over all $(a_1, \ldots, a_n) \in F_i$, and b_j for $j \neq i$ is defined above.

Farquhar [1, 2] proves that the relation $(\succ_i | x_{\bar{i}}, F_i)$ is a strict weak order on P_{X_i} for any $x_{\bar{i}} \in X_{\bar{i}}$ and any primal fraction F_i. He also establishes that the generator function $g_i(p_i | x_{\bar{i}}, F_i)$ is

linear: $g_i(\alpha p_i + (1 - \alpha) q_i | x_{\bar{i}}, F_i) =$

$$\alpha g_i(p_i | x_{\bar{i}}, F_i) + (1 - \alpha) g_i(q_i | x_{\bar{i}}, F_i), \tag{23a}$$

and *order-preserving:* $p_i^1 \ (\succ_i | x_{\bar{i}}, F_i) \ p_i^0$ iff

$$g_i(p_i^1 | x_{\bar{i}}, F_i) > g_i(p_i^0 | x_{\bar{i}}, F_i), \tag{23b}$$

for $p_i, q_i, p_i^0, p_i^1 \in P_{X_i}$ and $i = 1, \ldots, n$.

Define the *dual order* $\succ^*$ of $\succ$ on P by $p \succ^* q$ iff $q \succ p$ for all $p, q \in P$. A function is *order-reversing* for $\succ$ on P iff it is order-preserving for $\succ^*$ on P. Then the following result on transformations [2, 10] is applicable to fractional orders and generators, and is used to establish representation theorems.

LEMMA 1: Suppose $\succ$ is a strict weak order on P. Let g_1 and g_2 be real functions on P. If g_1 is linear and order-preserving for $\succ$ on P, and g_2 is linear and order-preserving (order-reversing) for $\succ$ on P, then g_1 and g_2 are related by a positive (negative) linear transformation: there exist constants α and β with $\beta > 0$ ($\beta < 0$) such that $g_2 \equiv \alpha + \beta g_1$ on P. Whenever $\succ = \phi$, g_1 and g_2 are constant, so let $\beta = 0$ above.

Two examples illustrate these definitions. For simplicity, let $X = Y \times Z$. Define the *apex fraction* by $F_i \equiv \{(1, 1)\}$ for $i = 1, 2$. Then the *apex order* on P_Y is given by the conditional preference order $\succ_z$ in (1), and the *apex generator* on P_Y is merely the expected utility function $u(p_Y, z) \equiv \Sigma_{y \in Y}\, p_Y(y)u(y, z)$. Similarly, define the *diagonal fraction* by $F_i \equiv \{(1, 1), (0, 0)\}$ for $i = 1, 2$. Then the *diagonal order* $(\succ_1 | z, F_1) \equiv \succ_{z,z^0}$ on P_Y for a given $z \in Z$ and a fixed $z^0 \in Z$ is defined as follows,

$$p_Y^1 \succ_{z,z^0} p_Y^0 \quad \text{iff}$$

$$\tfrac{1}{2}(p_Y^1, z) + \tfrac{1}{2}(p_Y^0, z^0) \succ \tfrac{1}{2}(p_Y^0, z) + \tfrac{1}{2}(p_Y^1, z^0), \tag{24}$$

for $p_Y^0, p_Y^1 \in P_Y$. The *diagonal generator* on P_Y is given by

$$g_1(p_Y | z, F_1) \equiv u(p_Y, z) - u(p_Y, z^0). \tag{25}$$

The diagonal order in (24) is motivated by the failure of additivity. For example, Fishburn [10] proves that for $z^0, z^1 \in Z$, $z^1(AE)z^0$ iff $\succ_{z^1,z^0} = \phi$. Hence Y and Z are additive independent if and only if $\succ_{z,z^0} = \phi$ for all $z \in Z$. In general, X_i and $X_{\bar{i}}$ are additive independent if and only if $(\succ_i | x_{\bar{i}}, F_i) = \phi$ for all $x_{\bar{i}} \in X_{\bar{i}}$, where F_i is any primal fraction with $|F_i| > 1$.

