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AN APPROXIMATE METHOD FOR THE CALCULATION OF NONSTATIONARY AIR FORCES AT SUBSONIC SPEEDS

HENRY E. FETTIS
FLIGHT RESEARCH LABORATORY

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WRIGHT AIR DEVELOPMENT CENTER

NOTICES

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Henry E. Fettis
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FOREWORD

This report was originally issued under the designation of OAR Technical Report #5, and contains the results of research on the problem of the calculation of oscillatory lift and moment coefficients which act on a two-dimensional airfoil moving at subsonic speeds. The work was begun and partly completed while the author was associated with the Dynamics Branch of the Aircraft Laboratory, Wright Air Development Center, under E. O. 459-41. The author, who was also the project engineer, completed the research while a member of the Applied Mathematics Research Section of the Flight Research Laboratory, Wright Air Development Center. The new edition is being issued for the purpose of correcting numerous errors in the original, as well as meeting the demand for additional copies.

The author wishes to acknowledge the assistance of Mr. Hewitt S. Toney, then of the Dynamics Branch, Aircraft Laboratory, and presently of the Computation Research Section, of the Flight Research Laboratory, in developing many of the formulae and in carrying out the numerical calculations.

ABSTRACT

The present report explains and illustrates a method of computing the non-stationary forces and moments on an oscillating airfoil at subsonic speeds. The process is based on the well known Possio integral equation relating the pressure on the airfoil to the normal velocity.

Part I of the report contains the theoretical development which leads to the required equations for determining the lift and moment.

In Part II the method of Part I is applied to the computation of the aerodynamic lift and moment coefficients for four principal degrees of freedom of the airfoil, these being:

a. Translation of the complete chord of the airfoil in a direction normal to the forward velocity (positive down).

b. Rotation of the entire chord about the forward quarter-chord point (positive for increasing angle of attack).

c. Translation of the portion of the airfoil extending from an arbitrary point to the trailing edge, in a direction normal to the forward velocity.

d. Rotation about an arbitrary point of that portion of the airfoil extending from that point to the trailing edge.

The appendices contain the detailed mathematical derivation of the various formulae involved in the problem.

PUBLICATION REVIEW

Manuscript Copy of this report has been reviewed and found satisfactory for publication.

FOR THE COMMANDING GENERAL:

Leslie B Williams

LESLIE B. WILLIAMS
Colonel, USAF
Chief, Flight Research Laboratory
Research Division

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TABLE OF SYMBOLS

Constants

ω	-	"Reduced frequency", $\omega = \nu b / V$ where ν is the frequency of oscillation, b the semichord, and V the forward velocity.
λ	-	Mach number; $\lambda = V/V_s$ where V_s is velocity of sound.
ρ	-	Air density.
μ	-	Parameter defined by $\mu = \frac{\lambda^2 \omega}{1-\lambda^2}$
x	-	Chordwise coordinate, referred to semichord as unity.
ξ	-	Chordwise variable of integration.
e	-	Coordinate of control surface leading edge.
w	-	Parameter defined by $w = \frac{\lambda \omega (x-\xi)}{1-\lambda^2}$.
z	-	Variable of integration. Also used in Appendix VII to designate $\omega(x-\xi)$
φ	-	Variable defined by $\xi = \cos \varphi$.
θ	-	Variable defined by $x = \cos \theta$.
ϵ	-	Variable defined by $e = \cos \epsilon$.
α_n	-	Coefficient of $\omega(x-\xi)^n$ in polynomial approximation of $\bar{K}$ (cf. equation 1.16).
a_n^m	-	Constants defined by equation (A2.03).
P_n^m	-	Constants defined by equation (A2.04).

Functions

$T(\omega)$ - Function defined by Kussner; $T(\omega) = \dfrac{H_1^{(2)}(\omega) + iH_0^{(2)}(\omega)}{H_1^{(2)}(\omega) - iH_0^{(2)}(\omega)}$ (Ref. 1)

$\Lambda(\theta,\varphi)$ - Function defined by Schwartz: $\Lambda(\theta,\varphi) = \frac{1}{2} Ln \dfrac{1-\cos(\theta+\varphi)}{1-\cos(\theta-\varphi)}$ (Ref. 2)

$\Pi(x)$ - Pressure distribution across chord.

$\Pi_0(x)$ - Pressure distribution across chord when $\lambda = 0$.

$W(x)$ - Downwash distribution across chord.

$K(\lambda,z)$ - Kernel of the Possio integral equation.

$K(0,z)$ - Kernel of the Possio integral equation when $\lambda = 0$.

$K_1(\lambda,z)$ - Non-singular kernel as defined by Schwartz (Ref. 16).

$\bar{K}(\lambda,z)$ - Non-singular kernel as defined by equation (1.04).

$G(x,\xi)$ - Inversion kernel of Kussner as defined by equation (1.08).

$\dot{J}_p(\mu,\theta)$ - Function defined by: $\dot{J}_p(\mu,\theta) = \int_0^\theta e^{i\mu\cos\varphi}\cos p\varphi \, d\varphi$

$I_p(\mu,x,e)$ - Function defined by $I_p(\mu,x,e) = \int_x^1 e^{-i\mu\xi}(\xi-e)^p \Lambda(\cos^{-1}x, \cos^{-1}\xi)\,d\xi$

$F(\mu,\theta,\epsilon)$ - Function defined by $F(\mu,\theta,\epsilon) = \int_0^\theta \dfrac{1-e^{-i\mu(\cos\varphi-\cos\epsilon)}}{\cos\varphi-\cos\epsilon}\,d\varphi$

Functions (continued)

$Q_k(\omega, e)$	-	Coefficients defined by equation (A3.01)
$R_k(\omega, e)$	-	Coefficients defined by equation (A3.04)
$U_k(\theta, \epsilon)$	-	Function defined by equation (A5.32)
$A_{ij}(x)$	-	Function defined by equation (1.22)
A_{ij}	-	$A_{ij} = A_{ij}(-1)$
$B_j(x)$	-	Function defined by equation (1.22)
B_j	-	$B_j = B_j(-1)$
$A_k(\mu)$	-	Coefficients in series expansion of $J_p^j(\mu, \theta)$ (See equations A4.17, 21, 24, 26)
$B_k(\mu)$	-	
$C_k(\mu)$	-	
$D_n(\mu)$	-	
$U_n(z)$	-	Polynomial defined by equation (1.17)
$\Phi_n(x)$	-	Function defined by equation (1.18)

INTRODUCTION

The question of the effect of compressibility on flutter calculations has been the subject of numerous investigations. The first approach to the problem was the use of the well known Prandtl-Glauert correction factor by which the aerodynamic force is increased in the ratio $1 : \sqrt{1-\lambda^2}$ where λ is the Mach number. Since this correction changes the magnitude of the aerodynamic force but not the phase, it is evident that such a correction, while satisfactory for the stationary case, cannot be relied upon in the non-stationary case where the phase change is one of the most important factors.

In 1938 Possio (Ref. 11) wrote down the relation between the pressure distribution over a chordwise element of the airfoil and the total normal velocity at any point (downwash), taking into account the compressibility of the medium, in the form of an integral equation of the first kind which now bears his name. The same equation was derived independently by Küssner in 1940 (Ref. 4). Since no explicit solution of the equation was evident (nor has since been found) recourse to an approximate solution was made. Possio obtained a solution by assuming that if the equation were satisfied at a finite number of points on the chord, the results should approximate the exact values. Using this method, Possio was able to calculate total lift and moment coefficients for values of the reduced frequency less than .6, and for motions of the airfoil corresponding to rigid translation and rotation of the complete chord. Possio's results were later checked and extended by Frazer and Skan (Ref. 5). This method, now known as the collocation method, results in a system of linear equations with as many unknowns as points for which the equation is satisfied. Since the coefficients in these equations are complex numbers, it is evident that such a solution is long and tedious if a large range of parameters is to be considered. Further, it is not possible to duplicate accurately by this method the conditions of a discontinuous downwash which occurs when a control surface is added to the airfoil.

A different approach to the problem was made by Schade (Ref. 12) and Eichler (Ref. 13), in which expansions of both sides of the equations were made in terms of known functions. Schade employed Legendre functions while Eichler used a trigonometric series. By limiting the expansions to a finite number of terms and equating like coefficients, the problem was again reduced to the solution of a system of linear equations. Thus this method, while perhaps more accurate than the collocation method, was still too laborious to be practical. (Schade indicated that the case of a discontinuous downwash could be handled by the introduction of the proper singularity in the pressure distribution. He did not, however, present any numerical results for the non-stationary case).

In 1943 a new departure was made by Dietze (Ref. 3) who noted that the difference between the kernel of the integral equation in the compressible case and that of the incompressible case was small compared to the actual value of the kernel. Thus, using the known incompressible solution as a starting point, Dietze was able to compute by an iterative process the solution to the compressible problem. By this method Dietze obtained a number of results for the case of control surface rotation. While the details of Dietze's calculations were not available to the author, it appears that a large amount of labor would be required to obtain a complete set of aerodynamic coefficients covering the range of parameters required for conventional aircraft.

The present method resembles Dietze's in that the incompressible solution is used as a starting point. It is, however, not an iterative process but results in a closed solution based on replacing the non-singular portion of the kernel by a polynomial. The question of the rapidity of the convergence of the pressure distribution series is no longer of any concern, and the only discrepancy between the solution obtained and the exact solution lies in the difference existing between the actual kernel remainder and the approximation. The remainder is approximated over the required interval by minimizing the total "mean square" error over the interval. The numerical results indicate that this approximation is satisfactory even when an apparently large discrepancy exists between the kernel difference and the polynomial approximation.

PART I

STATEMENT OF THE PROBLEM AND METHOD OF SOLUTION

The problem of determining the lift and moment on an oscillating airfoil in compressible subsonic flow was reduced by Possio to the solution of an integral equation of the first kind, which relates the pressure differential over the airfoil chord to the downwash at any point on the chord. The equation may be written in the form

$$(1.01) \qquad W(x) = \omega \int_{-1}^{1} K\left[\lambda, \omega(x-\xi)\right] \Pi(\xi)\, d\xi \quad ,$$

where λ is the Mach number, $W(x)$ is the downwash at any point, expressed as a function of the distance x (positive aft) of the point from the mid-chord, ω is the "reduced frequency" and $\Pi(\xi)$ is equal to $1/\rho V$ times the pressure distribution across the chord.

The distances x and ξ are non-dimensional with the semi-chord taken as unity. The explicit form of the nucleus K is given elsewhere (see for example, App. 7). In the present treatment only the singularities and the numerical values of K are required. As shown in other investigations of the subject, $K\left[\lambda, \omega(x-\xi)\right]$ has the following form near $x = \xi$:

$$(1.02) \qquad K\left[\lambda, \omega(x-\xi)\right] = \frac{\sqrt{1-\lambda^2}}{2\pi} \frac{1}{\omega(x-\xi)} + \frac{i}{2\pi\sqrt{1-\lambda^2}} \operatorname{Log}\left|\omega(x-\xi)\right| + K_1\left[\lambda, \omega(x-\xi)\right].$$

where K_1 has no singularities. For $\lambda = 0$,

$$(1.03) \qquad K\left[0, \omega(x-\xi)\right] = \frac{1}{2\pi} \frac{1}{\omega(x-\xi)} + \frac{i}{2\pi} \operatorname{Log}\left|\omega(x-\xi)\right| + K_1\left[0, \omega(x-\xi)\right].$$

Since the solution of equation (1.01) is known for the incompressible case where $\lambda = 0$, it is logical to attempt a solution for $\lambda \neq 0$ by making use of the results already obtained for the incompressible case.

To this end, the nucleus is written in the form

$$(1.04)\quad K[\lambda, \omega(x-\xi)] = \frac{1}{\sqrt{1-\lambda^2}} K[0, \omega(x-\xi)] + \frac{\lambda^2}{2\pi\sqrt{1-\lambda^2}}\left[\frac{1}{\omega(x-\xi)}\right] + \bar{K}[\lambda, \omega(x-\xi)]$$

Clearly, $\bar{K}$ is non-singular and may be approximated to a good degree of accuracy by a polynomial. The advantage obtained by the present treatment lies in the fact that all singularities in the pressure distribution are taken care of by means of the incompressible solution, and that the resulting solution is obtained without the use of the usual series representation for $\Pi(\xi)$. Thus the question of the rapidity of convergence of such a series does not enter. Further, for low values of ω, a satisfactory approximation to $\bar{K}$ is obtained by retaining only the constant and linear terms of the polynomial.

Placing for K the alternative expression as given by (1.04)

$$(1.05)\quad W(x) = \frac{\omega}{\sqrt{1-\lambda^2}}\int_{-1}^{1} K[0, \omega(x-\xi)]\Pi(\xi)\,d\xi + \frac{\lambda^2}{2\pi\sqrt{1-\lambda^2}}\int_{-1}^{1}\frac{\Pi(\xi)\,d\xi}{x-\xi} + \omega\int_{-1}^{1}\bar{K}[\lambda, \omega(x-\xi)]\Pi(\xi)\,d\xi$$

According to the work of Kussner (Ref. 1), the solution to the equation

$$(1.06)\qquad \omega\int_{-1}^{1} K[0, \omega(x-\xi)]\,\Pi(\xi)\,d\xi = f(x)$$

subject to the Kutta condition that $\Pi(\xi)$ remain finite at $\xi = 1$ is given by

$$(1.07)\qquad \Pi(x) = \int_{-1}^{1} G(x,z) f(z)\,dz$$

where $$G(x,z) = -\frac{2}{\pi}\left[i\omega\Lambda(x,z) + \sqrt{\frac{1-x}{1+x}}\sqrt{\frac{1+z}{1-z}}\left\{C(\omega) + \frac{1}{z-x}\right\}\right]$$

$$(1.08)\quad \Lambda(x,z) = \frac{1}{2}\operatorname{Log}\frac{1-xz+\sqrt{1-x^2}\sqrt{1-z^2}}{1-xz-\sqrt{1-x^2}\sqrt{1-z^2}}\,;\quad C(\omega) = \frac{T(\omega)-1}{2} = \frac{-H_0^{(2)}(\omega)}{H_0^{(2)}(\omega) - i H_1^{(2)}(\omega)}$$

Since $T(\omega) = 1$ for $\omega = 0$, this gives the additional result that if

(1.09) $$\frac{1}{\pi}\int_{-1}^{1}\frac{\Pi(\xi)\,d\xi}{x-\xi} = f(x)$$

and $\Pi(1)$ is finite, then

(1.10) $$\sqrt{\frac{1+x}{1-x}}\,\Pi(x) = \frac{2}{\pi}\int_{-1}^{1}\frac{f(z)}{x-z}\sqrt{\frac{1+z}{1-z}}\,dz$$

or, what is equivalent

(1.11) $$\int_{-1}^{1}\sqrt{\frac{1+z}{1-z}}\,\frac{dz}{z-x}\int_{-1}^{1}\frac{\Pi(\xi)}{z-\xi}\,d\xi \equiv \pi^2\sqrt{\frac{1+x}{1-x}}\,\Pi(x)$$

It can be seen that if equation (1.05) is rewritten in the form

(1.12) $$\frac{\omega}{\sqrt{1-\lambda^2}}\int_{-1}^{1}K\left[0,\omega(x-\xi)\right]\Pi(\xi)\,d\xi = W(x) - \frac{\lambda^2}{2\pi\sqrt{1-\lambda^2}}\int_{-1}^{1}\frac{\Pi(\xi)\,d\xi}{x-\xi} - \omega\int_{-1}^{1}\bar{K}\left[\lambda,\omega(x-\xi)\right]\Pi(\xi)\,d\xi$$

it can be formally regarded as a special case of equation (1.06) with $f(x)$ replaced by the entire right side of (1.12). Thus application of equation (1.07) gives the result

(1.13) $$\frac{\Pi(x)}{\sqrt{1-\lambda^2}} = \Pi_o(x) + \frac{\lambda^2}{\pi^2\sqrt{1-\lambda^2}}\int_{-1}^{1}\left[i\omega\Lambda(x,z) + \sqrt{\frac{1-x}{1+x}}\sqrt{\frac{1+z}{1-z}}\left\{\frac{1}{z-x} + C(\omega)\right\}\right]dz\int_{-1}^{1}\frac{\Pi(\xi)\,d\xi}{x-\xi}$$
$$-\,\omega\int_{-1}^{1}G(x,z)\,dz\int_{-1}^{1}\bar{K}\left[\lambda,\omega(z-\xi)\right]\Pi(\xi)\,d\xi$$

where $\Pi_o(x) = \int_{-1}^{1}G(x,z)\,W(z)\,dz$ is the incompressible pressure distribution corresponding to the given downwash $W(z)$. Making use of equation (1.11) and the following results established in Appendix I.

(1.14A) $$\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+z}{1-z}}\,dz\int_{-1}^{1}\frac{\Pi(\xi)\,d\xi}{z-\xi} = \int_{-1}^{1}\Pi(\xi)\,d\xi$$

$$(1.14B)\qquad \frac{1}{\pi}\int_{-1}^{1}\Lambda(x,z)\,dz\int_{-1}^{1}\frac{\Pi(\xi)\,d\xi}{z-\xi}=\cos^{-1}x\int_{-1}^{1}\Pi(\xi)\,d\xi-\pi\int_{z}^{1}\Pi(\xi)\,d\xi$$

gives

$$\Pi(x)+i\mu\int_{x}^{1}\Pi(\xi)\,d\xi=\frac{\Pi_0(x)}{\sqrt{1-\lambda^2}}+\left\{\frac{\lambda^2 C(\omega)}{\pi(1-\lambda^2)}\sqrt{\frac{1-x}{1+x}}+\frac{i\mu}{\pi}\cos^{-1}x\right\}\int_{-1}^{1}\Pi(\xi)\,d\xi$$

(1.15)

$$-\frac{\omega}{\sqrt{1-\lambda^2}}\int_{-1}^{1}G(x,z)\,dz\int_{-1}^{1}\bar{K}[\lambda,\omega(z-\xi)]\,\Pi(\xi)\,d\xi,$$

with $\mu=\frac{\lambda^2\omega}{1-\lambda^2}$.

