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THE AERODYNAMICS OF SUPERSONIC BIPLANES OF FINITE SPAN

H. S. TAN
CORNELL UNIVERSITY

DECEMBER 1950

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THE AERODYNAMICS OF SUPERSONIC BIPLANES OF FINITE SPAN

H. S. Tan
Cornell University

December 1950

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Air Research and Development Command
United States Air Force
Wright-Patterson Air Force Base, Ohio

Foreword

This report is submitted as part of Contract AF33(038)-9832, which was administered by the eronautical Research Laboratory, Wright Air Development Center, under RDO No. 465-1, Aerodynamics of Compressible Fluids, with Mr. Lee S. Wasserman acting as project engineer. The work reported was done at the Graduate School of Aeronautical Engineering at Cornell University during 1950.

The investigation is based on the author's thesis (Reference 3) and on a paper by Professor W.R. Sears and the author (Reference 6). Here the work is extended and is put into more usable form.

The author wishes to acknowledge the assistance of both, Professor W.R. Sears for supervising the work and Mr. H.K. Cheng for carefully checking the formulas and carrying out the tedious computation work.

Abstract

Linearized supersonic-Flow theory is employed to evaluate the lift and drag of biplane cellules having the "Busemann-biplane" configuration. The lift and drag are explicitly expressed as functions of the thickness ratio and the angle of attack; the coefficients involved are universal for all Busemann biplane. Interpretation of the results for various Mach numbers is afforded by a similarity rule.

Most of the results are presented graphically. It is found that the wave drag due to thickness of finite-span Busemann biplanes is small.

Publication Review

The publication of this report does not constitute approval by the Air Force of the findings or the conclusions contained therein. It is published only for the exchange and stimulation of ideas.

FOR THE COMMANDER:

Leslie B. Williams

LESLIE B. WILLIAMS, Colonel, USAF
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Introduction

A study of the linearized equation of motion of supersonic flow, with the velocity potential ϕ as dependent variable, reveals that the differential equation is of hyperbolic type. As is well known, a Cauchy's boundary value problem is then properly set if initial data over an open surface is given. This is exactly the case with a supersonic wing. Thus, following the exhaustive study by J. Hadamard, by using the singular fundamental solution, introducing the notion of finite part of an improper integral, and generalizing the Green's formula to a space with non-positive-definite metric, a solution for the differential system is obtained which bears exactly the same relation to the distributed boundary value as the potential to its corresponding source distribution in Dirichlet's problem in the elliptic case. Accordingly, the terms "supersonic source" and "supersonic distance" were introduced.

Further investigation of the problem shows that we can always introduce a fictitious plane separating the retrogressive characteristic cone, so that by suitably distributing the boundary value on the two surfaces of the plane, two distinct solutions can be obtained for the two separated regions. Thus the introduction of a diaphragm and the exclusive use of source distribution for the determination of potential, following Evvard, can be justified. The detailed proofs of this "supersonic source-distribution theory" are successively developed in Refs. 1, 2, 3.

Thus, the study of supersonic wings resolves itself essentially into the same study of the potential due to plane supersonic source distributions.

In the linearized theory, the Busemann biplane arrangement becomes the one shown in Fig. 1, i.e., the top and bottom surfaces are flat, the leading-edge Mach wave of either wing intersects the other wing at mid-chord, and the airfoil slopes are related by the formulas, for $x > c/2$

$$Y_1'(x) = -Y_2'(x - \tfrac{c}{2})$$

$$Y_2'(x) = -Y_1'(x - \tfrac{c}{2})$$

The typical case is then simply that of two isosceles triangles pointing at each other.

In this investigation, the Busemann relationship between gap, chord, and Mach angles shown in Fig. 1 will be assumed, but it will not be necessary to specify the shape of the profile in deriving some general results. It will be shown that the velocity potential, including all interaction effects, can be calculated by means of integrations involving the wing surface slopes only. The general results will be applied to the numerical evaluation of the wave lift and drag coefficients of the typical Busemann arrangement having triangular wing sections.

This investigation, conducted at Cornell University, was sponsored by Aeronautical Research Laboratory, Wright Air Development Center, Wright-Patterson Air Force Base, Ohio

Notation

Coordinates

Cartesian $x, y, z; \xi, \eta, \zeta$

Mach $u_1, v_1, z; u, v, \zeta$

U uniform free stream velocity in x -direction

ϕ disturbance velocity potential

p, ρ pressure, density (p_0 , ρ_0 for free stream)

M free-stream Mach number

β $\sqrt{M^2-1}$

C_p pressure coefficient

n normal

σ local slope of wing surface in flow direction

λ local slope of diaphragm in flow direction

q supersonic source intensity

δ wedge angle

$2a$ chord

$2b$ span

c gap $= a/\beta$

S integration area over surface being considered

S' integration area over interacting surface of other wing

Subscripts:

T, B top and bottom surface

u, l upper and lower wing

I, II of wing and of diaphragm

PART I. THEORY

A. THEORY OF SUPERSONIC BIPLANE

Formulas for Source Distributions

The equation satisfied by the disturbance velocity potential ϕ in the linearized theory is

$$\beta^2 \phi_{xx} - \phi_{yy} - \phi_{zz} = 0 \;; \qquad \beta^2 = M^2 - 1 \tag{1}$$

where subscripts denote partial differentiation with respect to the rectangular Cartesian coordinates x, y, z. Here M denotes the free-stream Mach number, and the coordinate x is taken in the direction of the undisturbed stream. It has been assumed in deriving Eq. (1) that ϕ_x, ϕ_y, and ϕ_z are small compared to the stream speed, U. A consistent approximate formula for the pressure coefficient is

$$C_p = \frac{p - p_0}{\frac{1}{2} \rho_0 U^2} = -2 \phi_x / U \tag{2}$$

where p_0, ρ_0 are the pressure and density of the undisturbed stream.

An elementary solution of Eq. (1) is the so-called supersonic source, $\phi(x, y, z) = [(x - \xi)^2 - \beta^2 (y - \eta)^2 - \beta^2 (z - \zeta)^2]^{-1/2}$, provided that the value zero is taken outside of the Mach cone that originates at the point ξ, η, ζ. For brevity, we shall adopt the following notation:

$$\mu(z) \equiv [(x - \xi)^2 - \beta^2 (y - \eta)^2 - \beta^2 z^2]^{-1/2}$$

It is well known (Refs. 2, 3) that a continuous distribution q of these singularities over a surface parallelto the flow yields a solution satisfying Eq. (1) and the boundary condition $\partial\phi/\partial n = \pi q$ on the surface. Moreover, Evvard (Ref. 4) has shown how a distribution of these sources over a fictitious diaphragm at a wing tip can be used to account for the interaction of upper and lower surfaces of a monoplane wing.

We shall adopt Evvard's scheme here for the calculation of tip effects for both upper and lower wings, placing a diaphragm at each wing tip and introducing the conditions that these diaphragms are stream surfaces of the flow. The potential at points on the top (T) and bottom (B) surfaces of the upper (u) wings is given by

$$\phi_{uT}(x,y) = -\int_S q_{uT}\,\mu(0)\,dS \tag{3}$$

$$\phi_{uB}(x,y) = -\int_S q_{uB}\,\mu(0)\,dS - \int_{S'} q_{LT}\,\mu(c)\,dS \tag{4}$$

and there are analogous formulas for the lower (l) wing. The areas of integration S, on the wing under consideration, and S' on the other wing, are shown in Fig. 2.

Now the integrations over portions of S and S' can be simplified immediately by use of monoplane results. First of all, it is clear that, in all areas unaffected by biplane interaction, the wing-surface boundary condition requires that $q = U\sigma/\pi$ where σ is the slope of the wing profile in the x direction.

Moreover, Evvard has shown, that for monoplanes - and therefore for biplane regions unaffected by interwing interaction - the integration over the diaphragm can be replaced by another integration over part of the wing. For any point forward of mid-chord, i.e., $x < a$, there can be no biplane interaction, hence it is convenient to write the relatively simple expressions for these points before going on to treat the interacting regions.

$x < a$: no biplane interaction: Here monoplane results are applicable. For both upper and lower wings, we have

$$\phi_T(x,y) = -\frac{U}{\pi}\int_{S_I} \sigma_T \mu(o)\, dS - \frac{U}{\pi}\int_{S_{I_o}} \frac{\sigma_B - \sigma_T}{2}\mu(o)\, dS \tag{5}$$

$$\phi_B(x,y) = -\frac{U}{\pi}\int_{S_I} \sigma_B \mu(o)\, dS - \frac{U}{\pi}\int_{S_{I_o}} \frac{\sigma_T - \sigma_B}{2}\mu(o)\, dS \tag{6}$$

$x > a$: We consider now a point on the upper wing, top surface. If the point lies forward of the Mach line from the tip mid-chord (outside of area N in Fig. 3), there is again no biplane interference and Eqs. (5) and (6) apply. For a point in N , however, there exists an effect of the lower wing, transmitted through the interaction regions of the tip diaphragm. We can write

$$\phi_{uT}(x,y) = -\frac{U}{\pi}\int_{S_I} \sigma_{uT} \mu(o)\, dS - \frac{U}{\pi}\int_{S_{II}} \lambda_u \mu(o)\, dS \tag{7}$$

Here, and in subsequent formulas, we denote by $\lambda(\xi,\eta)$ the slopes of the tip diaphragms of upper and lower wings. Thus, for any point on the top of the upper-wing diaphragm, q_{uT} is equal to $U\lambda_u/\pi$, and this value has been used in eq. (7). In regions unaffected by biplane interaction (e.g. for $\xi < a$), $\lambda(\xi,\eta)$ is the same as for a monoplane and Evvard's results will be used for such regions. In interacting regions λ is still unknown, of course; its determination constitutes the main problem of this investigation. We shall postpone this to the next section, after writing an analogous formula for points on the bottom surface of the upper wing.

All points of the bottom surface of the upper wing, for which $x > a$ are affected by biplane interaction. Let S_I' and S_{II}' denote the areas of the lower wing and its diaphragm that affect the point (x, y) . The wing-surface boundary condition is

$$q_{uB}(x,y) + \frac{U}{\pi^2}\frac{\partial}{\partial c}\left[\int_{S_I'} \sigma_{LT}\,\mu(c)\,dS + \int_{S_{II}'} \lambda_L\,\mu(c)\,dS\right] = \frac{U}{\pi}\sigma_{uB}(x,y) \qquad (8)$$

This is an explicit formula for $q_{uB}(x,y)$, involving only known quantities. It may be noted that in the region S_{II}', q_{LT} has been put equal to $U\lambda_L/\pi$. Moreover, here λ_L is a monoplane value unaffected by biplane interference, and is therefore known from Evvard's work. We now have

$$\phi_{uB}(x,y) = -\int_{S_I + S_{II}} q_{uB}\,\mu(o)\,dS$$
$$-\frac{U}{\pi}\int_{S_I'} \sigma_{LT}\,\mu(c)\,dS - \frac{U}{\pi}\int_{S_{II}'} \lambda_L\,\mu(c)\,dS \qquad (9)$$

where q_{uB} in S_I is known from Eq. (8) and λ_L in S_{II}' is known from monoplane theory. Again the calculation of the diaphragm source distribution, q_{uB} in S_{II}, is postponed to the next section.

For the lower wing there are formulas exactly analogous to Eqs. (7), (8) and (9), which will not be written out here.

2. Calculation of Diaphragm Distributions

The conditions that insure that the tip diaphragms will be stream surfaces are the conditions of equal slope and equal pressure on top and bottom. Since, as Evvard has pointed out (Ref. 5), the diaphragms of a rectangular wing tip are not vortex sheets, equal pressures imply equal values of ϕ, the perturbation velocity potential. We have, then, in region S_{II},

$$\frac{\partial \phi_T}{\partial z} = \frac{\partial \phi_B}{\partial z} \quad \text{and} \quad \phi_T = \phi_B \tag{10}$$

The first of these equations leads to

$$\begin{aligned} \frac{U}{\pi}\lambda_u(x,y) &= q_{uT}(x,y) \\ &= -q_{uB}(x,y) - \frac{U}{\pi^2}\frac{\partial}{\partial c}\Big[\int_{S_I'}\sigma_{LT}\,\mu(c)\,dS + \int_{S_{II}'}\lambda_L\,\mu(c)\,dS\Big] \end{aligned} \tag{11}$$

The second Eq. (10) states that, in S_{II} ,

$$-\frac{U}{\pi}\int_{S_I}\sigma_{uT}\,\mu(0)\,dS - \frac{U}{\pi}\int_{S_{II}}\lambda_u\,\mu(0)\,dS$$

$$= -\int_{S_I+S_{II}} q_{uB}\,\mu(0)\,dS - \frac{U}{\pi}\int_{S_I'}\sigma_{LT}\,\mu(c)\,dS - \frac{U}{\pi}\int_{S_{II}'}\lambda_L\,\mu(c)\,dS \qquad (12)$$

where q_{uB} in S_I and S_{II} is given by Eqs. (8) and (11), respectively. We have now an integral equation for the diaphragm slope λ_u : for points x , y in S_{II} ,

$$2\int_{S_{II}}\lambda_u\,\mu(0)\,dS = \int_{S_I}(\sigma_{uB}-\sigma_{uT})\,\mu(0)\,dS + \int_{S_I'}\sigma_{LT}\,\mu(c)\,dS + \int_{S_{II}'}\lambda_L\,\mu(c)\,dS$$

$$-\frac{1}{\pi}\int_{S_I+S_{II}}\mu(0)\,\frac{\partial}{\partial c}\left[\int_{S_I'}\sigma_{LT}\,\mu(c)\,dS^* + \int_{S_{II}'}\lambda_L\,\mu(c)\,dS^*\right]dS \qquad (13)$$

There is an analogous equation for λ_L , which will not be written out.

Eq. (13) is to be satisfied for all points x , y on the upper-wing diaphragm. For some areas, there is no biplane interaction, i.e., S_I' and S_{II}' vanish, so that the second and third integrals on the right side of Eq. (13) disappear. It is clear that for these points the third integral vanishes as well, since S_I and S_{II} do not contain any points ξ, η affected by interaction. Consequently, for non-interacting points x, y, Eq. (13) reduces to Evvard's integral equation for the diaphragm slope of a monoplane (Ref. 4).

3. Solution of the Integral Equation

Eq. (13) can be written in the form

$$\int_{S_{II}} \lambda_u \, \mu(0) \, dS = F_1(x, y) \tag{14}$$

for points x, y in S_{II}, where $2F_1(x, y)$ denotes the entire right-hand side of Eq. (13), and involves only known functions. We now introduce the new coordinates, u, v measured along the two families of Mach lines on the wing in question:

$$\begin{aligned} u &= \frac{M}{2\beta}(\xi + \beta\eta) \qquad v = \frac{M}{2\beta}(\xi - \beta\eta) \\ \xi &= \frac{\beta}{M}(u + v) \qquad \eta = \frac{1}{M}(u - v) \\ |J| &= \left| \frac{\partial(\xi, \eta)}{\partial(u, v)} \right| = \frac{2\beta}{M^2} \\ \mu(z) &\equiv \left\{ (x - \xi)^2 - \beta^2(y - \eta)^2 - \beta^2 z^2 \right\}^{-1/2} \\ &= \frac{M}{2\beta} \left\{ (u_1 - u)(v_1 - v) - M^2 z^2/4 \right\}^{-1/2} \end{aligned} \tag{15}$$

Our integral equation now takes the form

$$\int_0^{u_1} \frac{du}{\sqrt{u_1 - u}} \int_u^{v_1} \frac{\lambda(u, v) \, dv}{\sqrt{v_1 - v}} = F(u_1, v_1) \tag{16}$$

for points u_1, v_1 in S_{II}.

The solution can now be found by means of the following process:

$$\int_0^{u'} \frac{F(u_1,v_1)\,du_1}{\sqrt{u'-u_1}} = \int_0^{u'} \frac{du_1}{\sqrt{u'-u_1}} \int_0^{u_1} \frac{du}{\sqrt{u_1-u}} \int_u^{v_1} \frac{\lambda(u,v)\,dv}{\sqrt{v_1-v}}$$

$$= \int_0^{u'} \frac{du_1}{\sqrt{u'-u_1}} \int_0^{u_1} \frac{H(u,v_1)}{\sqrt{u_1-u}}\,du, \qquad \text{say}$$

$$= \int_0^{u'} H(u,v_1)\,du \int_u^{u'} \frac{du_1}{\sqrt{u'-u_1}\sqrt{u_1-u}}$$

$$= \pi \int_0^{u'} H(u,v_1)\,du \tag{17}$$

Differentiating this result with respect to u', we have

$$\frac{\partial}{\partial u'} \int_0^{u'} \frac{F(u_1,v_1)}{\sqrt{u'-u_1}}\,du_1 = \pi \int_{u'}^{v_1} \frac{\lambda(u',v)}{\sqrt{v_1-v}}\,dv \tag{18}$$

We now multiply both sides of Eq. (18) by $1/\sqrt{v'-v_1}$ integrate with respect to v_1, and exchange order of integration in a manner similar to that just employed. The result is

$$\int_{u'}^{v'} \frac{dv_1}{\sqrt{v'-v_1}} \left(\frac{\partial}{\partial u'} \int_0^{u'} \frac{F(u_1,v_1)\,du_1}{\sqrt{u'-u_1}} \right) = \pi^2 \int_{u'}^{v'} \lambda(u',v)\,dv \tag{19}$$

which implies (dropping the primes)

$$\lambda(\mu, \nu) = \frac{1}{\pi^2}\frac{\partial}{\partial \nu}\int_{\mu}^{\nu}\left(\frac{\partial}{\partial \mu}\int_{0}^{\mu}\frac{F(\mu_1, \nu_1)\, d\mu_1}{\sqrt{\mu - \mu_1}}\right)\frac{d\nu_1}{\sqrt{\nu - \nu_1}} \tag{20}$$

This solution can be used to calculate the slopes λ_μ in regions of interaction. This completes Eq. (7) for ϕ_{uT}, and, by use of Eq. (11), also completes Eq. (9) for ϕ_{uB}. Eq. (20) constitutes a generalization of Evvard's expression for the tip-diaphragm slope, to which, in fact, it immediately reduces when μ, ν lie in a region free of biplane interaction.

4. Calculation of the Potential

Although the biplane problem is now completely solved in principle, the straightforward calculation of ϕ, especially for regions of biplane interference, by substitution in Eqs. (7) and (9), is extremely tedious. Fortunately, as will now be shown, it is possible to eliminate entirely the integration involving λ_μ in these two formulas.

In both Eqs. (7) and (9), the term involving λ_u is

$$\int_{S_{II}} \lambda_\mu \mu(0)\, dS = \frac{1}{M}\int_{0}^{\nu_1}\frac{du}{\sqrt{\mu_1 - u}}\int_{\mu}^{\nu_1}\frac{\lambda u(u,v)\, dv}{\sqrt{\nu_1 - \nu}} \tag{21}$$

where now μ_1, ν_1 lie in region S_I.

We return to Eq. (13), which holds for points in S_{II}, and write it in the form

$$\int_0^{u_1} \frac{du}{\sqrt{u_1 - u}}\, G(u, v_1) = -\frac{M\pi}{2U}\, \phi'(u_1, v_1) \tag{22}$$

where

$$G(u, v_1) = \int_u^{v_1} \frac{\lambda_u(u,v)\,dv}{\sqrt{v_1 - v}} - \frac{1}{2}\int_{-u}^{u} \frac{(\sigma_{uB} - \sigma_{uT})\,dv}{\sqrt{v_1 - v}} - \frac{1}{2U}\int_{-u}^{v_1} \frac{dv}{\sqrt{v_1 - v}}\, \frac{\partial}{\partial c}\phi'(u,v) \tag{23}$$

$$\phi'(u_1, v_1) = \frac{U}{\pi}\left\{ \int_{S_I'} \sigma_{LT}\, \mu(c)\, dS + \int_{S_{II}'} \lambda_L\, \mu(c)\, dS \right\} \tag{24}$$

Actually, ϕ' is the potential contributed at u_1, v_1 by the lower wing.

The solution of Eq. (22) can be written down immediately (Ref. 6); viz.,

$$G(u, v_1) = -\frac{M}{2U}\frac{\partial}{\partial u}\int_0^{u} \frac{\phi'(u', v_1)}{\sqrt{u - u'}}\, du' \tag{25}$$

Since Eq. (22) is correct only for points u_1 , v_1 in S_I - i.e., for $u_1 \leq v_1$ - we must restrict u in Eq. (25) as indicated.

Now for the points outside of the interaction region, i.e., for $u' \leq M^2c^2/4v_1$, the interaction potential $\phi'(u', v_1)$ is zero. Thus $G(u, v_1)$ is also zero for $u < M^2c^2/4v_1$.

We can now consider an integral involving $G(u, v_1)$; i.e.,

$$I(u_1, v_1, \kappa) \equiv \int_0^{\kappa} \frac{du}{\sqrt{u_1 - u}} G(u, v_1) = \int_{M^2c^2/4v_1}^{\kappa} \frac{du}{\sqrt{u_1 - u}} G(u, v_1)$$

where $\kappa \leq u_1$.

If $\kappa \leq v_1$ also, $G(u, v_1)$ can be taken from Eq. (25):

$$I(u_1, v_1, \kappa) = -\frac{M}{2U} \int_{M^2c^2/4v_1}^{\kappa} \frac{du}{\sqrt{u_1 - u}} \left\{ \frac{\partial}{\partial u} \int_{M^2c^2/4v_1}^{u} \frac{\phi'(u', v_1)\, du'}{\sqrt{u - u'}} \right\}$$

$$= -\frac{M}{2U} \int_{M^2c^2/4v_1}^{\kappa} \phi'(u', v_1) \sqrt{\frac{\kappa - u'}{u_1 - \kappa}} \left[\frac{1}{\kappa - u'} - \frac{1}{u_1 - u'} \right] du' \qquad (26)$$

after some manipulation.* Recalling the meaning of $G(u, v_1)$, (Eq. (23)), we can write Eq. (26) as

---

* Ref. 3, p. 36-40.

$$\int_0^{\kappa}\frac{du}{\sqrt{u_1-u}}\int_u^{v_1}\frac{\lambda_u(u,v)\,dv}{\sqrt{v_1-v}} = -\frac{M}{2U}\int_{M^2c^2/4v_1}^{\kappa}\phi'(u',v_1)\sqrt{\frac{\kappa-u'}{u_1-\kappa}}\left[\frac{1}{\kappa-u'}-\frac{1}{u_1-u'}\right]du'$$

$$+\frac{1}{2}\int_0^{\kappa}\frac{du}{\sqrt{u_1-u}}\left\{\int_{-u}^{u}\frac{(\sigma_{uB}-\sigma_{uT})dv}{\sqrt{v_1-v}}+\frac{1}{U}\int_{-u}^{v_1}\frac{dv}{\sqrt{v_1-v}}\frac{\partial}{\partial c}\phi'(u,v)\right\} \tag{27}$$

Since the only restrictions on Eq. (27) are $\kappa \leq u_1$, and $\kappa \leq v_1$, it is exactly the result we need for Eq. (21), in which $\kappa \leq v_1 \leq u_1$.

We are now prepared to write complete expressions for the potential on top and bottom surfaces of the upper wing, by substitution in Eqs. (7) and (9). Let S_{I_0} be the portion of S_I for which $u \leq v_1$, as indicated in Fig. 3; then

$$\phi_{uT}(x,y) = -\frac{U}{\pi}\int_{S_I}\sigma_{uT}\,\mu(0)\,dS - \frac{U}{2\pi}\int_{S_{I_0}}(\sigma_{uB}-\sigma_{uT})\,\mu(0)\,dS$$

$$-\frac{1}{2\pi}\int_{S_{I_0}+S_{II}}\mu(0)\frac{\partial}{\partial c}\phi'(u,v)\,dS$$

$$+\frac{1}{2\pi}\int_{M^2c^2/4v_1}^{v_1}\phi'(u',v_1)\sqrt{\frac{v_1-u'}{u_1-v_1}}\left[\frac{1}{v_1-u'}-\frac{1}{u_1-u'}\right]du' \tag{28}$$

$$\phi_{uB}(x,y) = -\frac{U}{\pi}\int_{S_I}\sigma_{uB}\,\mu(0)\,dS - \frac{1}{\pi}\int_{S_I+S_{II}}\mu(0)\frac{\partial}{\partial c}\phi'(u,v)\,dS$$

$$+\frac{U}{2\pi}\int_{S_{I_0}}(\sigma_{uB}-\sigma_{uT})\,\mu(0)\,dS + \frac{1}{2\pi}\int_{S_{I_0}+S_{II}}\mu(0)\frac{\partial}{\partial c}\phi'(u,v)\,dS$$

$$-\frac{1}{2\pi}\int_{M^2c^2/4v_1}^{v_1}\phi'(u',v_1)\sqrt{\frac{v_1-u'}{u_1-v_1}}\left[\frac{1}{v_1-u'}-\frac{1}{u_1-u'}\right]du' \tag{29}$$

$$+\phi'(x,y)$$

Formulas (28) and (29) permit the calculation of the potential, and consequently the pressure distribution, on the biplane. It is seen that, whereas we have succeeded in eliminating the integrals involving λ_u, for the upper wing, we are left with integrals involving λ_l, to be taken over certain interaction-free areas. In fact, if interplane interaction of a higher order were encountered, such as an area of the lower wing influenced by interacting regions of the upper wing, it would always be possible to eliminate the λ integral expressing the last stage of tip interaction.

B. SIMILARITY RULE FOR SUPERSONIC BIPLANE THEORY

(1) Conical and Pseudo-conical Flow Field

It is well known that the flow field in the tip region of a supersonic flat rectangular wing is conical. If instead of being flat, we have, denoting by σ the surface slope along flow direction x, that $\sigma(x, y) = \sigma(x)$ only - i.e., along each chord station x = constant, the surface slope is spanwise constant - then we have, in place of a conical flow field, a flow field which is "pseudo-conical." Thus, denoting by ϕ the disturbance velocity potential, the conical flow field will be characterized by

$$\phi = \kappa x \oint\left(\frac{\beta y}{x}\right) \tag{1}$$

while the "pseudo-conical" flow field will be characterized by

$$\phi = \kappa x \oint\left(\frac{\beta y}{x}, \frac{a}{x}\right) \tag{2}$$

where $\beta = \sqrt{M^2 - 1}$, M = free stream Mach number.