Next, we define an equivalence relation and a multivalent independence axiom for fractional hypercubes.

Definition 8: Suppose F_i is a primal fraction for a given $i = 1, \ldots, n$. The relation *fractional equivalence* (FE) on $X_{\bar{i}}$ with respect to F_i is defined by

$x_{\bar{i}}'(FE)x_{\bar{i}}''$ iff there exist constants a and b with $b > 0$ such that

$$g_i(x_i | x_{\bar{i}}'', F_i) = a + b g_i(x_i | x_{\bar{i}}', F_i) \quad \text{for all } x_i \in X_i. \tag{26}$$

We note that utility equivalence in Definition 2 is a special case of fractional equivalence. We observe also that Lemma 1 and the properties in (23) imply that $x_{\bar{i}}'(FE)x_{\bar{i}}''$ iff $(\succ_i | x_{\bar{i}}', F_i) = (\succ_i | x_{\bar{i}}'', F_i)$.

Definition 9: Suppose F_i is a primal fraction for a given $i = 1, \ldots, n$. Then X_i is *multivalent fractionally independent* of $[X_{\bar{i}}]$ with respect to F_i, denoted by $X_i(FI)[X_{\bar{i}}]$, if and only if $[X_{\bar{i}}]$ is a partition of $X_{\bar{i}}$ such that for all $\hat{X}_{\bar{i}} \in [X_{\bar{i}}]$,

$$x_{\bar{i}} \in \hat{X}_{\bar{i}} \text{ iff } g_i(x_i | x_{\bar{i}}, F_i) = \alpha_i(x_{\bar{i}}) + \beta_i(x_{\bar{i}}) g_i(x_i | \hat{x}_{\bar{i}}, F_i), \tag{27}$$

for all $x_i \in X_i$, where $\hat{x}_{\bar{i}}$ is fixed arbitrarily in $\hat{X}_{\bar{i}}$, and α_i and β_i are real functions on $X_{\bar{i}}$ with $\beta_i > 0$.

Univalent fractional independence is the same as ordinary fractional independence [1-3]. Although multivalent fractional independence axioms can lead to many interesting new representations, we consider only the diagonal fraction here and illustrate the results for two attributes [2, 9-10].

THEOREM 5: Let u be a von Neumann-Morgenstern utility function on the outcome space $Y \times Z$. Suppose that Y and Z are multivalent diagonally independent of [Z] and [Y], respectively. Then u has a *multivalent diagonal representation* on each $\hat{Y} \times \hat{Z}$, for $\hat{Y} \in [Y]$ and $\hat{Z} \in [Z]$,

$$u(y, z) = u(y, z^0) + u(y^0, z) + \hat{c}\beta_1(y)\beta_2(z), \tag{28}$$

where $\hat{c} \equiv u(\hat{y}, \hat{z}) - u(\hat{y}, z^0) - u(y^0, \hat{z})$ for arbitrarily fixed $\hat{y}$ in $\hat{Y}$ and $\hat{z}$ in $\hat{Z}$, and

$$\beta_1(y) = \begin{cases} 1 & \text{if } \quad z^1(AE)z^0 \text{ for } y \in \hat{Y}, \\ \dfrac{u(y, z^1) - u(y, z^0) - u(y^0, z^1)}{u(\hat{y}, z^1) - u(\hat{y}, z^0) - u(y^0, z^1)} & \text{if not } z^1(AE)z^0 \text{ for } y \in \hat{Y}, \end{cases} \tag{29a}$$

$$\beta_2(z) = \begin{cases} 1 & \text{if } \quad y^1(AE)y^0 \text{ for } z \in \hat{Z}, \\ \dfrac{u(y^1, z) - u(y^0, z) - u(y^1, z^0)}{u(y^1, \hat{z}) - u(y^0, \hat{z}) - u(y^1, z^0)} & \text{if not } y^1(AE)y^0 \text{ for } z \in \hat{Z}, \end{cases} \tag{29b}$$

for distinct $y^0, y^1 \in Y$, and $z^0, z^1 \in Z$, with $u(y^0, z^0) \equiv 0$.