The non-singular nucleus may evidently be approximated to the desired degree of accuracy by a polynomial of degree n:

$$\bar{K}[\lambda,\omega(z-\xi)]=\alpha_0+\alpha_1\omega(z-\xi)+\alpha_2\omega^2(z-\xi)^2+\cdots\cdots+\alpha_n\omega^n(z-\xi)^n$$

(1.16)

$$=u_0(z)+u_1(z)\xi+u_2(z)\xi^2+\cdots\cdots+u_n(z)\xi^n$$

where

$$u_0(z)=\alpha_0+\alpha_1\omega z+\alpha_2\omega^2 z^2+\cdots\cdots+\alpha_n\omega^n z^n$$

$$(1.17)\qquad u_1(z)=-\alpha_1\omega-2\alpha_2\omega^2 z-3\alpha_3\omega^3 z^2+\cdots\cdots-n\alpha_n\omega^n z^{n-1}$$

$$u_2(z)=\alpha_2\omega^2+3\alpha_3\omega^3 z+\cdots\cdots+\frac{n(n-1)}{2!}\alpha_n\omega^n z^{n-2}$$

$$u_3(z)=-\alpha_3\omega^3-4\alpha_4\omega^4 z+\cdots\cdots-\frac{n(n-1)(n-2)}{3!}\alpha_n\omega^n z^{n-3}$$

The last term in equation (1.15) then becomes

$$(1.18)\quad \Phi_0(x)\int_{-1}^{1}\Pi(t)\,dt+\Phi_1(x)\int_{-1}^{1}\Pi(t)\,t\,dt+\cdots\cdots+\Phi_n(x)\int_{-1}^{1}\Pi(t)\,t^n\,dt$$

with $\Phi_m(x)=\int_{-1}^{1}G(x,z)\,u_m(z)\,dz\,,\quad m=0,1,2\cdots n.$

Equation (1.15) may now be written

$$\Pi(x)+i\mu\int_{x}^{1}\Pi(t)\,dt=\frac{\Pi_0(x)}{\sqrt{1-\lambda^2}}+\left\{\frac{\lambda^2 c(\omega)}{\pi(1-\lambda^2)}\sqrt{\frac{1-x}{1+x}}+\frac{i\mu}{\pi}\cos^{-1}x-\omega\frac{\Phi_0(x)}{\sqrt{1-\lambda^2}}\int_{-1}^{1}\Pi(t)\,dt\right.$$

$$(1.19)\qquad -\omega\frac{\Phi_1(x)}{\sqrt{1-\lambda^2}}\int_{-1}^{1}\Pi(t)\,t\,dt+\cdots\cdots-\omega\frac{\Phi_n(x)}{\sqrt{1-\lambda^2}}\int_{-1}^{1}\Pi(t)\,t^n\,dt$$

This equation may be solved as a differential equation in $y=\int_x^1\Pi(t)\,dt$ by introducing the integrating factor $e^{-i\mu x}$, viz.

$$e^{-i\mu x}\int_{x}^{1}\Pi(t)\,dt=\frac{1}{\sqrt{1-\lambda^2}}\int_{x}^{1}\Pi_0(t)\,e^{-i\mu t}\,dt$$

$$(1.20)\qquad +\left\{\frac{\lambda^2 c(\omega)}{\pi(1-\lambda^2)}\int_{x}^{1}\sqrt{\frac{1-t}{1+t}}\,e^{-i\mu t}\,dt+\frac{i\mu}{\pi}\int_{x}^{1}\cos^{-1}t\,e^{-i\mu t}\,dt-\frac{\omega}{\sqrt{1-\lambda^2}}\int_{x}^{1}e^{-i\mu t}\,\Phi_0(t)\,dt\right\}\int_{-1}^{1}\Pi(t)\,dt$$

$$-\frac{\omega}{\sqrt{1-\lambda^2}}\int_{x}^{1}e^{-i\mu t}\,\Phi_1(t)\,dt\int_{-1}^{1}\Pi(t)\,t\,dt+\cdots\cdots-\frac{\omega}{\sqrt{1-\lambda^2}}\int_{x}^{1}e^{-i\mu t}\,\Phi_m(t)\,dt\int_{-1}^{1}\Pi(t)\,t^m\,dt$$

Setting $x = -1$ in the above expression, we obtain a linear equation relating the $(n+1)$ unknowns $\int_{-1}^{1}\Pi(t)\,dt,\ \int_{-1}^{1}\Pi(t)\,t\,dt, \cdots, \int_{-1}^{1}\Pi(t)\,t^{n}dt$.

$$(1.21)\quad A_{00}\int_{-1}^{1}\Pi(t)\,dt + A_{01}\int_{-1}^{1}\Pi(t)\,t\,dt + \cdots\cdots + A_{0n}\int_{-1}^{1}\Pi(t)\,t^{n}dt = B_0/\sqrt{1-\lambda^2}$$

where

$$A_{00} = e^{i\mu} - \frac{\lambda^2 C(\omega)}{\pi(1-\lambda^2)}\int_{-1}^{1} e^{-i\mu t}\sqrt{\frac{1-t}{1+t}}\,dt - \frac{i\mu}{\pi}\int_{-1}^{1} e^{-i\mu t}\cos^{-1} t\,dt + \frac{\omega}{\sqrt{1-\lambda^2}}\int_{-1}^{1} e^{-i\mu t}\phi_0(t)\,dt.$$

$$(1.22)\quad A_{0n} = \frac{\omega}{\sqrt{1-\lambda^2}}\int_{-1}^{1} e^{-i\mu t}\phi_n(t)\,dt, \quad n > 0$$

$$B_0 = \int_{-1}^{1} e^{-i\mu t}\,\Pi_0(t)\,dt$$

Expressions for these coefficients are obtainable in terms of Bessel and other known functions (See Appendix I, II, III).

The n additional equations required for the complete determination of the unknowns are obtained by multiplying equation (1.19) by $x^{k}\ [k = 0, 1, 2 \cdots (n-1)]$ and integrating between the limits -1 and 1. Then since

$$(1.23)\quad \int_{-1}^{1} x^{k}dx\int_{x}^{1}\Pi(t)\,dt = \frac{(-)^{k}}{k+1}\int_{-1}^{1}\Pi(t)\,dt + \int_{-1}^{1}\frac{t^{k+1}\,\Pi(t)\,dt}{k+1}$$

the following equations result

$$(1.24)\quad \begin{cases} A_{00}X_0 + A_{01}X_1 + \cdots\cdots + A_{0n}X_n = B_0/\sqrt{1-\lambda^2} \\ A_{10}X_0 + A_{11}X_1 + \cdots\cdots + A_{1n}X_n = B_1/\sqrt{1-\lambda^2} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ A_{n0}X_0 + A_{n1}X_1 + \cdots\cdots + A_{nn}X_n = B_n/\sqrt{1-\lambda^2} \end{cases}$$

with $\mathrm{X}_n = \int_{-1}^{1} \Pi(\xi)\xi^n \, d\xi$, and

$$A_{10} = 1 + i\mu - \frac{\lambda^2 C(\omega)}{\pi(1-\lambda^2)}\int_{-1}^{1}\sqrt{\frac{1-\xi}{1+\xi}}\, d\xi - \frac{i\mu}{\pi}\int_{-1}^{1}[\cos^{-1}\xi]\, d\xi + \frac{\omega}{\sqrt{1-\lambda^2}}\int_{-1}^{1}\Phi_0(\xi)\, d\xi \, ,$$

$$A_{11} = i\mu + \frac{\omega}{\sqrt{1-\lambda^2}}\int_{-1}^{1}\Phi_1(\xi)\, d\xi \, , \qquad A_{1n} = \frac{\omega}{\sqrt{1-\lambda^2}}\int_{-1}^{1}\Phi_n(\xi)\, d\xi \, , \quad n > 1$$

$$A_{20} = -\frac{i\mu}{2} - \frac{\lambda^2 C(\omega)}{\pi(1-\lambda^2)}\int_{-1}^{1}\sqrt{\frac{1-\xi}{1+\xi}}\,\xi\, d\xi - \frac{i\mu}{\pi}\int_{-1}^{1}[\cos^{-1}\xi]\,\xi\, d\xi + \frac{\omega}{\sqrt{1-\lambda^2}}\int_{-1}^{1}\Phi_0(\xi)\,\xi\, d\xi$$

$$A_{21} = 1 + \frac{\omega}{\sqrt{1-\lambda^2}}\int_{-1}^{1}\Phi_1(\xi)\,\xi\, d\xi \, , \quad A_{22} = \frac{i\mu}{2} + \frac{\omega}{\sqrt{1-\lambda^2}}\int_{-1}^{1}\Phi_2(\xi)\,\xi\, d\xi$$

$$A_{2n} = \frac{\omega}{\sqrt{1-\lambda^2}}\int_{-1}^{1}\Phi_n(\xi)\,\xi\, d\xi \, , \quad n > 2$$

$$A_{30} = \frac{i\mu}{3} - \frac{\lambda^2 C(\omega)}{1-\lambda^2}\int_{-1}^{1}\sqrt{\frac{1-\xi}{1+\xi}}\,\xi^2 d\xi - \frac{i\mu}{\pi}\int_{-1}^{1}[\cos^{-1}\xi]\,\xi^2 d\xi + \frac{\omega}{\sqrt{1-\lambda^2}}\int_{-1}^{1}\Phi_0(\xi)\,\xi^2 d\xi$$

$$A_{31} = \frac{\omega}{\sqrt{1-\lambda^2}}\int_{-1}^{1}\Phi_1(\xi)\,\xi^2 d\xi \, , \quad A_{32} = 1 + \frac{\omega}{\sqrt{1-\lambda^2}}\int_{-1}^{1}\Phi_2(\xi)\,\xi^2 d\xi \, , \quad A_{33} = \frac{i\mu}{3} + \frac{\omega}{\sqrt{1-\lambda^2}}\int_{-1}^{1}\Phi_3(\xi)\,\xi^2 d\xi$$

$$A_{3n} = \frac{\omega}{\sqrt{1-\lambda^2}}\int_{-1}^{1}\Phi_n(\xi)\,\xi^2 d\xi \, , \text{ etc.}$$

$$B_n = \int_{-1}^{1}\Pi_0(\xi)\,\xi^{n-1} d\xi \, , \quad n > 1 \, .$$

The $(n+1)$ equations may be solved simultaneously to determine the quantities

$$\mathrm{X}_0, \mathrm{X}_1, ----, \mathrm{X}_n$$

The first two of these give directly the lift and mid-chord moment over the entire chord. In the case of a wing-aileron combination, the lift on the control surface extending from $\xi = x$ to $\xi = 1$ is also required, which is found from equation (1.20) after substitution of the known values for $\mathrm{X}_0, \mathrm{X}_1, --, \mathrm{X}_n$

The partial mid-chord moment is found in a similar manner by integrating equation (1.19) from x to 1, with the aid of the relation

(1.25) $$\int_x^1 dx \int_x^1 \Pi(\xi)\, d\xi = \int_x^1 (\xi - x)\Pi(\xi)\, d\xi$$

Equation (1.25) may be applied to a wing-control surface combination by regarding x as the coordinate of the control surface leading edge, since the right side is precisely the moment about the leading edge.

The required relations are summarized by the following equations:

Total lift on control surface -

$$e^{-i\mu x}\int_x^1 \Pi(\xi)\,d\xi = \frac{1}{\sqrt{1-\lambda^2}} B_0(x) - A_{00}(x)\mathbf{X}_0 - A_{01}(x)\mathbf{X}_1 + \cdots - A_{0n}(x)\mathbf{X}_n . \tag{1.26}$$

Total moment about the control surface leading edge -

$$i\mu\int_x^1 (\xi - x)\Pi(\xi)\,d\xi = \frac{1}{\sqrt{1-\lambda^2}} B_1(x) - \int_x^1 \Pi(\xi)\,d\xi - A_{10}(x)\mathbf{X}_0 - A_{11}(x)\mathbf{X}_1 + \cdots - A_{1n}(x)\mathbf{X}_n . \tag{1.27}$$

Where $A_{0j}(x)$ and $B_0(x)$ designate the respective quantities appearing in equation (1.20); $A_{ij}(x)$ and $B_1(x)$ denote the corresponding quantities obtained from equation (1.19) by integration from x to 1.

The expressions for the A_{ij} $[i, j = 0, 1, 2, 3]$ and $A_{ij}(x)[i = 0, 1; j = 0, 1, 2, 3]$ are listed in Table (1.01). The various integrals involved are evalued in Appendix I, II, while the evaluation of the B_j is carried out in Appendix III for the following four types of motion:

a. Translation of the entire chord.

b. Rotation of the entire chord about the forward quarter chord point.

c. Translation of the control surface.

d. Rotation of the control surface about its leading edge.

For routine calculations the following procedure is suggested:

a. Calculation of all quantities which do not depend on the polynomial approximation. These include the following:

$\bar{A}_{00}$	$\bar{A}_{00}(x)$	$\mathcal{B}_0$	$\mathcal{B}_0(x)$
$\bar{A}_{10}$	$\bar{A}_{10}(x)$	$\mathcal{B}_1$	$\mathcal{B}_a(x)$

and the coefficients of a_k^m in equation (A2.08).

b. Determination of the approximating coefficients $\alpha_0, \alpha_1 \cdots\cdots \alpha_n$ as defined by equation (2.03).

c. Calculation of $P_k^m \left[m = 0, 1, \cdots n-1, \; k = 0, 1, \cdots n \right]$ as defined by (A2.05).

d. Calculation of a_k^m from equation (A2.03).

e. Calculation of the integrals involving Φ_m from equations (A2.08) - (A2.12).

f. Calculation of A_{ij}, $A_{ij}(x)$ as given in Table (A1.01).

g. Solution of the basic equations (A1.24) to determine the unknown quantities $X_0, X_1, \cdots, X_n$.
The first two of these are proportional respectively to the total lift and total moment about the mid-chord.

h. Resubstitution of the above quantities into equations (1.25) and (1.27) to determine the lift and moment over any portion of the chord.

TABLE (1.01)

SUMMARY OF THE COEFFICIENTS A_{ij}, $A_{ij}(x)$

$$A_{ij} = \bar{A}_{ij} + \bar{\bar{A}}_{ij}, \text{ where}$$

$$\bar{A}_{00} = J_0(\mu) - \frac{\lambda^2 C(\omega)}{1-\lambda^2}\left[J_0(\mu) + i J_1(\mu)\right] ; \quad \bar{A}_{0j} = 0 \quad , j > 0$$

$$\bar{A}_{10} = 1 - \frac{\lambda^2 C(\omega)}{1-\lambda^2} ; \quad \bar{A}_{11} = i\mu \quad \bar{A}_{1j} = 0 \ , \ j > 1$$

$$\bar{A}_{20} = -\frac{i\mu}{4} \quad ; \quad \bar{A}_{21} = 1 \quad ; \bar{A}_{22} = \frac{i\mu}{2} \quad ; \bar{A}_{2j} = 0, \ j > 2$$

$$\bar{A}_{30} = -\frac{\lambda^2 C(\omega)}{2(1-\lambda^2)} \ ; \ \bar{A}_{31} = 0 \ ; \ \bar{A}_{32} = 1 \ ; \ \bar{A}_{33} = \frac{i\mu}{3} \ ; \ \bar{A}_{3n} = 0 \ , \ j > 3$$

$$\bar{\bar{A}}_{0j} = \frac{\omega}{\sqrt{1-\lambda^2}} \int_{-1}^{1} e^{-i\mu\xi} \Phi_j(\xi)\, d\xi \quad ; \quad \bar{\bar{A}}_{ij} = \frac{\omega}{\sqrt{1-\lambda^2}} \int_{-1}^{1} \xi^{i-1} \Phi_j(\xi)\, d\xi \quad , \ i > 0$$

$$\bar{A}_{00}(x) = \frac{1}{\pi}\left[j_0(-\mu, \cos^{-1}x) - e^{-i\mu x}\cos^{-1}x\right] - \frac{\lambda^2 C(\omega)}{\pi(1-\lambda^2)}\left[j_0(-\mu, \cos^{-1}x) - j_1(-\mu, \cos^{-1}x)\right]$$

$$\bar{A}_{0j}(x) = 0 \ , \ j > 0$$

$$\bar{A}_{10}(x) = \frac{i\mu}{\pi}\left[x\cos^{-1}x - \sqrt{1-x^2}\right] - \frac{\lambda^2 C(\omega)}{\pi(1-\lambda^2)}\left[\cos^{-1}x - \sqrt{1-x^2}\right]$$

$$\bar{A}_{ij}(x) = 0 \ , \ j > 0$$

$$\bar{\bar{A}}_{0j}(x) = \frac{\omega}{\sqrt{1-\lambda^2}} \int_{x}^{1} e^{-i\mu\xi} \Phi_j(\xi)\, d\xi \ ; \ A_{ij}(x) = \frac{\omega}{\sqrt{1-\lambda^2}} \int_{x}^{1} \Phi_j(\xi)\, d\xi \quad , \ j > 0$$

For the integrals involving Φ_n see Appendix II.

PART II

RESULTS OF NUMERICAL CALCULATIONS

In the present calculations, the lift and moment coefficients have been calculated for the case where $\lambda = .7$ and $\omega \leq .5$. Only the constant and linear terms have been retained in the kernel difference $\bar{K}$ as defined by equation (1.04). The difference is plotted in figure (A7.01) as a function of $z = \omega(x - \xi)$, from which it can be seen that for the imaginary part, a straight line approximation is adequate over the interval $[-1 \leq z \leq 1]$ corresponding to the range $[0 \leq \omega \leq .5]$ for the reduced frequency. For the real part, this approximation is also considered satisfactory since the magnitude of this portion over the interval is small compared to the magnitude of the actual kernel. The line used is that one which gives for $\bar{K}$ the best approximation in the "least squares" sense, i.e., for each ω the coefficients α_0 and α_1 are so determined that

(2.01) $$E(\omega) = \int_{-a}^{a} [\bar{K} - (\alpha_0 + \alpha_1 z)]^2 dz$$

is a minimum, where $a = 2\omega$. This condition is satisfied provided

(2.02) $$\frac{\partial E}{\partial \alpha_0} = 0 \quad , \quad \frac{\partial E}{\partial \alpha_1} = 0 \, ,$$

which leads for the determination of α_0 and α_1 to the relations

(2.03) $$\alpha_0 = \frac{1}{2a} \int_{-a}^{a} \bar{K}(z)\, dz$$
$$\alpha_1 = \frac{3}{2a^3} \int_{-a}^{a} \bar{K}(z)\, z\, dz$$

These equations define α_0 and α_1 as continuous functions of ω. The integrations are carried by the procedure described in Appendix VII and the results are listed in Table (2.01). α_0 and α_1 are plotted in figures (A7.02) and (A7.03) as functions of ω.

TABLE 2:01

VALUES OF α_0 AND α_1

ω	$\omega\alpha_0$		$\omega^2\alpha_1$	
	REAL	IMAGINARY	REAL	IMAGINARY
.05	0	-.0025331	.000795	-.0004201
.10	-.000005	-.0050791	.002425	-.0016711
.20	-.000041	-.0102771	.006601	-.0065781
.30	-.000161	-.0157031	.010565	-.0143771
.40	-.000453	-.0214621	.013190	-.0245231
.50	-.001053	-.0276411	.013779	-.0363091

The next step is the evaluation of the coefficients A_{00}, A_{01}, A_{10} and A_{11}. These are tabulated in Table (2.02) for $\omega = .05, .1, .2, .3, .4, .5$.

In the evaluation of the $\mathcal{D}_i$, the factor $\pi \rho b^3 v^2$ times the non-dimensional amplitude is removed in order to obtain lift and moment coefficients comparable to those tabulated in reference 6. Values of these quantities are listed in Table (2.03).