While the conical flow field finds great application in the monoplane supersonic wing theory, it will be shown that a rectangular supersonic biplane flow consists in addition, in the interaction region, of a "pseudo-conical" flow field. Thus the success of developing a biplane wing theory depends much on the existence of the latter type flow.

(2) Case of Busemann Biplane

Let us take a Busemann biplane, and follow the result and notation of Part I where the potential at any point (x, y) of upper wing top and bottom surface respectively are given as follows:

$$
\begin{aligned}
\phi_{uT}(x,y) = & -\frac{U}{\pi}\int_{S_I}\sigma_{uT}\,\mu(0)\,dS - \frac{U}{2\pi}\int_{S_{I_0}}(\sigma_{uB}-\sigma_{uT})\,\mu(0)\,dS \\
& -\frac{1}{2\pi}\int_{S_{I_0}+S_{II}}\mu(0)\frac{\partial}{\partial c}\phi'(u,v)\,dS \\
& +\frac{1}{2\pi}\int_{M^2c^2/4v_1}^{v_1}\phi'(u',v_1)\sqrt{\frac{v_1-u'}{u_1-v_1}}\left[\frac{1}{v_1-u'}-\frac{1}{u_1-u'}\right]du'
\end{aligned}
\qquad (3)
$$

$$
\begin{aligned}
\phi_{uB}(x,y) = & -\frac{U}{\pi}\int_{S_I}\sigma_{uB}\,\mu(0)\,dS - \frac{1}{\pi}\int_{S_I+S_{II}}\mu(0)\frac{\partial}{\partial c}\phi'(u,v)\,dS \\
& +\frac{U}{2\pi}\int_{S_{I_0}}(\sigma_{uB}-\sigma_{uT})\,\mu(0)\,dS + \frac{1}{2\pi}\int_{S_{I_0}+S_{II}}\mu(0)\frac{\partial}{\partial c}\phi'(u,v)\,dS \\
& -\frac{1}{2\pi}\int_{M^2c^2/4v_1}^{v_1}\phi'(u',v_1)\sqrt{\frac{v_1-u'}{u_1-u'}}\left[\frac{1}{v_1-u'}-\frac{1}{u_1-u'}\right]du' + \phi'(x,y)
\end{aligned}
\qquad (4)
$$

in which

$$\sigma_{uT} = -\sigma_{LB} = -\alpha$$

$$
\begin{aligned}
\sigma_{uB} = \sigma_{LT} &= \alpha+\delta & x<a \\
&= -\alpha+\delta & x>a
\end{aligned}
$$

It was shown in Ref. 3 that

a) $$\phi_u' = \frac{U\sigma_{LT}}{M\pi} I_1 - \frac{U(\sigma_{LB}-\sigma_{LT})}{M\pi} I_2 \tag{5a}$$

$$I_1 = -2\left[\begin{array}{l} \frac{u_1+v_1}{2}\left(\frac{\pi}{2} + \sin^{-1}\frac{u_1-v_1}{\sqrt{(u_1+v_1)^2-M^2C^2}}\right) + \frac{u_1-v_1}{2}\cosh^{-1}\frac{u_1+v_1}{\sqrt{(u_1-v_1)^2+M^2C^2}} \\ -\frac{MC}{2}\left(\sin^{-1}\frac{u_1(u_1+v_1)-M^2C^2/2}{u_1\sqrt{(u_1+v_1)^2-M^2C^2}} + \sin^{-1}\frac{u_1(u_1-v_1)+M^2C^2/2}{u_1\sqrt{(u_1-v_1)^2+M^2C^2}}\right) \end{array}\right]$$

$$I_2 = -\left\{\begin{array}{l} 2\sqrt{u_1v_1 - M^2C^2/4} - \frac{u_1-v_1}{2}\cosh^{-1}\frac{u_1+v_1}{\sqrt{(u_1-v_1)^2+M^2C^2}} \\ + \frac{u_1+v_1}{2}\left[\sin^{-1}\frac{u_1-v_1}{\sqrt{(u_1+v_1)^2-M^2C^2}} + \sin^{-1}\frac{2v_1-\sqrt{(u_1-v_1)^2+M^2C^2}}{\sqrt{(u_1+v_1)^2-M^2C^2}}\right] \\ -\frac{MC}{2}\left[\begin{array}{l}\frac{\pi}{2} - \sin^{-1}\frac{u_1^2-v_1^2-M^2C^2+(u_1+v_1)\sqrt{(u_1-v_1)^2+M^2C^2}}{\sqrt{(u_1+v_1)^2-M^2C^2}\,[u_1-v_1+\sqrt{(u_1-v_1)^2+M^2C^2}]} \\ + \sin^{-1}\frac{u_1(u_1+v_1)-M^2C^2/2}{u_1\sqrt{(u_1+v_1)^2-M^2C^2}} - \sin^{-1}\frac{u_1(u_1-v_1)+M^2C^2/2}{u_1\sqrt{(u_1-v_1)^2+M^2C^2}}\end{array}\right] \\ -\sqrt{v_1(u_1-v_1) - M^2C^2/2 + v_1\sqrt{(u_1-v_1)^2+M^2C^2}} \end{array}\right\}$$

Using the relation

$$u_1 = \frac{M}{2\beta}(x+\beta y) \qquad v_1 = \frac{M}{2\beta}(x-\beta y)$$

$$|J| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{2\beta}{M^2}$$

$$\mu(Z) \equiv \left[(x-\xi)^2-\beta^2(y-\eta)^2-\beta^2 z^2\right]^{-1/2}$$

$$= \frac{M}{2\beta}\left[(u_1-u)(v_1-v) - M^2z^2/4\right]^{-1/2}$$

(5a) can be put in form

$$\phi_u' = \frac{Ux}{\pi\beta}\left[\alpha\,\Phi_1\!\left(\frac{\beta y}{x},\frac{a}{x}\right) + \delta\,\Phi_2\!\left(\frac{\beta y}{x},\frac{a}{x}\right)\right] \tag{5}$$

b) $$\frac{\partial\phi_u'}{\partial C} = \frac{U}{\pi}\left[\sigma_{LT}I_3 + (\sigma_{LB}-\sigma_{LT})I_4\right] \tag{6a}$$

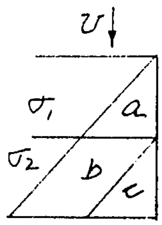
$$I_3 = \frac{\pi}{2} + \tan^{-1}\frac{u_1^2-v_1^2}{2MC\sqrt{u_1v_1 - M^2C^2/4}}$$

$$I_4 = -\frac{1}{2}\left(\begin{array}{l} \frac{\pi}{2} - \sin^{-1}\frac{(u_1-v_1)u_1+M^2C^2/2}{u_1\sqrt{(u_1-v_1)^2+M^2C^2}} + \sin^{-1}\frac{(u_1+v_1)u_1-M^2C^2/2}{u_1\sqrt{(u_1+v_1)^2-M^2C^2}} \\ -\sin^{-1}\frac{(u_1^2-v_1^2)-M^2C^2+(u_1+v_1)\sqrt{(u_1-v_1)^2+M^2C^2}}{\sqrt{(u_1+v_1)^2-M^2C^2}\,[u_1-v_1+\sqrt{(u_1-v_1)^2+M^2C^2}]} \\ +\frac{2MC}{\sqrt{(u_1-v_1)^2+M^2C^2}}\,\frac{\sqrt{v_1(u_1-v_1)-M^2C^2/2+v_1\sqrt{(u_1-v_1)^2+M^2C^2}}}{u_1-v_1+\sqrt{(u_1-v_1)^2+M^2C^2}} \end{array}\right)$$

(6a) can be put in form

$$\frac{\partial \phi_u'}{\partial c} = \frac{U}{\pi}\left[\alpha\,\Phi_3\left(\frac{v}{u}, \frac{Mc}{u}\right) + \delta\,\Phi_4\left(\frac{v}{u}, \frac{Mc}{u}\right)\right] \tag{6}$$

c) $-\frac{U}{2\pi}\int_{S_I}\sigma\,\mu(0)\,dS =$



$$-\frac{U\sigma_1 x}{\pi\beta}\left[\frac{\beta y}{x}\cosh^{-1}\frac{x}{\beta y} + \sin^{-1}\frac{\beta y}{x} + \frac{\pi}{2}\right]$$

$$-\frac{U\sigma_1 x}{\pi\beta}\left[\frac{\beta y}{x}\cosh^{-1}\frac{x}{\beta y} + \sin^{-1}\frac{\beta y}{x} + \frac{\pi}{2}\left(2\frac{a}{x} - 1\right)\right] - \frac{U\sigma_2 x}{\pi\beta}\left(1 - \frac{a}{x}\right)$$

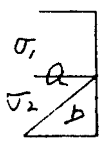
$$-\frac{U\sigma_1 x}{\pi\beta}\left[\sin^{-1}\frac{\beta y}{x} + \frac{\beta y}{x}\cosh^{-1}\frac{x}{\beta y} - \left(1 - \frac{a}{x}\right)\sin^{-1}\frac{\beta y/x}{1 - a/x} - \frac{\beta y}{x(1 - a/x)}\cosh^{-1}\frac{x(1 - a/x)}{\beta y}\right]$$

$$-\frac{U\sigma_2 x}{\pi\beta}\left[\left(1 - \frac{a}{x}\right)\sin^{-1}\frac{\beta y}{x\left(1 - \frac{a}{x}\right)} + \frac{\beta y}{x}\cosh^{-1}\frac{x\left(1 - \frac{a}{x}\right)}{\beta y} + \frac{\pi}{2}\left(1 - \frac{a}{x}\right)\right] \tag{7a}$$

(7a) can be put in form

$$\frac{U}{\pi}\int_{S_I}\sigma\,\mu(0)\,dS = \frac{Ux}{\beta\pi}\left[\alpha\,\Phi_5\left(\frac{\beta y}{x}, \frac{a}{x}\right) + \delta\,\Phi_6\left(\frac{\beta y}{x}, \frac{a}{x}\right)\right] \tag{7}$$

d) $-\frac{U}{2\pi}\int_{S_{I_0}}(\sigma_{uB} - \sigma_{uT})\,\mu(0)\,dS$



$$-\frac{U(\sigma_{uB} - \sigma_{uT})_1}{2\pi}\int_0^{x-\beta y} d\xi \int_0^{\frac{x-\xi}{\beta} - y} \frac{d\eta}{\sqrt{(x-\xi)^2 - \beta^2(y-\eta)^2}}$$

$$= -\frac{U(\sigma_{uB} - \sigma_{uT})_1}{2\pi\beta}\left[\int_0^{x-\beta y}\sin^{-1}\frac{\beta y}{x-\xi}\,d\xi - \int_0^{x-\beta y}\sin^{-1}\left(\frac{2\beta y}{x-\xi} - 1\right)d\xi\right]^*$$

$$= -\frac{U(\sigma_{uB} - \sigma_{uT})_1}{2\pi\beta}x\left[\begin{array}{l}\frac{\beta y}{x}\cosh^{-1}\frac{x}{\beta y} + \sin^{-1}\frac{\beta y}{x} - \frac{\pi}{2}\frac{\beta y}{x} \\ -2\sqrt{\frac{\beta y}{x}}\sqrt{1 - \frac{\beta y}{x}} + 2\tan^{-1}\sqrt{\frac{x}{\beta y} - 1} - \frac{\pi}{2}\left(1 - \frac{\beta y}{x}\right)\end{array}\right]$$

$$-\frac{U}{2\pi\beta}\left[(\sigma_{uB} - \sigma_{uT})_2\int_0^{x-\beta y}[\,''\,]\,d\xi - (\sigma_{uB} - \sigma_{uT})\Big|_1^2\int_0^a[\,''\,]\,d\xi\right]^*$$

$$= -\frac{U(\sigma_{uB} - \sigma_{uT})_2}{2\pi\beta}x\left[\begin{array}{l}\frac{\beta y}{x}\cosh^{-1}\frac{x}{\beta y} + \sin^{-1}\frac{\beta y}{x} - \frac{\pi}{2}\frac{\beta y}{x} \\ -2\sqrt{\frac{\beta y}{x}}\sqrt{1 - \frac{\beta y}{x}} + 2\tan^{-1}\sqrt{\frac{x}{\beta y} - 1} - \frac{\pi}{2}\left(1 - \frac{\beta y}{x}\right)\end{array}\right]$$

---

* Ref. 3. Appendix 5.

$$+\frac{U(\sigma_{uB}-\sigma_{uT})|_1^2}{2\pi\beta}x\left[\begin{array}{l}\sin^{-1}\frac{\beta y}{x}+\frac{\beta y}{x}\cosh^{-1}\frac{x}{\beta y}-(1-\frac{a}{x})\sin^{-1}\frac{\beta y}{x(1-a/x)}\\ -\frac{\beta y}{x(1-\frac{a}{x})}\cosh^{-1}\frac{x(1-\frac{a}{x})}{\beta y}-\frac{a}{x}\sin^{-1}\left(2\frac{\beta y}{x(1-\frac{a}{x})}-1\right)\\ -2\sqrt{\frac{\beta y}{x}}\left(\sqrt{1-\frac{\beta y}{x}}-\sqrt{1-\frac{a}{x}-\frac{\beta y}{x}}\right)+2\left(\tan^{-1}\sqrt{\frac{x}{\beta y}-1}-\tan^{-1}\sqrt{\frac{x(1-\frac{a}{x})}{\beta y}-1}\right)\end{array}\right] \tag{8a}$$

so (8a) again be put in form

$$-\frac{U}{2\pi}\int_{S_{I_0}}(\sigma_{uB}-\sigma_{uT})\mu(0)\,dS=\frac{Ux}{\beta\pi}\left[\alpha\Phi_7\left(\frac{\beta y}{x},\frac{a}{x}\right)+\delta\Phi_8\left(\frac{\beta y}{x},\frac{a}{x}\right)\right] \tag{8}$$

Now by (6b)

$$\int_{S_{I_0}+S_{II}}\mu(0)\frac{\partial\phi'}{\partial c}\,dS=\frac{U}{\pi}\int_0^{v_1}du\int_{-u}^{v_1}\frac{f\left(\frac{v}{u},\frac{Mc}{u}\right)dv}{M\sqrt{(u_1-u)(v_1-v)}}$$

$$=\frac{U}{\pi M}\int_0^{v_1}du\int_{-1}^{\frac{v_1}{u}}\frac{\sqrt{u}\,f\left(\frac{v}{u},\frac{Mc}{u}\right)d\left(\frac{v}{u}\right)}{\sqrt{u_1-u}\sqrt{\frac{v_1}{u}-\frac{v}{u}}}=\frac{Uv_1}{M\pi}\int_0^{v_1}\frac{\sqrt{\frac{u}{v_1}}\,G\left(\frac{v_1}{u},\frac{Mc}{u}\right)d\left(\frac{u}{v_1}\right)}{\sqrt{\frac{u_1}{v_1}-\frac{u}{v_1}}}$$

$$=\frac{Uv_1}{M\pi}\,\phi\left(\frac{u}{v_1},\frac{Mc}{u}\right)\Big|_0^{v_1}=\frac{Uv_1}{M\pi}\,\phi\left(1,\frac{v_1\beta}{Ma}\right)$$

$$=\frac{Ux}{2\pi\beta}\,\psi\left(\frac{\beta y}{x},\frac{a}{x}\right)$$

so (6a) takes the form

$$-\frac{U}{2\pi}\int_{S_{I_0}+S_{II}}(\sigma_{uB}-\sigma_{uT})\mu(0)\,dS=\frac{Ux}{\pi\beta}\left[\alpha\Phi_3\left(\frac{\beta y}{x},\frac{a}{x}\right)+\delta\Phi_4\left(\frac{\beta y}{x},\frac{a}{x}\right)\right] \tag{6}$$

Moreover,

$$\int_0^{v_1}\phi'(u',v_1)\sqrt{\frac{v_1-u'}{u_1-v_1}}\left(\frac{1}{v_1-u'}-\frac{1}{u_1-u'}\right)du'$$

$$=\frac{Uv_1}{\pi M}\int_0^{v_1}\phi\left(\frac{v_1}{u'},\frac{Ma}{\beta u'}\right)\frac{\sqrt{\frac{v_1}{u'}-1}}{\sqrt{\frac{u_1}{u'}-\frac{v_1}{u'}}}\left[\frac{1}{\frac{v_1}{u'}-1}-\frac{1}{\frac{u_1}{u'}-1}\right]d\left(\frac{u'}{v_1}\right)$$

$$=\frac{Uv_1}{\pi M}\int_0^{v_1}\chi\left(\frac{v_1}{u'},\frac{u_1}{v_1},\frac{Ma}{\beta v_1}\right)d\left(\frac{u'}{v_1}\right)=\frac{Uv_1}{\pi M}\,X\left(\frac{u'}{v_1},\frac{v_1}{u_1},\frac{v_1\beta}{Ma}\right)\Big|^{v_1}$$

but $\frac{v_1}{u_1}=\frac{1-\beta y/x}{1+\beta y/x}$ $\qquad \frac{\beta v_1}{Ma}=\frac{x-\beta y}{a}=\frac{x}{a}\left(1-\frac{\beta y}{x}\right)$

$$v_1=\frac{M}{\beta}x\left(1-\frac{\beta y}{x}\right)$$

so

$$\int_0^{v_1} \phi'(u',v_1)\sqrt{\frac{v_1-u'}{u_1-v_1}}\left(\frac{1}{v_1-u'}-\frac{1}{u_1-u'}\right)du' = \frac{Ux}{\pi\beta}\left[\alpha\Phi_9\left(\frac{\beta y}{x},\frac{a}{x}\right)+\delta\Phi_{10}\left(\frac{\beta y}{x},\frac{a}{x}\right)\right] \quad (9)$$

e) $\frac{U}{\pi}\int_{S_I+S_{II}} \mu(0)\frac{\partial\phi'}{\partial c}\,dS$

$$= \frac{U}{\pi}\int_0^{u_1} du\int_{-u}^{v_1}\frac{f\left(\frac{v}{u},\frac{Mc}{u}\right)dv}{M\sqrt{(u_1-u)(v_1-v)}} = \frac{U}{\pi M}\int_0^{u_1}du\int_{-1}^{\frac{v_1}{u}}\frac{\sqrt{u}\,f\left(\frac{v}{u},\frac{Mc}{u}\right)d\left(\frac{v}{u}\right)}{\sqrt{u_1-u}\sqrt{\frac{v_1}{u}-\frac{v}{u}}}$$

$$= \frac{Uv_1}{M\pi}\int_0^{u_1}\frac{\sqrt{\frac{u}{v_1}}\,G\left(\frac{v_1}{u},\frac{Mc}{u}\right)d\left(\frac{u}{v_1}\right)}{\sqrt{\frac{u_1}{v_1}-\frac{u}{v_1}}}$$

$$= \frac{Uv_1}{M\pi}\Phi\left(\frac{u}{v_1},\frac{Mc}{u}\right)\Big|_0^{u_1} = \frac{Uv_1}{M\pi}\Phi\left(\frac{u_1}{v_1},\frac{Mc}{u_1}\right)$$

$$= \frac{Ux}{2\pi\beta}\Psi\left(\frac{\beta y}{x},\frac{a}{x}\right) \quad (10a)$$

(10a) can be put in form

$$\frac{U}{\pi}\int_{S_I+S_{II}}\mu(0)\frac{\partial\phi'}{\partial c}\,dS = \frac{Ux}{\pi\beta}\left[\alpha\Phi_{11}\left(\frac{\beta y}{x},\frac{a}{x}\right)+\delta\Phi_{12}\left(\frac{\beta y}{x},\frac{a}{x}\right)\right] \quad (10)$$

Substituting (5),(6), (7), (8), (9), and (10) into (3) and (4), we see that

$$\phi_i(x,y) = \frac{Ux}{\pi\beta}\, \Phi_i\left(\frac{\beta y}{x}, \frac{a}{x} \,\middle|\, a_i\alpha,\, b_i\delta\right) \tag{11}$$

and

$$\frac{\partial}{\partial x}\phi_i(x,y) = \frac{U}{\beta}\, \Psi_i\left(\frac{\beta y}{x}, \frac{a}{x} \,\middle|\, a_i\alpha,\, b_i\delta\right) \tag{12}$$

$i = 1, 2, 3,$ etc. denotes regions of interaction, which is actually a "pseudo conical" flow field.

3. Similarity Rule

It is seen from above that in eq. (11), (12): for $x < a$, a/x term does not occur, so flow field is conical. for $x > a$, a/x terms occur, so flow field is "pseudo conical." However, for both regions, the same similarity rule holds, i.e.:

> "For different values of Mach number, $\partial\phi_i/\partial x$ differs by a constant factor $1/\beta$, provided the ratios $\beta y/x$ and a/x are maintained constant."

This constitutes a "similarity rule" for supersonic flow at different Mach numbers. It leads directly to the determination of a Mach number correction factor for converting the biplane property calculated at one Mach number into that at other Mach numbers.

Expressing our similarity rule in terms of C_p , we have

for $x_M = x_{\sqrt{2}}$, $y_M = \frac{1}{\beta} y_{\sqrt{2}}$,

simply

$$C_p = \frac{1}{\beta} C_{p\sqrt{2}} \qquad (12)$$

To evaluate the correction factor for lift and drag over wing tip region surface, since

$$L = \frac{\rho U^2}{2}\iint C_p \, dy \, dx = -\rho U \int^{y} \phi(2a)\, dy = -\frac{\rho U^2 2a}{\beta}\int_0^{\frac{2a}{\beta}} \phi\left(\frac{\beta y}{2a}\right) dy = \frac{4a^2 \rho U^2 \delta}{\beta^2} K(\alpha/\delta)$$

$$D = -\rho U \delta \int_0^{\frac{2a}{\beta}} (2\phi(a) - \phi(2a))\, dy = \frac{4a^2 \rho U^2 \delta^2}{\beta^2} J(\alpha/\delta)$$

we obtain: (14)

$$L, D = \frac{1}{\beta^2} L_{\sqrt{2}}, D_{\sqrt{2}} \,.$$

Thus, in Part II, below, it will be found that at zero incidence, $M = \sqrt{2}$, $C_{D_0} = 0.823\, \delta^2 / A\!\!R$. It follows that at any other Mach number

$$C_{D_0} = \frac{0.823\, \delta^2}{\beta^2 A\!\!R} \qquad (15)$$

4. Remarks

It might be noted that the above Similarity Rule is not limited to bi-plane application only. The same holds for delta wings as long as the leading edges lie either outside of Mach cone or in flow direction, and $\sigma = \sigma(x)$. However, as soon as the leading edge goes inside the Mach cone, the Similarity Rule ceases to hold. Indeed, even in the simple case of conical flow, it is easily seen that Tschaplygin's transformation will give for each Mach number an independent boundary condition for the Laplace equation, which much complicates the problem.

PART II. APPLICATION

A. POTENTIAL, LIFT AND DRAG

(1) Potential

With the Similarity Rule established, the study of supersonic biplane is greatly simplified. Indeed, the only computation needed will be for the case of $M = \sqrt{2}$. For all other Mach numbers, the required conversion relationship will be furnished by this Similarity Rule.