Remarks: The multivalent diagonal representation in (28) requires at most the assessment of four conditional utility functions, $u(y, z^0)$, $u(y, z^1)$, $u(y^0, z)$, and $u(y^1, z)$, and the utilities assigned to $(\hat{y}, \hat{z})$ in $\hat{Y} \times \hat{Z}$. This assessment requirement is identical to that of the additive-multiplicative representations in Theorem 1 when certain *uniform preferability* assumptions are met (see Corollary 1 in [6]). Special cases of the multivalent diagonal representation yield the forms in (5). The ordinary diagonal representation occurs if $[Y] = \{Y\}$ and $[Z] = \{Z\}$ in Theorem 5.

For consistency with other results involving two attributes, the subscripts of α and β in (27) are denoted by $\bar{i}$ instead of i. Also, F_i is suppressed in writing generator functions.

Proof: Since Y and Z are multivalent diagonally independent of $[Z]$ and $[Y]$, respectively, Definition 9 implies

$$g_1(y|z) = \alpha_2(z) + \beta_2(z)g_1(y|\hat{z}) \quad \text{for all } y \in Y,\ z \in \hat{Z}, \tag{30a}$$

$$g_2(z|y) = \alpha_1(y) + \beta_1(y)g_2(z|\hat{y}) \quad \text{for all } y \in \hat{Y},\ z \in Z. \tag{30b}$$

From (25), we note that $g_1(y|z) = u(y, z) - u(y, z^0)$ and $g_2(z|y) = u(y, z) - u(y^0, z)$. Therefore, (30a) and (30b) give

$$u(y, z) = \alpha_2(z) + \beta_2(z)u(y, \hat{z}) + (1-\beta_2(z))u(y, z^0), \tag{31a}$$

$$u(y, z) = \alpha_1(y) + \beta_1(y)u(\hat{y}, z) + (1-\beta_1(y))u(y^0, z). \tag{31b}$$

From (31a) we get expressions for $u(\hat{y}, z)$ and $u(y^0, z)$ to substitute into (31b); after some simplification, we obtain the following intermediate result.

$$\begin{aligned} u(y, z) = {} & \alpha_1(y) + \alpha_2(z) + \beta_1(y)\beta_2(z)u(\hat{y}, \hat{z}) \\ & + \beta_1(y)(1-\beta_2(z))u(\hat{y}, z^0) + (1-\beta_1(y))\beta_2(z)u(y^0, \hat{z}) \\ & + (1-\beta_1(y))(1-\beta_2(z))u(y^0, z^0) \quad \text{for all } y \in \hat{Y},\ z \in \hat{Z}. \end{aligned} \tag{32}$$

If we scale the utility function so that $u(y^0, z^0) = 0$, the last term vanishes and (32) gives

$$u(y, z) = [\alpha_1(y) + \beta_1(y)u(\hat{y}, z^0)] + [\alpha_2(z) + \beta_2(z)u(y^0, \hat{z})]$$

$$+ \beta_1(y)\beta_2(z)[u(\hat{y}, \hat{z}) - u(\hat{y}, z^0) - u(y^0, \hat{z})], \quad (33)$$

for all $y \in \hat{Y}$ and $z \in \hat{Z}$. The first term on the right in (33) equals $u(y, z^0)$, which follows from (31b) with $z = z^0$; likewise, the second term above equals $u(y^0, z)$ from (31a) with $y = y^0$. If $\hat{c} \equiv [u(\hat{y}, z) - u(\hat{y}, z^0) - u(y^0, \hat{z})]$, then (33) yields the desired representation in (28).

The expressions for $\beta_1(y)$ and $\beta_2(z)$ are obtained by solving the equations generated by (30a) with distinct y^0, $y^1 \in Y$, and the equations generated by (30b) with z^0, $z^1 \in Z$, respectively. The results in (29a,b) follow directly by substitution of each generator function. ■

Theorem 5 generalizes to n attributes in a straightforward manner using the assumption of correlative multivalent diagonal independence. Further representation theorems are possible with multivalent independence axioms based on other fractional hypercubes.