TABLE (2.02)

THE COEFFICIENTS A_{ij} FOR VARIOUS VALUES OF ω -

$\lambda = .7$

ω	A_{00}	A_{01}	A_{10}	A_{11}
.05	1.08269+.15000i	.006078-.003930i	1.086884+.14789i	.005980+.043966i
.10	1.14694+.21851i	.016659-.014081i	1.159400+.21071i	.015933+.081234i
.20	1.21915+.29921i	.041736-.043799i	1.254400+.27350i	.037007+.144937i
.30	1.24357+.36271i	.070906-.079484i	1.307580+.31446i	.057738+.201426i
.40	1.23995+.42664i	.106130-.119225i	1.337930+.35374i	.079972+.253358i
.50	1.21895+.49421i	.149389-.163119i	1.355730+.39613i	.104048+.300627i

TABLE (2.03)

VALUES OF B_i FOR VARIOUS ω; $\lambda = .9$

MODE OF MOTION →	h TRANSLATION OF ENTIRE CHORD		α ROTATION ABOUT QUARTER CHORD POINT	
ω	$B_{0h}/\pi\rho b^2 v^2 h_0$	$B_{1h}/\pi\rho b^2 v^2 h_0$	$B_{0\alpha}/\pi\rho b^3 v^2 \alpha_0$	$B_{1\alpha}/\pi\rho b^3 v^2 \alpha_0$
.05	-3.3500-36.4649i	-4.2256-36.3604	-733.628+30.52290i	-731.933+48.1552i
.10	-1.6408-16.7655i	-2.4460-16.6384i	-170.274-.38025i	-169.339+7.8216i
.20	-.1762-7.3892i	-.8862-7.2758i	-38.0994-6.54790i	-37.765-2.8448i
.30	.45113-4.1710i	-.1955-4.4331i	-15.5592-6.08606i	-15.4726-3.78158i
.40	.77636-3.1662i	.1751-3.1249i	-8.10142-5.20012i	-8.13715-3.56265i
.50	.96094-2.3964i	.3972-2.3918i	-4.78424-4.43375i	-4.88636-3.18608i

MODE OF MOTION →	z CONTROL SURFACE TRANSLATION $e = .5$		β CONTROL SURFACE ROTATION $e = .5$	
ω	$B_{0z}/\pi\rho b^2 v^2 z_0$	$B_{1z}/\pi\rho b^2 v^2 z_0$	$B_{0\beta}/\pi\rho b^3 v^2 \beta_0$	$B_{1\beta}/\pi\rho b^3 v^2 \beta_0$
.05	-2.85070-22.21350	-2.98699-22.14336	-445.28940+49.47210	-443.90690+52.22120
.10	-1.80925-10.22309	-1.90310-10.13279	-102.89650+14.62350	-101.99990+15.59110
.20	-.915826-4.525884	-.95319-4.43095	-22.99155+3.04556	-22.50450+3.26170
.30	-.529040-2.786501	-.53255-2.69978	-9.51866+.82165	-9.20616+.85853
.40	-.325568-1.979462	-.30687-1.90305	-5.10957+.14755	-4.88799+.12105
.50	-.206246-1.523000	-.17162-1.45655	-3.16595-.09751	-2.99768+.15123

The equations for the total lift and moment are

$$
(2.04)\quad \begin{cases} A_{00}\displaystyle\int_{-1}^{1}\Pi(\xi)\,d\xi + A_{01}\int_{-1}^{1}\Pi(\xi)\,\xi\,d\xi = B_0\Big/\sqrt{1-\lambda^2} \\ A_{10}\displaystyle\int_{-1}^{1}\Pi(\xi)\,d\xi + A_{11}\int_{-1}^{1}\Pi(\xi)\,\xi\,d\xi = B_1\Big/\sqrt{1-\lambda^2} \end{cases}
$$

In order to obtain directly the moment about the quarter chord, these equations may be re-written in the form

$$
(2.05)\quad \begin{cases} \left[A_{00}-\frac{1}{2}A_{01}\right]\displaystyle\int_{-1}^{1}\Pi(\xi)\,d\xi + A_{01}\int_{-1}^{1}\Pi(\xi)\left[\tfrac{1}{2}+\xi\right]d\xi = B_0\Big/\sqrt{1-\lambda^2} \\ \left[A_{10}-\frac{1}{2}A_{11}\right]\displaystyle\int_{-1}^{1}\Pi(\xi)\,d\xi + A_{11}\int_{-1}^{1}\Pi(\xi)\left[\tfrac{1}{2}+\xi\right]d\xi = B_1\Big/\sqrt{1-\lambda^2} \end{cases}
$$

The solution of the above equations results in values for the total lift and moment coefficients as listed in Table (2.04).

TABLE (2.04)

COEFFICIENTS OF TOTAL LIFT AND MOMENT

$\lambda = .7$

MODE OF MOTION →	h TRANSLATION OF ENTIRE CHORD		α ROTATION ABOUT QUARTER CHORD POINT	
ω	$L_h = \dfrac{\int_{-1}^{1} \Pi_h(t)\,dt}{\pi \rho b^2 v^2 h_0}$	$M_h = \dfrac{\int_{-1}^{1} \Pi_h(t)\left[\frac{1}{2}+t\right]dt}{\pi \rho b^3 v^2 h_0}$	$L_\alpha = \dfrac{\int_{-1}^{1} \Pi_\alpha(t)\,dt}{\pi \rho b^3 v^2 \alpha_0}$	$M_\alpha = \dfrac{\int_{-1}^{1} \Pi_\alpha(t)\left[\frac{1}{2}+t\right]dt}{\pi \rho b^4 v^2 \alpha_0}$
.05	-10.793-45.769i	1.196-.244i	-927.4+170.32i	-4.0381-38.168i
.10	-5.880-19.437i	1.057-.279i	-201.3+39.64i	-2.1010-17.8649i
.20	-2.361-7.958i	.918-.299i	-43.07+4.147i	-.9635-8.4004i
.30	-1.102-4.789i	.842-.310i	-17.91-.807i	-.5962-5.4471i
.40	-.520-3.414i	.788-.326i	-9.847-1.787i	-.4451-4.0128i
.50	-.217-2.672i	.745-.346i	-6.306-1.893i	-.3783-3.1697i

MODE OF MOTION →	$e = .5$ β - ROTATION OF CONTROL SURFACE		$e = .5$ z - TRANSLATION OF CONTROL SURFACE	
ω	$L_\beta = \dfrac{\int_{-1}^{1} \Pi_\beta(t)\,dt}{\pi \rho b^2 v^2 \beta_0}$	$M_\beta = \dfrac{\int_{-1}^{1} \Pi_\beta(t)\left[\frac{1}{2}+t\right]dt}{\pi \rho b^3 v^2 \beta_0}$	$L_z = \dfrac{\int_{-1}^{1} \Pi_z(t)\,dt}{\pi \rho b^3 v^2 z_0}$	$M_z = \dfrac{\int_{-1}^{1} \Pi_z(t)\left[\frac{1}{2}+t\right]dt}{\pi \rho b^4 v^2 z_0}$
.05	-556.226+141.666i	-234.688-11.6283i	-7.55258-27.680i	.34425-11.7377i
.10	-117.566+40.606i	-59.689-4.9078i	-4.4527-11.6000i	.25098-5.9715i
.20	-23.733+9.4235i	-15.4232-1.9616i	-2.1923-4.5886i	.14680-3.0828i
.30	-9.3262+3.6604i	-7.0532-1.1120i	-1.3613-2.6470i	.08426-2.1067i
.40	-4.8473+1.8168i	-4.0644-.72841i	-.96113-1.7938i	.04039-1.60748i
.50	-2.9295+1.0432i	-2.6554-.51346i	-.73411-1.32381i	.00645-1.29947i

Equations (1.25) and (1.27) may now be employed to calculate the forces and moments on the control surface, after first evaluating the quantities $A_{00}(x)$, $A_{01}(x)$, $A_{10}(x)$, $A_{11}(x)$, $B_0(x)$ and $B_1(x)$. In the example, the value $x = e = .5$ was used.* The basic quantities are listed in Tables (2.05) and (2.06) and the results in Table (2.07).

* Here e denotes the coordinate of the control surface leading edge in the notation of Ref. 7.

TABLE (2.05)

THE COEFFICIENTS $A_{ij}(x)$ FOR VARIOUS VALUES OF ω

$x=e^{-.5}$, $\lambda=.7$

ω	$A_{00}(x)\times 10^{+2}$	$A_{01}(x)\times 10^{+2}$
.05	.45570+.33840i	.03502-.02226i
.10	.75200+.21250i	.09707-.07758i
.20	.8738-.2752i	.2650-.2300i
.30	.5970-.6760i	.5211-.4225i
.40	.1105-.8557i	.9039-.7020i
.50	-.4518-.7912i	1.4050-1.1508i

ω	$A_{10}(x)\times 10^{2}$	$A_{11}(x)\times 10^{+2}$
.05	.44420+.35330i	.03575-.02110i
.10	.73640+.26200i	.10200-.07090i
.20	.9035-.1580i	.2932-.1925i
.30	.7205-.5489i	.5944-.3095i
.40	.3320-.8083i	1.0567-.4358i
.50	-.1714-.9179i	1.7051-.6206i

TABLE (2.06)

VALUES OF $\frac{\mathcal{B}(\omega)}{\sqrt{1-\lambda^2}}$ FOR VARIOUS ω; $x = e = .5$, $\lambda = .7$

MODE OF → MOTION	h TRANSLATION OF ENTIRE CHORD		α ROTATION ABOUT QUARTER CHORD POINT	
ω	$\mathcal{B}_0(\omega) / \pi\rho b^3 v^2 h_0 \sqrt{1-\lambda^2}$	$\mathcal{B}_1(\omega) / \pi\rho b^3 v^2 h_0 \sqrt{1-\lambda^2}$	$\mathcal{B}_0(\omega) / \pi\rho b^3 v^2 \alpha_0 \sqrt{1-\lambda^2}$	$\mathcal{B}_1(\omega) / \pi\rho b^3 v^2 \alpha_0 \sqrt{1-\lambda^2}$
.05	-.246179-2.929692i	-.148238-2.936189i	-59.01179-1.862665i	-58.91258-3.834638i
.10	-.094170-1.340463i	-.004521-1.343598i	-13.670495-2.321586i	-13.48092-3.22843i
.20	-.042037-.598815i	.121436-.587539i	-3.123135-1.759751i	-2.85663-2.15783i
.30	.102201-.386589i	.177219-.357986i	-1.356597-1.349293i	-1.05645-1.59206i
.40	.132741-.298704i	.207142-.252342i	-.782840-1.084747i	-.46409-1.25271i
.50	.148618-.256968i	.225075-.193139i	-.532428-.904972i	-.20158-1.0293i

MODE OF → MOTION	z TRANSLATION OF CONTROL SURFACE		β ROTATION OF CONTROL SURFACE	
ω	$\mathcal{B}_0(\omega) / \pi\rho b^3 v^2 z_0 \sqrt{1-\lambda^2}$	$\mathcal{B}_1(\omega) / \pi\rho b^3 v^2 z_0 \sqrt{1-\lambda^2}$	$\mathcal{B}_0(\omega) / \pi\rho b^3 v^2 \beta_0 \sqrt{1-\lambda^2}$	$\mathcal{B}_1(\omega) / \pi\rho b^3 v^2 \beta_0 \sqrt{1-\lambda^2}$
.05	-.317587-6.03730i	-.12367-6.04447i	-120.9059+3.39750i	-120.9469-.48457i
.10	-.224897-2.938094i	-.03614-2.94642i	-29.5092+.80110i	-29.4919-109175i
.20	-.141457-1.415948i	.040566-1.42190i	-7.17947+,005054i	-7.11113-.91199i
.30	-.104009-.925652i	.074534-.92740i	-3.17216-.11118i	-3.08145-.71426i
.40	-.084281-.688476i	.092758-.68572i	-1.80065-.12651i	-1.69822-.57792i
.50	-.073168-.55042i	.10368-.54325i	-1.17588-.11899i	-1.06674-.48237i

TABLE (2.07)

CONTROL SURFACE LIFT AND MOMENT COEFFICIENTS

[illegible] $e = .5$

MODE OF MOTION →	h TRANSLATION OF ENTIRE CHORD		α ROTATION ABOUT QUARTER CHORD POINT	
$\omega \downarrow$	$P_h = \frac{\int_e^1 \Pi_h(t)\,dt}{\pi\rho b^2 v^2 h_0}$	$T_h = \frac{\int_e^1 (t-e)\Pi_h(t)\,dt}{\pi\rho b^2 v^2 h_0}$	$P_\alpha = \frac{\int_e^1 \Pi_\alpha(t)\,dt}{\pi\rho b^3 v^2 \alpha_0}$	$T_\alpha = \frac{\int_e^1 (t-e)\Pi_\alpha(t)\,dt}{\pi\rho b^3 v^2 \alpha_0}$
.05	-.29362-2.698891	-.05339-.509751	-54.3428-.660451	-10.685-.269781
.10	-.04530-1.191421	-.00456-.232961	-12.02504-2.659421	-2.34416-.588851
.20	.12209-.581441	.02770-.104161	-2.59931-2.098481	-.49679-.442901
.30	.17699-.343621	.03820-.068421	-1.05165-1.571271	-.19924-.328181
.40	.19888-.271531	.04238-.053331	-.55034-1.235241	-.10058-.259291
.50	.20661-.232361	.04387-.045661	-.32912-1.007471	-.05793-.209401

MODE OF MOTION →	z TRANSLATION OF CONTROL SURFACE		β ROTATION OF CONTROL SURFACE	
$\omega \downarrow$	$P_z = \frac{\int_e^1 \Pi_z(t)\,dt}{\pi\rho b^2 v^2 z_0}$	$T_z = \frac{\int_e^1 (t-e)\Pi_z(t)\,dt}{\pi\rho b^3 v^2 z_0}$	$P_\beta = \frac{\int_e^1 \Pi_\beta(t)\,dt}{\pi\rho b^3 v^2 \beta_0}$	$T_\beta = \frac{\int_e^1 (t-e)\Pi_\beta(t)\,dt}{\pi\rho b^3 v^2 \beta_0}$
.05	-.23729-5.892821	-.03940-.987931	-117.967+1.84001	-19.7773+.143751
.10	-.08174-2.346521	-.00915-.474591	-28.5224-.601031	-4.7529-.223731
.20	.02145-1.381251	.01069-.229221	-6.92832-.801441	-1.14727-.208181
.30	.053103-.913591	.01667-.151551	-3.05662-.636431	-.50412-.158141
.40	.064404-.684951	.01872-.113541	-1.71933-.504451	-.28241-.123561
.50	.068505-.548401	.01942-.090881	-1.10263-.410721	-.13004-.1[illegible]0071

For comparison, some values of the above coefficients as computed in Reference 17 from Dietze's original data of Reference 3 are shown in Table (2.08). The control surface coefficients are somewhat incomplete and also correspond to a value of $\mathcal{C} = .52$ so that only a qualitative comparison can be made. However, it appears that reasonably good agreement exists even for the higher values of ω where only a rough approximation to the actual kernel was used.

In a subsequent report, an extended range of control surface chords is to be considered. It is also planned to check the present results for the higher values of ω by employing a cubic approximation for $\overline{K}$. This approximation should hold for values of ω as high as 1.

TABLE (2.08)

LIFT AND MOMENT COEFFICIENTS COMPUTED FROM DIETZE

$\lambda = .7$

ω	L_h	M_h	L_α	M_α
.5	-.21520 - 2.72800j	+.81104 - .35860j	-6.48240 - 1.94480j	-.39900 - 3.40760j
.4	-.51688 - 3.45188j	+.84081 - .32444j	-9.98938 - 1.83750j	-.43375 - 4.22250j
.3	-1.10111 - 4.81333j	+.87389 - .29900j	-18.03222 - .84078j	-.55589 - 5.60667j
.2	-2.36250 - 7.96500j	+0.93050 - 0.28700j	-43.15500 + 4.14500j	-.89950 - 8.50750j
.1	-5.89200 -19.41000j	+1.05600 - .25200j	-201.37000 + 39.69000j	-1.99900 -17.92000j

TABLE (2.08)

LIFT AND MOMENT COEFFICIENTS COMPUTED FROM DIETZE

ω	M_β	L_β	T_h
.5	- 2.79499 - .42520j	- 2.88932 + 1.15560j	+ .04516 - .03880j
.4	- 4.19739 - .65562j	- 4.78622 + 1.90688j	+ .04216 - .04562j
.3	- 7.15183 - 1.05489j	- 9.18150 + 3.70666j	+ .03695 - .05956j
.2	-15.47739 - 1.83875j	- 23.24622 + 9.42250j	+ 0.02653 -0.09225j
.1	-59.63739 - 4.63400j	-115.34372 + 40.23000j	- .00472 .20800j

ω	T_α	T_β
.5	- .04399 - .20532j	- .17621 - .09144j
.4	- .08201 - .24556j	- .26972 - .11350j
.3	- .17006 - .30722j	- .47263 - .14489j
.2	- 0.44214 - 0.40850j	- 1.06466 - 0.18750j
.1	- 2.09939 - .54100j	- 4.38041 - .20100j

CONCLUSIONS

The method described in this report is suitable for the computation of compressible, non-stationary, aerodynamic lift and moment coefficients for values of the reduced frequency less than 1.

The subject method is better adapted to routine computation than are those previously known.

RECOMMENDATIONS

It is recommended that:

1. The results of this report be extended by the method described therein to cover a larger range of values of the control surface chord.

2. The accuracy of the linear approximation for the value .5 of the reduced frequency be checked using a cubic approximation for the nuclear difference.