Using the general expression for potential derived above, we have

$$\phi_{uT} = -\frac{U}{\pi}\int_{S_I}\sigma_{uT}\,\mu(0)\,dS - \frac{U}{2\pi}\int_{S_{I_0}}(\sigma_{uB}-\sigma_{uT})\,\mu(0)\,dS$$

$$-\frac{1}{2\pi}\int_{S_{I_0}+S_{II}}\mu(0)\frac{\partial}{\partial c}\phi_u'\,dS + \frac{1}{2\pi}\int_{M^2c^2/4v_1}^{v_1}\phi_u'(u',v_1)\sqrt{\frac{v_1-u'}{u_1-v_1}}\left[\frac{1}{v_1-u'}-\frac{1}{u_1-u'}\right]du'$$

$$\phi_{uB} = -\frac{U}{\pi}\int_{S_I}\sigma_{uB}\,\mu(0)\,dS - \frac{1}{\pi}\int_{S_I+S_{II}}\mu(0)\frac{\partial\phi_u'}{\partial c}\,dS + \frac{U}{2\pi}\int_{S_{I_0}}(\sigma_{uB}-\sigma_{uT})\,\mu(0)\,dS$$

$$+\frac{1}{2\pi}\int_{S_{I_0}+S_{II}}\mu(0)\frac{\partial\phi_u'}{\partial c}\,dS - \frac{1}{2\pi}\int_{M^2c^2/4v_1}^{v_1}\phi_u'(u',v_1)\sqrt{\frac{v_1-u'}{u_1-v_1}}\left[\frac{1}{v_1-u'}-\frac{1}{u_1-u'}\right]du' + \phi_u'$$

$$\phi_{LT} = -\frac{U}{\pi}\int_{S_I}\sigma_{LT}\,\mu(0)\,dS - \frac{1}{\pi}\int_{S_I+S_{II}}\mu(0)\frac{\partial\phi_L'}{\partial c}\,dS - \frac{U}{2\pi}\int_{S_{I_0}}(\sigma_{LB}-\sigma_{LT})\,\mu(0)\,dS$$

$$+\frac{1}{2\pi}\int_{S_{I_0}+S_{II}}\mu(0)\frac{\partial\phi_L'}{\partial c}\,dS - \frac{1}{2\pi}\int_{M^2c^2/4v_1}^{v_1}\phi_L'(u',v_1)\sqrt{\frac{v_1-u'}{u_1-v_1}}\left[\frac{1}{v_1-u'}-\frac{1}{u_1-u'}\right]du' + \phi_L'$$

$$\phi_{LB} = -\frac{U}{\pi}\int_{S_I}\sigma_{LB}\,\mu(0)\,dS + \frac{U}{2\pi}\int_{S_{I_0}}(\sigma_{LB}-\sigma_{LT})\,\mu(0)\,dS$$

$$-\frac{1}{2\pi}\int_{S_{I_0}+S_{II}}\mu(0)\frac{\partial\phi_L'}{\partial c}\,dS + \frac{1}{2\pi}\int_{M^2c^2/4v_1}\phi_L'(u',v_1)\sqrt{\frac{v_1-u'}{u_1-v_1}}\left[\frac{1}{v_1-u'}-\frac{1}{u_1-u'}\right]du'$$

where it is understood that ϕ', $\partial\phi'/\partial c$ and S' come in only where the interaction is effected. By carrying out the integration as given in Ref. 3, we have

$$\phi_u' = -\frac{v\,\sigma_{LT}}{M\pi} I_1 + \frac{v(\sigma_{LB}-\sigma_{LT})}{M\pi} I_2$$

$$\phi_L' = -\frac{v\,\sigma_{uB}}{M\pi} I_1 - \frac{v(\sigma_{uB}-\sigma_{uT})}{M\pi} I_2$$

$$I_1 = 2\left[\int_0^{\tilde{u}} \frac{\sqrt{(u_1-u)(v_1+u)-M^2c^2/4}}{u_1-u}\,du - \int_0^{u_*} \frac{\sqrt{(u_1-u)(v_1-u)-M^2c^2/4}}{u_1-u}\,du\right]$$

$$= 2\left(\begin{array}{l} \frac{u_1+v_1}{2}\frac{\pi}{2} - \frac{Mc}{2}\frac{\pi}{2} + \sqrt{u_1v_1-\frac{M^2c^2}{4}} + \frac{u_1+v_1}{2}\sin^{-1}\frac{u_1-v_1}{\sqrt{(u_1+v_1)^2-M^2c^2}} \\ -\frac{Mc}{2}\sin^{-1}\frac{u_1(u_1+v_1)-M^2c^2/2}{u_1\sqrt{(u_1+v_1)^2-M^2c^2}} + \frac{Mc}{2}\frac{\pi}{2} - \sqrt{u_1v_1-\frac{M^2c^2}{4}} \\ +\frac{u_1-v_1}{2}\cosh^{-1}\frac{u_1+v_1}{\sqrt{(u_1-v_1)^2+M^2c^2}} - \frac{Mc}{2}\sin^{-1}\frac{u_1(u_1-v_1)+M^2c^2/2}{u_1\sqrt{(u_1-v_1)^2+M^2c^2}} \end{array}\right)$$

$$= 2\left(\begin{array}{l} \frac{u_1+v_1}{2}\left(\frac{\pi}{2}+\sin^{-1}\frac{u_1-v_1}{\sqrt{(u_1+v_1)^2-M^2c^2}}\right) + \frac{u_1-v_1}{2}\cosh^{-1}\frac{u_1+v_1}{\sqrt{(u_1-v_1)^2+M^2c^2}} \\ -\frac{Mc}{2}\left(\sin^{-1}\frac{u_1(u_1+v_1)-M^2c^2/2}{u_1\sqrt{(u_1+v_1)^2-M^2c^2}} + \sin^{-1}\frac{u_1(u_1-v_1)+M^2c^2/2}{u_1\sqrt{(u_1-v_1)^2+M^2c^2}}\right. \end{array}\right)$$

$$I_2 = \int_0^{u_*} \frac{\sqrt{(u_1-u)(v_1-u)-M^2c^2/4}}{u_1-u}\,du + \int_0^{u_*} \frac{\sqrt{(u_1-u)(v_1+u)-M^2c^2/4}}{u_1-u}\,du$$

$$= 2\sqrt{u_1v_1-\frac{M^2c^2}{4}} - \frac{u_1-v_1}{2}\cosh^{-1}\frac{u_1+v_1}{\sqrt{(u_1-v_1)^2+M^2c^2}}$$

$$+\frac{u_1+v_1}{2}\left[\sin^{-1}\frac{u_1-v_1}{\sqrt{(u_1+v_1)^2-M^2c^2}} + \sin^{-1}\frac{2v_1-\sqrt{(u_1-v_1)^2+M^2c^2}}{\sqrt{(u_1+v_1)^2-M^2c^2}}\right]$$

$$-\frac{Mc}{2}\left[\begin{array}{l} \frac{\pi}{2} - \sin^{-1}\frac{u_1^2-v_1^2-M^2c^2+(u_1+v_1)\sqrt{(u_1-v_1)^2+M^2c^2}}{\sqrt{(u_1+v_1)^2-M^2c^2}\,\left[u_1-v_1+\sqrt{(u_1-v_1)^2+M^2c^2}\right]} \\ +\sin^{-1}\frac{u_1(u_1+v_1)-M^2c^2/2}{u_1\sqrt{(u_1+v_1)^2-M^2c^2}} - \sin^{-1}\frac{u_1(u_1-v_1)+M^2c^2/2}{u_1\sqrt{(u_1-v_1)^2+M^2c^2}} \end{array}\right]$$

$$-\sqrt{v_1(u_1-v_1)-\frac{M^2c^2}{2}+v_1\sqrt{(u_1-v_1)^2+M^2c^2}}$$

$$\frac{\partial \phi_u'}{\partial c} = \frac{U}{\pi}\left[\sigma_{LT} I_1' + (\sigma_{LB} - \sigma_{LT}) I_2'\right]$$

$$\frac{\partial \phi_L'}{\partial c} = \frac{U}{\pi}\left[\sigma_{uB} I_1' - (\sigma_{uB} - \sigma_{uT}) I_2'\right]$$

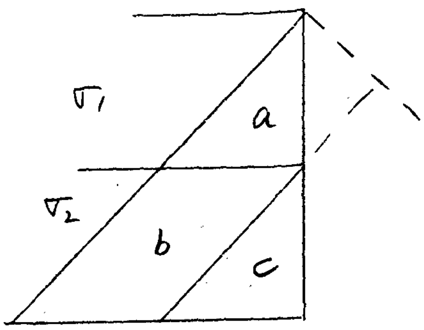
$$I_1' = \frac{\pi}{2} + \tan^{-1}\frac{u_1^2 - v_1^2}{2Mc\sqrt{u_1 v_1 - M^2c^2/4}}$$

$$I_2' = -\frac{1}{2}\left(\frac{\pi}{2} - \sin^{-1}\frac{(u_1-v_1)u_1 + M^2c^2/2}{u_1\sqrt{(u_1-v_1)^2+M^2c^2}} + \sin^{-1}\frac{(u_1+v_1)u_1 - M^2c^2/2}{u_1\sqrt{(u_1+v_1)^2-M^2c^2}} \right.$$
$$- \sin^{-1}\frac{(u_1^2-v_1^2) - M^2c^2 + (u_1+v_1)\sqrt{(u_1-v_1)^2+M^2c^2}}{\sqrt{(u_1+v_1)^2-M^2c^2}\,[u_1 - v_1 + \sqrt{(u_1-v_1)^2+M^2c^2}}$$
$$\left. + \frac{2Mc}{\sqrt{(u_1-v_1)^2+M^2c^2}}\,\frac{\sqrt{v_1(u_1-v_1) - M^2c^2/2 + v_1\sqrt{(u_1-v_1)^2+M^2c^2}}}{u_1 - v_1 + \sqrt{(u_1-v_1)^2+M^2c^2}} \right)$$

Tip region potential of a symmetric wing for

(i) $\phi = -\frac{U}{\pi}\int_{S_I} \sigma\,\mu(0)\,dS$

$\phi_a,\ \phi_b,\ \phi_c$



(Appendix 5, Ref. 3)

$\underline{x < a}$

(a)
$$\varphi_a = -\frac{U}{\pi}\int_{S_I}\sigma\mu(0)\,dS = -\frac{\sigma U}{\pi}\int_0^{x-\beta y} d\xi \int_0^{y-\frac{x-\xi}{\beta}} \mu(0)\,d\eta - U\sigma y$$
$$= -\frac{U\sigma}{\pi\beta}\int_0^{x-\beta y}\left[\sin^{-1}\frac{\beta y}{x-\xi} + \frac{\pi}{2}\right]d\xi - U\sigma y$$

$\underline{x > a}$

(b)
$$\varphi_b = -\frac{U}{\pi}\int_0^{x-\beta y} d\xi \int_0^{y-\frac{x-\xi}{\beta}} \sigma_1\mu(0)\,d\eta - U\sigma_1 y + \frac{U(\sigma_1-\sigma_2)}{\beta}\int_a^x d\xi$$
$$= -\frac{U\sigma_1}{\pi\beta}\int_0^{x-\beta y}\left[\sin^{-1}\frac{\beta y}{x-\xi} + \frac{\pi}{2}\right]d\xi - \frac{U\sigma_1}{\beta}(a-x+\beta y) - \frac{U\sigma_2}{\beta}(x-a)$$

(c)
$$\varphi_c = -\frac{U}{\pi}\int_0^{x-\beta y} d\xi \int_0^{y-\frac{x-\xi}{\beta}} \sigma_1\mu(0)\,d\eta - U\sigma_1 y + \frac{U}{\pi}\int_a^{x-\beta y} d\xi \int_0^{y-\frac{x-\xi}{\beta}} (\sigma_1-\sigma_2)\mu(0)\,d\eta + U(\sigma_1-\sigma_2)y$$
$$= -\frac{U\sigma_1}{\pi\beta}\int_0^{x-\beta y}\left[\sin^{-1}\frac{\beta y}{x-\xi} + \frac{\pi}{2}\right]d\xi + \frac{U(\sigma_1-\sigma_2)}{\pi\beta}\int_a^{x-\beta y}\left[\sin^{-1}\frac{\beta y}{x-\xi} + \frac{\pi}{2}\right]d\xi - U\sigma_2 y$$
$$= -\frac{U\sigma_1}{\pi\beta}\int_0^{a}\left[\sin^{-1}\frac{\beta y}{x-\xi} + \frac{\pi}{2}\right]d\xi - \frac{U\sigma_2}{\pi\beta}\int_a^{x-\beta y}\left[\sin^{-1}\frac{\beta y}{x-\xi} + \frac{\pi}{2}\right]d\xi - U\sigma_2 y$$

where by Appendix 5: Ref. 3

(1) $$\int_0^{x-\beta y}\left[\sin^{-1}\frac{\beta y}{x-\xi} + \frac{\pi}{2}\right]d\xi = \beta y \cosh^{-1}\frac{x}{\beta y} + x\sin^{-1}\frac{\beta y}{x} + \frac{\pi}{2}(x-2\beta y)$$

(2) $$\int_0^{a}\left[\sin^{-1}\frac{\beta y}{x-\xi} + \frac{\pi}{2}\right]d\xi = \left[\xi\sin^{-1}\frac{\beta y}{\xi} + \beta y\cosh^{-1}\frac{\xi}{\beta y}\right]_{x-a}^{x} + \frac{\pi}{2}a$$

(3) $$\int_a^{x-\beta y}\left[\sin^{-1}\frac{\beta y}{x-\xi} + \frac{\pi}{2}\right]d\xi = (1) - (2)$$
$$= \frac{\pi}{2}(x-2\beta y) + (x-a)\sin^{-1}\frac{\beta y}{x-a} + \beta y\cosh^{-1}\frac{x-a}{\beta y} - \frac{\pi}{2}a$$

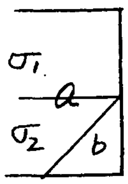
when $y \to 0$

(a) $x<a$ $\qquad \varphi_a = -\frac{U\sigma_1}{\pi\beta}\frac{\pi}{2}x = -\frac{U\sigma_1 x}{2\beta}$

(b) $\varphi_b = -\frac{U\sigma_1}{\pi\beta}\frac{\pi}{2}x - \frac{U}{\beta}\sigma_1(a-x) - \frac{U}{\beta}\sigma_2(x-a) = \frac{U\sigma_1 x}{2\beta} - \frac{U\sigma_2 x}{\beta} - \frac{U}{\beta}a(\sigma_1-\sigma_2)$

(c) $\varphi_c = -\frac{\pi}{2}a\frac{U\sigma_1}{\pi\beta} - \frac{U\sigma_2}{\pi\beta}\frac{\pi}{2}(x-a) = -\frac{U\sigma_2}{2\beta}x - \frac{Ua}{2\beta}(\sigma_1-\sigma_2)$

(ii) $-\frac{U}{2\pi}\iint_{I_0}(\sigma_{uB}-\sigma_{uT})\mu(0)\,dS$



a): $$= -\frac{U}{2\pi}(\sigma_{uB}-\sigma_{uT})_1\int_0^{x-\beta y}d\xi\int_0^{\frac{x-\xi}{\beta}-y}\frac{d\eta}{\sqrt{(x-\xi)^2-\beta^2(y-\eta)^2}}$$
$$= -\frac{U}{2\pi\beta}(\sigma_{uB}-\sigma_{uT})_1\int_0^{x-\beta y}\left[\sin^{-1}\frac{\beta y}{x-\xi} - \sin^{-1}\left(\frac{2\beta y}{x-\xi}-1\right)\right]d\xi$$
$$= -\frac{U}{2\pi\beta}(\sigma_{uB}-\sigma_{uT})_1\, I_1$$

b): $$= -\frac{U}{2\pi}(\sigma_{uB}-\sigma_{uT})_2\int_0^{x-\beta y}d\xi\int_0^{\frac{x-\xi}{\beta}-y}\frac{d\eta}{\sqrt{(x-\xi)^2-\beta^2(y-\eta)^2}}$$
$$+ \frac{U}{2\pi}(\sigma_{uB}-\sigma_{uT})\Big|_1^2\int_0^{a}d\xi\int_0^{\frac{x-\xi}{\beta}-y}\frac{d\eta}{\sqrt{(x-\xi)^2-\beta^2(y-\eta)^2}}$$
$$= -\frac{U}{2\pi\beta}(\sigma_{uB}-\sigma_{uT})_2 I_1 + \frac{U}{2\pi\beta}(\sigma_{uB}-\sigma_{uT})\Big|_1^2 I_2$$

$$I_1 = \int_0^{x-\beta y}\left[\sin^{-1}\frac{\beta y}{x-\xi} - \sin^{-1}\left(\frac{2\beta y}{x-\xi}-1\right)\right]d\xi$$
$$= x\sin^{-1}\frac{\beta y}{x} + \beta y\cosh^{-1}\frac{x}{\beta y}$$
$$+ 2x\tan^{-1}\sqrt{\frac{x-\beta y}{\beta y}} - 2\sqrt{\beta y}\sqrt{x-\beta y} - \frac{\pi}{2}x$$

$$I_2 = x\sin^{-1}\frac{\beta y}{x} + \beta y\cosh^{-1}\frac{x}{\beta y} - (x-a)\sin^{-1}\frac{\beta y}{x-a} - \beta y\cosh^{-1}\frac{x-a}{\beta y}$$
$$+ 2x\left(\tan^{-1}\sqrt{\frac{x}{\beta y}-1} - \tan^{-1}\sqrt{\frac{x-a}{\beta y}-1}\right) - 2\sqrt{\beta y}\left(\sqrt{x-\beta y} - \sqrt{x-a-\beta y}\right)$$
$$- a\sin^{-1}\left(\frac{2\beta y}{x-a}-1\right)$$

*Appendix 5. Check p. 115,116 Ref. 3.

(2) Lift and Drag

With the velocity potential determined, it remains a simple matter to compute the pressure coefficient, lift and drag. By the linearized theory, the pressure coefficient is given as

$$C_p = \frac{\Delta p}{\frac{1}{2}\rho_0 U^2} = -\frac{2u}{U} = -\frac{2}{U}\frac{\partial \phi}{\partial x}$$

The total lift and drag over a surface are obtained from the C_p distribution by simple integration

$$L = \frac{\rho_0 U^2}{2}\int C_p \, dS$$

$$D = \frac{\rho_0 U^2}{2}\int C_p \sigma \, dS$$

With linearized theory, the lift computation can further be simplified to the following form, which is especially useful when curved surfaces are involved:

$$L = \frac{\rho_0 U^2}{2}\iint C_p \, dy \, dx = \frac{\rho_0 U^2}{2}\left(-\frac{2}{U}\right)\int^{y} \phi \Big|_0^{2a} dy$$

$$= -\rho_0 U \int^{y} \phi(2a) \, dy$$

In case the wing has an isoceles triangular or "diamond" profile, since the constant surface slope can be taken out of the integral sign, the determination of drag simplifies to the following form

$$D = \frac{\rho_0 U^2}{2}\iint C_p \sigma \, dx \, dy = \frac{\rho_0 U^2}{2}\left(-\frac{2}{U}\right)\delta\int^{y}\left[\phi\Big|_0^{a} - \phi\Big|_a^{2a}\right] dy$$

$$= -\rho_0 U \delta \int^{y}\left[2\phi(a) - \phi(2a)\right] dy$$

For a multiplane system, with projected area S in plan form, the lift and drag coefficients of the system are obtained as follows:

$$C_L = \frac{\sum_i (-1)^i \int_{S_i} C_p \, dS}{S}$$

$$C_D = \frac{\sum_i \int_{S_i} C_p \sigma_i \, dS}{S}$$

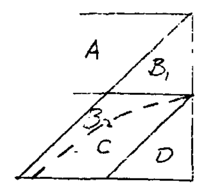
B. COMPUTATION AND RESULTS

(1) General Forms of Velocity Potential

The potential at a point (x, y) or (u, v) in region D of the upper wing is

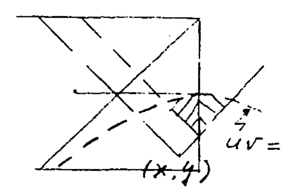
$$\varphi_{uT} = -\frac{U}{\pi}\int_{S_I} \sigma_{uT}\,\mu(0)\,dS - \frac{U}{2\pi}\int_{S_{I_0}} (\sigma_{uB}-\sigma_{uT})\,\mu(0)\,dS$$

$$-\frac{1}{2\pi}\int_{S_{I_0}+S_{II}} \mu(0)\,\frac{\partial \varphi_u'}{\partial c}\,dS + \frac{1}{2\pi}\int_{M^2c^2/4v_1}^{v_1} \varphi_u'(u',v_1)\sqrt{\frac{v_1-u'}{u_1-v_1}}\left[\frac{1}{v_1-u'}-\frac{1}{u_1-u'}\right]du' \qquad (1,$$



$$\varphi_{uL} = -\frac{U}{\pi}\int_{S_I} \sigma_{uB}\,\mu(0)\,dS - \frac{1}{\pi}\int_{S_I+S_{II}} \mu(0)\,\frac{\partial \varphi_u'}{\partial c}\,dS$$

$$+\frac{U}{2\pi}\int_{S_{I_0}} (\sigma_{uB}-\sigma_{uT})\,\mu(0)\,dS + \frac{1}{2\pi}\int_{S_{I_0}+S_{II}} \mu(0)\,\frac{\partial \varphi_u'}{\partial c}\,dS$$



$$-\frac{1}{2\pi}\int_{M^2c^2/4v_1}^{v_1} \varphi_u'(u',v_1)\sqrt{\frac{v_1-u'}{u_1-v_1}}\left[\frac{1}{v_1-u'}-\frac{1}{u_1-u'}\right]du' + \varphi_u' \qquad (1$$

In fact, (1.1) and 1.2) are general forms of the potential in all regions. In region C, the point (x, y) or $(u, v$ considered is beyond the influence of the rear half diaphragm, and so the term

$$\frac{1}{2\pi}\int_{M^2c^2/4v_1}^{v_1} \varphi_u'(u',v_1)\sqrt{\frac{v_1-u'}{u_1-v_1}}\left[\frac{1}{v_1-u'}-\frac{1}{u_1-u'}\right]du'$$

vanishes in passing from region D to C, and remains nil on the rest of the wing surface. Further, since $\frac{\partial \varphi_u'}{\partial c}$ differs from zero only for $\xi > a$ on the wing and $uv > \frac{M^2c^2}{4}$ on diaphragm, the integral

$$\frac{1}{2\pi}\int_{S_{I_0}+S_{II}} \mu(0)\,\frac{\partial \varphi_u'}{\partial c}\,dS$$

goes to zero as (x, y) goes to the hyperbolic boundary separating regions D and C. In fact, in all the regions A, B, C it contributes nothing, as the effective area for $\frac{\partial \varphi_u'}{\partial c}$ in $S_{I_0} + S_{II}$ disappears. On the other hand, the integral

$$\frac{1}{\pi}\int_{S_I + S_{II}} \mu(0) \frac{\partial \varphi_u'}{\partial c} dS$$

degenerates to

$$\frac{1}{\pi}\int_{S_I} \mu(0) \frac{\partial \varphi_u'}{\partial c} dS$$

in regions A, B, C. In the regions A and B, i.e., those areas forward of the hyperbolic $uv = M^2c^2/4$, the effect of the tip region of the neighbouring surface cannot be felt and the analytic expressions for the integrals are available.

There is a difference in the surface slope σ for the region $\xi < a$ and $\xi > a$, but the formulas (1.1) and (1.2) remain true.

In region A, the tip effect of the supersonic monoplane disappears, the flow reduces to that of a two-dimensional Busemann biplane.

The structure of the present theory is based on the linearized theory of potential flow. As a consequence, the potential will be linear in the angle of attack α, and the thickness ratio δ, in the form

$$\varphi_{uT} = \varphi_{uT}^{(\alpha)} \cdot \alpha + \varphi_{uT}^{(\delta)} \cdot \delta \qquad (1.3)$$

$$\varphi_{uB} = \varphi_{uB}^{(\alpha)} \cdot \alpha + \varphi_{uB}^{(\delta)} \cdot \delta \qquad (1.4)$$

For the same point (x,y) on the lower wing, similarly, we have

$$\varphi_{LT} = -\varphi_{uB}^{(\alpha)}\cdot\alpha + \varphi_{uB}^{(\delta)}\cdot\delta \tag{1.5}$$

$$\varphi_{LB} = -\varphi_{uT}^{(\alpha)}\cdot\alpha + \varphi_{uT}^{(\delta)}\cdot\delta \tag{1.6}$$

(2) Lift and Drag in Terms of Potentials

In the linearized theory, as shown before, we have

$$\pm L_i/\rho U = \int \varphi_i\Big]_{x_1}^{x_2} dy = \alpha\cdot\int \varphi_i^{(\alpha)}\Big]_{x_1}^{x_2} dy + \delta\cdot\int \varphi_i^{(\delta)}\Big]_{x_1}^{x_2} dy \tag{2.1}$$

$$-D_i/\rho U = \int \sigma_i\cdot\varphi_i\Big]_{x_1}^{x_2} dy = \alpha\cdot\int \sigma_i\cdot\varphi_i^{(\alpha)}\Big]_{x_1}^{x_2} dy + \delta\cdot\int \sigma_i\cdot\varphi_i^{(\delta)}\Big]_{x_1}^{x_2} dy \tag{2.2}$$

where the (+) sign stands for top surface and the (-) sign for bottom surface.