Generalizations

Fractional equivalence and multivalent fractional independence involve positive, linear transformations. These definitions, however, can be extended by considering nonnull, linear transformations. Thus, *generalized fractional equivalence* is derived from Definition 8 by replacing "$b > 0$" with "$b \neq 0$." Likewise, *generalized multivalent fractional independence* is derived from Definition 9 by replacing "$\beta_i > 0$" with "$\beta_i \neq 0$." There is no effect on the form of the representations obtained from the generalized axioms, as indicated by Lemma 1. The orbital structure is different though:

$x_{\bar{i}} \in \hat{X}_{\bar{i}}$ iff $(\succ_i | x_{\bar{i}}, F_i) \in \{(\succ_i | \hat{x}_{\bar{i}}, F_i), (\succ_i | \hat{x}_{\bar{i}}, F_i)^*\}$. For example, z' and z'' are generalized utility equivalent if and only if either $\succ_{z'} = \succ_{z''}$ or $\succ_{z'} = \succ^*_{z''}$. Also note that removing the nonnull restriction in the generalized version of Definition 8 gives a relation which need not be an equivalence. Further discussion on generalized independence and its uses is in [2-4, 10-12].

The assumption that u is a von Neumann-Morgenstern utility function is unnecessary in all the theorems presented so far. We need only assume that u is a real function on $X_1 \times \ldots \times X_n$. Regarding u as a utility function, however, helps in interpretation.

4. CONTINUITY AND MULTIVALENT PREFERENCES

The equivalence relations and representation theorems in previous sections do not presuppose any special structure on the function u or the attributes $X_1, \ldots, X_n$. Our concern with continuity, however, requires that these attributes be topological spaces. To focus on a simple case, we consider two attributes Y and Z which are each subsets of finite-dimensional Euclidean spaces.

The function u is *continuous* on Z iff for each fixed $y \in Y$ and each sequence $\{z_t\}$ in Z, it follows that $u(y, z_t) \to u(y, z)$ whenever $z_t \to z$ and $z \in Z$. The element γ is a *limit point* of a subset A of Z iff there is a sequence $\{z_t\}$ in A such that $z_t \to \gamma$. If A and B are nonempty subsets of Z, then A *touches* B, denoted A(T)B, iff for every $\delta > 0$ there exist $z_a \in A$ and $z_b \in B$ such that $||z_a - z_b|| < \delta$, where $||\ldots||$ denotes the Euclidean metric. Analogous definitions hold on the attribute Y.

Continuity and additive equivalence

THEOREM 6: Suppose u is continuous on Z and $z, z_1, z_2, \ldots \in Z$. If $z_t \to z$ and $z_s(AE)z_t$ for all s, t, then $z(AE)z_t$ for all $t = 1, 2, \ldots$. Similarly, suppose u is continuous on Y and $y, y_1, y_2, \ldots \in Y$. If $y_t \to y$ and $y_s(AE)y_t$ for all s, t, then $y(AE)y_t$ for all $t = 1, 2, \ldots$.

Note: Henceforth in our theorems, we shall omit the latter halves pertaining to Y since they are symmetric to the first halves pertaining to Z.

Proof: From Definition 4, $z(AE)z_t$ iff $u(y, z) - u(y, z_t) = u(y^0, z) - u(y^0, z_t)$ for all $y \in Y$, where y^0 is fixed arbitrarily in Y, and $t = 1, 2, \ldots$. Continuity on Z implies $u(y, z) - u(y, z_t) = \lim[u(y, z_s) - u(y, z_t)]$ as $s \to \infty$. Since $z_s(AE)z_t$ for all s, t, $\lim[u(y, z_s) - u(y, z_t)] = \lim[u(y^0, z_s) - u(y^0, z_t)] = u(y^0, z) - u(y^0, z_t)$. The proof for $y(AE)y_t$ merely interchanges the roles of y and z. ■

Additive equivalence (AE) *holds* on a subset A of Z iff $z'(AE)z''$ for all $z', z'' \in A$. Theorem 6 and the transitivity of (AE) therefore imply

COROLLARY 1: Suppose u is continuous on Z. Let A and B be nonempty subsets of Z such that A(T)B, and for some $z \in Z$, let z be a limit point of both A and B. If (AE) holds on A and (AE) holds on B, then (AE) holds on $A \cup B$.