3. Calculations be made using the cubic approximation for values of the reduced frequency between .5 and 1.0.

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APPENDIX I

EVALUATION OF CERTAIN INTEGRALS

A. Proof of Equation (1.13)

The relation as stated in equation (1.13) is

$$\int_{-1}^{1}\sqrt{\frac{1+z}{1-z}}\,dz\int_{-1}^{1}\frac{\Pi(\xi)\,d\xi}{z-\xi} = \pi\int_{-1}^{1}\Pi(\xi)\,d\xi \tag{A1.01}$$

Schwartz (Ref. 2) has established the validity of the interchange of the order of integration in the double integral above, and other integrals of this type; thus

$$\int_{-1}^{1}\sqrt{\frac{1+z}{1-z}}\,dz\int_{-1}^{1}\frac{\Pi(\xi)\,d\xi}{z-\xi} = \int_{-1}^{1}\Pi(\xi)\,d\xi\int_{-1}^{1}\sqrt{\frac{1+z}{1-z}}\,\frac{dz}{z-\xi} \tag{A1.02}$$

Now $\int_{-1}^{1}\sqrt{\frac{1+z}{1-z}}\,\frac{dz}{z-\xi}$ becomes upon setting $z=\cos\varphi$, $\xi=\cos\theta$

$$\int_{0}^{\pi}\left(\frac{1+\cos\varphi}{\sin\varphi}\right)\frac{\sin\varphi\,d\varphi}{\cos\varphi-\cos\theta} = (1+\cos\theta)\int_{0}^{\pi}\frac{d\varphi}{\cos\varphi-\cos\theta}+\int_{0}^{\pi}d\varphi \tag{A1.03}$$

The above integral is of the Cauchy "principal-value" type, and it is well known that

$$\int_{0}^{\pi}\frac{d\varphi}{\cos\varphi-\cos\theta} = 0 \tag{A1.04}$$

Thus

$$\int_{-1}^{1}\sqrt{\frac{1+z}{1-z}}\,dz\int_{-1}^{1}\frac{\Pi(\xi)\,d\xi}{z-\xi} = \pi\int_{-1}^{1}\Pi(\xi)\,d\xi \tag{A1.05}$$

B. Proof of Equation (1.14)

The relation to be proved is

$$(A1.06)\quad \frac{1}{\pi}\int_{-1}^{1}\Lambda(x,z)\,dz\int_{-1}^{1}\frac{\Pi(\xi)\,d\xi}{z-\xi}=\cos^{-1}x\int_{-1}^{1}\Pi(\xi)\,d\xi-\pi\int_{x}^{1}\Pi(\xi)\,d\xi$$

Designating the right side by $u(x)$, then since (Ref. 2, Equation 62)

$$(A1.07)\quad \frac{d}{dx}\Lambda(x,z)=\sqrt{\frac{1+z}{1-z}}\left\{\sqrt{\frac{1-x}{1+x}}\,\frac{1}{z-x}-\frac{1}{\sqrt{1-x^2}}\right\}$$

$$(A1.08)\quad \pi\frac{du}{dx}=\sqrt{\frac{1-x}{1+x}}\int_{-1}^{1}\sqrt{\frac{1+z}{1-z}}\,\frac{dz}{z-x}\int_{-1}^{1}\frac{\Pi(\xi)\,d\xi}{z-\xi}-\frac{1}{\sqrt{1-x^2}}\int_{-1}^{1}\sqrt{\frac{1+z}{1-z}}\,dz\int_{-1}^{1}\frac{\Pi(\xi)\,d\xi}{z-\xi}$$

By equations (1.11) and (A1.05)

$$(A1.09)\quad \frac{du}{dx}=\pi\left[\Pi(x)-\frac{1}{\sqrt{1-x^2}}\int_{-1}^{1}\Pi(\xi)\,d\xi\right]$$

Further $\Lambda(1,z)=0$ so that $u(1)=0$. Therefore

$$(A1.10)\quad u(x)=\int_{x}^{1}\frac{d\xi}{\sqrt{1-\xi^2}}\left[\int_{-1}^{1}\Pi(\xi)\,d\xi\right]-\pi\int_{x}^{1}\Pi(\xi)\,d\xi$$

or

$$(A1.11)\quad \frac{1}{\pi}\int_{x}^{1}\Lambda(x,z)\,dz\int_{-1}^{1}\frac{\Pi(\xi)\,d\xi}{z-\xi}=\cos^{-1}x\int_{-1}^{1}\Pi(\xi)\,d\xi-\pi\int_{x}^{1}\Pi(\xi)\,d\xi$$

C. Evaluation of the integral $\int_{x}^{1}\sqrt{\frac{1-\xi}{1+\xi}}\,e^{-i\mu\xi}\,d\xi$

Set $\xi=\cos\varphi$, $x=\cos\theta$ Then

$$(A1.12)\quad \int_{x}^{1}\sqrt{\frac{1-\xi}{1+\xi}}\,e^{-i\mu\xi}\,d\xi=\int_{0}^{\theta}(1-\cos\varphi)\,e^{-i\mu\cos\varphi}\,d\varphi$$

The function

$$\int_0^\theta e^{i\mu\cos\varphi}\cos n\varphi\,d\varphi \tag{A1.13}$$

is designated by $j_n(\mu,\theta)$. Properties of this function are discussed further in Appendix IV. Thus

$$\int_x^1 \sqrt{\frac{1-\xi}{1+\xi}}\,e^{-i\mu\xi}d\xi = j_0(-\mu,\cos^{-1}x)-j_1(-\mu,\cos^{-1}x) \tag{A1.14}$$

For $x=-1$, since $j_n(\mu,\pi)=\pi(i)^n J_n(\mu)$, this gives

$$\int_{-1}^1 \sqrt{\frac{1-\xi}{1+\xi}}\,e^{-i\mu\xi}d\xi = \pi\left[J_0(\mu)+iJ_1(\mu)\right] \tag{A1.14A}$$

D. <u>Evaluation of the integral</u> $\int_x^1 e^{-i\mu\xi}\cos^{-1}\xi\,d\xi$

Making the change of variables: $\xi=\cos\varphi$, $x=\cos\theta$,

$$\int_x^1 e^{-i\mu\xi}\cos^{-1}\xi\,d\xi = \int_0^\theta e^{-i\mu\cos\varphi}\sin\varphi\,\varphi\,d\varphi \tag{A1.15}$$

Integrating by parts, with

$$u=\varphi,\quad dv=e^{-i\mu\cos\varphi}\sin\varphi\,d\varphi$$

whence

$$du=d\varphi,\quad v=(e^{-i\mu\cos\varphi})/i\mu$$

$$\int_0^\theta e^{-i\mu\cos\varphi}\sin\varphi\,\varphi\,d\varphi = \frac{1}{i\mu}\left[\theta e^{-i\mu\cos\theta}-\int_0^\theta e^{-i\mu\cos\varphi}d\varphi\right] \tag{A1.16}$$

or

$$i\mu\int_x^1 e^{-i\mu\xi}\cos^{-1}\xi\,d\xi = e^{-i\mu x}\cos^{-1}x - j_0(-\mu,\cos^{-1}x) \tag{A1.17}$$

and for $x = -1$

(A1.18) $$i\mu\int_{-1}^{1} e^{-i\mu\xi}\cos\xi\, d\xi = \pi\left[e^{i\mu} - J_0(\mu)\right].$$

E. <u>Evaluation of the integral</u> $\int_x^1 \sqrt{1-\xi^2}\, e^{-i\mu\xi} d\xi$

By the usual change of variables,

(A1.19) $$\begin{aligned}\int_x^1 \sqrt{1-\xi^2}\, e^{-i\mu\xi} d\xi &= \int_0^\theta e^{-i\mu\cos\phi} \sin^2\phi\, d\phi \\ &= \frac{1}{2}\int_0^\theta e^{-i\mu\cos\phi}\left[1-\cos 2\phi\right] d\phi \\ &= \frac{1}{2}\left[j_0(\mu,\theta) - j_2(\mu,\theta)\right]\end{aligned}$$

But equation (A4.07)

$$j_0(\mu,\theta) - j_2(\mu,\theta) = \frac{2i}{\mu}\left[e^{i\mu\cos\theta}\sin\theta - j_1(\mu,\theta)\right]$$

Therefore,

(A1.20) $$\int_x^1 \sqrt{1-\xi^2}\, e^{-i\mu\xi} d\xi = \frac{i}{\mu}\left[j_1(\mu,\cos^{-1}x) - e^{-i\mu x}\sqrt{1-x^2}\right]$$

and for $x = -1$

(A1.21) $$\int_{-1}^1 \sqrt{1-\xi^2}\, e^{-i\mu\xi} d\xi = \frac{\pi}{\mu} J_1(\mu)$$

F. <u>Evaluation of the integral</u> $\int_x^1 \xi\sqrt{1-\xi^2}\, e^{-i\mu\xi} d\xi$

(A1.22) $$\int_x^1 \xi\sqrt{1-\xi^2}\, e^{-i\mu\xi} d\xi = \int_0^\theta e^{-i\mu\cos\phi} \sin^2\phi\cos\phi\, d\phi$$

Integrating by parts with

$$u = \sin\phi\cos\phi = \tfrac{1}{2}\sin 2\phi, \quad dv = e^{-i\mu\cos\phi}\sin\phi\, d\phi$$

$$du = \cos 2\phi\, d\phi \qquad\qquad v = \left(e^{-i\mu\cos\phi}\right)/i\mu$$

gives

$$\text{(A1.23)}\quad \int_0^\theta e^{-i\mu\cos\phi}\sin^2\phi\cos\phi\, d\phi = \frac{1}{i\mu}\left[e^{-i\mu\cos\theta}\sin\theta\cos\theta - \int_0^\theta e^{-i\mu\cos\phi}\cos 2\phi\, d\phi\right]$$

$$= \frac{1}{i\mu}\left[e^{-i\mu\cos\theta}\sin\theta\cos\theta - \hat{J}_2(-\mu,\theta)\right]$$

Therefore

$$\text{(A1.24)}\quad \int_x^1 t\sqrt{1-t^2}\, e^{-i\mu t}\, dt = \frac{i}{\mu}\left[\hat{J}_2(-\mu, \cos^{-1}x) - x\sqrt{1-x^2}\, e^{-i\mu x}\right]$$

and if $x = -1$

$$\text{(A1.25)}\quad \int_{-1}^1 t\sqrt{1-t^2}\, e^{-i\mu t}\, dt = -\frac{i\pi}{\mu} J_2(\mu)$$

TABLE (A1.01)

VALUES OF $\int_x^1 e^{-i\mu\xi}\sqrt{\frac{1+\xi}{1-\xi}}\,d\xi$

ω	$x = e = .5$
.0	.1811722-0i
.05	.1810678-.0060478i
.10	.1807542-.0120883i
.20	.1795027-.0241168i
.30	.1774241-.0360264i
.40	.1745291-.0477582i
.50	.1708338-.0592547i

TABLE (A1.02)

VALUES OF $i\mu\int_x^1 e^{-i\mu\xi}\cos^{-1}\xi\,d\xi$

ω	$x = e = .5$
.0	.0
.05	.000552+.016440i
.10	.002206+.032823i
.20	.008801+.065188i
.30	.019721+.096638i
.40	.034855+.120728i
.50	.054053+.155022i

TABLE (A1.03)

VALUES OF $\int_x^1 e^{-i\mu\xi}\sqrt{1-\xi^2}\,d\xi$

ω	$x = e = .5$
.0	.3070925 0
.05	.3069102-.0103986i
.10	.3063637-.0207840i
.20	.3041812-.0414625i
.30	.3005572-.0619307i
.40	.2955115-.0820851i
.50	.2890695-.1018226i

TABLE (A1.04)

VALUES OF $\int_x^1 e^{-i\mu\xi}\sqrt{1-\xi^2}\,\xi\,d\xi$

ω	$x = e = .5$
.0	.2165064-0i
.05	.2163671-.0075876i
.10	.2159580-.0151642i
.20	.2143126-.0302427i
.30	.2115818-.0451616i
.40	.2077791-.0598393i
.50	.2029216-.0741873i

APPENDIX II

EVALUATION OF $\int_x^1 e^{-i\mu\xi}\Phi_m(\xi)d\xi,\ \int_x^1 \xi^p \Phi_m(\xi)d\xi$

By definition [equation (1.18)]

(A2.01) $$\Phi_m(x) = \int_{-1}^{1} G(x,z)u_m(z)dz\ ,\quad m=0,1,\cdots m.$$

According to Küssner (Ref. 1), this expression may be written as

(A2.02) $$\Phi_m(-\cos\theta) = -2\left[a_0^m\left(\frac{1+\cos\theta}{\sin\theta}\right)+2\sum_{n=1}^{n-m+1} a_n^m \sin n\theta\right]$$

with $x=-\cos\theta$, where

(A2.03)
$$a_0^m = \frac{1+T(\omega)}{2}\left[P_0^m - P_1^m\right] + P_r^m$$
$$a_n^m = \frac{i\omega}{2n}\left[P_{n-1}^m - P_{n+1}^m\right] - P_n^m\ ,\quad n>0$$

and $u_m(z)\ [z=-\cos\varphi]$ has the form

(A2.04) $$u_m(-\cos\varphi) = P_0^m + 2\sum_{n=1}^{n-m} P_n^m \cos n\varphi\,.$$

Expressions for the P's in terms of the coefficients α may be found by converting the powers of $\cos\varphi$ into cosines of multiples of φ. For $n=3$,

(A2.05)
$$P_0^0 = \alpha_0 + \frac{\omega^2\alpha_2}{2},\ P_1^0 = -\frac{\omega\alpha_1}{2} - \frac{3\omega^3\alpha_3}{8},\ P_2^0 = \frac{\omega^2\alpha_2}{4},\ P_3^0 = -\frac{\omega^3\alpha_3}{4}$$
$$P_0^1 = -\omega\alpha_1 - \frac{3\omega^3\alpha_3}{2},\ P_1^1 = \omega^2\alpha_2\ ,\ P_2^1 = -\frac{3\omega^3\alpha_3}{2}$$
$$P_0^2 = \omega^2\alpha_2\ ,\ P_1^2 = \frac{3\omega^3\alpha_3}{2}$$
$$P_0^3 = -\omega^3\alpha_3$$

Finally,

$$\int_x^1 e^{-i\mu\zeta}\phi_m(\zeta)\,d\zeta = \int_0^\theta e^{-i\mu\cos\varphi}\phi_m(\cos\varphi)\sin\varphi\,d\varphi \quad , \quad x=\cos\theta$$

(A2.06)
$$= -2\int_0^\theta e^{-i\mu\cos\varphi}\Big\{[a_0^m + a_1^m] - [a_0^m + a_2^m]\cos\varphi + [a_3^m - a_1^m]\cos 2\varphi + (-)^k[a_{k+1}^m - a_{k-1}^m]\cos k\varphi\Big\}\,d\varphi$$

Since $\int_0^\theta e^{-i\mu\cos\varphi}\cos n\varphi\,d\varphi = j_n(-\mu,\theta)$, and since by equation (A4.07)

(A2.07)
$$j_{n-1}(-\mu,\theta) - j_{n+1}(-\mu,\theta) = \frac{2in}{\mu} j_n(-\mu,\theta) - \frac{2i}{\mu} e^{i\mu\cos\theta}\sin\theta$$

it follows that

(A2.08)
$$\int_x^1 e^{-i\mu\zeta}\phi_m(\zeta)\,d\zeta = -2\Big\{a_0^m[j_0(-\mu,\theta) - j_1(-\mu,\theta)] - \frac{2i}{\mu}\sum_{k=1}^{n-m+1}(-)^k k\, a_k^m j_k(-\mu,\theta) + \frac{2i}{\mu} e^{-i\mu\cos\theta}\sum_{k=1}^{n-m+1}(-)^k a_k^m \sin k\theta\Big\}, \quad m=0,1,2,\cdots(n+1)$$

For $x=-1$ this gives

(A2.09)
$$\int_{-1}^1 e^{-i\mu\zeta}\phi_m(\zeta)\,d\zeta = -2\pi\Big\{a_0^m[J_0(\mu) + iJ_1(\mu)] - \frac{2i}{\mu}\sum_{k=1}^{n-m+1}(i)^k k\, a_k^m J_k(\mu)\Big\}$$

If $\mu = 0$ we obtain

(A2.10) $$\int_x^1 \varphi_m(\xi)\,d\xi = -2\left\{\theta\left[a_0^m + a_1^m\right] - \sin\theta\left[a_2^m + a_0^m\right] + \frac{\sin 2\theta}{2}\left[a_3^m - a_1^m\right] - \cdots\right\},$$

$(x = \cos\theta)$

and for $\mu = 0$, $x = -1$ $[\theta = \pi]$

(A2.11) $$\int_{-1}^1 \varphi_m(\xi)\,d\xi = -2\pi\left[a_0^m + a_1^m\right]$$

Similarly,

(A2.12) $$\int_{-1}^1 \varphi_m(\xi)\,\xi\,d\xi = \pi\left[a_0^m + a_2^m\right], \int_{-1}^1 \varphi_m(\xi)\,\xi^2\,d\xi = -\pi\left[a_0^m + \frac{1}{2}(a_1^m + a_3^m)\right], \text{etc.}$$

APPENDIX III

EVALUATION OF $\mathcal{B}_0(x) = \int_x^1 e^{-i\mu\xi}\Pi_0(\xi)d\xi$ and $\mathcal{B}_n(x) = \int_x^1 \xi^{n-1}\Pi_0(\xi)d\xi$

For practical purposes the above integrals must be evaluated for four types of motions, these are:

1. Translation of the entire chord.
2. Rotation of the entire chord about the forward quarterchord point.
3. Translation of the control surface.
4. Rotation of the control surface.

It has been shown by Küssner (Ref. 1) that in each of the first two cases, the pressure distribution $\Pi_0(\xi)$ is expressible in the form

$$\text{(A3.01)}\quad \Pi_0(\xi) = \rho v^2 b^2 A e^{i\nu t}\left\{Q_1\sqrt{\frac{1-\xi}{1+\xi}} + Q_2\sqrt{1-\xi^2} + Q_3\,\xi\sqrt{1-\xi^2}\right\}$$

where A is the non-dimensional amplitude of the motion and the Q_i are functions only of the reduced frequency, as given in Table A.3.01. Thus the evaluation of $\int_x^1 e^{-i\mu\xi}\Pi_0(\xi)d\xi$ is in these cases reduced to the determination of the integrals

$$\text{(A3.02)}\quad \int_x^1 e^{-i\mu\xi}\sqrt{\frac{1-\xi}{1+\xi}}\,d\xi,\quad \int_x^1 e^{-i\mu\xi}\sqrt{1-\xi^2}\,d\xi,\quad \int_x^1 e^{-i\mu\xi}\,\xi\sqrt{1-\xi^2}\,d\xi$$

These have already been shown in Appendix I to be respectively

$$\int_x^1 e^{-i\mu\xi}\sqrt{\frac{1-\xi}{1+\xi}}\,d\xi = j_0(-\mu,\cos^{-1}x) - j_1(-\mu,\cos^{-1}x)$$

$$\text{(A3.03)}\quad \int_x^1 e^{-i\mu\xi}\sqrt{1-\xi^2}\,d\xi = \frac{i}{\mu}\left[j_1(-\mu,\cos^{-1}x) - e^{-i\mu x}\sqrt{1-x^2}\right]$$

$$\int_x^1 e^{-i\mu\xi}\,\xi\sqrt{1-\xi^2}\,d\xi = \frac{i}{\mu}\left[j_2(-\mu,\cos^{-1}x) - e^{-i\mu x}x\sqrt{1-x^2}\right]$$

In the last two cases the incompressible pressure distribution consists of terms of the above type and additional terms of the form

(A3.04) $$R_p(\xi-e)^p\Lambda(\xi,e), \quad p=0,1,2,$$

where e is the coordinate of the control surface leading edge. The integral

(A3.05) $$I_p(\mu,x,e)=\int_x^1 e^{-i\mu\xi}(\xi-e)^p\Lambda(\xi,e)\,d\xi$$

is discussed in Appendix V. Thus the general expressions for B_o become as follows:

(A.306)
$$\begin{aligned}\frac{1}{\sqrt{1-\lambda^2}}B_o(x)=\frac{\pi\rho v^2b^2}{\sqrt{1-\lambda^2}}Ae^{ivt}\Big\{&\frac{Q_1(\omega,e)}{\pi}\Big[j_o(-\mu,\cos^{-1}x)-j_1(-\mu,\cos^{-1}x)\Big]\\&+\frac{i}{\mu}\frac{Q_2(\omega,e)}{\pi}\Big[j_1(-\mu,\cos^{-1}x)-e^{-i\mu x}\sqrt{1-x^2}\Big]\\&+\frac{i}{\mu}\frac{Q_3(\omega,e)}{\pi}\Big[j_2(-\mu,\cos^{-1}x)-e^{-i\mu x}x\sqrt{1-x^2}\Big]\\&+\frac{R_o}{\pi}I_o(\mu,x,e)+\frac{R_1}{\pi}I_1(\mu,x,e)+\frac{R_2}{\pi}I_2(\mu,x,e)\Big\}\end{aligned}$$

$$\frac{1}{\sqrt{1-\lambda^2}} B_0(-1) = \frac{\pi \rho v^2 b^2}{\sqrt{1-\lambda^2}} A e^{i\nu t} \Big\{ Q_1 \left[J_0(\mu) + i J_1(\mu) \right] + Q_2 \left[J_1(\mu)/\mu \right]$$

(A3.07)

$$+ Q_3 \left[\frac{-i J_1(\mu)}{\mu} \right] + \frac{R_0}{\pi} I_0(\mu,-1,e) + \frac{R_1}{\pi} I_1(\mu,-1,e) + \frac{R_2}{\pi} I_2(\mu,-1,e) \Big\}$$

where the Q_i and R_i are listed in Table A (3.01) for the four types of motion to be considered.