With the distribution of σ_i in the Busemann biplane,

$$\begin{array}{lcc} & 0<x<a & a<x<2a \\ \sigma_{uT} = -\sigma_{LB} & -\alpha & -\alpha \\ \sigma_{uB} & \alpha+\delta & \alpha-\delta \\ \sigma_{LT} & -\alpha+\delta & -\alpha-\delta \end{array} \tag{2.3}$$

and noting that $\varphi(0) = 0$, we have

$$\begin{aligned} L_{uT,\,LB}/\rho U &= \alpha\cdot\int \varphi_{uT}^{(\alpha)}(2a)\,dy \pm \delta\cdot\int \varphi_{uT}^{(\delta)}(2a)\,dy \\ L_{uB,\,LT}/\rho U &= -\alpha\cdot\int \varphi_{uB}^{(\alpha)}(2a)\,dy \mp \delta\cdot\int \varphi_{uB}^{(\delta)}(2a)\,dy \\ D_{uT,\,LB}/\rho U &= \alpha^2\cdot\int \varphi_{uT}^{(\alpha)}(2a)\,dy \pm \alpha\delta\cdot\int \varphi_{uT}^{(\delta)}(2a)\,dy \\ D_{uB,\,LT}/\rho U &= -\alpha^2\cdot\int \varphi_{uB}^{(\alpha)}(2a)\,dy - \delta^2\cdot\int\left[2\varphi_{uB}^{(\delta)}(a) - \varphi_{uB}^{(\delta)}(2a)\right]dy \\ &\quad \mp \alpha\delta\cdot\int\left[2\varphi_{uB}^{(\alpha)}(a) - \varphi_{uB}^{(\alpha)}(2a) + \varphi_{uB}^{(\delta)}(2a)\right]dy \end{aligned} \tag{2.4}$$

The total lift and drag of the biplane becomes

$$L/2\rho U = \alpha \cdot \int [\varphi_{UT}^{(\alpha)}(2a) - \varphi_{UB}^{(\alpha)}(2a)]\,dy$$

$$D/2\rho U = \alpha^2 \cdot \int [\varphi_{UT}^{(\alpha)}(2a) - \varphi_{UB}^{(\alpha)}(2a)]\,dy \tag{2.5}$$

$$+ \delta^2 \int [\varphi_{UB}^{(\delta)}(2a) - 2\varphi_{UB}^{(\delta)}(a)]\,dy$$

$$Q = \tfrac{1}{2}\rho U^2 \tag{2.6}$$

$$L/\tfrac{4}{\pi}Q = \alpha \cdot \int [\tfrac{\pi}{U}\varphi_{UT}^{(\alpha)}(2a) - \tfrac{\pi}{U}\varphi_{UB}^{(\alpha)}(2a)]\,dy$$

$$D/\tfrac{4}{\pi}Q = \alpha^2 \cdot \int [\tfrac{\pi}{U}\varphi_{UT}^{(\alpha)}(2a) - \tfrac{\pi}{U}\varphi_{UB}^{(\alpha)}(2a)]\,dy \tag{2.7}$$

$$+ \delta^2 \cdot \int [\tfrac{\pi}{U}\varphi_{UB}^{(\delta)}(2a) - \tfrac{2\pi}{U}\varphi_{UB}^{(\delta)}(a)]\,dy \tag{2.8}$$

3. Consequence of Similarity Rules

a. Lift and Drag over the Tip Region

From the Similarity Rule for the tip flow of a Busemann biplane

$$\frac{\pi}{U}\varphi_i(x,y) = \frac{x}{\beta}\Phi_i\left(\frac{a}{x}, \frac{\beta y}{x}; \alpha, \delta\right)$$

and since α and δ can be separated, we have

$$\frac{\pi}{U}\varphi_i^{(j)}(x,y) = \frac{x}{\beta}\Phi_i^{(j)}\left(\frac{a}{x}, \frac{\beta y}{x}\right) \tag{3.1}$$

where $j = \alpha, \delta$

$i = UT, UB, LT, LB.$

Introducing $\zeta = \beta y / 2a$, we obtain

$$L_{tip} = \alpha \left(\frac{2a}{\beta}\right)^2 \frac{4}{\pi} Q \int_0^1 \left[\Phi_{uT}^{(\alpha)}\left(\tfrac{1}{2}, \zeta\right) - \Phi_{uB}^{(\alpha)}\left(\tfrac{1}{2}, \zeta\right) \right] d\zeta \tag{3.2}$$

$$= Q \cdot \left(\frac{2a}{\beta}\right)^2 \cdot \bar{L} \cdot \alpha$$

$$D_{tip} = Q \cdot \left(\frac{2a}{\beta}\right)^2 \left[\bar{L} \cdot \alpha^2 + \bar{D} \cdot \delta^2 \right] \tag{3.3}$$

$$\bar{D} = \frac{4}{\pi} \int_0^1 \left[\Phi_{uB}^{(\delta)}\left(\tfrac{1}{2}, \zeta\right) - \Phi_{uB}^{(\delta)}(1, \zeta) \right] d\zeta$$

$$\bar{L} = \frac{4}{\pi} \int_0^1 \left[\Phi_{uT}^{(\alpha)}\left(\tfrac{1}{2}, \zeta\right) - \Phi_{uB}^{(\alpha)}\left(\tfrac{1}{2}, \zeta\right) \right] d\zeta \tag{3.4}$$

where $\bar{L}$ and $\bar{D}$ are universal constants for all Busemann biplanes.

b. Lift and Drag for Off-Tip Region

Beyond the tip region, the flow is two dimensional. So

$$\begin{aligned}
&\frac{\pi}{U} \varphi_{uB}^{(\alpha)}(a) = -\frac{\pi a}{\beta} & &\frac{\pi}{U} \varphi_{uB}^{(\delta)}(a) = -\frac{\pi a}{\beta} \\
&\frac{\pi}{U} \varphi_{uB}^{(\alpha)}(2a) = 0 & &\frac{\pi}{U} \varphi_{uB}^{(\delta)}(2a) = -\frac{2a\pi}{\beta} \\
&\frac{\pi}{U} \varphi_{uT}^{(\alpha)}(2a) = 2a\frac{\pi}{\beta} & &\frac{\pi}{U} \varphi_{uT}^{(\delta)}(2a) = 0
\end{aligned} \tag{3.5}$$

and we have

$$\frac{\pi}{U} \varphi_{uT}^{(\alpha)}(2a) - \frac{\pi}{U} \varphi_{uB}^{(\alpha)}(2a) = \frac{2\pi a}{\beta} \tag{3.6}$$

$$\frac{\pi}{U} \varphi_{uB}^{(\delta)}(2a) - \frac{2\pi}{U} \varphi_{uB}^{(\delta)}(a) = 0 \tag{3.7}$$

Let L_{off} denote the lift of half biplane system beyond the tip,

$$L_{off} = \frac{4}{\pi} Q \cdot \alpha \cdot \int_{\frac{2a}{\beta}}^{b} \left(\frac{2\pi a}{\beta} \right) dy$$

$$= \frac{4}{\pi} Q \cdot \frac{2\pi a}{\beta} \left(b - \frac{2a}{\beta} \right) \cdot \alpha$$

$$= Q \cdot \left(\frac{2a}{\beta} \right)^2 \cdot \left(2\beta AR - 4 \right) \cdot \alpha \tag{3.8}$$

$$D_{off} = Q \cdot \left(\frac{2a}{\beta} \right)^2 \cdot \left(2\beta AR - 4 \right) \cdot \alpha^2 \tag{3.9}$$

where $AR = b/a$.

(4) Lift Coefficient, Drag Coefficient, and L/D Ratio

In view of (3.3) and (3.5), the total lift and drag of a biplane is

$$L = 2 \left(L_{off} + L_{tip} \right)$$

$$= \left(\frac{2a}{\beta} \right)^2 Q \cdot \left(4\beta AR - 8 + 2\bar{L} \right) \cdot \alpha \tag{4.1}$$

$$D = 2 \left(D_{off} + D_{tip} \right)$$

$$= \left(\frac{2a}{\beta} \right)^2 Q \cdot \left[\left(4\beta AR - 8 + 2\bar{L} \right) \cdot \alpha^2 + 2\bar{D} \cdot \delta^2 \right] \tag{4.2}$$

Defining the lift and drag coefficient based on the projected area of the biplane

$$C_{L,D} = \frac{L, D}{4ab \cdot Q} \tag{4.3}$$

we get $$c_L = \frac{1}{\beta^2 AR} \left[4\beta AR + 2\bar{L} - 8 \right] \tag{4.4}$$

$$C_D = \frac{1}{\beta^2 AR}\left[(4\beta AR + 2\bar{L} - 8)\cdot\alpha^2 + 2\bar{D}\cdot\delta^2\right] \tag{4.5}$$

and

$$L/D = \frac{1}{\alpha}\ \frac{1}{1 + \frac{\bar{D}}{2\beta AR + \bar{L} - 4}\left(\frac{\delta}{\alpha}\right)^2} \tag{4.6}$$

(5) Computation of $\bar{L}$ and $\bar{D}$.

As shown in (3),

$$\bar{L} = \frac{1}{\pi}\frac{\beta^2}{a^2}\int_0^{\frac{2a}{\beta}}\left[\frac{\pi}{U}\varphi_{uT}^{(\alpha)}(2a) - \frac{\pi}{U}\varphi_{uB}^{(\alpha)}(2a)\right]dy \tag{5.1}$$

$$\bar{D} = \frac{1}{\pi}\frac{\beta^2}{a^2}\int_0^{\frac{2a}{\beta}}\left[\frac{\pi}{U}\varphi_{uB}^{(\delta)}(2a) - \frac{\pi}{U}2\varphi_{uB}^{(\delta)}(a)\right]dy \tag{5.2}$$

Setting $\beta = 1$, $a = 10$, we have

$$\bar{L} = \frac{1}{100\pi}\int_0^{20}\left[\frac{\pi}{U}\varphi_{uT}^{(\alpha)}(20) - \frac{\pi}{U}\varphi_{uB}^{(\alpha)}(20)\right]dy \tag{5.3}$$

$$\bar{D} = \frac{1}{100\pi}\int_0^{20}\left[\frac{\pi}{U}\varphi_{uB}^{(\delta)}(20) - \frac{\pi}{U}2\varphi_{uB}^{(\delta)}(10)\right]dy \tag{5.4}$$

Let $-\int_{S_I}\sigma_{uB}\,\mu(0)\,dS = -A_\alpha\cdot\alpha + A_\delta\cdot\delta$

$$\frac{1}{2}\int_{S'_{I_0}}(\sigma_{uB} - \sigma_{uT})\,\mu(0)\,dS = B_\alpha\cdot\alpha + B_\delta\cdot\delta$$

$$\frac{\sqrt{2}\pi}{U}\varphi_u' = J_\alpha\cdot\alpha - J_\delta\cdot\delta$$

$$-\sqrt{2}\int_{S_I+S_{II}}\mu(0)\frac{\pi}{U}\frac{\partial\Phi_u'}{\partial c}dS = \iint_{S_I+S_{II}}\{J_\alpha'\}\cdot\alpha - \iint_{S_I+S_{II}}\{J_\delta'\}\cdot\delta$$

$$\sqrt{2}\int_{S_{I_0}+S_{II}}\mu(0)\frac{\pi}{U}\frac{\partial\Phi_u'}{\partial c}dS = -\iint_{S_{I_0}+S_{II}}\{J_\alpha'\}\cdot\alpha - \iint_{S_{I_0}+S_{II}}\{J_\delta'\}\cdot\delta \tag{5.5}$$

$$\int_{M^2c^2/4v_1}^{v_1}\frac{\sqrt{2}\pi}{U}\varphi'(u',v_1)\sqrt{\frac{v_1-u'}{u_1-v_1}}\left[\frac{1}{v_1-u'} - \frac{1}{u_1-u'}\right]du' = -\int\{J_\alpha\}\cdot\alpha + \int\{J_\delta\}\cdot\delta$$

where all the coefficients of α and δ are functions of (x, y) .

Through (5.1), the general forms of potential (1.1) and (1.2) become respectively

$$\frac{\pi}{U}\varphi_{uT} = \alpha \cdot \frac{1}{\sqrt{2}}\left[\sqrt{2}A_\alpha - \sqrt{2}B_\alpha + \frac{1}{2\pi}\iint_{S_{I_0}+S_{II}}\{J_\alpha'\} + \frac{1}{2\pi}\int\{J_\alpha\}\right]$$

$$+\delta \frac{1}{\sqrt{2}}\left[0 - \sqrt{2}B_\delta - \frac{1}{2\pi}\iint_{S_{I_0}+S_{II}}\{J_\delta'\} - \frac{1}{2\pi}\int\{J_\delta\}\right] \quad (5.6)$$

$$\frac{\pi}{U}\varphi_{uB} = \alpha \cdot \frac{1}{\sqrt{2}}\left[-\sqrt{2}A_\alpha + \sqrt{2}B_\alpha + J_\alpha + \frac{1}{\pi}\iint_{S_I+S_{II}}\{J_\alpha'\} - \frac{1}{2\pi}\iint_{S_{I_0}+S_{II}}\{J_\alpha'\} - \frac{1}{2\pi}\int\{J_\alpha\}\right]$$

$$+\delta \cdot \frac{1}{\sqrt{2}}\Big[\sqrt{2}A_\delta + \sqrt{2}B_\delta - J_\delta - \frac{1}{\pi}\iint_{S_I+S_{II}}\{J_\delta'\} \quad (5.7)$$

$$+\frac{1}{2\pi}\iint_{S_{I_0}+S_{II}}\{J_\delta'\} + \frac{1}{2\pi}\int\{J_\delta\}\Big]$$

For regions C and $(B)_2$, we have

$$\frac{\pi}{U}\varphi_{uT} = \alpha \cdot \frac{1}{\sqrt{2}}\left[\sqrt{2}A_\alpha - \sqrt{2}B_\alpha\right] + \delta \cdot \frac{1}{\sqrt{2}}\left[0 - \sqrt{2}B_\delta\right] \quad (5.8)$$

$$\frac{\pi}{U}\varphi_{uB} = \alpha \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}-\sqrt{2}A_\alpha + \sqrt{2}B_\alpha \\ + J_\alpha + \frac{1}{\pi}\iint_{S_I}\{J_\alpha'\}\end{bmatrix} + \delta \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{2}A_\delta + \sqrt{2}B_\delta \\ -J_\delta - \frac{1}{\pi}\iint_{S_I}\{J_\delta'\}\end{bmatrix} \quad (5.9)$$

For region $(A)_2$, the tip effect becomes nil, so

$$\frac{\pi}{U}\varphi_{uT} = \alpha \cdot A_\alpha + \delta \cdot 0 \quad (5.10)$$

$$\frac{\pi}{U}\varphi_{uB} = \alpha\begin{bmatrix}-A_\alpha + \frac{1}{\sqrt{2}}J_\alpha \\ + \frac{1}{\sqrt{2}\pi}\iint_{S_I}\{J_\alpha'\}\end{bmatrix} + \delta\begin{bmatrix}A_\delta - \frac{1}{\sqrt{2}}J_\delta \\ -\frac{1}{\sqrt{2}\pi}\iint_{S_I}\{J_\delta'\}\end{bmatrix} \quad (5.11)$$

For region $(B)_1$, which differs from $(B)_2$ by a difference in surface slope and by the absence of interaction between surfaces, we have

$$\frac{\pi}{U}\varphi_{uT} = \alpha \cdot \left[A_\alpha - B_\alpha\right] + \delta\left[0 - B_\delta\right] \tag{5.12}$$

$$\frac{\pi}{U}\varphi_{uB} = \alpha \cdot \left[-A_\alpha + B_\alpha\right] + \delta\left[-A_\delta + B_\delta\right] \tag{5.13}$$

For region $(A)_1$

$$\frac{\pi}{U}\varphi_{uT} = \alpha \cdot A_\alpha + \delta \cdot 0 \tag{5.14}$$

$$\frac{\pi}{U}\cdot\varphi_{uB} = \alpha \cdot \left[-A_\alpha\right] + \delta\left[-A_\delta\right] \tag{5.15}$$

In fact, the coefficients A_α, B_α, J_α, A_δ, B_δ, J_δ as well as the integrals

$$\frac{1}{\sqrt{2}\pi}\iint_{S_I}\{J_\alpha'\} \qquad \text{and} \qquad \frac{1}{\sqrt{2}\pi}\iint_{S_I}\{J_\delta'\}$$

can all be analytically evaluated. Therefore the potentials in region A and B can be put in purely analytic form

$$\phi = \phi^{(\alpha)}\cdot\alpha + \phi^{(\delta)}\cdot\delta$$

For the rest of the integrals

$$\iint_{S_I + S_{II}}\{J_i'\}, \quad \iint_{S_{I_0}+S_{II}}\{J_i'\}, \quad \int\{J_i\}, \qquad (i = \alpha, \delta)$$

numerical computation and graphical integration becomes necessary.

(5.1) Computation of $\frac{\sqrt{2}\pi}{U}\varphi_u'$ (for region D and C)

The potential contributed by the neighbouring surface is

$$\varphi_u' = -\frac{U}{\pi}\left\{\int_{S_I'}\sigma_{LT}\,\mu(c)\,dS + \int_{S_{II}'}\lambda_L\,\mu(c)\,dS\right\} \tag{5.16}$$

where λ_L is the monoplane value given by (Ref. 3):

$$\lambda_L(u,v) = \frac{\sigma_{LB}-\sigma_{LT}}{\pi}\left(\sqrt{\frac{2u}{v-u}} - \tan^{-1}\sqrt{\frac{2u}{v-u}}\right) \tag{5.17}$$

Putting

$$I_1 = M\int_{S_I'}\mu(c)\,dS'$$

$$I_2 = -\frac{M}{\pi}\int_{S_{II}'}\mu(c)\left(\sqrt{\frac{2u}{v-u}} - \tan^{-1}\sqrt{\frac{2u}{v-u}}\right)dS \tag{5.18}$$

Thus

$$\frac{\pi}{U}\varphi_u' = -\sigma_{LT}\frac{I_1}{M} + (\sigma_{LB}-\sigma_{LT})\frac{I_2}{M} \tag{5.19}$$

Here the quantities σ_{LT}, σ_{LB} belong to forward half of the wing, so that, with $\sigma_{LT} = -\alpha + \delta$, $\sigma_{LB} = \alpha$, and $\beta = 1$, we have

$$\begin{aligned}\sqrt{2}\,\frac{\pi}{U}\varphi_u' &= -(-\alpha+\delta)I_1 + (2\alpha-\delta)I_2 \\ &= (I_1 + 2I_2)\cdot\alpha - (I_1 + I_2)\cdot\delta \\ &= J_\alpha\cdot\alpha - J_\delta\cdot\delta\end{aligned} \tag{5.20}$$

By Appendix 4, Ref. 3:

$$I_1 = 2\left[\begin{aligned}&\frac{u_1+v_1}{2}\left(\frac{\pi}{2} + \sin^{-1}\frac{u_1-v_1}{\sqrt{(u_1+v_1)^2-M^2c^2}}\right) + \frac{u_1-v_1}{2}\cosh^{-1}\frac{u_1+v_1}{\sqrt{(u_1-v_1)^2+M^2c^2}} \\ &-\frac{Mc}{2}\left(\sin^{-1}\frac{u_1(u_1+v_1)-M^2c^2/2}{u_1\sqrt{(u_1+v_1)^2-M^2c^2}} + \sin^{-1}\frac{u_1(u_1-v_1)+M^2c^2/2}{u_1\sqrt{(u_1-v_1)^2+M^2c^2}}\right.\end{aligned}\right] \tag{5.21}$$

By Appendix 4, Ref. 3:

$$I_2 = 2\sqrt{u_1 v_1 - \frac{M^2c^2}{4}} - \frac{u_1 - v_1}{2}\cosh^{-1}\frac{u_1+v_1}{\sqrt{(u_1-v_1)^2+M^2c^2}}$$

$$+\frac{u_1+v_1}{2}\left[\sin^{-1}\frac{u_1-v_1}{\sqrt{(u_1+v_1)^2-M^2c^2}} + \sin^{-1}\frac{2v_1-\sqrt{(u_1-v_1)^2+M^2c^2}}{\sqrt{(u_1+v_1)^2-M^2c^2}}\right] \qquad (5.21)$$

$$-\frac{Mc}{2}\left[\frac{\pi}{2} - \sin^{-1}\frac{u_1^2-v_1^2-M^2c^2+(u_1+v_1)\sqrt{(u_1-v_1)^2+M^2c^2}}{\sqrt{(u_1+v_1)^2-M^2c^2}\left[u_1-v_1+\sqrt{(u_1-v_1)^2+M^2c^2}\right]}\right.$$

$$\left.+\sin^{-1}\frac{u_1(u_1+v_1)-M^2c^2/2}{u_1\sqrt{(u_1+v_1)^2-M^2c^2}} - \sin^{-1}\frac{u_1(u_1-v_1)+M^2c^2/2}{u_1\sqrt{(u_1-v_1)^2+M^2c^2}}\right] \qquad (5.22)$$

$$-\sqrt{v_1(u_1-v_1)-M^2c^2/2+v_1\sqrt{(u_1-v_1)^2+M^2c^2}}$$

Note that the limiting value of φ_u' goes to zero along the hyperbolic boundary $uv = \frac{M^2c^2}{4}$ off the wing and thus φ_u' is continuous across the hyperbola. On the other branch of the hyperbola on the wing, φ_u' satisfies the relation (Ref. 3 Appendix 4)

$$\varphi_u' = -\frac{U\sigma_{LB}}{\beta}(x-\beta c) = -\frac{U\sigma_{LB}}{\beta}(u+v) + U\sigma_{LB}c$$

which, in fact, is the expression for the potential φ_u' for region A and B. These remarks serve as a check to the numerical computations.

Since in region C, D and over the diaphragm the expression for φ_u' is complicated, we carry out the computations numerically for points in these regions. As φ_u' only enters into the integration over the regions extended by the inverse Mach cone, we need only compute for points bounded by the hyperbola $uv = M^2c^2/4$, the trailing edge of wing and the inverse Mach line on the diaphragm extending from trailing edge tip. Hence, using $\beta = 1$, $a = c = 10$, the range of computation will be as follows:

$\mu\nu = 50$

ν: 2.5, 3, 3.5, 4, 5, 6, 7, 9, 11, 13

μ: 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 19, 21, 22, 23, 24

Our computation starts with a chosen ν (say = 4), and proceeds with the formulas (5.21), (5.22) in computing I_1 and I_2 for a range of μ.

Fig. 4 & 5 present the results for I_1 and I_2. Fig. 6, 7 give $J_\alpha(\mu,\nu)$ and $J_\delta(\mu,\nu)$.

For the sake of convenience, we use the principal branch in the identity

$$\cosh^{-1}\mu = \ln(\mu + \sqrt{\mu^2-1})$$

for determining $\cosh^{-1}\mu$.

(5.2) Computation of $\frac{\pi}{U}\frac{\partial\Phi_u'}{\partial c}$ (for region D and C)

$$\frac{\pi}{U}\frac{\partial\Phi_u'}{\partial c} = -\frac{\partial}{\partial c}\int_{S_I'}\sigma_{cT}\,\mu(c)\,dS - \frac{\partial}{\partial c}\int_{S_{II}'}\lambda_c\,\mu(c)\,dS \tag{5.23}$$

$$= \sigma_{cT}\,I_1' + (\sigma_{cB} - \sigma_{cT})\,I_2' \tag{5.24}$$

$$= -J_\alpha'\cdot\alpha + J_\delta'\cdot\delta \tag{5.25}$$

where

$$\begin{aligned} J_\alpha' &\equiv I_1' - 2I_2' \\ J_\delta' &\equiv I_1' - I_2' \end{aligned} \tag{5 .26}$$

By Appendix 3. Ref. 3:

$$I_1' = \frac{\pi}{2} + \tan^{-1}\frac{\mu_1^2 - \nu_1^2}{2Mc\sqrt{\mu_1\nu_1 - M^2c^2/4}} \tag{5.27}$$

$$I_2' = -\frac{1}{2}\left\{\frac{\pi}{2} - \sin^{-1}\frac{(\mu_1-\nu_1)\mu_1 + M^2c^2/2}{\mu_1\sqrt{(\mu_1-\nu_1)^2+M^2c^2}} + \sin^{-1}\frac{(\mu_1+\nu_1)\mu_1 - M^2c^2/2}{\mu_1\sqrt{(\mu_1+\nu_1)^2-M^2c^2}}\right.$$
$$- \sin^{-1}\frac{\mu_1^2-\nu_1^2-M^2c^2+(\mu_1+\nu_1)\sqrt{(\mu_1-\nu_1)^2+M^2c^2}}{\sqrt{(\mu_1+\nu_1)^2-M^2c^2}\,[\,\mu_1-\nu_1+\sqrt{(\mu_1-\nu_1)^2+M^2c^2}}$$
$$\left.- \frac{2Mc\sqrt{\nu_1(\mu_1-\nu_1)-M^2c^2/2+\nu_1\sqrt{(\mu_1-\nu_1)^2+M^2c^2}}}{\sqrt{(\mu_1-\nu_1)^2+M^2c^2}\,[\,\mu_1-\nu_1+\sqrt{(\mu_1-\nu_1)^2+M^2c^2}\,]}\right\} \tag{5.28}$$

The limiting value of $\frac{\partial \varphi_u'}{\partial c}$ along the hyperbola $\mu\nu = 50$ is zero off wing and constant $= U\varphi_{LT}$ on the wing. So it is continuous across the curve $\mu\nu = 50$. $\frac{\partial \varphi_u'}{\partial c}$ is, however, discontinuous across the midchord. All these have been used as a check for the numerical result.

The numerical data and the computation procedure are similar to those for φ_u'. In

$$\cosh^{-1}\frac{1}{\sqrt{1+x^2}} = \tan^{-1} x$$

the principal branch of $\tan^{-1} x$ is used. In fact, much of the computed result for φ_u' can be taken over, and so the labor of calculation much reduced.

Fig. 8, 9 present the result of $J_\alpha'(u,v)$ and $J_\delta'(u,v)$.

(5.3) Graphical Integration of $\int\{J_\alpha\}$ and $\int\{J_\delta\}$

$$\text{i.e.} \int_{\frac{M^2c^2}{4v_1}}^{v_1} \frac{\sqrt{2}\,\pi}{U}\varphi_u'(u',v_1)\sqrt{\frac{v_1-u'}{u_1-v_1}}\left[\frac{1}{v_1-u'}-\frac{1}{u_1-u'}\right]du'$$

Since potentials in regions c and D enter into the expressions for lift and drag only as $\phi(2a)$, we are concerned, therefore, only with those points (μ_1, ν_1) along the trailing edge $x = 2a$.

For numerical computation, we choose the following four points:

$$(\mu_1,\nu_1) = (19.284, 9),\ (17.284, 11),\ (15.284, 13),\ (10\sqrt{2}, 10\sqrt{2})$$

For the last point at tip, the limit of the integral converges to (Ref. 3):

$$\pi J_\alpha(10\sqrt{2}, 10\sqrt{2})\cdot\alpha - \pi J_\delta(10\sqrt{2}, 10\sqrt{2})\cdot\delta$$

Writing the integral in the form

$$\sum_{i=\alpha,\delta} i \cdot f(J_i) = \sum i \cdot \int_{50/v_1}^{v_1} J_i(u',v_1)\sqrt{\frac{v_1-u'}{u_1-v_1}}\left[\frac{1}{v_1-u'}-\frac{1}{u_1-u'}\right]du'$$

$$= \sum i \cdot \sqrt{u_1-v_1} \cdot \int_{50/v_1}^{v_1} \frac{J_i(u',v_1)\ du'}{(u_1-u')\sqrt{v_1-u'}} \tag{5.29}$$

$$= \sum i \cdot \sqrt{u_1-v_1} \left[\int_{50/v_1}^{v_1-\epsilon} \frac{J_i(u',v_1)\,du'}{(u_1-u')\sqrt{v_1-u'}} + 2 \cdot \epsilon^{\frac{1}{2}} J_i(v_1-\epsilon,v_1) + \frac{4}{3}\epsilon^{3/2} J_i'(v_1-\epsilon,v_1)\right]$$

$\epsilon = 0.10$, we get

$$= \sum i \sqrt{u_1-v_1} \left[\int_{50/v_1}^{v_1-.1} \frac{J_i(u',v_1)\,du'}{(u_1-u')\sqrt{v_1-u'}} + .633 J_i(v_1-.1,v_1) + .042\left(\frac{d}{du'}J_i(u',v_1)\right)_{u'=v_1-.1}\right] \tag{5.30}$$

The error involved will be of the order $\epsilon^{5/2} = 0.003$.

After the point (u_1, v_1) was chosen, we simply have to form the integral

$$\frac{J_i(u',v_1)}{(u_1-u')\sqrt{v_1-u'}} \qquad (i = \alpha, \delta)$$

over the interval $(50/v_1,\ v_1 - 0.10)$ on the diaphragm off wing, with $J_\alpha(u', v_1)$ and $J_\delta(u', v_1)$ known from (6.1). The graphical integration is performed by using a planimeter. The last correction term $\frac{dJ_i}{du}$ is approximated by $\frac{\Delta J_i}{\Delta u}$.

The result is plotted in Fig. 10.

(5.4) Graphical Integration of

$$\iint_{S_I + S_{II}} \{ J'_{\alpha,\delta} \} = \int^{\mu_1} \frac{d\mu}{\sqrt{\mu_1 - \mu}} \int \frac{J'_{\alpha,\delta}}{\sqrt{v_1 - v}} dv$$

and

$$\iint_{S_{I_0} + S_{II}} \{ J'_{\alpha,\delta} \} = \int^{\mu_0} \frac{d\mu}{\sqrt{\mu_1 - \mu}} \int \frac{J'_{\alpha,\delta}}{\sqrt{v_1 - v}} dv$$

It is seen that the second integral is contained as a part of the first integral, and can easily be separated from the first at the last step of integration. The values J'_α and J'_δ have been computed in (6.2).

