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A NEWTON METHOD FOR THE PIES ENERGY MODEL.(U)

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MRC Technical Summary Report #1971

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June 1979

(Received May 25, 1979)

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National Science Foundation
Washington, D.C. 20550

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

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ABSTRACT

Newton's method is a well known and often applied technique for computing a zero of a nonlinear function. By using the theory of generalized equations, a Newton method has been developed to solve problems arising in both mathematical programming and mathematical economics. ~~(see Josephy [11])~~.

~~We show~~ It is shown how Newton's method for generalized equations can be applied to the economic equilibrium problem of the Project Independence Evaluation System (PIES) Energy Model. Solutions to a simplified version of PIES are obtained using a Newton method, and comparisons are made to solutions which have appeared in the literature. (Hogan [9]).

AMS (MOS) Subject Classifications: 90A15, 90C99, 49D99
Key Words: PIES Energy Model, Economic Equilibrium, Newton's Method
Work Unit Number 5 (Mathematical Programming and Operations Research)

Sponsored by the United States Army under Contract No. DAAG29-75-C-0024. This material is based upon work supported by the National Science Foundation under Grant Nos. DCR 74-20584 and MCS 74-20584 A02 and the Graduate School of the University of Wisconsin-Madison.

SIGNIFICANCE AND EXPLANATION

Newton's method is a well known and often applied technique for computing a zero of a nonlinear function. By using the theory of generalized equations, a Newton method has been developed to solve problems arising in both mathematical programming and mathematical economics (Josephy [11]).

Recent problems of practical importance in the study and computation of equilibria for large-scale economic systems motivate increased interest in the use of Newton's method for analyzing and computing economic equilibrium. One such problem is the economic equilibrium of the Project Independence Evaluation System (PIES) Energy Model. Developed by the Federal Energy Administration to model the energy sector of the United States economy, PIES formulates an equilibrium problem between the suppliers and consumers of energy. This problem can be modeled as a generalized equation, and thus can be analyzed using the methods developed in Josephy [11].

We present computational results of Newton's method applied to the economic equilibrium problem of the Project Independence Evaluation System (PIES) Energy Model. Solutions to a simplified version of PIES are obtained using a Newton method, and comparisons are made to solutions which have appeared in the literature.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A NEWTON METHOD FOR THE PIES ENERGY MODEL

Norman H. Josephy

1. Introduction

A generalized equation is a set-valued analogue of a single-valued equation. Many problems arising in mathematical programming and mathematical economics can be analyzed by generalizing, to the set-valued case, classical techniques used in the study of single-valued equations. Preliminary investigations of the properties and applications of generalized equations have been carried out by Robinson [16-20]. Josephy [11,12] develops Newton methods for computing the solution of generalized equations. This paper presents an application of generalized equations to the economic equilibrium problem of the Project Independence Evaluation System (PIES) Energy Model. Developed by the Federal Energy Administration to model the energy sector of the United States economy, PIES formulates an equilibrium problem between the suppliers and consumers of energy. This problem can be modeled as a generalized equation, and thus can be analyzed using the methods developed in Josephy [11,12].

Section 2 begins with a general discussion of the equilibrium problem generated by trying to balance supply and demand of consumable goods in a competitive market environment. An example published by Hogan [9] illustrating the major aspects of the PIES model is detailed in Section 3. A description of the solution procedure used by the Federal Energy Administration (FEA) in solving the equilibrium problem

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is given in Section 4. Newton's method for generalized equations is applied to the economic equilibrium problem in Section 5. The use of a Newton method to compute economic equilibrium is first suggested in Robinson [16]. An alternative approach to applying a Newton method to the PIES model can be found in Eaves [5].

Section 6 concludes the paper. We present computational results comparing solutions published by Hogan and the solutions obtained by using Lemke's algorithm to solve the linear complementarity subproblems of the Newton iterations. The data for Hogan's example are reproduced in an appendix.

2. Equilibrium in a Market Economy

The competitive market which underlies the PIES model consists of two classes of agents, suppliers and consumers, and two classes of goods, factors of production and consumable goods. The suppliers, faced with a perceived demand for consumable goods, convert factors of production into consumable goods, sustaining a cost for conversion to and delivery of consumable goods, and charging a price for those goods. The consumers purchase consumable goods at levels dependent upon the prices of all goods.