It is noted that for airfoils with a single control surface it is necessary to consider only the case where $x = e$.

The integrals $B_1(e) = \int_e^1 \Pi_0(t)\,dt$, $B_2(e) = \int_{-1}^1 \Pi_0(t)\,t\,dt$ are already known, being respectively the incompressible lift over the portion of the airfoil extending from $x = e$ to $x = 1$, and the total incompressible moment about the midchord. In the notation of Reference 6, these are:

$$\begin{cases} (1)\ B_1(e) = \pi \rho v^2 b^3 (h_0/b) P_h(e), & B_1(-1) = \pi \rho v^2 b^3 (h_0/b) L_h \\ (2)\ B_1(e) = \pi \rho v^2 b^3 \alpha_0 P_\alpha(e), & B_1(-1) = \pi \rho v^2 b^3 \alpha_0 L_\alpha \\ (3)\ B_1(e) = \pi \rho v^2 b^3 (z_0/b) P_z(e), & B_1(-1) = \pi \rho v^2 b^3 (z_0/b) L_z \\ (4)\ B_1(e) = \pi \rho v^2 b^3 \beta_0 P_\beta(e), & B_1(-1) = \pi \rho v^2 b^3 \beta_0 L_\beta \end{cases}$$

(A3.08)

$$\begin{cases} (1)\ B_2(-1) = \pi \rho v^2 b^4 (h_0/b) \left[M_h - \tfrac{1}{2} L_h \right] \\ (2)\ B_2(-1) = \pi \rho v^2 b^4 \alpha_0 \left[M_\alpha - \tfrac{1}{2} L_\alpha \right] \\ (3)\ B_2(-1) = \pi \rho v^2 b^4 (z_0/b) \left[M_z - \tfrac{1}{2} L_z \right] \\ (4)\ B_2(-1) = \pi \rho v^2 b^4 \beta_0 \left[M_\beta - \tfrac{1}{2} L_\beta \right] \end{cases}$$

The relations (A3.08) are useful as checks on the computation of $\mathcal{B}_0(x)$ since

$$\mathcal{B}_1(x) = \lim_{\mu \to 0} \mathcal{B}_0(\mu, x)$$

For the evaluation of $\mathcal{B}_3$ the following integrals are needed:

$$\int_{-1}^{1} \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\, \varepsilon^2 d\varepsilon \,, \quad \int_{-1}^{1} \sqrt{1-\varepsilon^2}\, \varepsilon^2 d\varepsilon \,, \quad \int_{-1}^{1} \sqrt{1-\varepsilon^2}\, \varepsilon^3 d\varepsilon$$

$$\int_{-1}^{1} \Lambda(\varepsilon, e)\, \varepsilon^2 d\varepsilon \,, \quad \int_{-1}^{1} \Lambda(\varepsilon, e)(\varepsilon - e)\varepsilon^2 d\varepsilon \,, \quad \int_{-1}^{1} \Lambda(\varepsilon, e)(\varepsilon - e)^2 \varepsilon^2 d\varepsilon$$

These are found to be respectively:

$$\int_{-1}^{1} \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\, \varepsilon^2 d\varepsilon = \pi/2 \qquad \int_{-1}^{1} \sqrt{1-\varepsilon^2}\, \varepsilon^2 d\varepsilon = \frac{\pi}{8}$$

$$\int_{-1}^{1} \sqrt{1-\varepsilon^2}\, \varepsilon^3 d\varepsilon = 0$$

(A3.09)
$$\int_{-1}^{1} \Lambda(\varepsilon, e)\varepsilon^2 d\varepsilon = \frac{\pi}{6}\sqrt{1-e^2}\,(1+2e^2)$$

$$\int_{-1}^{1} \Lambda(\varepsilon, e)(\varepsilon - e)\varepsilon^2 d\varepsilon = -\frac{\pi}{24}\sqrt{1-e^2}\,(e+2e^3)$$

$$\int_{-1}^{1} \Lambda(\varepsilon, e)(\varepsilon - e)^2 \varepsilon^2 d\varepsilon = \frac{\pi}{120}\sqrt{1-e^2}\,(9+2e^2+4e^4)$$

Thus

(A3.10)
$$\mathcal{B}_3 = \pi\rho b^3 v^2 \Lambda \Big\{ \frac{1}{2} Q_1 + \frac{1}{8} Q_2 + \frac{1}{6} R_0 \sqrt{1-e^2}\,(1+2e^2) - \frac{1}{24} R_1 \sqrt{1-e^2}\,(e+2e^3) + \frac{1}{120} R_2 \sqrt{1-e^2}\,(9+2e^2+4e^4) \Big\}$$

TABLE A3.01

Mode of Motion	Q_1	Q_2	Q_3	R_0	R_1	R_2
Translation of Entire Chord	$-\frac{i}{\omega}\left\{1+T(\omega)\right\}$	2	0	0	0	0
Rotation about Quarter Chord	$-\left\{\frac{1+T(\omega)}{\omega^2}+\frac{i\,T(\omega)}{\omega}\right\}$	$1-\frac{4i}{\omega}$	1	0	0	0
Translation of Control Surface	$\frac{i}{\pi}\left\{\frac{2\sqrt{1-e^2}}{\omega}-\left[\frac{1+T(\omega)}{\omega}\right]\Phi_1(e)\right\}$	$\frac{2}{\pi}\cos^{-1}e$	0	$-\frac{2i}{\pi\omega}$	$\frac{2}{\pi}$	0
Rotation of Control Surface	$\frac{1}{\pi}\left\{\frac{2\sqrt{1-e^2}}{\omega^2}+\frac{i}{\omega}\Phi_3(e)-\left[\frac{1+T(\omega)}{\omega^2}\right]\left[\Phi_1+\frac{i\omega}{2}\Phi_3\right]\right.$	$\frac{1}{\pi}\left\{\sqrt{1-e^2}-\frac{4i}{\omega}\cos^{-1}e-2e\cos^{-1}e\right\}$	$\frac{1}{\pi}\cos^{-1}e$	$-\frac{2}{\pi\omega^2}$	$-\frac{4i}{\pi\omega}$	$\frac{1}{\pi}$
Coefficients:						
For B_0	$J_0(\mu)+i\,J_1(\mu)$	$J_1(\mu)/\mu$	$-i\,J_2(\mu)/\mu$	$\frac{1}{\pi}I_0(\mu,-1,e)$	$\frac{1}{\pi}I_1(\mu,-1,e)$	$\frac{1}{\pi}I_2(\mu,-1,e)$
For $\pi B_0(x)$	$j_0(-\mu,\cos^{-1}x)-j_1(-\mu,\cos^{-1}x)$	$\frac{i}{\mu}\left\{j_1(-\mu,\cos^{-1}x)-e^{-i\mu x}\sqrt{1-x^2}\right\}$	$\frac{i}{\mu}\{j_2(-\mu,\cos^{-1}x-e^{-i\mu x}x\sqrt{1-x^2}$	$I_0(\mu,e,e)$	$I_1(\mu,e,e)$	$I_2(\mu,e,e)$
For B_3:	$-1/2$	$1/8$	0	$\frac{\sqrt{1-e^2}}{6}\left[1+2e^2\right]$	$-\frac{\sqrt{1-e^2}}{24}\left[e+2e^3\right]$	$\frac{\sqrt{1-e^2}}{120}\left[9+2e^2+e^4\right]$

The quantities $\Phi_1(e)$, $\Phi_2(e)$, $\Phi_3(e)$ are defined by Kussner in Ref. 1, and are not related to the function $\Phi_n(x)$ defined by equation 1.18. The function $T(\omega)$ is related to $C(\omega)$ according to equation 1.08.

APPENDIX IV

THE FUNCTION $\mathcal{J}_n(\mu,\theta)$

By definition

(A4.01)
$$\mathcal{J}_n(\mu,\theta) = \int_0^\theta e^{i\mu\cos\phi}\cos n\phi\, d\phi .$$

a. Special Values -

The function is seen to possess the following limiting values:

(A4.02)
$$\mathcal{J}_n(\mu,\pi) = \pi(i)^n J_n(\mu) \quad ;$$

(A4.03)
$$\mathcal{J}_n(\mu,\pi/2) = \frac{\pi}{2} e^{in\frac{\pi}{2}}\left[J_n(\mu) - i E_n(\mu)\right],$$

where E_n is the Lommel-Weber function of order n;

$$\mathcal{J}_n(0,\theta) = \frac{\sin(n\theta)}{n} \quad , n > 0 \quad ;$$

(A4.04)
$$\mathcal{J}_0(0,\theta) = \theta$$

b. Differential Relations -

By differentiation of equation (A4.01)

$$\frac{\partial}{\partial\mu}\left[J_n(\mu,\theta)\right] = i\int_0^\theta e^{i\mu\cos\varphi}\cos n\varphi\,\cos\varphi\,d\varphi$$

$$= \frac{i}{2}\int_0^\theta e^{i\mu\cos\varphi}\left[\cos(n-1)\varphi + \cos(n+1)\varphi\right]d\varphi .$$

Or,

(A4.05)
$$\frac{\partial}{\partial\mu}\left[J_n(\mu,\theta)\right] = \frac{i}{2}\left[J_{n+1}(\mu,\theta) + J_{n-1}(\mu,\theta)\right]$$

In particular, for $n = 0$,

(A4.06)
$$\frac{\partial}{\partial\mu}\left[J_0(\mu,\theta)\right] = i J_1(\mu,\theta)$$

c. **Recurrence Formulae** -

Integration of Equation (A4.01) by parts gives

$$nJ_n(\mu,\theta) = e^{i\mu\cos\theta}\sin(n\theta) + i\mu\int_0^\theta e^{i\mu\cos\varphi}\sin(n\varphi)\sin\varphi\,d\varphi$$

(A4.07)
$$= e^{i\mu\cos\theta}\sin(n\theta) + \frac{i\mu}{2}\int_0^\theta e^{i\mu\cos\varphi}\left[\cos(n-1)\varphi - \cos(n+1)\varphi\right]d\varphi,$$

or

(A4.08)
$$J_{n+1}(\mu,\theta) = \frac{2in}{\mu}J_n(\mu,\theta) + J_{n-1}(\mu,\theta) - \frac{2i}{\mu}e^{i\mu\cos\theta}\sin(n\theta)$$

d. The Differential Equation Satisfied By $\dot{J}_n(\mu,\theta)$

Differentiation of (A4.05), together with successive applications of equations (A4.05), together with successive applications of equations (A4.05) and (A4.08), shows that $y(\mu) = \dot{J}_n(\mu,\theta)$ satisfies the differential equations

$$\frac{d^2y}{d\mu^2} + \frac{1}{\mu}\frac{dy}{d\mu} + \left[1 - \frac{n^2}{\mu^2}\right]y = e^{i\mu\cos\theta}\left[\frac{i}{\mu}\cos(n\theta)\sin\theta - \frac{n}{\mu^2}\sin(n\theta)\right]. \tag{A4.09}$$

In particular, $\dot{J}_0(\mu,\theta)$ is a solution of

$$\frac{d^2y}{d\mu^2} + \frac{1}{\mu}\frac{dy}{d\mu} + y = \frac{i}{\mu}e^{i\mu\cos\theta}\sin\theta \tag{A4.10}$$

e. Expansion of $\dot{J}_0(\mu,\theta)$ and $\dot{J}_1(\mu,\theta)$ as Power Series in $\cos\theta$

It has already been shown that $\dot{J}_0(\mu,\theta)$ is a solution of the equation

$$\mu\frac{d^2y}{d\mu^2} + \frac{dy}{d\mu} + \mu y = i e^{i\mu\cos\theta}\sin\theta \tag{A4.11}$$

$$= i\sin\theta\sum_{k=0}^{\infty}\frac{(i\mu\cos\theta)^k}{k!}$$

It follows that $\dot{J}_0(\mu,\theta)$ can be written in the form

$$(A4.12) \qquad j_c(\mu,\theta) = A\,J_0(\mu) + B\,N_0(\mu) + i\sin\theta \sum_{k=0}^{\infty} (i)^n A_k(\mu)\,\frac{\cos^k\theta}{k!}$$

where $A_k(\mu)$ satisfies the equation

$$(A4.13) \qquad \mu\frac{d^2A_k}{d\mu^2} + \frac{dA_k}{d\mu} + \mu A_k = \mu^k$$

and it is easily shown that a solution of this equation is

$$(A4.14) \qquad A_k(\mu) = \frac{\mu^{k+1}}{(k+1)^2}\left[1 - \frac{\mu^2}{(k+3)^2} + \frac{\mu^4}{(k+3)^2(k+5)^2} + \cdots\right]$$

The A_k satisfies the recurrence relation

$$(A4.15) \qquad A_{k+1}(\mu) = \mu^k - k^2 A_{k-1}(\mu)$$

with the following initial values

$$(A4.16) \qquad \begin{aligned} A_0(\mu) &= -\frac{\pi}{2}E_0(\mu) \\ A_1(\mu) &= 1 - J_0(\mu) \end{aligned}$$

where E_0 is the Weber function of order zero.

As a check on the computation of the A's, the following relations are useful:

$$(A4.17) \qquad \left\{ \begin{aligned} A_1(\mu) - \frac{A_3(\mu)}{3!} + \frac{A_5(\mu)}{5!} &= J_0(\mu) - \cos(\mu) \\ A_0(\mu) - \frac{A_2(\mu)}{2!} + \frac{A_4(\mu)}{4!} &= \sin(\mu) \end{aligned} \right.$$

Further, since $j_0(0,\theta) = \theta$ and $A_n(0) = 0$, it follows that $B = 0$ and $A = \theta$ in Equation (A4.12). Thus, finally

$$\text{(A4.18)} \quad j_0(\mu,\theta) = \theta J_0(\mu) + i\sin\theta \sum_{n=0}^{\infty} (i)^n A_n(\mu) \frac{\cos^n\theta}{n!}$$

Separating real and imaginary parts,

$$\text{(A4.19)} \quad \begin{cases} j_{0R}(\mu,\theta) = \theta J_0(\mu) - \sin\theta \sum_{n=0}^{\infty} (-)^n A_{2n+1}(\mu) \dfrac{\cos^{2n+1}\theta}{(2n+1)!} \\ j_{0I}(\mu,\theta) = \sin\theta \sum_{n=0}^{\infty} (-)^n A_{2n}(\mu) \dfrac{\cos^{2n}\theta}{(2n)!} \end{cases}$$

Since $j_1(\mu,\theta) = -i \dfrac{d}{d\mu} j_0(\mu,\theta)$

$$\text{(A4.20)} \quad \begin{cases} j_{1R}(\mu,\theta) = \sin\theta \sum_{n=0}^{\infty} (-)^n B_{2n}(\mu) \dfrac{\cos^{2n}\theta}{(2n)!} \\ j_{1I}(\mu,\theta) = \theta J_1(\mu) + \sin\theta \sum_{n=0}^{\infty} (-)^n B_{2n+1}(\mu) \dfrac{\cos^{2n+1}}{(2n+1)!} \end{cases}$$

where

$$\text{(A4.21)} \quad B_n(\mu) = \frac{d}{d\mu} A_n(\mu) = \left[\frac{\mu^n}{n+1} - \frac{\mu^{n+2}}{(n+1)^2(n+3)} + \frac{\mu^{n+4}}{(n+1)^2(n+3)^2(n+5)} + \right]$$

For $n > 1$, $j_n(\mu,\theta)$ may be found from the relation

$$\text{(A4.22)} \quad j_{n+1}(\mu,\theta) = \frac{2in}{\mu} j_n(\mu,\theta) + j_{n-1}(\mu,\theta) - \frac{2i}{\mu} e^{i\mu\cos\theta} \sin(n\theta)$$

However, if $\mu \leq 1$, this formula may be subject to considerable error unless the lower order functions are carried to many more places than is required in the final results. In such cases it may be preferable to employ the following type of expansions obtained by applying Equation (A4.05). Thus for $j_2(\mu, \theta)$

$$\text{(A4.23)} \quad j_2(\mu,\theta) = \theta\left[J_0(\mu) - J_2(\mu)\right] - j_0(\mu,\theta) - 2i\sin\theta \sum_{n=0}^{\infty} (i)^n C_n(\mu) \frac{\cos^n\theta}{n!}$$

with

$$\text{(A4.24)} \quad C_n(\mu) = \frac{d}{d\mu}\left[B_n(\mu)\right] = \left[\frac{n}{n+1}\mu^{n-1} - \frac{n+2}{(n+1)^2(n+3)}\mu^{n+1} + \frac{n+4}{(n+1)^2(n+3)^2(n+5)}\mu^{n+3} + \cdots\right]$$

Similarly,

$$\text{(A4.25)} \quad j_3(\mu,\theta) = i\theta\left[3J_1(\mu) - J_3(\mu)\right] - 3j_1(\mu,\theta) - 4\sin\theta \sum_{n=0}^{\infty} (i)^n D_n(\mu) \frac{\cos^n\theta}{n!}$$

with

$$\text{(A4.26)} \quad D_n(\mu) = \frac{d}{d\mu}\left[C_n(\mu)\right] = \left[\frac{n(n-1)}{(n+1)}\mu^{n-2} - \frac{(n+2)}{(n+1)(n+3)}\mu^{n} + \frac{n+4}{(n+1)^2(n+3)(n+5)}\mu^{n+1} + \cdots\right]$$

TABLE (A4.01)

VALUES OF $J_n(\mu, \cos^{-1}x)$

$\lambda = .7, \quad \mu = \dfrac{\lambda^2 \omega}{1-\lambda^2}$

ω	$n=0, \ x=.5$		$n=1, \ x=.5$	
.0	1.0471976	− 0i	.8660254	− 0i
.05	1.0463437	−.0415911i	.8652759	−.0355433i
.10	1.0437836	−.0831104i	.8630294	−.0710221i
.20	1.0335647	−.1656458i	.8540640	−.1415290i
.30	1.0166208	−.2470356i	.8391967	−.2110092i
.40	.9930688	−.3267178i	.8185397	−.2789596i
.50	.9630815	−.4041429i	.7922477	−.3448882i