The integrals are evaluated graphically for points (μ_1, v_1): (10/2, 10/2), (15.284,13), (17.284,11), (19.284,9) in region D, and (21.284,7), (22.284,6), (23.284,5), (24.284,4) in region C for which the second integral disappears and the first reduces to

$$\int_{S_I} \frac{d\mu}{\sqrt{\mu_1 - \mu}} \int \frac{J'_{\alpha,\delta}\, dv}{\sqrt{v_1 - v}} .$$

In each of the stripwise integration with respect to v and μ, the same technique as explained in (6.3) is adopted for the neighborhood of the singular point. Results are plotted in Fig. 11,12.

(6.5) Computation of $-\int_{S_I} \sigma_{uB}\, \mu(0)\, dS$

As shown in Part II A(1), (i):

For region $(B)_2$ and C

$$-\int_{S_I} \sigma_{uB}\, \mu(0)\, dS = \pi \left[-\frac{\sigma_{uB}^{(1)}}{\pi\beta} \int_0^{x-\beta y} \left[\sin^{-1}\frac{\beta y}{x-\xi} + \frac{\pi}{2} \right] d\xi - \frac{\sigma_{uB}^{(1)}}{\beta}(a - x + \beta y) - \frac{\sigma_{uB}^{(2)}}{\beta}(x-a) \right]$$

$$= -\frac{\pi}{\beta}\left[\frac{1}{\pi}A + (a - x + \beta y) + (x-a) \right] \cdot \alpha$$

$$-\frac{\pi}{\beta}\left[\frac{1}{\pi}A + (a - x + \beta y) - (x-a) \right] \cdot \delta \qquad (5.31)$$

For region D

$$-\int_{S_I}\sigma_{uB}\mu(0)\,dS = \pi\left[\begin{array}{l} -\dfrac{\sigma_{uB}^{(1)}}{\pi\beta}\displaystyle\int_0^a\left[\sin^{-1}\frac{\beta y}{x-\xi}+\frac{\pi}{2}\right]d\xi \\ -\dfrac{\sigma_{uB}^{(2)}}{\pi\beta}\displaystyle\int_0^{x-\beta y}\left[\sin^{-1}\frac{\beta y}{x-\xi}+\frac{\pi}{2}\right]d\xi - \sigma_{uB}^{(2)}y \end{array}\right] \quad (5.32)$$

$$= -\frac{\pi}{\beta}\left[\frac{1}{\pi}B+\frac{1}{\pi}C+\beta y\right]\cdot\alpha$$

$$-\frac{\pi}{\beta}\left[\frac{1}{\pi}B-\frac{1}{\pi}C-\beta y\right]\cdot\delta$$

where

$$A = \int_0^{x-\beta y}\left[\sin^{-1}\frac{\beta y}{x-\xi}+\frac{\pi}{2}\right]d\xi$$

$$= \left[\cosh^{-1}\frac{x}{\beta y}\right]\beta y + x\sin^{-1}\frac{\beta y}{x}+\frac{\pi}{2}(x-2\beta y)$$

$$B = \int_0^a\left[\sin^{-1}\frac{\beta y}{x-\xi}+\frac{\pi}{2}\right]d\xi$$

$$= x\sin^{-1}\frac{\beta y}{x}+\beta y\cosh^{-1}\frac{x}{\beta y}-(x-a)\sin^{-1}\frac{\beta y}{x-a}-\beta y\cosh^{-1}\frac{x-a}{\beta y}+\frac{\pi}{2}a$$

$$C = A - B$$

and $\sigma_{uB}^{(1)} = \alpha+\delta$, $\sigma_{uB}^{(2)} = \alpha-\delta$ are noted.

Along $y = 0$

$$-\int_{S_I}\sigma_{uB}\mu(0)\,dS = \frac{\pi}{\beta}\left[-\frac{x}{2}\right]\cdot\alpha+\frac{\pi}{\beta}\left[\frac{x}{2}-a\right]\cdot\delta$$

By putting $\delta = 0$ and changing the sign of α in $-\int_{S_I}\sigma_{uB}\mu(0)\,dS$ we get $-\int_{S_I}\sigma_{uT}\mu(0)\,dS$

The points (x, y) computed are: $x = 20$; $y = 0$, 1.615, 4.444, 7.272, 10.101, 11.515, 12.929, and 14.343. The results are shown in Fig. 13.

(5.6) Computations of $\frac{1}{2}\int_{S_{I_0}}(\sigma_{uB}-\sigma_{uT})\mu(0)\,dS$

From Part IIA(1), (ii), with $\sigma_{uB}^{(1)}-\sigma_{uT}^{(1)} = 2\alpha+\delta$, $\sigma_{uB}^{(2)}-\sigma_{uT}^{(2)} = 2\alpha-\delta$, we have:

for region B_2, C

$$\frac{1}{2}\int_{S_{I_0}}(\sigma_{uB}-\sigma_{uT})\mu(0)\,dS = \frac{1}{2\beta}(\sigma_{uB}^{(1)}-\sigma_{uT}^{(1)})I_1 = \frac{I_1}{\beta}\cdot\alpha+\frac{I_1}{2\beta}\cdot\delta \quad (5.33)$$

for region D

$$\frac{1}{2}\int_{S_{I_D}}(\sigma_{uB}-\sigma_{uT})\mu(0)\,dS = \frac{1}{2\beta}(\sigma_{uB}^{(2)}-\sigma_{uT}^{(2)})\,I_1 - \frac{1}{2\beta}(\sigma_{uB}-\sigma_{uT})\Big|_{(1)}^{(2)} I_2 \tag{5.34}$$
$$= \frac{I_1}{\beta}\cdot\alpha + \frac{I_2-\frac{1}{2}I_1}{\beta}\cdot\delta$$

where

$$I_1 = x\sin^{-1}\frac{\beta y}{x} + 2x\tan^{-1}\sqrt{\frac{x-\beta y}{\beta y}} + \beta y\cosh^{-1}\frac{x}{\beta y} - 2\sqrt{\beta y}\sqrt{x-\beta y} - \frac{\pi}{2}x$$

$$I_2 = x\sin^{-1}\frac{\beta y}{x} - (x-a)\sin^{-1}\frac{\beta y}{x-a} - a\sin^{-1}\left(\frac{2\beta y}{x-a}-1\right) + \beta y\left[\cosh^{-1}\frac{x}{\beta y} - \cosh^{-1}\frac{x-a}{\beta y}\right]$$
$$+ 2x\left[\tan^{-1}\sqrt{\frac{x}{\beta y}-1} - \tan^{-1}\sqrt{\frac{x-a}{\beta y}-1}\right] - 2\sqrt{\beta y}\left[\sqrt{x-\beta y} - \sqrt{x-a-\beta y}\right]$$

The same set of points on the trailing edge as in (6.5) is used. The result is shown in Fig. 14.

(5.7) Computation of Potential in Region B_1 and B_2

The potentials outside of tip interaction region (C and D) are

$$\varphi_{uT} = -\frac{U}{\pi}\int_{S_I}\sigma_{uT}\,\mu(0)\,dS + \frac{U}{2\pi}\int_{S_{I_b}}\mu(0)\left[\sigma_{uT} - \frac{\pi}{U}\zeta_{uB}\right]dS$$
$$\varphi_{uB} = -\int_{S_I}\zeta_{uB}\,\mu(0)\,dS - \frac{U}{2\pi}\int_{S_{I_b}}\mu(0)\left[\sigma_{uT} - \frac{\pi}{U}\zeta_{uB}\right]dS + \varphi_u' \tag{5.35}$$

where

$$\varphi_u' = -\frac{U}{\pi}\int_{S'}\sigma_{lT}\,\mu(0)\,dS \tag{5.36}$$
$$\zeta_{uB} = \frac{U}{\pi}\left[\sigma_{uB}^{(2)} + \frac{1}{\pi}\sigma_{lT}^{(1)}I_1' + \frac{1}{\pi}(\sigma_{lB}^{(1)} - \sigma_{lT}^{(1)})I_2'\right]$$

(see Part IA(8)).

i) For B_1, i.e., $0 < x < a$, $I_1' = I_2' = 0$, so

$$\zeta_{uB} = \frac{U}{\pi}\sigma_{uB}^{(1)} = \frac{U}{\pi}(\alpha+\delta)$$
$$\zeta_{lT} = \frac{U}{\pi}\sigma_{lT}^{(1)} = \frac{U}{\pi}(-\alpha+\delta) \tag{5.37}$$
$$\frac{\pi}{U}\varphi_{uT} = \left[\int_{S_{I_a}}\mu(0)\,dS\right]\cdot\alpha - \left[\frac{1}{2}\int_{S_{I_b}}\mu(0)\,dS\right]\cdot\delta$$

$$\frac{\pi}{U}\varphi_{uB} = -\left[\int_{S_{Ia}}\mu(o)\,dS\right]\cdot\alpha - \left[\int_{S_{Ia}}\mu(o)\,dS + \frac{1}{2}\int_{S_{Ib}}\mu(o)\,dS\right]\cdot\delta \qquad (5.38)$$

Now, with $\beta = 1$

$$\int_{S_{Ia}}\mu(o)\,dS = \pi x + 2\sqrt{y}\sqrt{x-y} - 2x\tan^{-1}\sqrt{\frac{x-y}{y}} \qquad (5.39)$$

$$\frac{1}{2}\int_{S_{Ib}}\mu(o)\,dS = -\sqrt{y}\sqrt{x-y} + x\tan^{-1}\sqrt{\frac{x-y}{y}} + \frac{1}{2}y\cosh^{-1}\frac{x}{y} - \frac{\pi}{4}x + \frac{1}{2}x\sin^{-1}\frac{y}{x} \qquad (5.40)$$

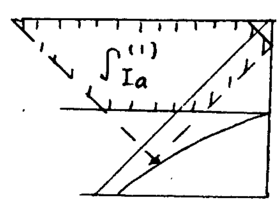
So in $\quad \frac{\pi}{U}\varphi_{i'} = \frac{\pi}{U}\varphi_{i'}^{(\alpha)}\cdot\alpha + \frac{\pi}{U}\varphi_{i'}^{(\delta)}\cdot\delta \qquad (i' = uT, uB) \qquad (5.41)$

$$\frac{\pi}{U}\varphi_{uT}^{(\alpha)} = \int_{S_{Ia}}\mu(o)\,dS \qquad \frac{\pi}{U}\varphi_{uT}^{(\delta)} = -\frac{1}{2}\int_{S_{Ib}}\mu(o)\,dS$$
$$\frac{\pi}{U}\varphi_{uB}^{(\alpha)} = -\int_{S_{Ia}}\mu(o)\,dS \qquad \frac{\pi}{U}\varphi_{uB}^{(\delta)} = -\int_{S_{Ia}}\mu(o)\,dS - \frac{1}{2}\int_{S_{Ib}}\mu(o)\,dS \qquad (5.42)$$

$\varphi_{i'}^{(\alpha)}$ etc. are in analytic forms, i.e.

ii) For B_2, i.e., $a < x < 2a$, $\beta y \geq \sqrt{x^2 - a^2}$

$$I_1' = \pi \qquad I_2' = 0$$
$$q_{uB} = \frac{U}{\pi}(\sigma_{uB}^{(2)} + \sigma_{cT}^{(1)}) = 0$$



$$\frac{\pi}{U}\varphi_{uT} = \alpha\cdot\int_{S_{Ia}}\mu(o)\,dS - \delta\cdot\frac{1}{2}\int_{S_{Ib}}\mu(o)\,dS \qquad (5.43)$$

$$\frac{\pi}{U}\varphi_{uB} = \alpha\cdot\left[-\int_{S_{Ia}^{(1)}}\mu(o)\,dS + \int_{S_{I'}}\mu(o)\,dS\right] + \delta\cdot\left[-\int_{S_{Ia}^{(1)}}\mu(o)\,dS - \frac{1}{2}\int_{S_{Ib}}\mu(o)\,dS - \int_{S_{I'}}\mu(o)\,dS\right] \qquad (5.44)$$

where, with

$$\int_{S_{Ia}} \mu(0)\, dS = \pi x + 2\sqrt{y}\sqrt{x-y} - 2x \tan^{-1}\sqrt{\frac{x-y}{y}}$$

$$\frac{1}{2}\int_{S_{Ib}} \mu(0)\, dS = -\sqrt{y}\sqrt{x-y} + x \tan^{-1}\sqrt{\frac{x-y}{y}} + \frac{1}{2} y \cosh^{-1}\frac{x}{y}$$
$$- \frac{\pi}{4} x + \frac{1}{2} \sin^{-1}\frac{y}{x}$$

$$\int_{S_{Ia}^{(1)}} \mu(0)\, dS = \pi a + 2\sqrt{y}\sqrt{x-y} - 2x \tan^{-1}\sqrt{\frac{x-y}{y}}$$

$$\int_{S_{I'}} \mu(c)\, dS = \pi (x-a)$$

So $\varphi_{uB}^{(\alpha)}$ and $\varphi_{uB}^{(\delta)}$ are again analytic.

The limiting values of these integrals on the boundary separating A and B are

$$\lim_{y\to x} \int_{S_{Ia}} \mu(0)\, dS = \pi x \qquad \lim_{y\to x} \int_{S_{Ia}^{(1)}} \mu(0)\, dS = \pi a$$

$$\lim_{y\to x} \frac{1}{2}\int_{S_{Ib}} \mu(0)\, dS = 0 \qquad \lim_{y\to x} \int_{S_{I'}} \mu(c)\, dS = \pi(x-a)$$

which check the computed results.

In the computation, the integrals such as (5.37), (5.38) etc. are first evaluated, then $\varphi_{uT}^{(\alpha)}(a)$ etc. The computations are made for points (x, y) as follows:

for $\varphi(2a)$: $x = 20$

$y = \sqrt{300},\ 26\sqrt{2}-20,\ 27\sqrt{2}-20,\ 20.$

for $\varphi(a)$: $x = 10$

$y = 0,\ 1.615,\ 4.444,\ 7.272$

The point $(20, \sqrt{300})$ is the intersection of hyperbola $uv = 50$ and the trailing edge $x = 2a = 20$. Results are plotted in Fig. 15, 16.

(5.8) Computation of Potential For A_1 and A_2

For A_1: $$\frac{\pi}{U}\varphi_{uT} = \alpha \cdot \int_{S_I} \mu(0)\, dS = \pi x \alpha \qquad (5.45)$$

$$\frac{\pi}{U}\varphi_{uB} = -(\alpha+\delta)\int_{S_I} \mu(0)\, dS = -\pi x(\alpha+\delta)$$

 (5.46)

For A_2:

$$\frac{\pi}{U}\varphi_{uT} = \alpha\cdot\int_{S_I}\mu(0)\,dS = \pi\chi\alpha \tag{5.47}$$

$$\begin{aligned}\frac{\pi}{U}\varphi_{uB} &= -\int_{S_I^{(1)}}\mu(0)q_{uB}^{(1)}\,dS - \int_{S_I^{(2)}}\mu(0)\,q_{uB}^{(2)}\,dS - \int_{S_{I'}}\mu(c)\sigma_{cT}^{(1)}\,dS \\ &= \left[-\int_{S_I^{(1)}}\mu(0)\,dS + \int_{S_{I'}}\mu(c)\,dS\right]\cdot\alpha \\ &\quad - \left[\int_{S_I^{(1)}}\mu(0)\,dS + \int_{S_{I'}}\mu(c)\,dS\right]\cdot\delta \\ &= -(2a-\chi)\pi\alpha - \pi\chi\delta\end{aligned} \tag{5.48}$$

So

$$\frac{\pi}{U}\varphi(a):\quad \begin{aligned}&\frac{\pi}{U}\varphi_{uB}^{(\alpha)}(a) = \frac{\pi}{U}\varphi_{uB}^{(\delta)}(a) = -\frac{\pi}{U}\varphi_{uT}^{(\alpha)}(a) = -\pi a\\ &\frac{\pi}{U}\varphi_{uT}^{(\delta)}(a) = 0\end{aligned} \tag{5.49}$$

$$\frac{\pi}{U}\varphi(2a):\quad \begin{aligned}&\frac{\pi}{U}\varphi_{uB}^{(\alpha)}(2a) = \frac{\pi}{U}\varphi_{uT}^{(\delta)}(2a) = 0\\ &\frac{\pi}{U}\varphi_{uB}^{(\delta)}(2a) = -\frac{\pi}{U}\varphi_{uT}^{(\alpha)}(2a) = -2a\pi\end{aligned} \tag{5.50}$$

(5.9) Resume of Results of Computation

The results of computation for potentials in region D and C are combined according to (5.6), (5.7), (5.8), (5.9), (5.10) (5.11) etc. to give the final form of potentials, i.e.

$$\varphi_i(x) = \varphi_i^{(\alpha)}(x)\cdot\alpha + \varphi_i^{(\delta)}(x)\cdot\delta$$

where $i = uT,\ uB$; $x = a,\ 2a$

The resultant $\varphi_i^{(\alpha)}(x)$ etc. are plotted in Fig. 17.

Finally, the functions

$$\varphi_{uT}^{(\alpha)}(2a) - \varphi_{uB}^{(\alpha)}(2a)$$

and

$$\varphi_{uB}^{(\delta)}(2a) - 2\varphi_{uB}^{(\delta)}(a)$$

are computed for points along the span over the tip region, from $y = 0$ to $y = 2a/\beta = 20$, and therefore the contributions by the angle of attack α and the thickness ratio δ on lift and drag along the span are obtained, as shown in Fig. 18.

Integration by planimeter gives

$$\int_0^{20} \left\{ \varphi_{UT}^{(\alpha)}(20) - \varphi_{UB}^{(\alpha)}(20) \right\} dy = 731.0$$

$$\int_0^{20} \left\{ \varphi_{UB}^{(\delta)}(20) - 2\varphi_{UB}^{(\delta)}(10) \right\} dy = 129.2$$

So in (5.3) and (5.4)

$$\bar{L} = \frac{731.0}{100\pi} = 2.327 \tag{5.51}$$

$$\bar{D} = \frac{129.2}{100\pi} = 0.4113 \tag{5.52}$$

C. RESULTS AND CONCLUSION

From the results of computation, it is obtained that

$$\bar{L} = 2.327$$
$$\bar{D} = 0.4113$$

Thus the lift and drag coefficients of a Busemann biplane with chord length $2a$, span $2b$, wedge angle δ at Mach number M, and angle of attack α are given by (4.4), (4.5) and (4.6) as follows:

$$C_L = \frac{L}{4ab\cdot Q} = \frac{1}{\beta^2 AR}[4\beta AR + 2\bar{L} - 8]\alpha = \frac{1}{\beta^2 AR}[4\beta AR - 3.346]\alpha \quad (1)$$

$$C_D = \frac{D}{4ab\cdot Q} = \frac{1}{\beta^2 AR}[(4\beta AR + 2\bar{L} - 8)\alpha^2 + 2\bar{D}\cdot\delta^2]$$
$$= \frac{1}{\beta^2 AR}[(4\beta AR - 3.346)\alpha^2 + 0.823\,\delta^2] \quad (2)$$

$$L/D = \frac{1}{\alpha}\,\frac{1}{1 + \frac{0.823}{4\beta AR - 3.346}\left(\frac{\delta}{\alpha}\right)^2} \quad (3)$$

where $\beta = \sqrt{M^2 - 1}$, $AR = b/a$.

Figure (19) gives t a plot for C_L/α (which is also $C_D^{(\alpha^2)}/\alpha^2$) against Mach number M with different aspect ratios. Figure (20) gives the same plot for $C_D^{(\delta^2)}/\delta^2$. Thus, for any given aspect ratio of the biplane at a certain Mach number, C_L and C_D can immediately be obtained from the curves by following relations:

$$C_L = \left(\frac{C_L}{\alpha}\right)\cdot\alpha \qquad C_D = \left(\frac{C_D^{(\alpha^2)}}{\alpha^2}\right)\cdot\alpha^2 + \left(\frac{C_D^{(\delta^2)}}{\delta^2}\right)\cdot\delta^2$$

Figure (21) gives a plot of L/D against βAR for different δ/α ratios.

The limiting case of the present theory occurs when i.e., $\beta A\!\!R = 2$, in which case the two tip Mach cones on same wing meet. These limiting cases are either given in plots by the point $\otimes$, or indicated with incorporated tables. It may further be mentioned that these limiting cases lead to a unique result for L/D ratio as follows:

$$\left(L/D \right)_{Limit} = \frac{1}{\alpha} \frac{1}{1 + 0.177 \, (\delta/\alpha)^2}$$

From the computed result, it is easily seen that a Busemann biplane of finite span highly approximates a flat plate, as the wave drag due to wing thickness is inherently low.

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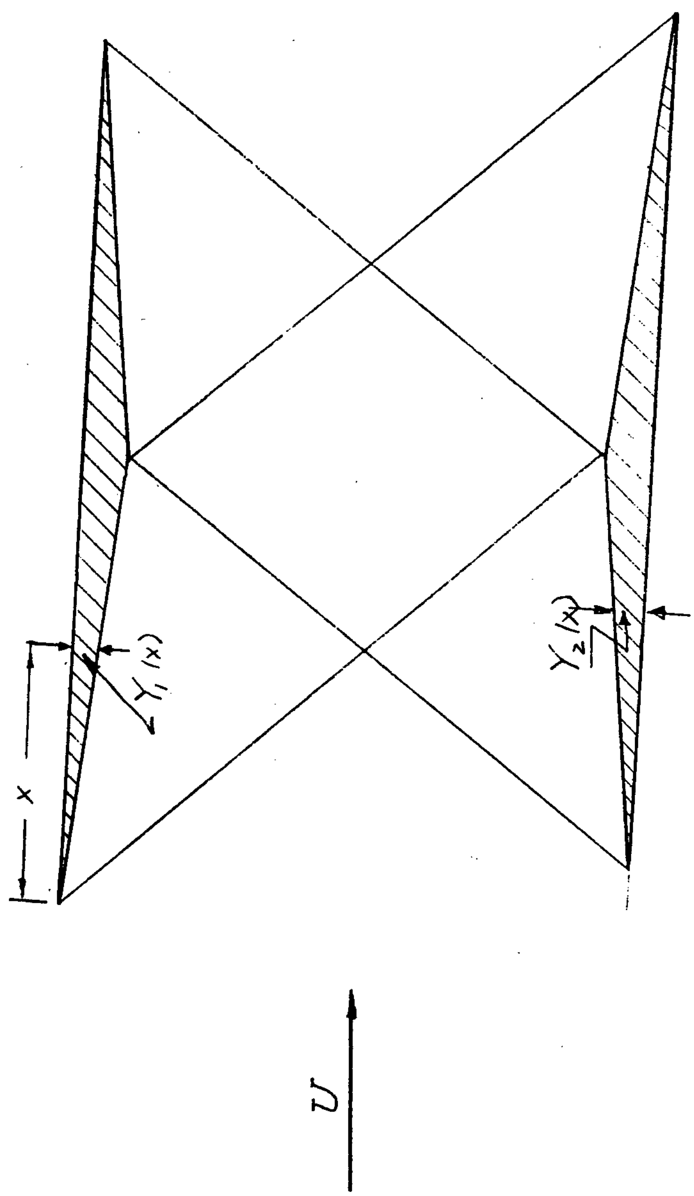


Fig. 1

The Busemann biplane arrangement

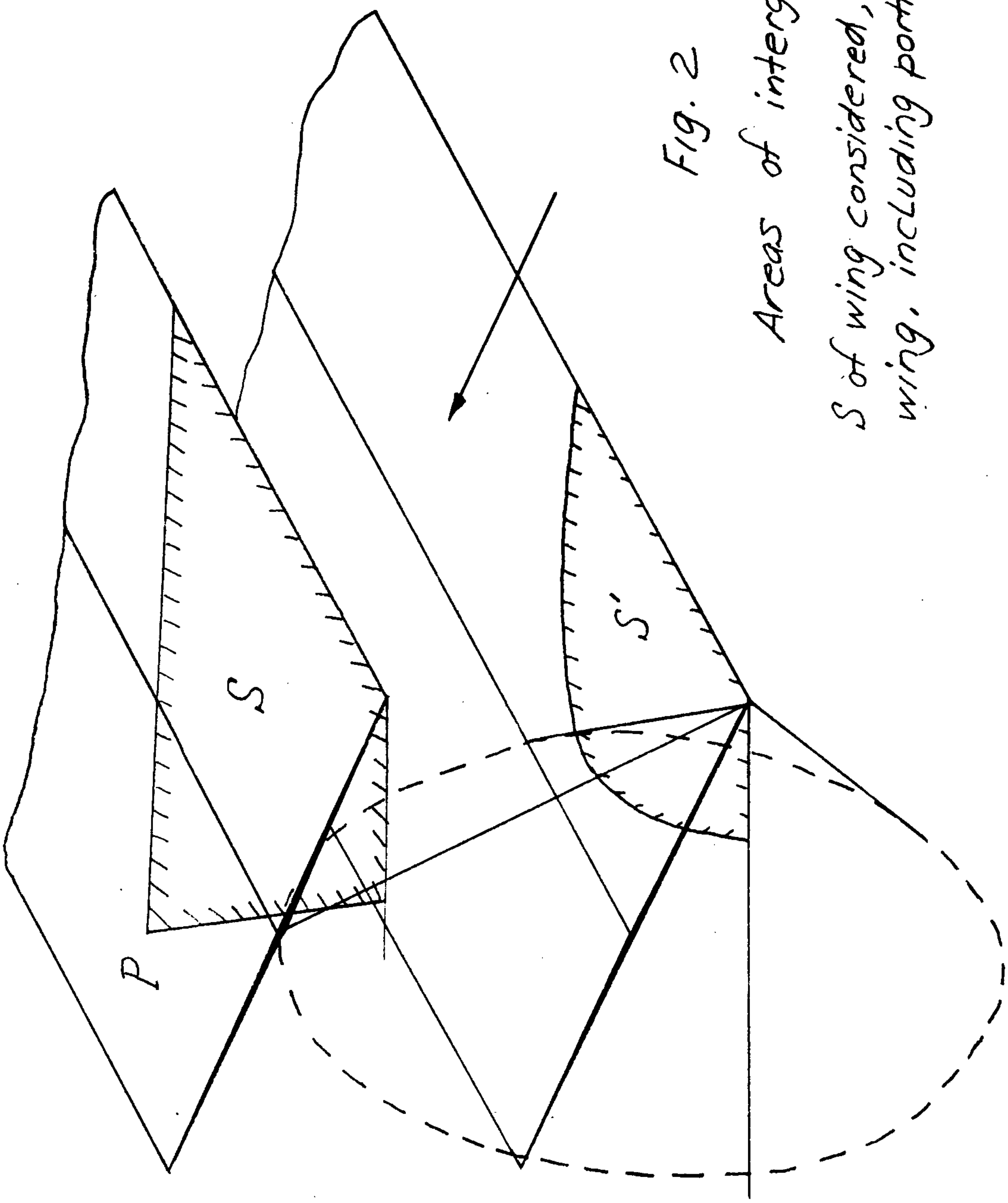


Fig. 2

Areas of intergration

S of wing considered, S' of other wing, including portions of diaphragms

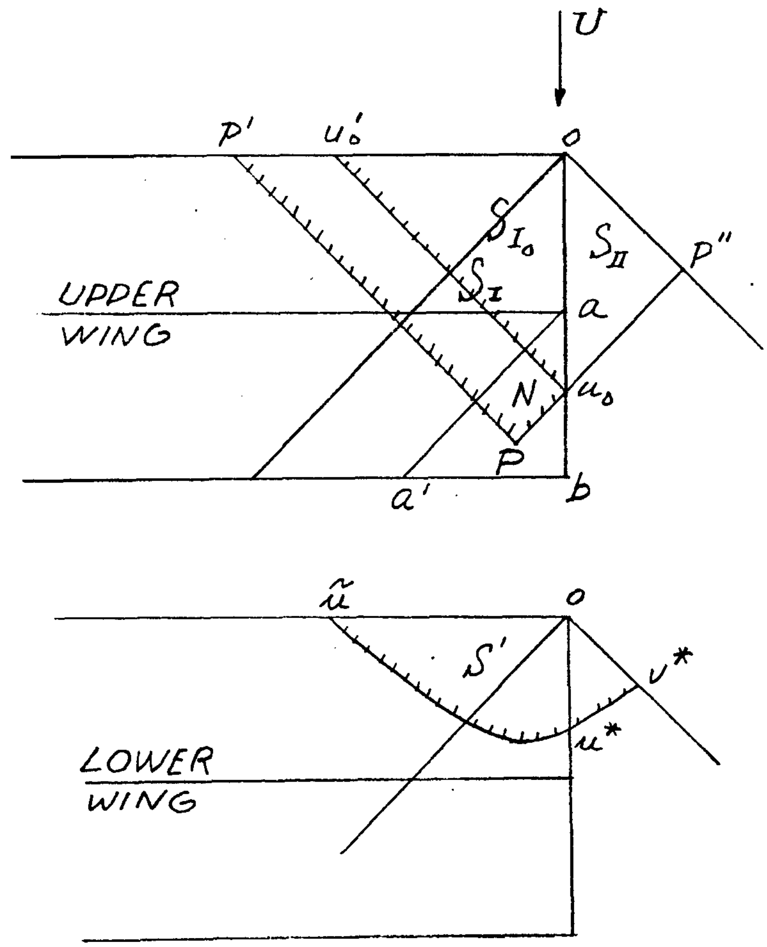


Fig. 3.