If u is uniformly continuous on Z, one does not have to assume the existence of a common limit point for A and B in Corollary 1: if A(T)B, the conclusions follow for additive equivalence. This stronger result relies on the special form of additive equivalence, and does not extend to utility and fractional equivalence.

Corollary 1 shows, for example, that if $Z = [0, 1]$ and u is continuous on Z, then an open subinterval of Z or a half-open subinterval of Z cannot be an orbital of [Z] under additive equivalence. It is impossible to have $[Z] = \{[0, 1/2), [1/2, 1]\}$, for instance, when u is continuous and Y(AI)[Z].

One might conjecture that if u is continuous on Z and Y(AI)[Z], then [Z] is a trivial partition of Z, that is, either $[Z] = \{Z\}$ or [Z] consists of all the single-element subsets of Z. The following example shows this conjecture is false: let $Y = Z = [0, 1]$ and define u by

$$u(y, z) = z + y(z - 1/2)^2 \quad \text{for } 0 \le y, z \le 1. \tag{34}$$

It is easily seen that u is continuous on Z and each orbital in [Z]--other than $\{1/2\}$--has two elements, namely $1/2 - \delta$ and $1/2 + \delta$ for $0 < \delta \le 1/2$. (On the other hand, [Y] consists of all the unit subsets, since $y'(AE)y''$ iff $y' = y''$ in (34).) Many other examples of nontrivial orbital structures can be given, too.

Continuity and utility equivalence

We say that $z \in Z$ is *essential* iff the conditional preference order $\succ_z \neq \emptyset$; in other words, z is essential iff there exist some $y^0, y^1 \in Y$ such that $u(y^1, z) > u(y^0, z)$.

THEOREM 7: Suppose u is continuous on Z and $z, z_1, z_2, \ldots \in Z$. If z is essential, $z_t \to z$, and $z_s(UE)z_t$ for all s, t, then $z(UE)z_t$ for all $t = 1, 2, \ldots$.

Proof: With the above assumptions, utility equivalence implies that there are sequences of constants $a_1, a_2, \ldots$ and positive constants $b_1, b_2, \ldots$ such that $u(y, z_t) = a_t + b_t u(y, z_1)$ for all $y \in Y$ and $t = 1, 2, \ldots$. Given

$u(y^1, z) > u(y^0, z)$ by the essentiality of z, and given the other assumptions above, $u(y^1, z) - u(y^0, z) = \lim[u(y^1, z_t) - u(y^0, z_t)] =$ $\lim(b_t[u(y^1, z_1) - u(y^0, z_1)]) = (\lim b_t)[u(y^1, z_1) - u(y^0, z_1)]$. Hence,

$$\lim_{t\to\infty} b_t = \frac{u(y^1, z) - u(y^0, z)}{u(y^1, z_1) - u(y^0, z_1)} > 0.$$

With $b \equiv \lim b_t$ it then follows that $\lim a_t = u(y, z) - bu(y, z_1)$ for all $y \in Y$. Therefore, $u(y, z) = a + bu(y, z_1)$ for all $y \in Y$, where $a \equiv \lim a_t$, so $z(UE)z_1$. By the transitivity of (UE), $z(UE)z_t$ for all $t = 1, 2, \ldots$. ■