ω	$n=2, \ x=.5$		$n=3, \ x=.5$	
.0	.4330127	− 0i	0	− 0i
.05	.4325233	−.0207939i	−.0001923	−.0051931i
.10	.4310562	−.0415425i	−.0008026	−.0133654i
.20	.4252043	−.0827208i	−.0031865	−.0205581i
.30	.4155065	−.1231743i	−.0071306	−.0303630i
.40	.4020469	−.1625476i	−.0125766	−.0396025i
.50	.3849425	−.2004977i	−.0194387	−.0481390i

APPENDIX V

THE FUNCTION $\mathcal{I}_n(\mu, x, e)$

a. Recurrence Relationships -

By definition

(A5.01) $$\mathcal{I}_n(\mu,x,e) = \int_x^1 e^{-i\mu\xi}(\xi - e)^n \Lambda(\xi,e)\, d\xi$$

whence

(A5.02) $$\frac{d}{d\mu}\left[e^{i\mu e}\mathcal{I}_n\right] = -i\int_x^1 e^{-i\mu(\xi-e)}(\xi-e)^{n+1}\Lambda(\xi,e)\,d\xi$$

or

(A5.03) $$\frac{d}{d\mu}\left[e^{i\mu e}\mathcal{I}_n\right] = -i\, e^{i\mu e}\mathcal{I}_{n+1}$$

b. Value of $\mathcal{I}_o(\mu, x, e)$

Setting $x = \cos\theta$, $e = \cos\epsilon$, $\xi = \cos\varphi$, in (A5.01), we have, for $n = 0$,

(A5.04) $$e^{i\mu\cos\epsilon}\mathcal{I}_o = \int_o^\theta e^{i\mu(\cos\epsilon - \cos\varphi)}\Lambda(\varphi,\epsilon)\sin\varphi\, d\varphi$$

Since (Ref. 1),

$$\frac{d}{d\varphi}\Lambda(\varphi,\epsilon) = \frac{\sin\epsilon}{\cos\varphi - \cos\epsilon},$$

integration of (A5.03) by parts gives

$$e^{i\mu\cos\epsilon} I_o = \frac{\Lambda(\theta,\epsilon)\left[e^{i\mu(\cos\epsilon - \cos\theta)} - 1\right]}{i\mu} + \frac{\sin\epsilon}{i\mu}\int_0^\theta \frac{1 - e^{-i\mu(\cos\varphi - \cos\epsilon)}}{\cos\varphi - \cos\epsilon} d\varphi \tag{A5.05}$$

Introducing the notation

$$F(\mu,\theta,\epsilon) = \int_0^\theta \frac{1 - e^{-i\mu(\cos\varphi - \cos\epsilon)}}{\cos\varphi - \cos\epsilon} d\varphi \tag{A5.06}$$

gives

$$i\mu e^{i\mu\cos\epsilon} I_o(\mu,\chi,e) = \Lambda(\theta,\epsilon)\left[e^{i\mu(\cos\epsilon - \cos\theta)} - 1\right] + \sin\epsilon\, F(\mu,\theta,\epsilon) \tag{A5.07}$$

The function $F(\mu,\theta,\epsilon)$ is discussed further in Section f. of this Appendix. For the present it is sufficient to note that

$$\frac{\partial F}{\partial \mu} = i\int_0^\theta e^{-i\mu(\cos\varphi - \cos\epsilon)} d\varphi = i e^{i\mu\cos\epsilon} J_o(-\mu,\theta) \tag{A5.08}$$

Thus since $F(0,\theta,\epsilon) = 0$,

$$F(\mu,\theta,\epsilon) = i\int_0^\mu e^{it\cos\epsilon} J_o(-t,\theta)\, dt \tag{A5.09}$$

c. Value of $I_1(\mu,\chi,e)$

By (A5.03),

$$\frac{d}{d\mu}\left[i\mu e^{i\mu\cos\epsilon} I_o\right] = \mu e^{i\mu\cos\epsilon} I_1 + i e^{i\mu\cos\epsilon} I_o \tag{A5.10}$$

But also from (A5.07) and (A5.08)

$$\text{(A5.11)} \quad \frac{d}{d\mu}\left[i\mu e^{i\mu\cos\epsilon} I_0\right] = i\Lambda(\theta,\epsilon)\, e^{i\mu(\cos\epsilon-\cos\theta)}(\cos\epsilon-\cos\theta) + i\sin\epsilon\, e^{i\mu\cos\epsilon}\dot{J}_0(-\mu,\theta).$$

Thus

$$\text{(A5.12)} \quad i\mu e^{i\mu\cos\epsilon} I_1 = e^{i\mu\cos\epsilon} I_0 - (\cos\epsilon-\cos\theta)\Lambda(\theta,\epsilon)\, e^{i\mu(\cos\epsilon-\cos\theta)}$$
$$- \sin\epsilon\, e^{i\mu\cos\epsilon}\dot{J}_0(-\mu,\theta)$$

or

$$\text{(A5.13)} \quad i\mu I_1(\mu,x,e) = I_0(\mu,x,e) + (x-e)\Lambda(\cos^{-1}x, \cos^{-1}e)\, e^{i\mu x} - \sqrt{1-e^2}\,\dot{J}_0(-\mu,\cos^{-1}x)$$

d. Value of $I_2(\mu,x,e)$

Differentiation of (A5.12) gives

$$\text{(A5.14)} \quad \frac{d}{d\mu}\left[i\mu e^{i\mu\cos\epsilon} I_1\right] = \frac{d}{d\mu}\left[e^{i\mu\cos\epsilon} I_0\right] - i(\cos\epsilon-\cos\theta)^2\Lambda(\theta,\epsilon)\, e^{i\mu(\cos\epsilon-\cos\theta)}$$
$$- i\sin\epsilon\cos\epsilon\, e^{i\mu\cos\epsilon}\dot{J}_0(-\mu,\theta) + i\sin\epsilon\, e^{i\mu\cos\epsilon}\dot{J}_1(-\mu,\theta)$$

and, applying (A5.03)

$$\text{(A5.15)} \quad i e^{i\mu\cos\epsilon} I_1 + i\mu\left[-i e^{i\mu\cos\epsilon} I_2\right] = -i e^{i\mu\cos\epsilon} I_1 + i\sin\epsilon\, e^{i\mu\cos\epsilon}\dot{J}_1(-\mu,\theta)$$
$$- i(\cos\epsilon-\cos\theta)^2\Lambda(\theta,\epsilon)\, e^{i\mu(\cos\epsilon-\cos\theta)} - i\sin\epsilon\cos\epsilon\, e^{i\mu\cos\epsilon}\dot{J}_0(-\mu,\theta)$$

Whence

(A5.16) $$i\mu e^{i\mu\cos\epsilon} I_2 = 2e^{i\mu\cos\epsilon} I_1 + (\cos\epsilon - \cos\theta)^2 \Lambda(\theta,\epsilon) e^{i\mu(\cos\epsilon - \cos\theta)}$$

$$+ \sin\epsilon\cos\epsilon\, e^{i\mu\cos\epsilon} \dot{J}_0(-\mu,\theta) - \sin\epsilon\, e^{i\mu\cos\epsilon} \dot{J}_1(-\mu,\theta),$$

or

(A5.17) $$i\mu I_2(\mu,x,e) = 2I_1(\mu,x,e) + (x-e)^2 \Lambda(\cos^{-1}x, \cos^{-1}e)\, e^{-i\mu x}$$

$$+ e\sqrt{1-e^2}\, \dot{J}_0(-\mu, \cos^{-1}x) - \sqrt{1-e^2}\, \dot{J}_1(-\mu, \cos^{-1}x).$$

e. Special Values -

For x = -1, $\theta = \pi$, $\Lambda(\theta,\epsilon) = 0$, and the following expressions result:

(A5.18) $$\begin{cases} i\mu I_0(\mu,-1,e) = e^{-i\mu e}\sqrt{1-e^2}\, F(\mu,\pi,\cos^{-1}e) \\ i\mu I_1(\mu,-1,e) = I_0(\mu,-1,e) - \pi\sqrt{1-e^2}\, J_0(\mu) \\ i\mu I_2(\mu,-1,e) = 2I_1(\mu,-1,e) + \pi e\sqrt{1-e^2}\, J_0(\mu) + i\pi\sqrt{1-e^2}\, J_1(\mu) \end{cases}$$

and for $\theta = e$, since $\lim_{\theta\to\epsilon} (\cos\theta - \cos\epsilon)\Lambda(\theta,\epsilon) = 0$

(A5.19) $$\begin{cases} i\mu I_0(\mu,e,e) = e^{-i\mu e}\sqrt{1-e^2}\, F(\mu,\cos^{-1}e,\cos^{-1}e) \\ i\mu I_1(\mu,e,e) = I_0(\mu,e,e) - \sqrt{1-e^2}\, \dot{J}_0(-\mu,\cos^{-1}e) \\ i\mu I_2(\mu,e,e) = 2I_1(\mu,e,e) + e\sqrt{1-e^2}\, \dot{J}_0(-\mu,\cos^{-1}e) \\ \qquad - \sqrt{1-e^2}\, \dot{J}_1(-\mu,\cos^{-1}e) \end{cases}$$

The following limiting values may also be deduced:

$$\text{(A5.20)}\quad \begin{cases} \lim\limits_{\mu\to 0} I_0(\mu,x,e) = \sqrt{1-e^2}\cos^{-1}x - (x-e)\Lambda(\cos^{-1}x, \cos^{-1}e) \\ \lim\limits_{\mu\to 0} I_1(\mu,x,e) = -\frac{1}{2}\left\{(x-e)^2\Lambda(\cos^{-1}x,\cos^{-1}e) + \sqrt{1-e^2}\left[e\cos^{-1}x - \sqrt{1-x^2}\right]\right\} \\ \lim\limits_{\mu\to 0} I_2(\mu,x,e) = \frac{1}{3}\left\{-(x-e)^3\Lambda(\cos^{-1}x,\cos^{-1}e) \right. \\ \qquad \left. + \sqrt{1-e^2}\left[\left(\frac{1}{2}+e^2\right)\cos^{-1}x - \sqrt{1-x^2}\left(2e-\frac{1}{2}x\right)\right]\right\} \end{cases}$$

f. Properties of the function $F(\mu,\theta,\epsilon)$

It has already been noted (Equation A5.09) that

$$F(\mu,\theta,\epsilon) = i\int_0^{\mu} e^{it\cos\epsilon} J_0(-t,\theta)\,dt$$

or

$$\text{(A5.21)}\qquad F(-\mu,\theta,\epsilon) = -i\int_0^{\mu} e^{-it\cos\epsilon} J_0(t,\theta)\,dt$$

Now from Equation (A3.09)

$$t\frac{d^2J_0}{dt^2} + \frac{dJ_0}{dt} + tJ_0 = i\sin\theta\, e^{it\cos\theta}$$

Multiplying both sides by $e^{-it\cos\epsilon}$ and integrating,

$$\text{(A5.22)}\quad \int_0^{\mu} e^{-it\cos\epsilon}\frac{d^2J_0}{dt^2}t\,dt + \int_0^{\mu} e^{-it\cos\epsilon}\frac{dJ_0}{dt}dt$$

$$+\int_0^{\mu} e^{-it\cos\epsilon} J_0\, t\,dt = i\sin\theta\int_0^{\mu} e^{it(\cos\theta-\cos\epsilon)}dt$$

The left side after several integrations by parts may be reduced to:

$$-i\cos\epsilon\int_0^{\mu} e^{-it\cos\epsilon}\dot{J}_0\,dt+\sin^2\epsilon\int_0^{\mu} e^{-it\cos\epsilon}\dot{J}_1\,t\,dt+i\mu\, e^{-i\mu\cos\epsilon}\left(\dot{J}_1+\cos\epsilon\,\dot{J}_0\right)$$

and the right side becomes

$$\sin\theta\left[\frac{e^{i\mu(\cos\theta-\cos\epsilon)}-1}{\cos\theta-\cos\epsilon}\right]$$

But

$$\frac{d}{d\epsilon}F^1(-\mu,\theta,\epsilon)=\sin\epsilon\int_0^{\mu} e^{-it\cos\epsilon}\dot{J}_0\,t\,dt \tag{A5.23}$$

Thus

$$\cos\epsilon\, F^1(-\mu,\theta,\epsilon)+\sin\epsilon\frac{d}{d\epsilon}F^1(-\mu,\theta,\epsilon)=$$
$$-i\mu\, e^{-i\mu\cos\epsilon}\left[\dot{J}_1+\cos\epsilon\,\dot{J}_0\right]+\sin\theta\left[\frac{e^{i\mu(\cos\theta-\cos\epsilon)}-1}{\cos\theta-\cos\epsilon}\right] \tag{A5.24}$$

or

$$\frac{d}{d\epsilon}\left[\sin\epsilon\; F^1(-\mu,\theta,\epsilon)\right]=\sin\theta\left[\frac{e^{i\mu(\cos\theta-\cos\epsilon)}-1}{\cos\theta-\cos\epsilon}\right]$$
$$-i\mu\, e^{-i\mu\cos\epsilon}\left[\dot{J}_1(\mu,\theta)+\cos\epsilon\,\dot{J}_0(\mu,\theta)\right] \tag{A5.25}$$

Integrating,

$$\sin\epsilon\; F^1(-\mu,\theta,\epsilon)=\sin\theta\int_0^{\epsilon}\frac{e^{i\mu(\cos\theta-\cos\varphi)}-1}{\cos\theta-\cos\varphi}\,d\varphi$$
$$-i\mu\left[\dot{J}_0(-\mu,\epsilon)\dot{J}_1(\mu,\theta)+\dot{J}_1(-\mu,\epsilon)\dot{J}_0(\mu,\theta)\right] \tag{A5.26}$$

$$\sin\theta\, F'(\mu,\epsilon,\theta) - \sin\epsilon\, F'(-\mu,\theta,\epsilon)$$

(A5.27)

$$= -i\mu\left[J_0(\mu,\theta)\, J_1(-\mu,\epsilon) + J_0(-\mu,\epsilon)\, J_1(\mu,\theta)\right]$$

If $\theta = \pi$, this gives

(A5.28) $$-\sin\epsilon\, F'(-\mu,\pi,\epsilon) = i\pi\mu\left[J_0(\mu)\, J_1(-\mu,\epsilon) + i J_1(\mu)\, J_0(-\mu,\epsilon)\right]$$

or

(A5.29) $$\sin\epsilon\, F'(\mu,\pi,\epsilon) = i\pi\mu\left[J_0(\mu)\, J_1(\mu,\epsilon) - i J_1(\mu)\, J_0(\mu,\epsilon)\right]$$

and if $\theta = \epsilon$

(A5.30) $$\sin\epsilon\left[F'(\mu,\epsilon,\epsilon) - F'(-\mu,\epsilon,\epsilon)\right] = -i\mu\left[J_0(\mu,\epsilon)\, J_1(-\mu,\epsilon) + J_0(-\mu,\epsilon)\, J_1(\mu,\epsilon)\right]$$

Since $F'(\mu,\epsilon,\epsilon)$ and $F'(-\mu,\epsilon,\epsilon)$ are complex conjugates, the above relation gives only the imaginary part of $F'(\mu,\epsilon,\epsilon)$. Equations (A5.27) and (A5.30), while of little value for direct computation, are useful as numerical checks on results obtained from other methods.

g. Expansion of $F'(\mu,\theta,\epsilon)$ as a power series in μ.

By definition,

$$F'(\mu,\theta,\epsilon) = \int_0^{\theta} \frac{1 - e^{-i\mu(\cos\phi - \cos\epsilon)}}{\cos\phi - \cos\epsilon}\, d\phi$$

Expanding the numerator,

(A5.31) $$F'(\mu,\theta,\epsilon) = \int_0^{\theta} \frac{\left\{1 - \sum_0^{\infty} \frac{(-i\mu)^k}{k!}(\cos\phi - \cos\epsilon)^k\right\}}{\cos\phi - \cos\epsilon}\, d\phi$$

$$= -\sum_0^{\infty} \frac{(-i\mu)^k}{k!} \int_0^{\theta} (\cos\phi - \cos\epsilon)^{k-1}\, d\phi .$$

Then with the notation

(A5.32) $$U_k(\theta,\epsilon) = \int_0^{\theta} (\cos\varphi - \cos\epsilon)^k d\varphi \quad , k \geq 0 ,$$

(A5.33) $$F^1(\mu,\theta,\epsilon) = -\sum_1^{\infty} \frac{(-i\mu)^k}{k!} U_{k-1}(\theta,\epsilon)$$

The U_k may be computed successively from the following recurrence relationship:

(A5.34) $$U_k(\theta,\epsilon) = \frac{\sin\theta}{k}\left[\cos\theta - \cos\epsilon\right]^{k-1} + \left(\frac{k-1}{k}\right)\sin^2\epsilon \, U_{k-2}(\theta,\epsilon) - \left(\frac{2k-1}{k}\right)\cos\epsilon \, U_{k-1}(\theta,\epsilon), \qquad k \geq 2$$

starting with the initial values

(A5.35) $$U_0(\theta,\epsilon) = \theta \quad , \quad U_1(\theta,\epsilon) = \sin\theta - \theta\cos\epsilon$$

In particular,

(A5.36) $$U_k(\epsilon,\epsilon) = \left(\frac{k-1}{k}\right)\sin^2\epsilon \, U_{k-2}(\epsilon,\epsilon) - \left(\frac{2k-1}{k}\right)\cos\epsilon \, U_{k-1}(\epsilon,\epsilon)$$

with $U_0(\epsilon,\epsilon) = \epsilon$, $U_1(\epsilon,\epsilon) = \sin\epsilon - \epsilon\cos\epsilon$

Since $U_k(\pi/2,\pi/2) = \int_0^{\pi/2} \cos^k\varphi \, d\varphi$ and since $\sum_0^{\infty} \cos^k\varphi$ converges if $0<\varphi<\pi/2$, so also does $\sum U_k(\pi/2,\pi/2)$, or, $\lim_{k\to\infty} U_k(\pi/2,\pi/2) = 0$ Also, if $\epsilon < \pi/2$, $U_k(\epsilon,\epsilon) \leq U_k(\pi/2,\pi/2)$
Thus the series for $F^1(\mu,\epsilon,\epsilon)$ converges more rapidly if $\epsilon < \pi/2$ since for $\epsilon > \pi/2$

$$U_k(\epsilon,\epsilon) \leq U_k(\pi,\pi) = 2^k \int_0^\pi \cos^{2k}\frac{\phi}{2}\, d\rho$$

and although the series $\sum \frac{(-i\mu)^k}{k!} U_{k-1}(\epsilon,\epsilon)$ still converges, the convergence is poorer if $\pi/2 < \epsilon < \pi$. In this case the following relation is useful:

$$\text{(A5.37)} \quad \Gamma(\mu,\pi-\epsilon,\pi-\epsilon) = \frac{i\pi\mu}{\sin\epsilon}\left[J_0(\mu)\int_1(-\mu,\epsilon) + i\,J_1(\mu)\int_0(-\mu,\epsilon)\right] + \Gamma(-\mu,\epsilon,\epsilon)$$