Diagram defining notation used in calculations for the upper wing

$a\,a'b$ — N
$P\,p'\,o\,u_0$ — S_I
$u_0\,o\,p''$ — S_{II}
$u_0\,u_0'\,o$ — S_{I_0}
$o\,u^*\tilde{u}$ — S_I'
$o\,u^*v^*$ — S_{II}'

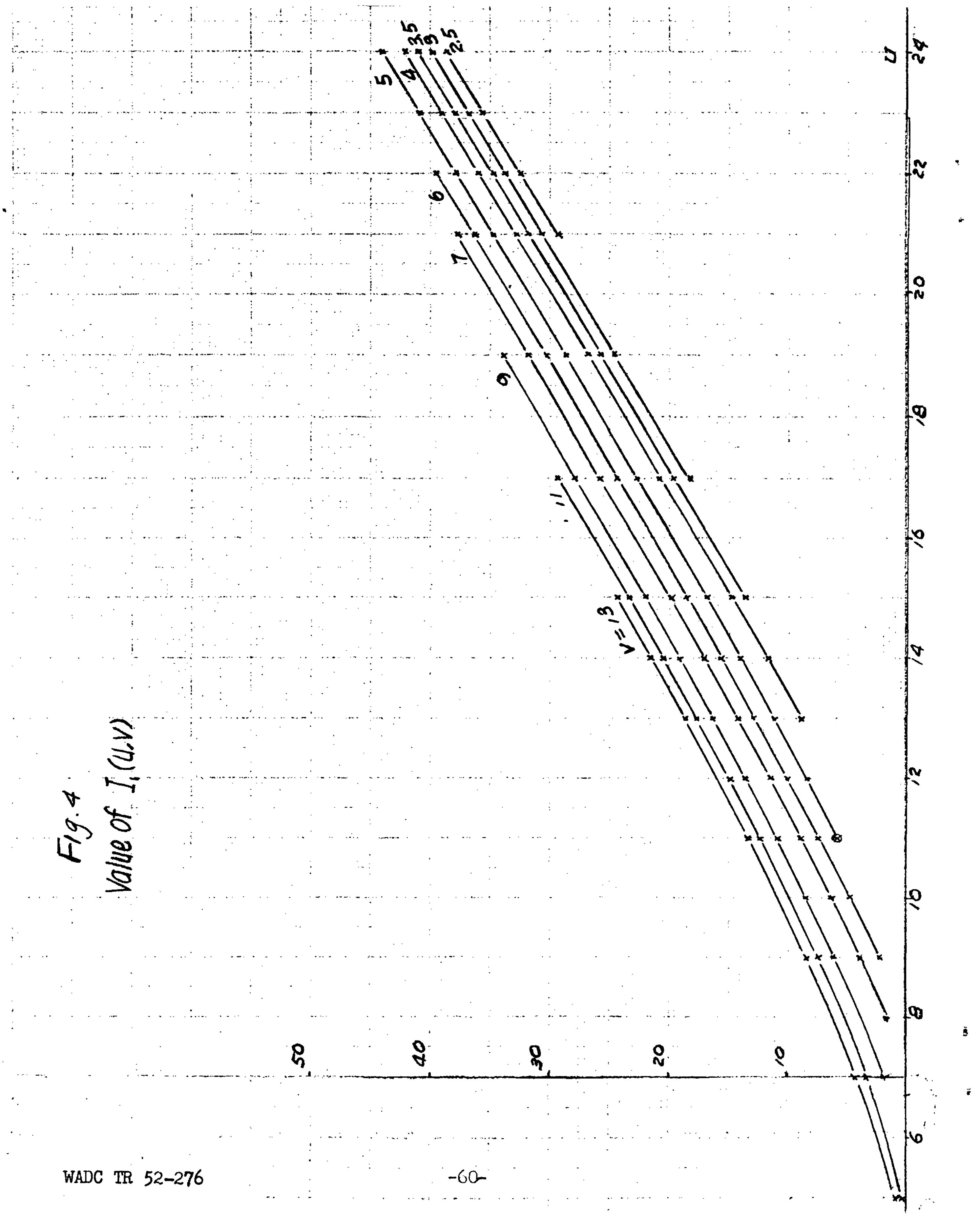


Fig. 4
Value of $I_1(U,V)$

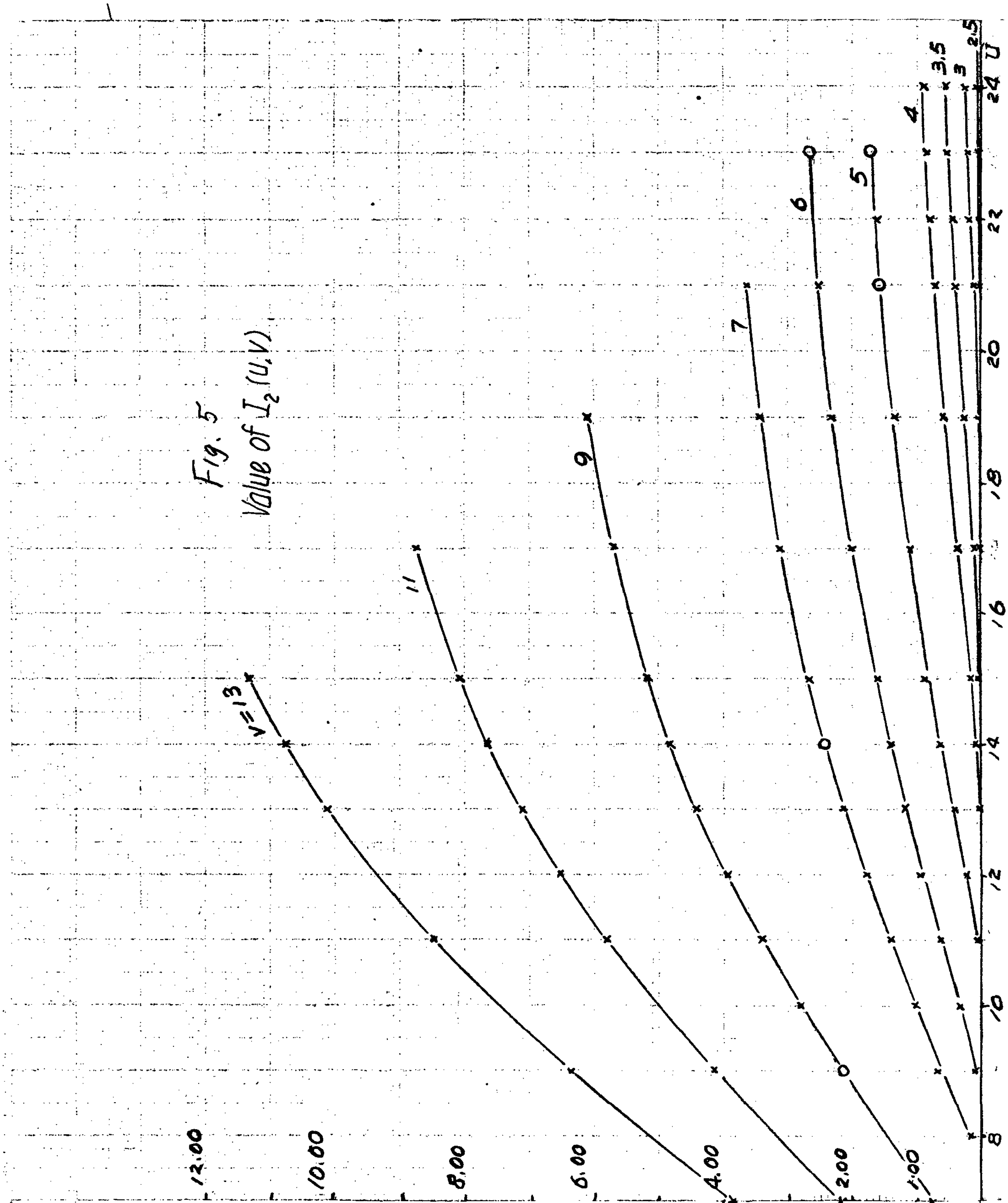


Fig. 5
Value of $I_2(U,V)$

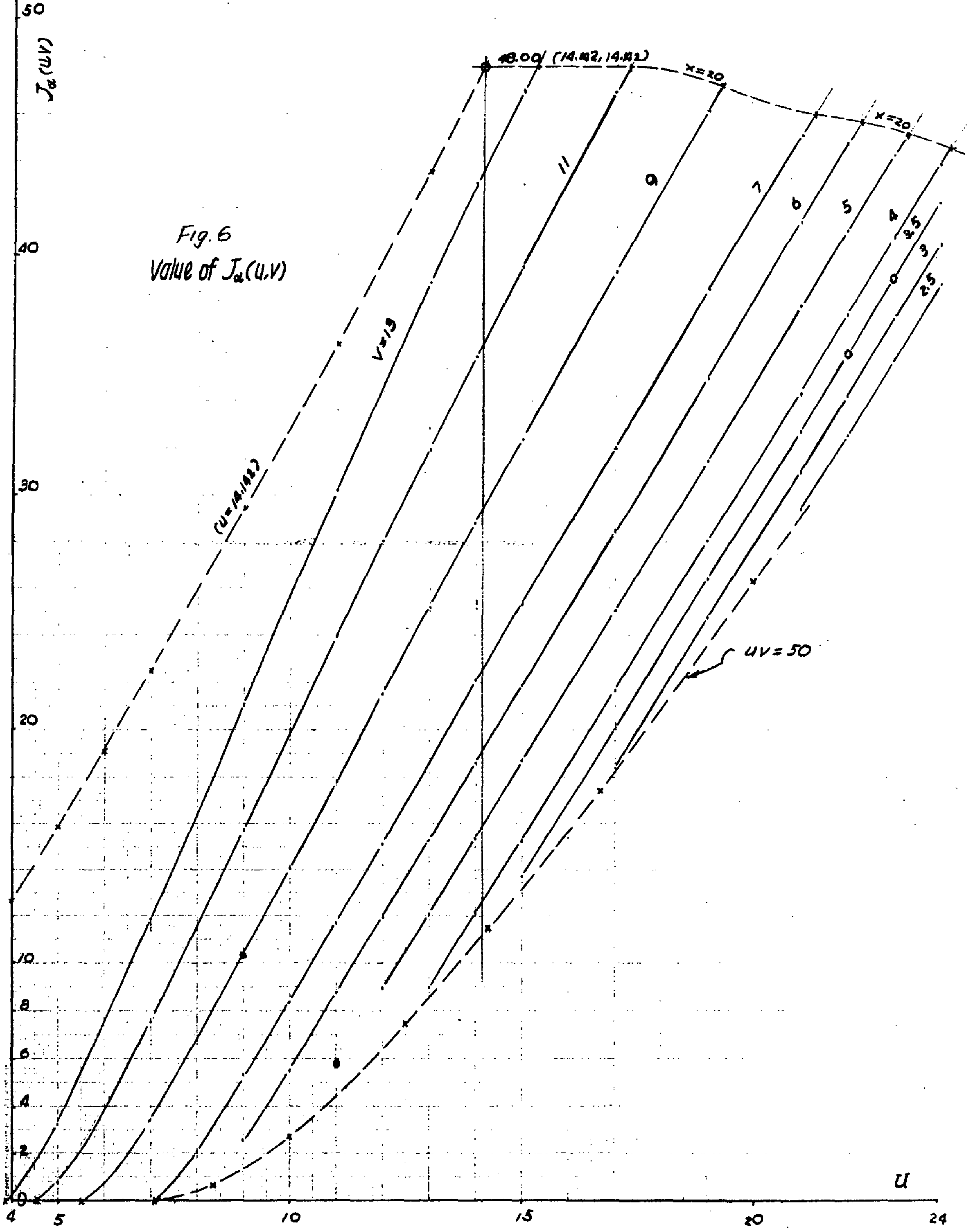


Fig. 6
Value of $J_\alpha(u,v)$
$J_\alpha(u,v)$
48.00 (14.142, 14.142)
x=20
x=20
(u=14.142)
V=13
11
9
7
6
5
4
3.5
3
2.5
uv=50
u
50
40
30
20
10
8
6
4
2
0
4
5
10
15
20
24