Suppose all of the assumptions of Theorem 7 hold except for essentiality, so that $u(y^0, z) = u(y^1, z)$ for all $y^0, y^1 \in Y$. Then $\lim(b_t[u(y^1, z_1) - u(y^0, z_1)] = 0$, for all $y^0, y^1 \in Y$, which implies either that $\lim b_t = 0$ or that $u(y^1, z_1) = u(y^0, z_1)$ for all $y^0, y^1 \in Y$. In the latter case, $u(y, z) = a + u(y, z_1)$ for some constant a and all $y \in Y$; hence $z(UE)z_1$ and consequently $z(UE)z_t$ for all $t = 1, 2, \ldots$. When $\lim b_t = 0$, however, one can only conclude that $u(y, z) = \lim a_t$ for all $y \in Y$. In particular, if $\lim b_t = 0$, $u(y, z_1)$ need not be constant over Y; when this happens, it is not true that $z(UE)z_1$.

The next result follows directly from Theorem 7 and the transitivity of (UE).

COROLLARY 2: Suppose u is continuous on Z. Let A and B be nonempty subsets of Z such that $A(T)B$, and for some $z \in Z$, let z be essential and a limit point of both A and B. If (UE) holds on A and (UE) holds on B, then (UE) holds on $A \cup B$.

Continuity and fractional equivalence

Suppose z^0 is fixed in Z and $(\succ_1 | z, F_1)$ is the fractional order on P_Y for a given $z \in Z$ and any primal fraction F_1 where $|F_1| > 1$. Then z is *nonadditive* (with respect to z^0) if and only if $(\succ_1 | z, F_1) \neq \phi$. From earlier remarks, note that $z(AE)z^0$ iff $(\succ_1 | z, F_1) = \phi$. Nonadditivity is an extension of the concept of essentiality to multiple-element conditional preference orders.

THEOREM 8: Suppose u is continuous on Z and $z, z_1, z_2, \ldots \in Z$. Suppose that z^0 is fixed in Z and $(\succ_1 | z, F_1)$ denotes the fractional order on P_Y for any primal fraction F_1 where $|F_1| > 1$. If z is nonadditive, $z_t \to z$, and $z_s(FE)z_t$ for all s, t, then $z(FE)z_t$ for all $t = 1, 2, \ldots$.

Proof: The proof is virtually identical to the proof of Theorem 7 with $u(y, z)$ and $u(y, z_t)$ replaced by the generator functions $g_1(y|z, F_1)$ and $g_1(y|z_t, F_1)$, respectively. Note that continuity of u on Z implies continuity of g_1 on Z. ■

The next result follows from Theorem 8 and the transitivity of (FE).

COROLLARY 3: Suppose u is continuous on Z. Let A and B be nonempty subsets of Z such that A(T)B, and for some $z \in Z$, let z be nonadditive and a limit point of both A and B. For any primal fraction F_1 where $|F_1| > 1$, if (FE) holds on A and (FE) holds on B, then (FE) holds on $A \cup B$.

Further remarks

The extension of the continuity theorems and corollaries to more than two attributes follows the development in Section 2. If all the equivalence relations in this section are replaced by generalized equivalence relations, the results still hold. As noted earlier, those parts of Theorems 7 and 8 and the corollaries dealing with equivalence and continuity on Y are omitted for brevity.

The effect of continuity of u on Z is exhibited in the orbital structure [Z]. The basic results of this section (given some minor regularity conditions) imply that if a sequence of elements in an orbital $\hat{Z}$ converges to an element $z \in Z$, then z also belongs to $\hat{Z}$. As a corollary, if two sequences of equivalent elements have a common limit point in Z, then both sequences belong to the same orbital. These results have strong implications on the admissibility of various orbital structures when Y is multivalent independent of [Z].

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In eliciting multiattribute utility functions, the valence approach partitions the elements of each attribute into equivalence classes on the basis of conditional preferences. This report establishes equivalence relations for multivalent forms of additive independence, utility independence, and fractional independence, which lead to several new utility representation theorems. When the utility function is continuous, however, some simple partitions are impossible in multivalent preference structures. These results should simplify testing and assessment procedures when attributes are interdependent.

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