TABLE (A5.01) - VALUES OF $\frac{1}{4\pi} F(\mu, x, e)$; $e = .5$

ω	$x = e$		$x = -1$	
0	1.0471976	-0	3.1415927	+0i
.05	1.0471453	-.0082247i	3.1406858	+.0377172i
.10	1.0469886	-.0164478i	3.1379702	+.0753583i
.20	1.0463617	-.0328827i	3.1271345	+.1501086i
.30	1.0453176	-.0492919i	3.1091812	+.2236506i
.40	1.0438574	-.0656626i	3.0842665	+.2953961i

TABLE (A5.02) - VALUES OF $I_0(\mu, x, e)$; $e = .5$

ω	$x = e$		$x = -1$	
0	.906899	- 0i	2.7206991	-.0i
.05	.9064218	-.0289009i	2.7199140	-.0326703i
.10	.9049887	-.0577691i	2.7175606	-.0653124i
.20	.8992647	-.1152759i	2.7081584	-.1313987i
.30	.8897558	-.1722599i	2.6925324	-.1950342i
.40	.8765085	-.2284629i	2.6707474	-.2589964i
.50	.8595872	-.2836311i	2.6428903	-.3220648i

TABLE (A5.03) - VALUES OF $I_1(\mu, x, e)$; $e = .5$

ω	$x = e$		$x = -1$	
0	.148275	- 0i	-.6801747	- 0i
.05	.1481707	-.0054456i	-.6800759	-.0163325i
.10	.1478647	-.0108828i	-.6797823	-.0326494i
.20	.1466385	-.0216995i	-.6786053	-.0651369i
.30	.1446013	-.0323916i	-.6766493	-.1600992i
.40	.1417670	-.0428965i	-.6739193	-.1290219i
.50	.1381515	-.0531528i	-.6704206	-.1600992i

TABLE (A5.04) - VALUES OF $I_2(\mu, x, e)$; $e = .5$

ω	$x = e$		$x = -1$	
0	.0392250	- 0i	.6801747	+ 0i
.05	.0391520	-.0014680i	.6799926	+.0122171i
.10	.0390673	-.0030755i	.6791402	+.0244863i
.20	.0387291	-.0061588i	.6761256	+.0488642i
.30	.0381164	-.0091821i	.6710323	+.0730556i
.40	.0372615	-.0121494i	.6639488	+.0969448i
.50	.0361732	-.0150354i	.6549472	+.1204249i

APPENDIX VI

A DERIVATION OF THE INVERSION FORMULA FOR THE EQUATION $f(x) = \frac{1}{\pi}\int_{-1}^{1}\frac{\Pi(t)\,dt}{x-t}$

A. Special integrals -

In the following demonstration it is necessary to recall the following:

(A6.01) $$\oint_0^{\pi}\frac{d\varphi}{\cos\varphi - \cos\theta} = 0$$

where the symbol $\oint$ denotes the Cauchy principal value, i.e.

(A6.02) $$\oint_0^{\pi}\frac{d\varphi}{\cos\varphi-\cos\theta} = \lim_{\epsilon\to 0}\left\{\int_0^{\theta-\epsilon}\frac{d\varphi}{\cos\varphi-\cos\theta} + \int_{\theta+\epsilon}^{\pi}\frac{d\varphi}{\cos\varphi-\cos\theta}\right\}$$

From this result it is easy to obtain by induction the following:

(A6.03A) $$\oint_0^{\pi}\frac{\cos(n\varphi)\,d\varphi}{\cos\varphi-\cos\theta} = \pi\,\frac{\sin(n\theta)}{\sin\theta}$$

(A6.03B) $$\oint_0^{\pi}\frac{\sin(n\varphi)\sin\varphi\,d\varphi}{\cos\varphi-\cos\theta} = \pi\cos(n\theta)$$

B. Solution of the equation

The equation to be solved is

(A6.04) $$f(x) = \frac{1}{\pi}\int_{-1}^{1}\frac{\Pi(t)\,dt}{x-t}$$

Making the substitution $x = \cos\theta,\ t = \cos\varphi$ in (A6.04)

(A6.05) $$f(\cos\theta) = \frac{1}{\pi}\int_0^{\pi}\frac{\Pi(\cos\varphi)\sin\varphi\,d\varphi}{\cos\varphi-\cos\theta}$$

Now assume a solution of (A6.05) of the form

$$\Pi(\cos\varphi)\sin\varphi = \sum_0^\infty a_n \cos(n\varphi). \tag{A6.06}$$

Then with the aid of equation (A6.03A),

$$\sin\theta f(\cos\theta) = \sum_1^\infty a_n \sin(n\theta) \tag{A6.07}$$

Multiplying both sides by $\left[\frac{\sin\theta}{\cos\theta - \cos\varphi}\right]$ and integrating with respect to θ,

$$\int_0^\pi \frac{f(\cos\theta)\sin^2\theta\, d\theta}{\cos\theta - \cos\varphi} = \sum_1^\infty a_n \int_0^\pi \frac{\sin(n\theta)\sin\theta\, d\theta}{\cos\theta - \cos\varphi} \tag{A6.08}$$

Or, from (A6.03B)

$$\begin{aligned}\int_0^\pi \frac{f(\cos\theta)\sin^2\theta\, d\theta}{\cos\theta - \cos\varphi} &= \pi \sum_1^\infty a_n \cos(n\varphi) \\ &= \pi\left[\Pi(\cos\varphi)\sin\varphi - a_0\right]\end{aligned} \tag{A6.09}$$

Returning to the original variables x and ξ,

$$\int_{-1}^{1} \frac{f(x)\sqrt{1-x^2}\, dx}{x-\xi} = \pi\left[\Pi(\xi)\sqrt{1-\xi^2} - a_0\right] \tag{A6.10}$$

or, interchanging x and ξ:

$$\sqrt{1-x^2}\,\Pi(x) = a_0 - \frac{1}{\pi}\int_{-1}^{1} \frac{f(\xi)\sqrt{1-\xi^2}\, d\xi}{x-\xi} \tag{A6.11}$$

It is noted that (A6.11) still contains the undetermined constant a_o, the presence of which may be attributed to equation (A6.01) which renders this quantity arbitrary. Thus a_o is analogous to a constant of integration in a differential equation and additional conditions must be imposed to determine it.

In thin airfoil theory where equation (A6.04) relates the downwash $f(x)$ to the pressure distribution $\Pi(\xi)$, the Kutta condition requires that a finite pressure exist at the trailing edge, x = 1. It is clear from inspection of (A6.11) that the only value of a_o for which this is possible is that one given by

$$a_o = \frac{1}{\pi}\int_{-1}^{1} \frac{f(\xi)\sqrt{1-\xi^2}}{1-\xi}\, d\xi \tag{A6.12}$$

Replacing a_o in equation (A6.11) by the above value and re-arranging gives

$$\Pi(x) = \frac{1}{\pi}\sqrt{\frac{1-x}{1+x}}\int_{-1}^{1} \frac{f(\xi)}{x-\xi}\sqrt{\frac{1+\xi}{1-\xi}}\, d\xi \tag{A6.13}$$

APPENDIX VII

THE POSSIO KERNEL AND ITS APPROXIMATE REPRESENTATION AS A POLYNOMIAL

1. Küssner (Ref. 4) gives for the kernel of the Possio integral equation the following expression:

$$K(\lambda, z) = \frac{i\lambda e^{-iz}}{4\sqrt{1-\lambda^2}} \int_{-\infty}^{z/1-\lambda^2} e^{iu} H_1^{(2)}[\lambda |u|] \frac{du}{u} \tag{A7.01}$$

where $H_1^{(2)}(x) = J_1(x) - iN_1(x)$, and J_1 and N_1 are the Bessel and Neumann functions of order unity in conventional notation.

The integral is only properly convergent when $z<0$, in which case

$$K(\lambda, -z) = \frac{-i\lambda e^{iz}}{4\sqrt{1-\lambda^2}} \int_{z/1-\lambda^2}^{\infty} e^{-iu} H_1^{(2)}[\lambda u] \frac{du}{u} \tag{A7.02}$$

In the case where $z>0$ it is necessary to form the Cauchy principal value under the integral sign in equation (1.01), whence it is found that

$$K(\lambda, z) = \frac{i\lambda e^{-iz}}{4\sqrt{1-\lambda^2}} \int_{z/1-\lambda^2}^{\infty} e^{iu} H_1^{(2)}[\lambda u] \frac{du}{u} \tag{A7.03}$$

By making use of the following identities which are easily proved by differentiation:

$$\text{(A7.04)} \quad \left\{ \begin{aligned} \int e^{iu} H_1^{(2)}(\lambda u) \frac{du}{u} &= \frac{\lambda^2-1}{\lambda} \int e^{iu} H_0^{(2)}(\lambda u)\, du - \frac{i e^{iu}}{\lambda}\left[H_0^{(2)}(\lambda u) - i\lambda H_1^{(2)}(\lambda u)\right] \\ \int e^{-iu} H_1^{(2)}(\lambda u) \frac{du}{u} &\doteq \frac{\lambda^2-1}{\lambda} \int e^{-iu} H_0^{(2)}(\lambda u)\, du + \frac{i e^{-iu}}{\lambda}\left[H_0^{(2)}(\lambda u) + i\lambda H_1^{(2)}(\lambda u)\right] \end{aligned} \right.$$

the following expanded forms of the Possio kernel are obtained:

$$
\text{(A7.05)}\quad
\begin{cases}
K(\lambda,z)=\dfrac{1}{4\sqrt{1-\lambda^2}}\left\{e^{i\lambda w}\left[-H_0^{(2)}(w)+i\lambda H_1^{(2)}(w)\right]-i(1-\lambda^2)e^{-iz}\displaystyle\int_{z/1-\lambda^2}^{\infty}e^{iu}H_0^{(2)}(\lambda u)\,du\right\}\\[2ex]
K(\lambda,-z)=\dfrac{1}{4\sqrt{1-\lambda^2}}\left\{e^{-i\lambda w}\left[-H_0^{(2)}(w)-i\lambda H_1^{(2)}(w)\right]+i(1-\lambda^2)e^{iz}\displaystyle\int_{z/1-\lambda^2}^{\infty}e^{-iu}H_0^{(2)}(\lambda u)\,du\right\}
\end{cases}
$$

where $z>0$, and $w=\dfrac{\lambda z}{1-\lambda^2}$. Since

$$
\text{(A7.06)}\quad \int_0^{\infty}e^{\pm iu}H_0(\lambda u)\,du=\frac{2}{\pi\sqrt{1-\lambda^2}}\operatorname{Log}\frac{1+\sqrt{1-\lambda^2}}{\lambda},
$$

the last integral in each expression may be written as

$$
\text{(A7.07)}\quad \frac{2}{\pi\sqrt{1-\lambda^2}}\operatorname{Log}\frac{1+\sqrt{1-\lambda^2}}{\lambda}-\int_0^{z/1-\lambda^2}e^{\pm iu}H_0^{(2)}(\lambda u)\,du,
$$

the latter form being more suitable for numerical computation. For $\lambda = 0$ it may be shown that

$$
\text{(A7.08)}\quad
\begin{cases}
z\pi K(0,z)=-\dfrac{1}{z}+i\,e^{-iz}\left[C+\operatorname{Log}z+i\pi/2-\displaystyle\int_0^{z}\frac{1-e^{iu}}{u}\,du\right]\\[2ex]
z\pi K(0,-z)=\dfrac{1}{z}+i\,e^{iz}\left[C+\operatorname{Log}z+i\dfrac{\pi}{2}-\displaystyle\int_0^{z}\frac{1-e^{-iu}}{u}\,du\right],
\end{cases}
$$

where C = .577216 , known as Euler's constant, is defined as

$$
\text{(A7.09)}\quad \lim_{n\to\infty}\left[1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\operatorname{Log}n\right]
$$

2. The singularities of $K(\lambda,z)$

It is well known that as $x\to 0$ $N_0(x)$ becomes infinite as log $|x|$ and $N_1(x)$ as $1/x$. Thus in the vicinity of x = 0,

$$H_0^{(2)}(x) = -\frac{2i}{\pi} \operatorname{Log}|x| + \text{non-singular terms} \; ;$$

(A7.10)

$$H_1^{(2)}(x) = \frac{2i}{\pi}\left[1/x\right] + \text{non-singular terms} \, .$$

Further, it is clear from the expanded form that these are the only singular terms. Affixing the proper coefficients to these terms as determined by equation (A7.05) gives

$$\text{(A7.11)} \quad K(\lambda, z) = \frac{i}{2\pi\sqrt{1-\lambda^2}} \operatorname{Log}|z| - \frac{\sqrt{1-\lambda^2}}{2\pi}\left[1/z\right] + K_1(\lambda, z)$$

where $K_1(\lambda z)$ has no singularities. In Ref. 16 Schwartz has tabulated the values of $K_1(\lambda, z)$ for $\lambda = 0 - .9$ in intervals of .1 and for $z = -2(1-\lambda^2)$ to $z = 2(1-\lambda^2)$ in intervals of (.02). Later in Ref. 17 the range of $|z|$ for $\lambda = .7$ was extended to 5.1; for λ = .8 and .9 to 2.00.

The singularities of $K(0, z)$ are, from (A7.11) given by

$$\text{(A7.12)} \quad K(0, z) = \frac{i}{2\pi} \operatorname{Log}|z| - \frac{1}{2\pi}\left[1/z\right] + K_1(0, z)$$

It follows that the difference

$$\text{(A7.13)} \quad \overline{K}(\lambda, z) = K(\lambda, z) - \frac{1}{\sqrt{1-\lambda^2}} K(0, z) - \frac{\lambda^2}{2\pi\sqrt{1-\lambda^2}}\left[1/z\right] \equiv K_1(\lambda, z) - \frac{1}{\sqrt{1-\lambda^2}} K_1(0, z)$$

is also non-singular. The real and imaginary parts of $\overline{K}$ are plotted in figure (A7.01) as functions of z for $\lambda = .7$.

2. Representation of $\overline{K}(\lambda, z)$ as a polynomial.

Since $\overline{K}$ has singular derivatives at $z = 0$, it is not possible to represent it by a Taylor series expansion about this point. Further, even if such an expansion were possible, the retention of a finite number of terms would give a good approximation only for small values of z, and more terms would be needed, the larger the z range became. The range over which the value of z extends is evidently -2ω to 2ω, since $z = \omega(x - \xi)$ and x and ξ each have the range -1 to 1. It, therefore, appears that a more systematic scheme would be to obtain for $\overline{K}$ a representation such that over any arbitrarily chosen interval, the difference between the actual value and the approximate value were made as small as possible. One means of accomplishing this is to represent $\bar{K}$ over (say) the interval $[-a$ to $a]$ by Legendre polynomials:

(A7.14) $$\bar{K}(z) = A_0 P_0(z/a) + A_1 P_1(z/a) + \cdots\cdots + A_n P_n(z/a)$$

The first few Legendre polynomials are:

(A7.15) $$P_0(x) = 1 \qquad P_2(x) = \tfrac{1}{2}(3x^2-1)$$
$$P_1(x) = x \qquad P_3(x) = \tfrac{1}{2}(5x^3-3x)\ .$$

The general expression for $P_n(x)$ may be found in numerous sources (for example, Refs. 14 and 15). In Ref. 15 it is shown that the coefficients A_n may be determined from

(A7.16) $$A_n = \frac{2n+1}{2a}\int_{-a}^{a} \bar{K}(z) P_n(z/a)\, dz$$

It can also be shown that with this representation, the mean square error, viz.

(A7.17) $$\epsilon(z) = \int_{-a}^{a} \left\{\bar{K} - \sum_{n=0}^{n} P_n(z/a)\right\}^2 dz$$

is made a minimum. It is further noted that since $a=2\omega$ equation (A7.16) defines the A_n as continuous functions of the reduced frequency, ω.

3. Evaluation of A_n

Since $P_n(z/a)$ is a polynomial of degree n in z, the evaluation of the A_n is reduced to the evaluation of integrals of the form

(A7.18) $$\int_{-a}^{a} K(0,z) z^p\, dz$$
$$\int_{-a}^{a} K(\lambda,z) z^p\, dz \qquad , \ p = 0, 1, 2, \cdots n$$

For $p = 0$ the value of the integral is defined as the Cauchy principal value, i. e.

(A7.19) $$\oint_{-a}^{a} K(\lambda,z)\,dz = \int_{-a}^{a}\left\{K(\lambda,z)-\frac{\sqrt{1-\lambda^2}}{2\pi}\left[1/z\right]\right\}dz$$

since $\oint_{-a}^{a}\frac{dz}{z} = 0$

The evaluation of the required integrals is easy for the case when $\lambda = 0$, since by differentiation of equation (A7.08),

(A7.20) $$i\frac{d}{dz}\left[K_1(0,z)\right] = K(0,z)+\frac{1}{2\pi}\left[1/z\right]$$

$$= K_1(0,z)+ i/2\pi\,\mathrm{Log}\,|z|$$

and so

(A7.21) $$\int_{-a}^{a}K(0,z)z^n dz = \int_{-a}^{a}\left\{K(0,z)+\frac{1}{2\pi}\left[1/z\right]\right\}dz - i/2\pi\int_{-a}^{a}\mathrm{Log}\,|z|\,z^n dz$$

$$= i\int_{-a}^{a}\frac{d}{dz}\left[K_1(0,z)\right]z^n dz - \frac{i}{2\pi}\int_{0}^{a}\left[(z)^n+(-z)^n\right]\mathrm{Log}\,z\,dz$$

(A7.22) $$\int_{-a}^{a}K_1(0,z)z^n dz = i\left[a^n K_1(0,a)-(-a)^n K_1(0,-a)\right]-in\int_{-a}^{a}K_1(0,z)z^{n-1}dz$$

$$-\frac{i}{2\pi}\,\frac{a^{n+1}-(-a)^{n+1}}{n+1}\left[\mathrm{Log}\,a-\frac{1}{n+1}\right]$$

For $\lambda \neq 0$, the expressions become more cumbersome. It is found after some calculation that

$$\int_{-a}^{a}K_1(\lambda,z)dz = i\left[K(\lambda,a)-K(\lambda,-a)\right]$$

(A7.23) $$+\frac{\lambda}{2\sqrt{1-\lambda^2}}\left\{\cos\frac{\lambda^2 a}{1-\lambda^2}H_1^{(2)}\left[\frac{\lambda a}{1-\lambda^2}\right]-\lambda\sin\frac{\lambda^2 a}{1-\lambda^2}H_0^{(2)}\left[\frac{\lambda a}{1-\lambda^2}\right]\right.$$

$$\left.-(1-\lambda^2)\int_0^{\frac{\lambda a}{1-\lambda^2}}H_0^{(2)}(u)\cos\lambda u\,du\right\}$$

For n 0 the integrals may be found by a recurrence relationship similar to (A7.22). The last integral constitutes the only unknown functions, but it can be shown by a process similar to that used in Appendix V, Section f that

$$(1-\lambda^2)\int_0^{\frac{\lambda a}{1-\lambda^2}} H_0^{(2)}(u)e^{i\lambda u}\,du = \frac{\lambda a}{\sqrt{1-\lambda^2}}\Big\{H_0^{(2)}\Big[\frac{\lambda a}{1-\lambda^2}\Big]J_1\Big(\frac{-\lambda a}{1-\lambda^2}, \cos^{-1}\lambda\Big)$$

A7.24

$$-iH_1^{(2)}\Big[\frac{\lambda a}{1-\lambda^2}\Big]J_0\Big(\frac{-\lambda a}{1-\lambda^2}, \cos^{-1}\lambda\Big)\Big\} + \frac{2}{\pi}\sqrt{1-\lambda^2}\cos^{-1}\lambda .$$

In view of the complexity of the expressions involved in the integration of $K(\lambda, z)$, the following alternative method was employed to obtain the results in the example of Part II of the report: In Reference 3, Dietze gives the following approximate expression for

A7.25 $$K_1(\lambda, z) = K_1(0, z) + R_{11} + z\big[R_{13} + R_{14}\,L_n|z|\big] + \sum_{n=2}^{9} R_{2n} z^n + \delta(\lambda, z)$$

where $\delta(\lambda, z)$ is a small regular remainder, the absolute value of which is never greater than (.2) in the range $-2 \leq z \leq 2$. For $-1 \leq z \leq 1$, the absolute value of δ is less than (.007). R_{11}, R_{13} and R_{14} are explicit functions of λ which are listed by Dietze on Page 26 of the reference above and are reproduced in Table (7.01). The numerical values of $R_{22} \cdots R_{29}$ are also given here for λ = .3, .4, .5, .6, .7.