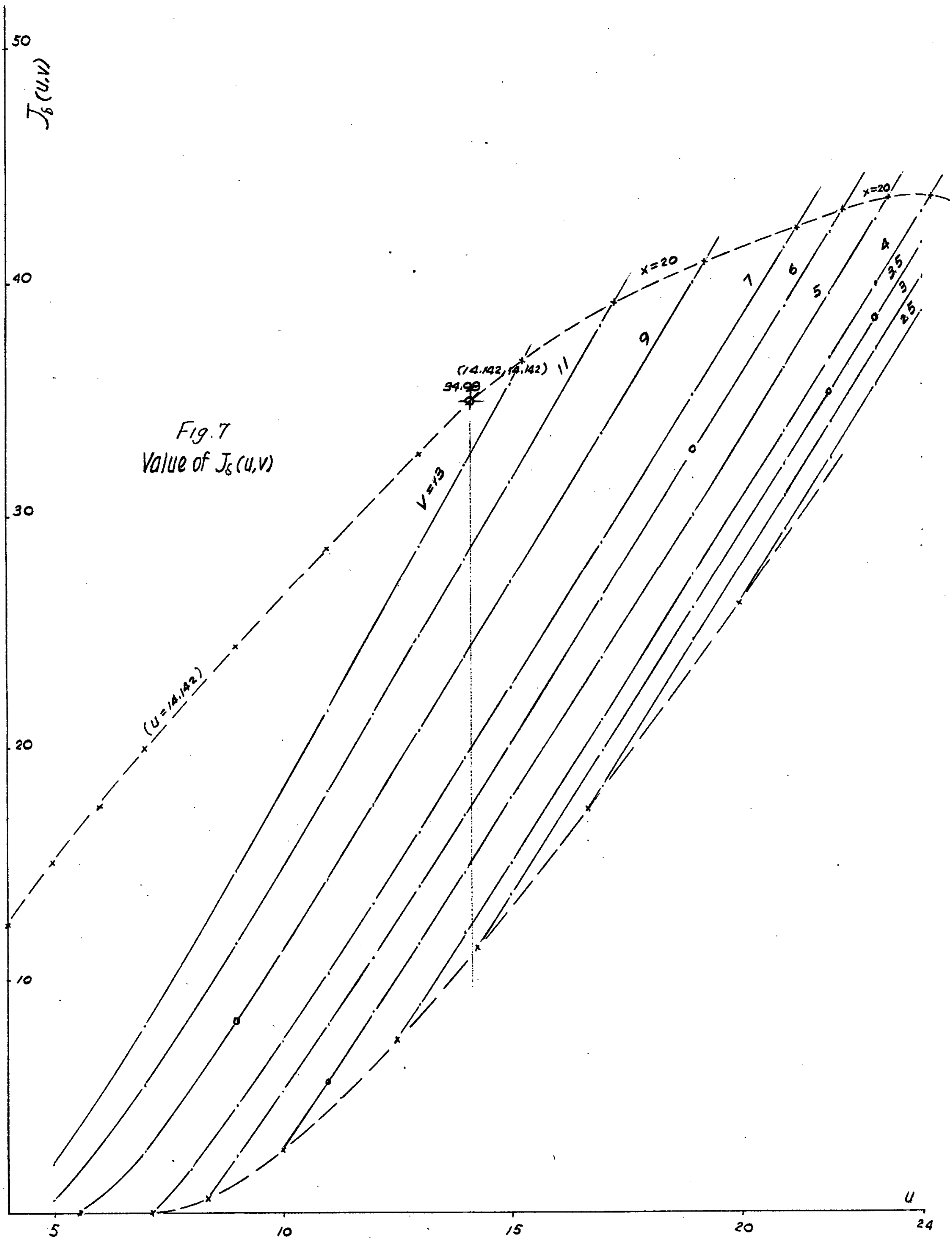


Fig. 7
Value of $J_6(u,v)$

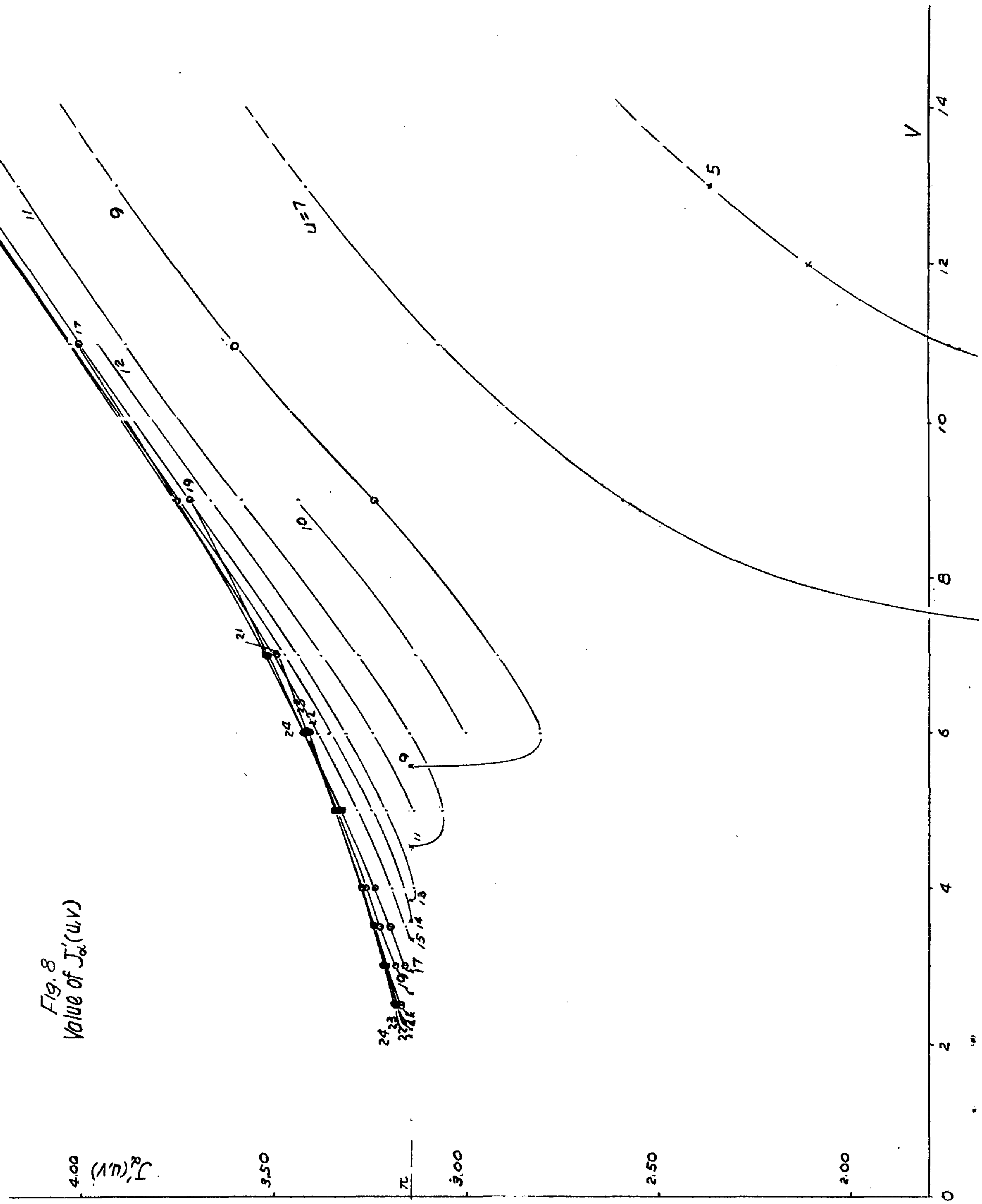


Fig. 8
Value of $J'_\alpha(u,v)$

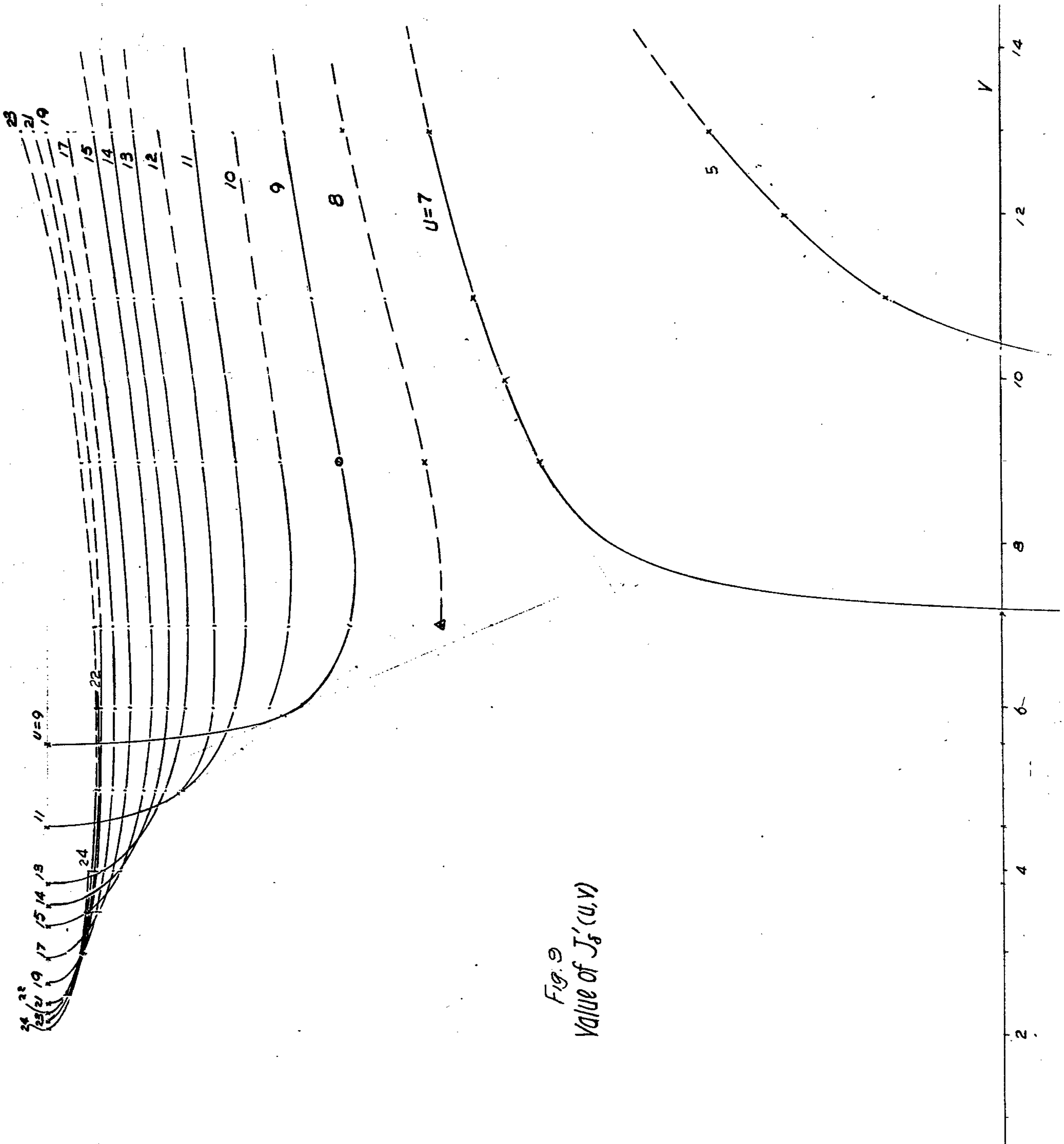
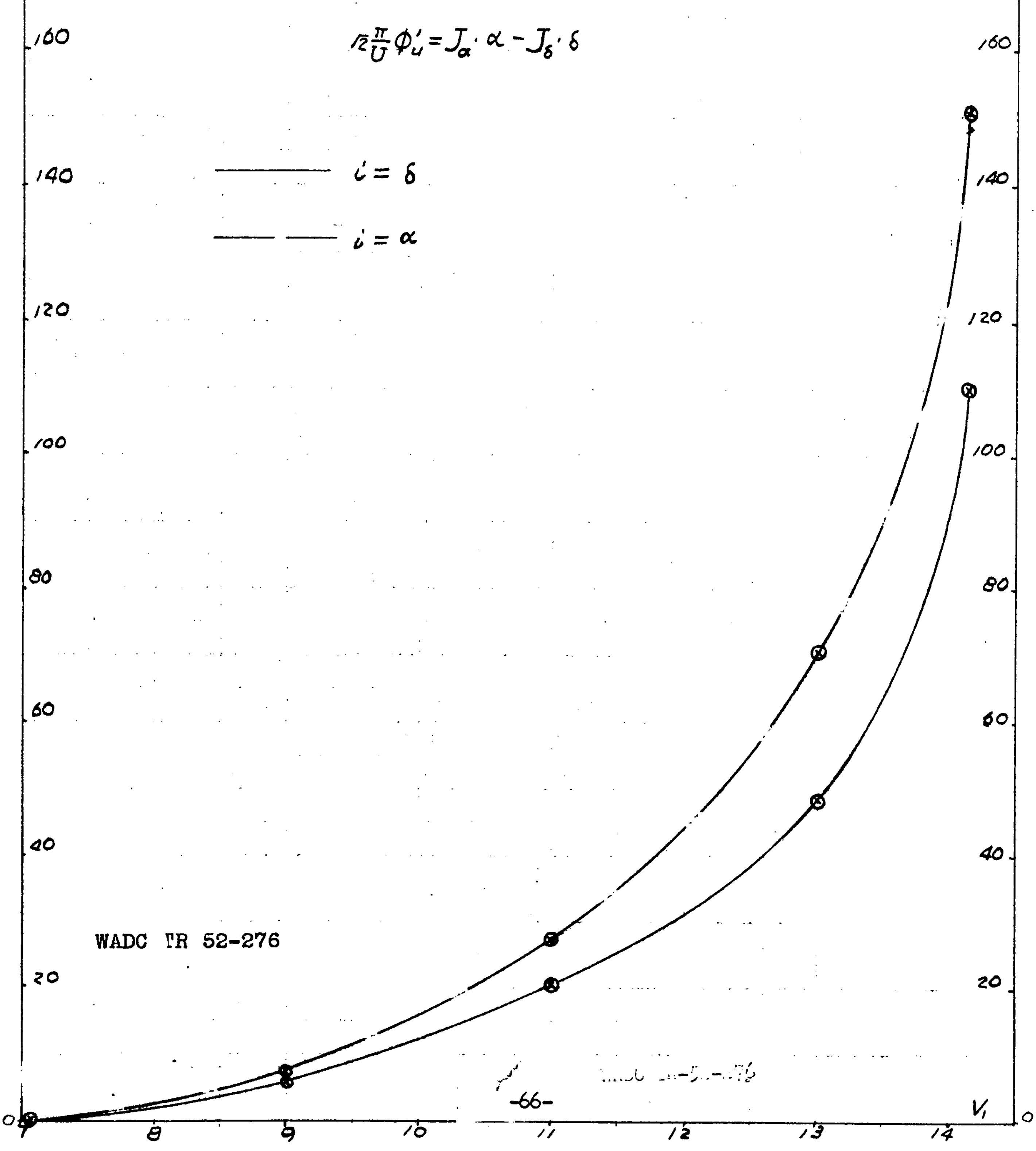


Fig. 9
Value of $J'_\delta(U,V)$

Fig. 10

Value of $\frac{1}{\sqrt{u_1-v_1}}\int_{50/v_1}^{v_1} J_i\,(u';v_1)\sqrt{v_1-u'}\left[\frac{1}{v_1-u'}-\frac{1}{u_1-u'}\right]du'$ along $x_1=2a=20$



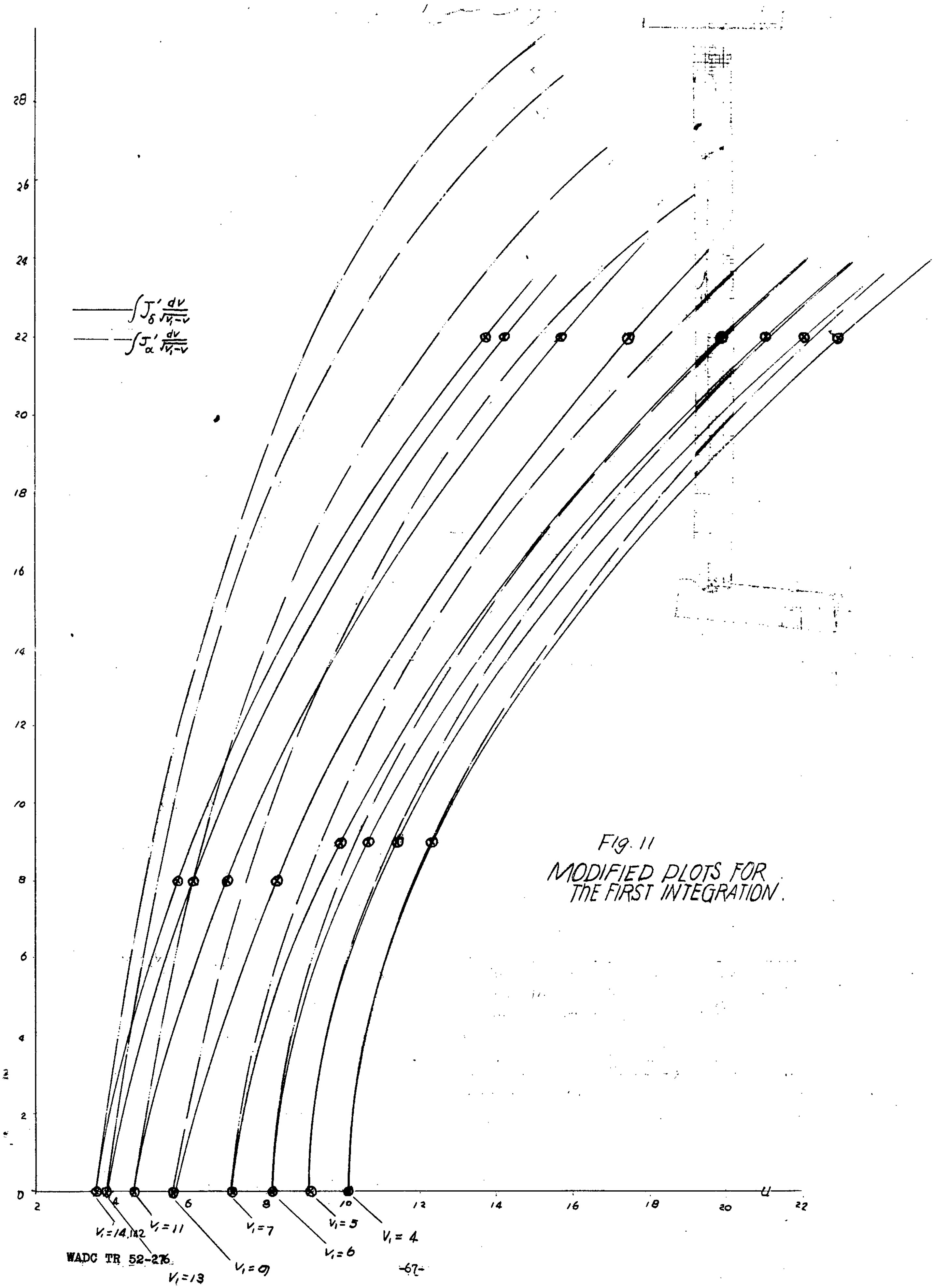


Fig. 11

MODIFIED PLOTS FOR THE FIRST INTEGRATION.

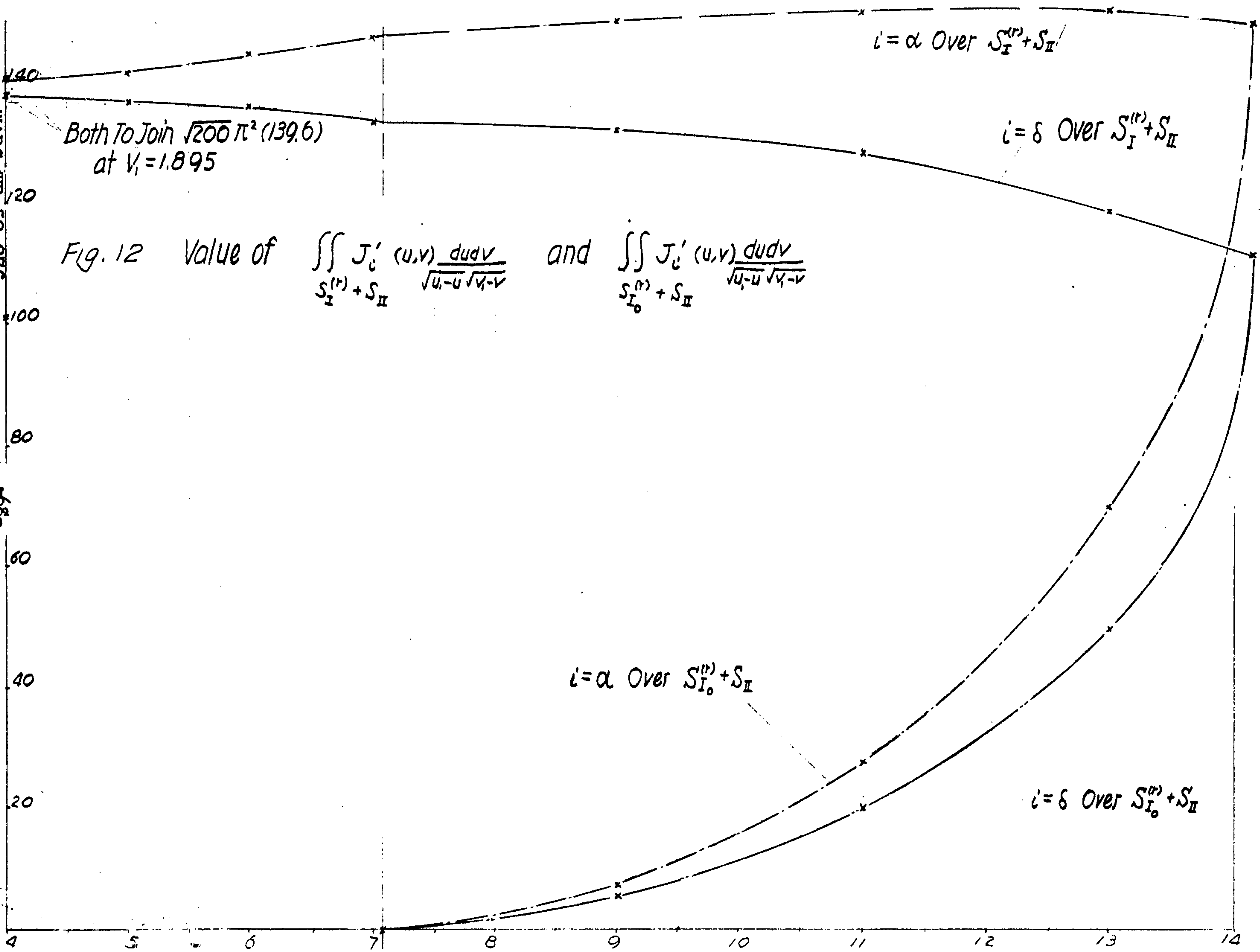


Fig. 12 Value of $\iint\limits_{S_I^{(r)} + S_{II}} J_i'(u,v) \frac{du\,dv}{\sqrt{u_1-u}\sqrt{v_1-v}}$ and $\iint\limits_{S_{I_0}^{(r)} + S_{II}} J_i'(u,v) \frac{du\,dv}{\sqrt{u_1-u}\sqrt{v_1-v}}$

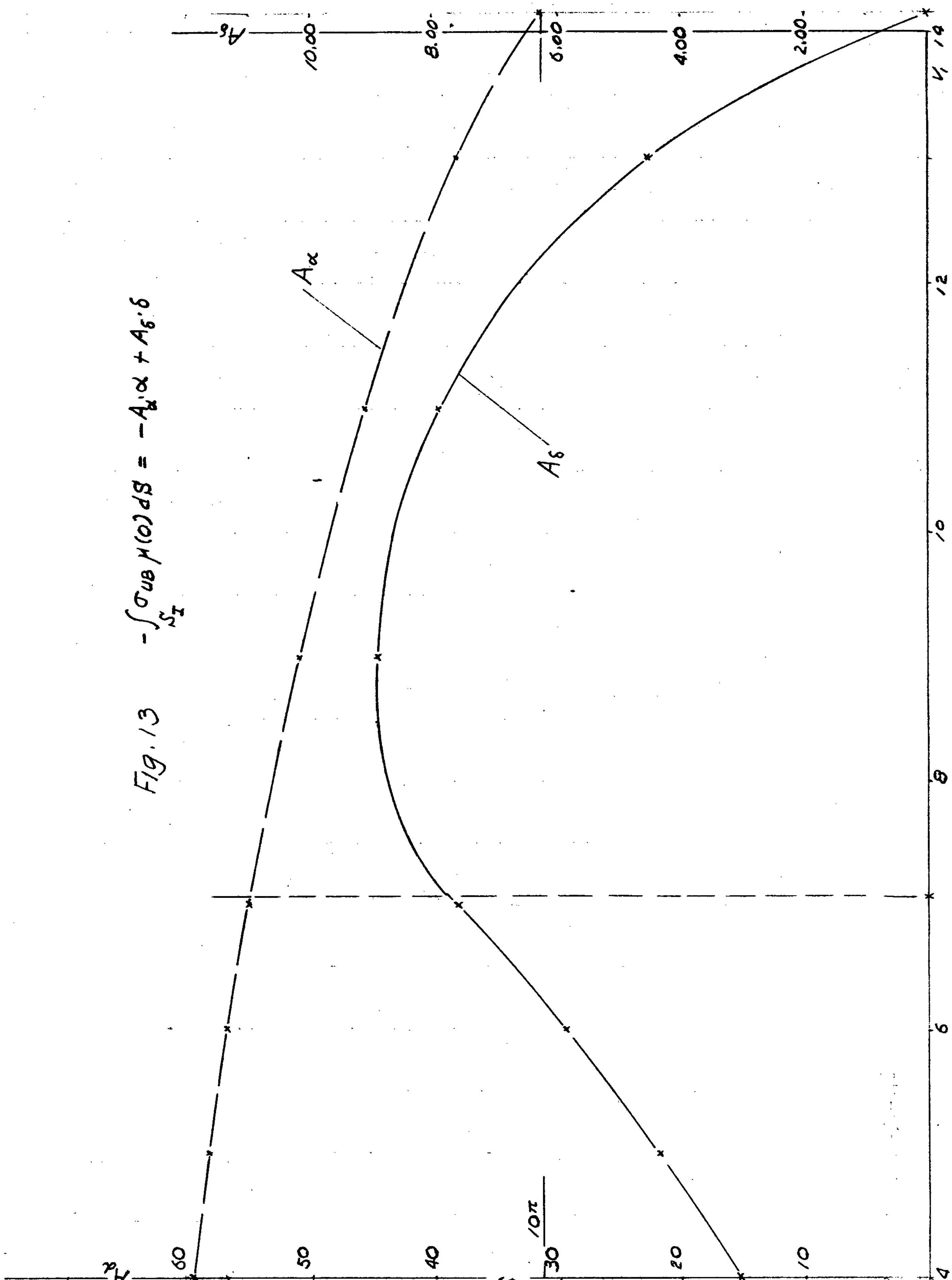


Fig. 13

$$-\int_{S_I} \sigma_{UB}\,\mu(0)\,dS = -A_\alpha \cdot \alpha + A_\delta \cdot \delta$$

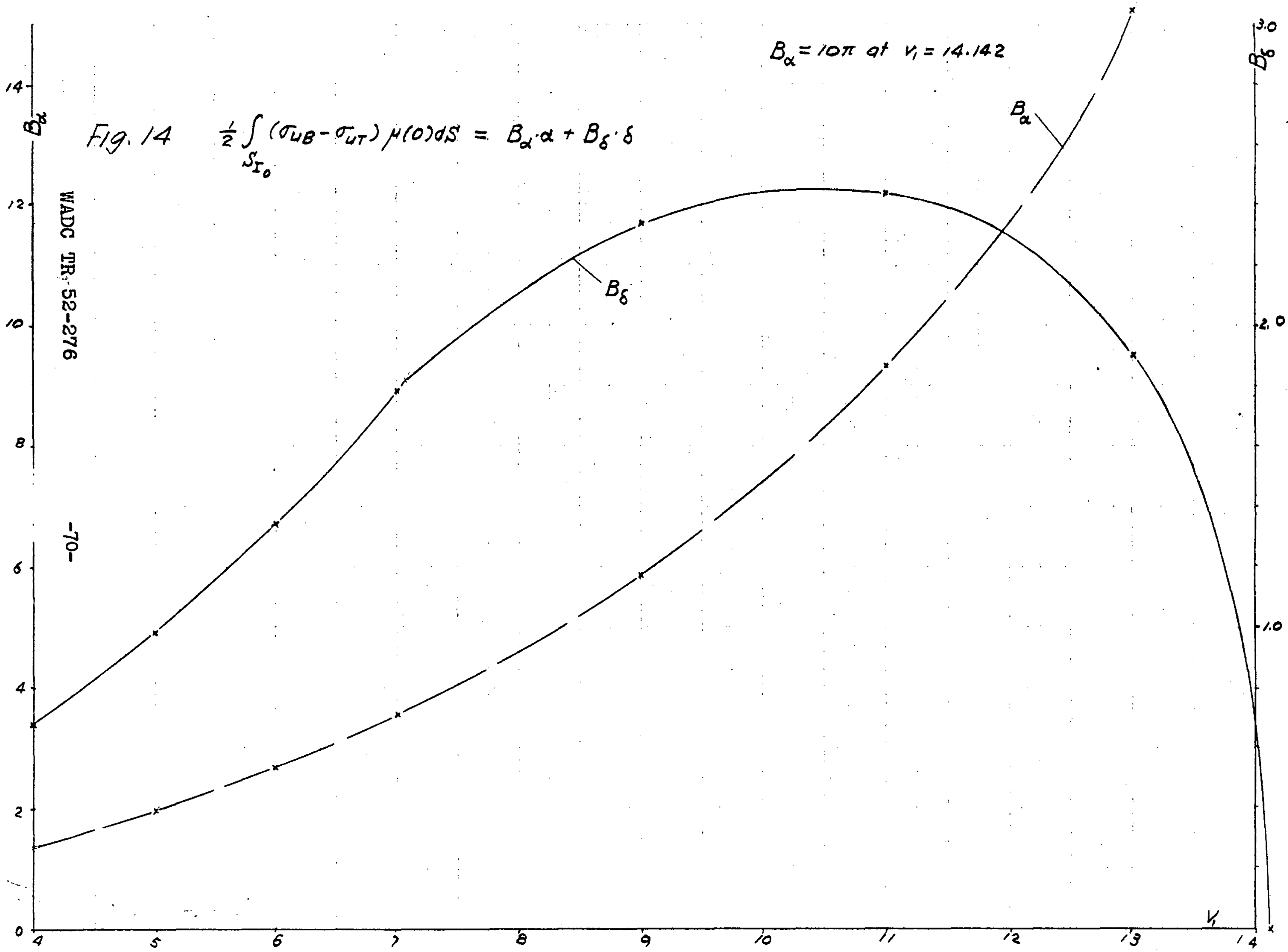


Fig. 14 $\quad \frac{1}{2}\int_{S_{I_0}} (\sigma_{uB} - \sigma_{uT})\,\mu(0)\,dS = B_\alpha \cdot \alpha + B_\delta \cdot \delta$

Fig. 15 Values of $\phi_{uT}(a)$ & $\phi_{uB}(a)$ in Region "B", $c=a=10$, $M=\sqrt{2}$

$$\frac{\pi}{U}\phi_{uT}(a) = \left[\iint_{S_{Ia}}\right]\cdot\alpha - \left[\frac{1}{2}\iint_{S_{Ib}}\right]\cdot\delta$$

$$\frac{\pi}{U}\phi_{uB}(a) = -\left[\iint_{S_{Ia}}\right]\cdot\alpha - \left[\iint_{S_{Ia}} + \frac{1}{2}\iint_{S_{Ib}}\right]\cdot\delta$$

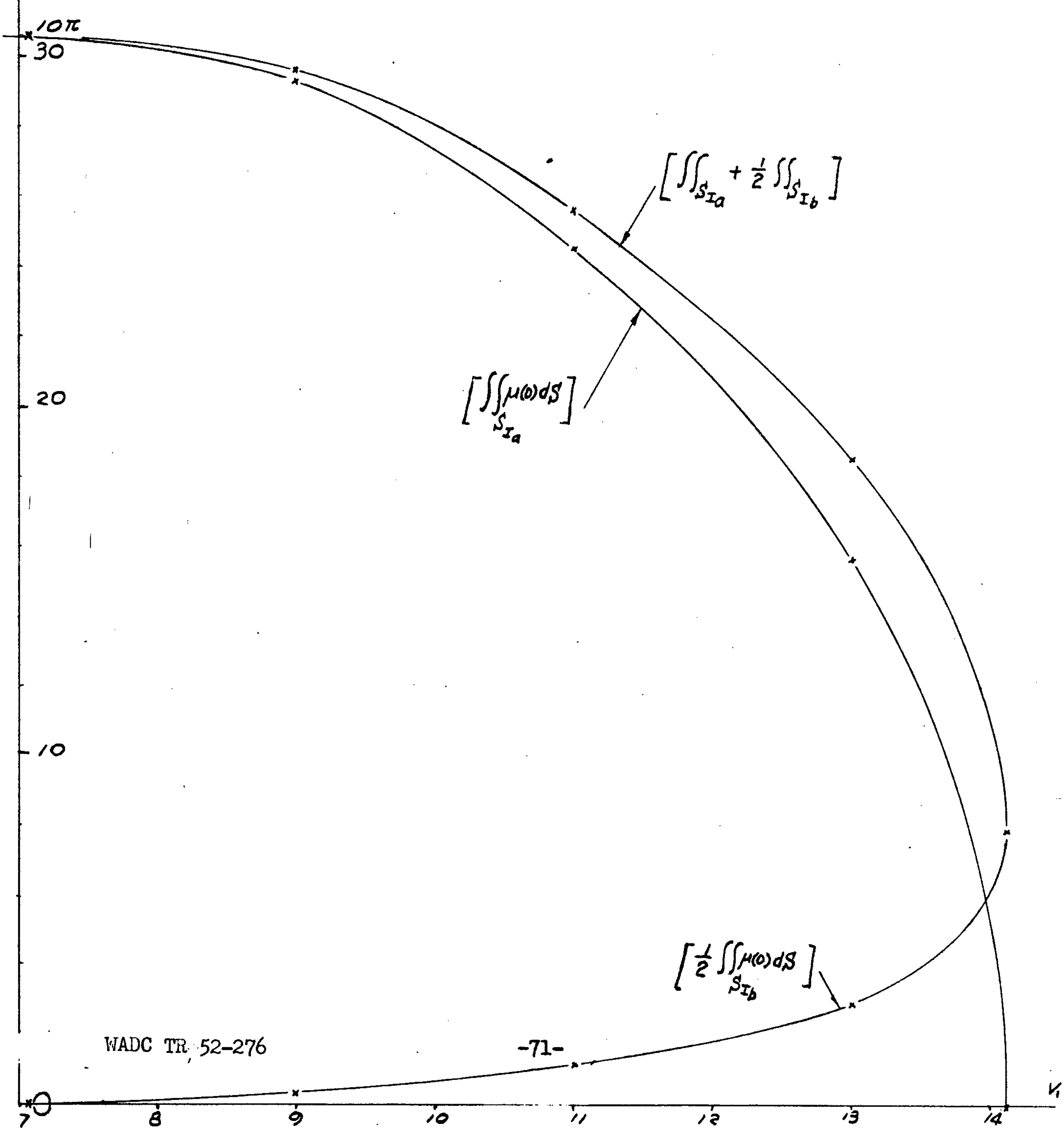
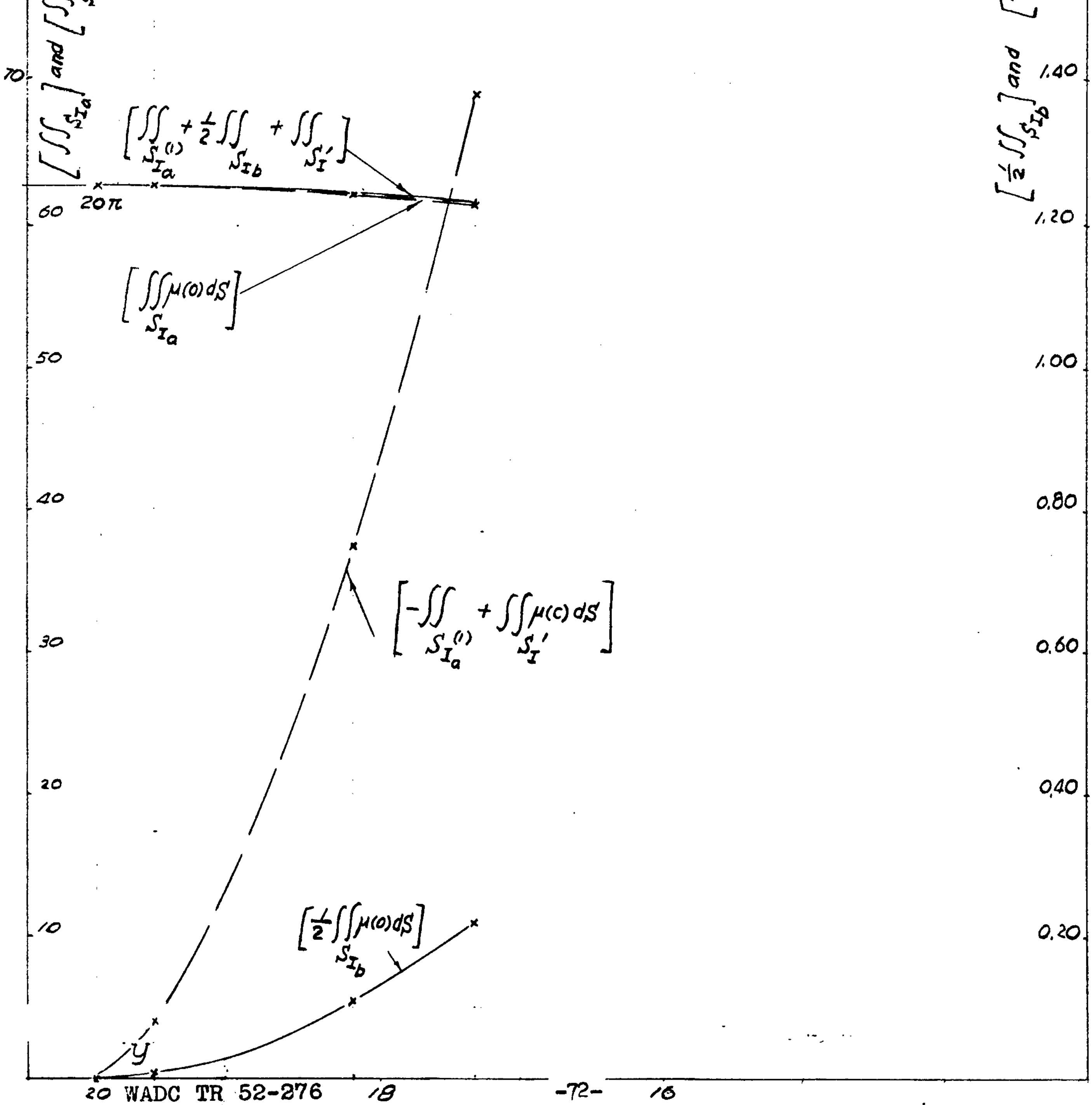