Since δ is small, the error introduced by employing an approximate quadrature formula such as Simpson's rule will be negligible, while the remainder of the terms involve only simple functions which can at once be integrated. Using this method the value of

$$\int_{-a}^{a} \overline{K}(\lambda, z) z^p\,dz, \; (p = 0, 1)$$

was calculated for λ = .7, and a = .1, .2, .3, .4, .5, .6, .7, .8, .9, 1.0. From these results the values of $\omega \alpha_0$ and $\omega^2 \alpha_1$ as listed in Table (2.01) were obtained.

TABLE (A7.01)
(Reproduced from Reference 3)

$$\Delta K(\lambda, z) = K(\lambda, z) - K(0, z)$$

$$= \Delta K_1(\lambda, z) + \Delta K_2(\lambda, z)$$

$$\boxed{\begin{aligned} &\Delta K_1(\lambda, z) = k_{10}/z + k_{11} + k_{12} \operatorname{Log}|z| + z(k_{13} + k_{14} \operatorname{Log} z) \\ &\Delta K_2(\lambda, z) = \sum_{n=2}^{9} k_{2n} z^n \end{aligned}}$$

$$k_{10} = \frac{1}{2\pi}\left(1 - \sqrt{1-\lambda^2}\right)$$

$$k_{11} = -\frac{1}{4}\left(1/\sqrt{1-\lambda^2} - 1\right) - i/2\pi \left[1/\sqrt{1-\lambda^2}\left\{\lambda^2 - \operatorname{Log}\left(\gamma\lambda/2(1-\lambda^2)\right)\right\} + \operatorname{Log}\left(\gamma\lambda/1+\sqrt{1-\lambda^2}\right)\right]$$

$$k_{12} = i/2\pi\left(1/\sqrt{1-\lambda^2} - 1\right)$$

$$k_{13} = -1/2\pi\left\{-1 + \operatorname{Log}\left(\gamma\lambda/1+\sqrt{1-\lambda^2}\right) + \left[1/(1-\lambda^2)^{3/2}\right]\left[1 - 3\lambda^2/4 - \lambda^4/2 - (1 - 3\lambda^2/2)\operatorname{Log}\left(\gamma\lambda/2(1-\lambda^2)\right)\right]\right\}$$
$$- i/4\left(1 + (3\lambda^2-2)/2(1-\lambda^2)^{3/2}\right)$$

$$k_{14} = -1/2\pi\left[1 + (3\lambda^2-2)/2(1-\lambda^2)^{3/2}\right] \quad ; \quad \operatorname{Log}\gamma = .57722 = \text{Euler's Constant}$$

λ	0,3	0,4	0,5	0,6	0,7
$+10^2 \cdot k'_{22}$	0,0195	0,0872	0,3274	1,1311	4,7319
$+10^2 \cdot k'_{23}$	-0,0032	-0,0237	-0,1197	-0,4493	-2,8593
$+10^2 \cdot k'_{24}$	0,0067	0,0021	-0,0047	-0,1054	-1,2914
$+10^2 \cdot k'_{25}$	-0,0075	0,0122	0,0894	0,3478	3,2183
$+10^2 \cdot k'_{26}$	-0,0013	0,0004	0,0033	0,0161	0,2452
$+10^2 \cdot k'_{27}$	0,0039	-0,0055	-0,0409	-0,1468	-1,5321
$+10^2 \cdot k'_{28}$	-0-0003	-0,0007	-0,0016	-0,0043	-0,0274
$+10^3 \cdot k'_{29}$	-0,0004	0,0011	0,0070	0,0238	0,2539
$+10^2 \cdot k''_{22}$	0,0419	0,1698	0,5484	1,6217	4,9165
$+10^2 \cdot k''_{23}$	-0,0026	0,0137	0,0745	0,4830	2,2234
$+10^2 \cdot k''_{24}$	-0,0206	-0,1266	-0,4694	-1,5934	-6,5805
$+10^2 \cdot k''_{25}$	0,0059	-0,0189	-0,0536	-0,3839	0,2119
$+10^2 \cdot k''_{26}$	0,0078	0,0553	0,2091	0,7325	3,0056
$+10^3 \cdot k''_{27}$	-0,0034	0,0119	0,0254	0,2153	-0,3464
$+10^2 \cdot k''_{28}$	-0,0011	-0,0086	-0,0326	-0,1184	-0,4719
$+10^2 \cdot k''_{29}$	0,0006	-0,0021	-0,0038	-0,0363	0,0677

$$k_{2n} = k'_{2n} + i\, k''_{2n}$$

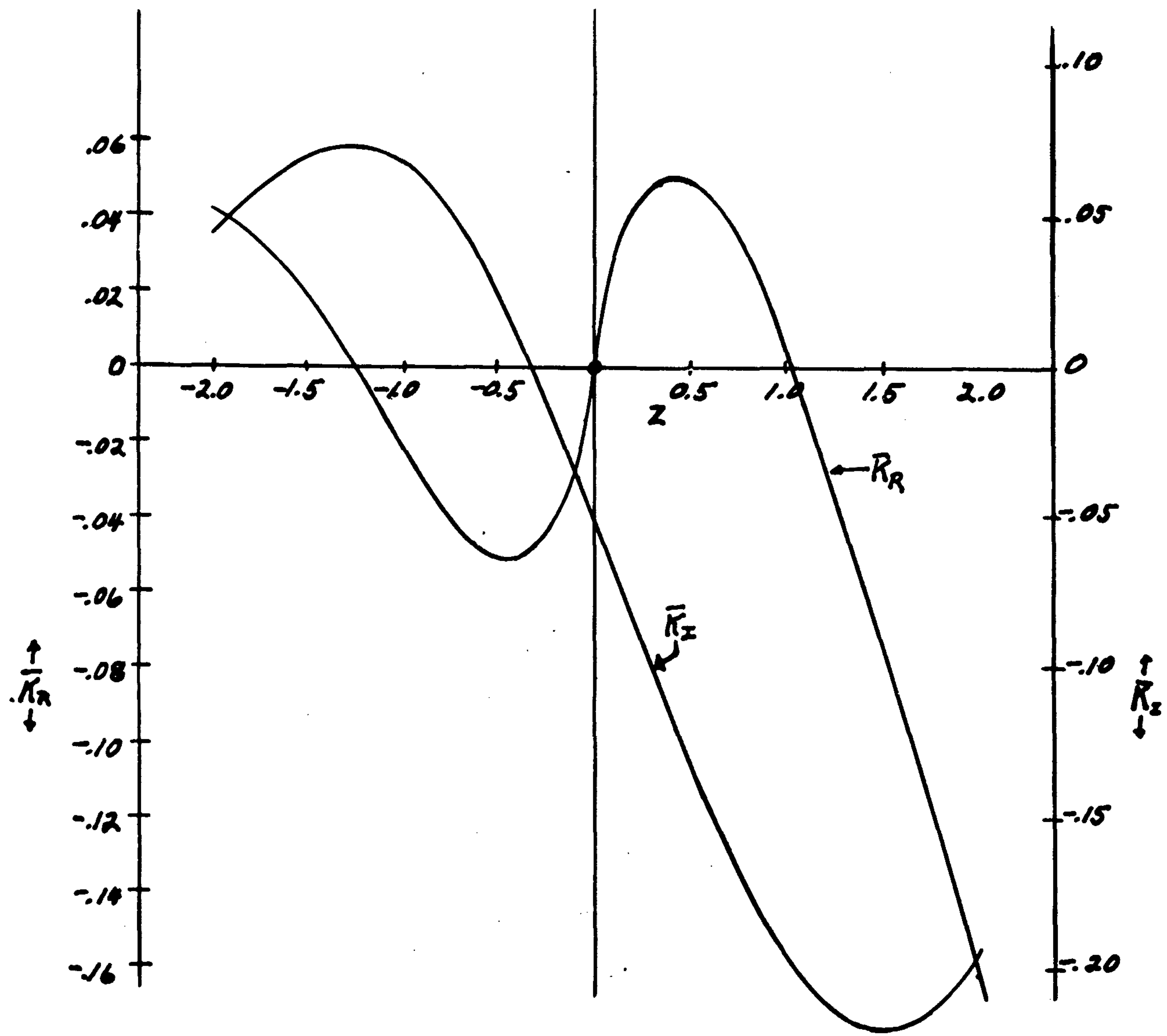

Figure A7:01

The Non-Singular Nucleus Difference: $\bar{K}(\lambda, z) = K(\lambda, z) - \frac{1}{\sqrt{1-\lambda^2}} K(0, z) - \frac{\lambda^2}{2\pi\sqrt{1-\lambda^2}} [1/z]$

$\lambda = .7$

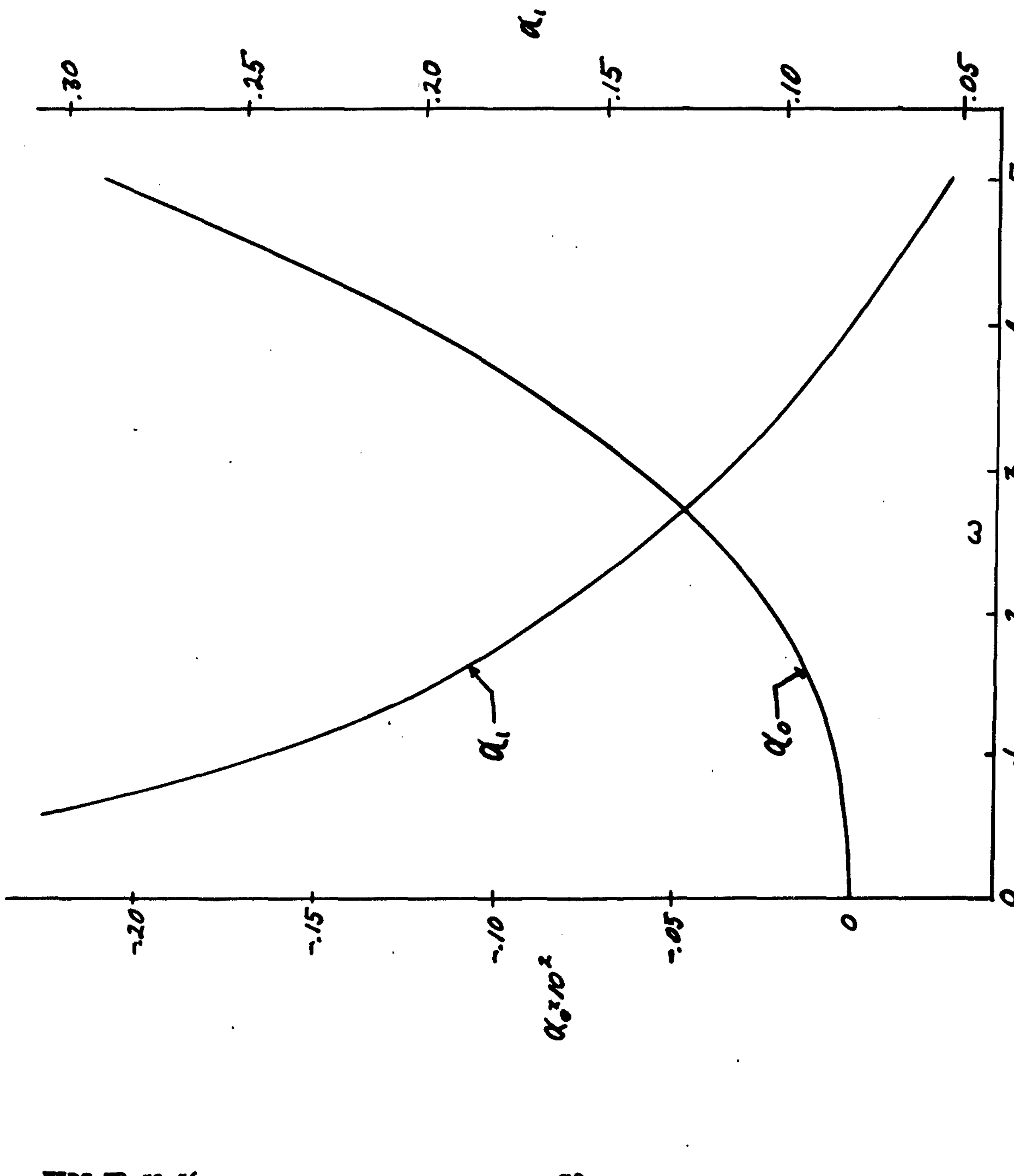

Figure A7:02

The Real Part of α_0 and α_1 vs. ω

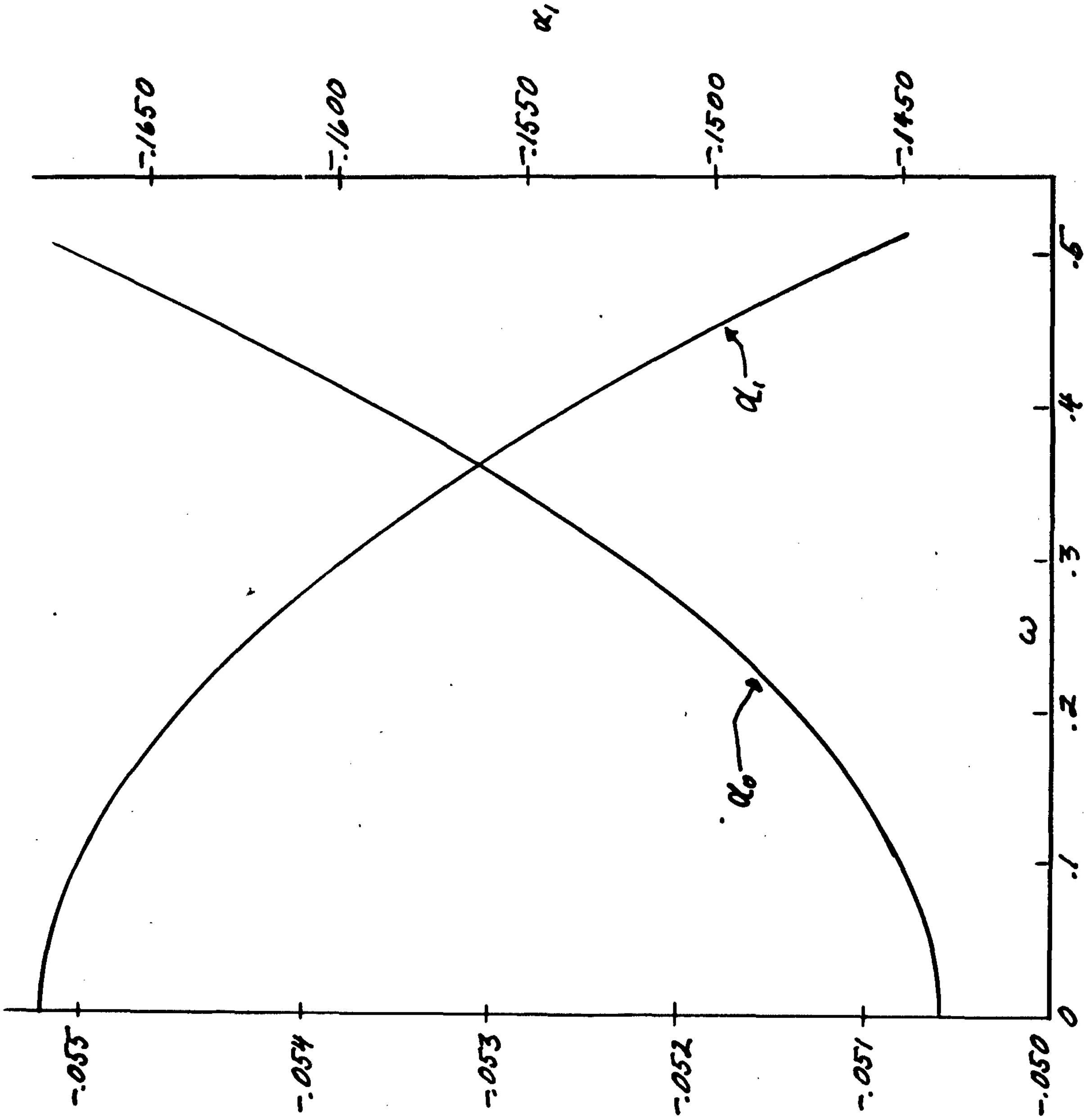

Figure A7:03

The Imaginary Part of α_0 and α_1 vs. ω

APPENDIX VIII

AUXILIARY TABLES

TABLE (A8.01)
THE FUNCTION $T(\omega)$ (REF. 1)

ω	$T(\omega)$
.05	.818018-.261289i
.10	.663848-.344604i
.20	.455160-.377248i
.30	.329942-.358638i
.40	.249952-.329968i
.50	.195872-.301419i

TABLE (A8.02)
$e^{i\mu e}$
$e = .5$, $\lambda = .7$

ω	μ	$e^{i\mu e}$
.05	.0480392	.9997115+.0240173i
.10	.0960784	.9988460+.048021i
.20	.1921569	.9953880+.095931i
.30	.2882353	.9896330+.143619i
.40	.3843137	.9815950+.190977i
.50	.4803922	.9712910+.237893i

TABLE (A8.03)

$J_p(\mu)$

$\lambda = .7$

ω	$p = 0$	$p = 1$	$p = 2$
.05	.9994231	.0240127	.000288
.10	.9976935	.0479838	.001153
.20	.9907902	.0956357	.004602
.30	.9793377	.1426262	.010313
.40	.9634151	.1886311	.018241
.50	.9431326	.2333334	.028282

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