Fig. 16

Values of $\Phi_{UT}(2a)$ & $\Phi_{UB}(2a)$ in Region "B", $c=a=10$, $M=\sqrt{2}$.

$$\frac{\pi}{U}\Phi_{UT}(2a) = \left[\iint_{S_{I_a}}\right]\cdot\alpha - \left[\frac{1}{2}\iint_{S_{I_b}}\right]\cdot\delta$$

$$\frac{\pi}{U}\Phi_{UB}(2a) = \left[-\iint_{S_{I_a}^{(1)}} + \iint_{S_I'}\right]\cdot\alpha - \left[\iint_{S_{I_a}^{(1)}} + \frac{1}{2}\iint_{S_{I_b}} + \iint_{S_I'}\right]\cdot\delta$$



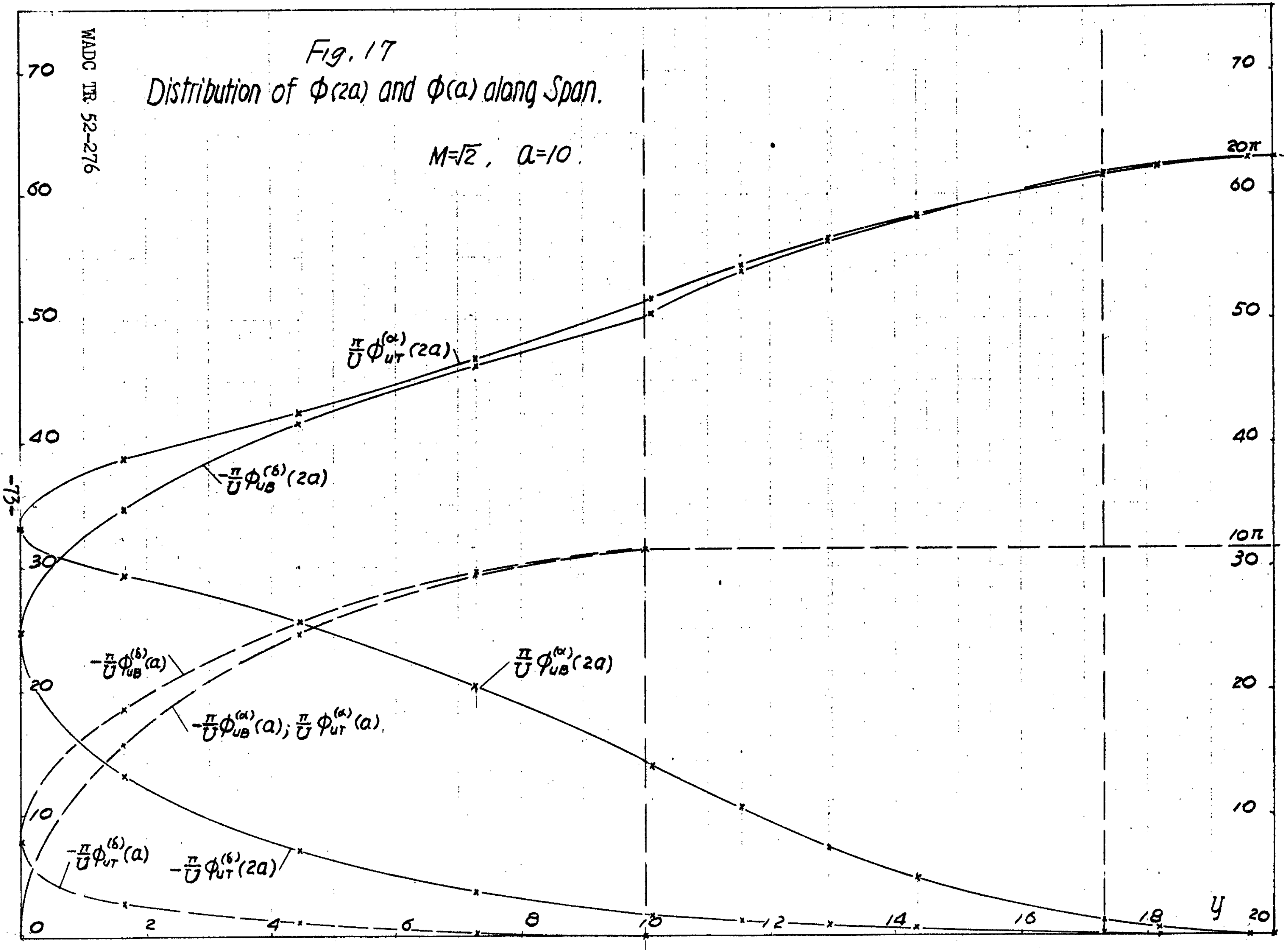
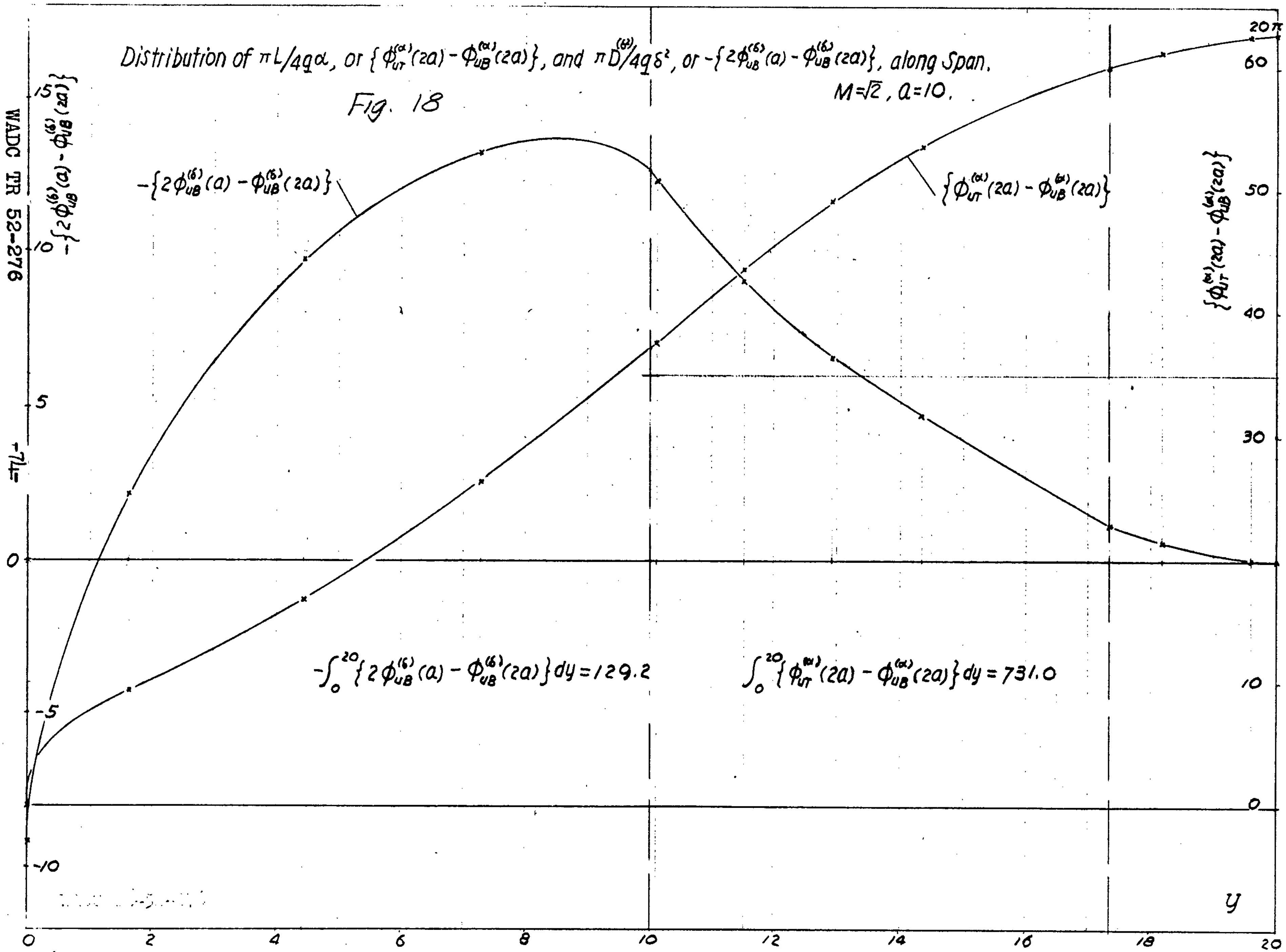


Fig. 17
Distribution of $\Phi(2a)$ and $\Phi(a)$ along Span.
$M=\sqrt{2}$, $a=10$.
$\frac{\pi}{U}\phi_{UT}^{(\alpha)}(2a)$
$-\frac{\pi}{U}\phi_{UB}^{(b)}(2a)$
$\frac{\pi}{U}\phi_{UB}^{(\alpha)}(2a)$
$-\frac{\pi}{U}\phi_{UB}^{(b)}(a)$
$-\frac{\pi}{U}\phi_{UB}^{(\alpha)}(a)$; $\frac{\pi}{U}\phi_{UT}^{(\alpha)}(a)$.
$-\frac{\pi}{U}\phi_{UT}^{(b)}(a)$
$-\frac{\pi}{U}\phi_{UT}^{(b)}(2a)$
20π
10π
y
0 2 4 6 8 10 12 14 16 18 20
0 10 20 30 40 50 60 70



Distribution of $\pi L/4q\alpha$, or $\{\phi_{UT}^{(\alpha)}(2a) - \phi_{UB}^{(\alpha)}(2a)\}$, and $\pi D^{(\delta)}/4q\delta^2$, or $-\{2\phi_{UB}^{(\delta)}(a) - \phi_{UB}^{(\delta)}(2a)\}$, along Span.

$M=\sqrt{2}$, $a=10$.

Fig. 18

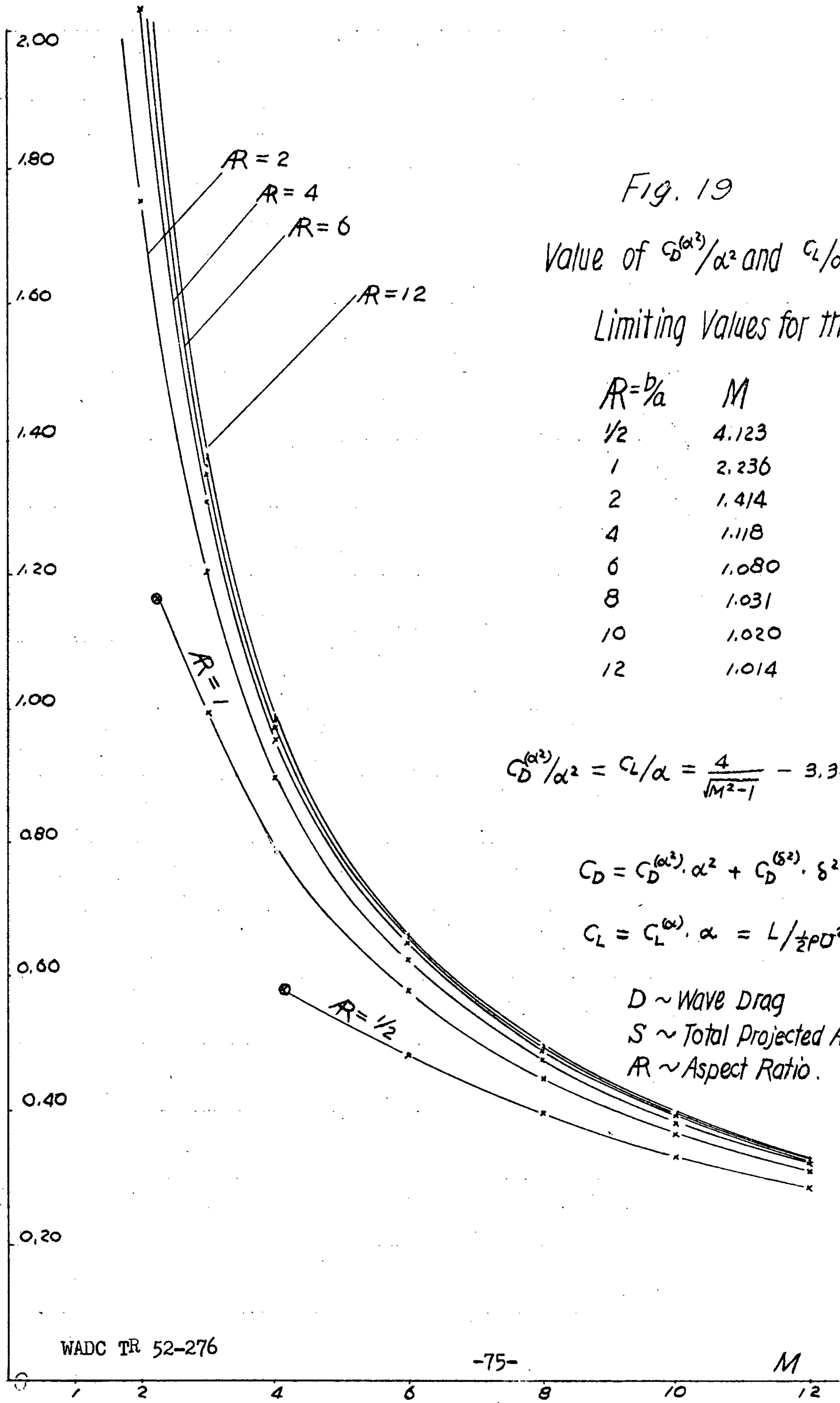


Fig. 19

Value of $C_D^{(\alpha^2)}/\alpha^2$ and C_L/α

Limiting Values for the Theory:

AR = b/a	M	$C_D^{(\alpha^2)}/\alpha^2$ or C_L/α
1/2	4.123	0.5817
1	2.236	1.164
2	1.414	2.327
4	1.118	4.654
6	1.080	6.980
8	1.031	9.308
10	1.020	11.64
12	1.014	13.96

$$C_D^{(\alpha^2)}/\alpha^2 = C_L/\alpha = \frac{4}{\sqrt{M^2-1}} - 3.346\,\frac{1}{(M^2-1)AR}$$

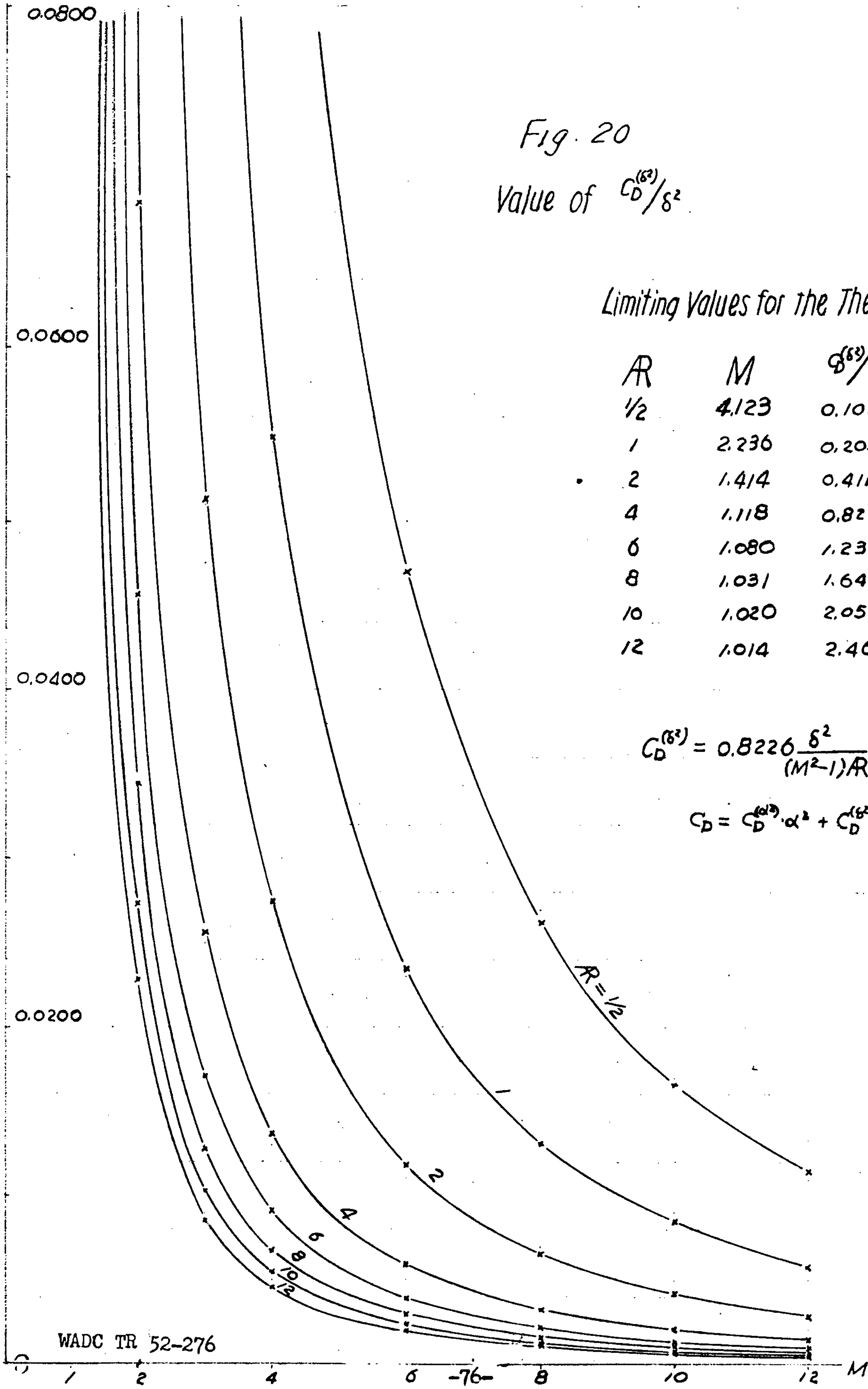
$$C_D = C_D^{(\alpha^2)}\cdot\alpha^2 + C_D^{(\delta^2)}\cdot\delta^2 = D/\tfrac{1}{2}\rho U^2 S$$

$$C_L = C_L^{(\alpha)}\cdot\alpha = L/\tfrac{1}{2}\rho U^2 S$$

D ~ Wave Drag
S ~ Total Projected Area.
AR ~ Aspect Ratio.

Fig. 20

Value of $C_D^{(\delta^2)}/\delta^2$



Limiting Values for the Theory:

$\mathcal{R}$	M	$C_D^{(\delta^2)}/\delta^2$
1/2	4.123	0.1028
1	2.236	0.2057
2	1.414	0.4113
4	1.118	0.8226
6	1.080	1.234
8	1.031	1.645
10	1.020	2.057
12	1.014	2.468

$$C_D^{(\delta^2)} = 0.8226 \frac{\delta^2}{(M^2-1)\mathcal{R}}.$$

$$C_D = C_D^{(\alpha^2)} \cdot \alpha^2 + C_D^{(\delta^2)} \cdot \delta^2.$$

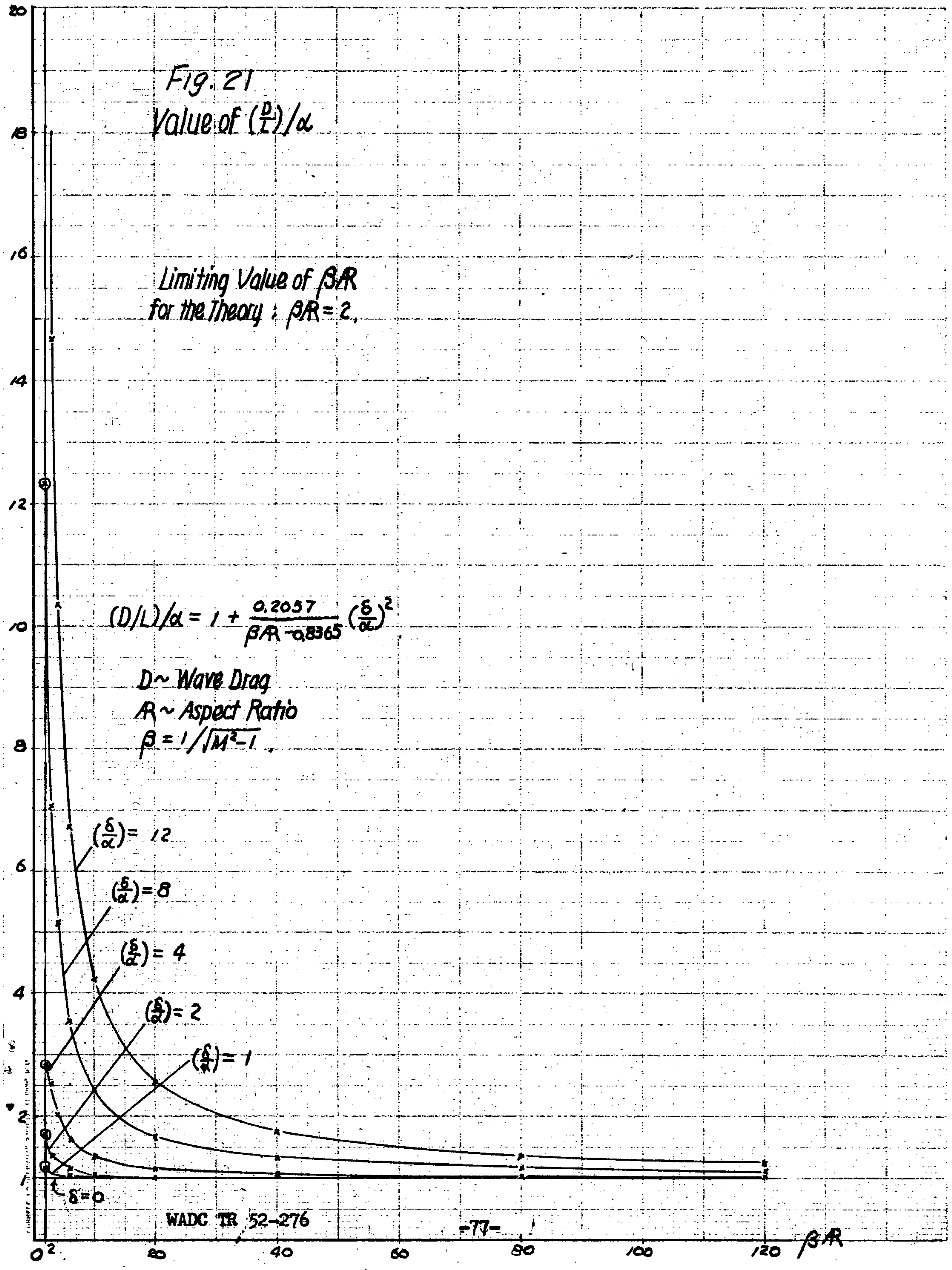


Fig. 21
Value of $(\frac{D}{L})/\alpha$
Limiting Value of $\beta A\!\!R$
for the Theory : $\beta A\!\!R = 2$,
$(D/L)/\alpha = 1 + \frac{0.2057}{\beta A\!\!R - 0.8365}(\frac{\delta}{\alpha})^2$
$D \sim$ Wave Drag
$A\!\!R \sim$ Aspect Ratio
$\beta = 1/\sqrt{M^2-1}$,
$(\frac{\delta}{\alpha}) = 12$
$(\frac{\delta}{\alpha}) = 8$
$(\frac{\delta}{\alpha}) = 4$
$(\frac{\delta}{\alpha}) = 2$
$(\frac{\delta}{\alpha}) = 1$
$\delta = 0$
20
18
16
14
12
10
8
6
4
2
1
0
2
20
40
60
80
100
120
$\beta A\!\!R$