Suppose there are n consumable goods. Let $q \in R^n_+$ and $p \in R^n_+$, where q_i is the consumption level of the ith good, and p_i is the price of the ith good. For a perceived demand q of consumable goods, the supplier will charge prices no lower than $p_S(q) \in R^n_+$. The consumer, desiring level q of consumable goods, is willing to pay no more than prices $p_D(q) \in R^n_+$. A market equilibrium is a quantity vector q and a price vector p such that $p = p_D(q) = p_S(q)$. That is, at price p, the market for all goods will clear. The producers will supply at price p a level of consumable goods q which the consumers are willing to purchase at price p. The supply relation p_S is typically determined by the solution of a cost minimization problem modeling the engineering/technological processes involved in conversion of factors of production to delivered consumable goods. The demand relation p_D is traditionally a behavioral model econometrically determined from historical data. The following example will illustrate the determination of the supply relation p_S.

Example 1. The supplier converts two factors of production into two consumable goods. The factors cost c_i dollars per unit, $i=1,2$, respectively. The total cost of converting x_1 units of factor one and x_2 units of factor two in $c_1x_1 + c_2x_2$. We assume a linear technology, that is, for factor levels x_1 and x_2, an amount $a_{11}x_1 + a_{12}x_2$ of good one and an amount $a_{21}x_1 + a_{22}x_2$ of good two will be produced. We assume there is a demand of q units of good one and 1 unit of good two. The cost minimization problem faced by the supplier is the following linear program:

$$\text{minimize } c_1x_1 + c_2x_2 \text{ subject to}$$

$$a_{11}x_1 + a_{12}x_2 \geq q ,$$

$$a_{21}x_1 + a_{22}x_2 \geq 1 ,$$

$$x_1 \geq 0 , x_2 \geq 0 .$$

The dual linear program is given by

$$\text{maximize } qp_1 + p_2 \text{ subject to}$$

$$a_{11}p_1 + a_{21}p_2 \leq c_1 \ ,$$

$$a_{12}p_1 + a_{22}p_2 \leq c_2 \ ,$$

$$p_1 \geq 0 \ , \ p_2 \geq 0 \ .$$

The dual problem is illustrated in Figure 1. It is assumed that the supplier will price the consumable goods at marginal price levels p_1, p_2. The dual linear objective function is a line $qp_1 + p_2$ in the $p_1 - p_2$ plane. For q=0, the optimal dual solution is the extreme point 1 , $(0,\hat{p}_2)$. This remains optimal as q is increased until the objective function is parallel to face 1 - 2 . At that value of q, every point on the line segment between $(0,\hat{p}_2)$ and $(\bar{p}_1,\bar{p}_2)$ is a dual optimal solution. As q is increased further, extreme point

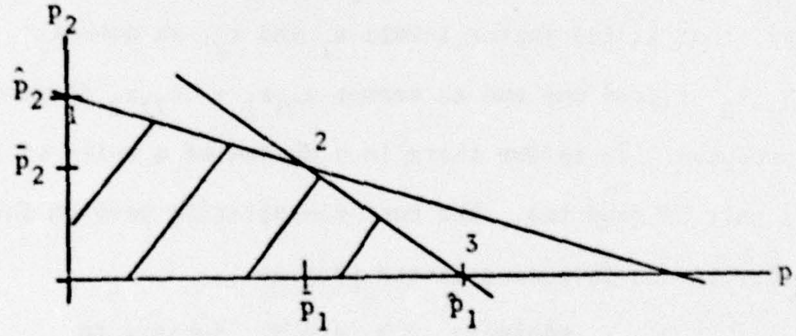

Figure 1. Supplier's Dual Linear Program

2 , $(\bar{p}_1,\bar{p}_2)$ becomes the unique dual optimal solution. This behavior is repeated with face 2 - 3 when q is increased sufficiently to make the objective function parallel to it. For larger values of q, extreme point 3 , $(\hat{p}_1,0)$ remains the unique dual optimal solution. The relationship between q and p_1 is illustrated in Figure 2.

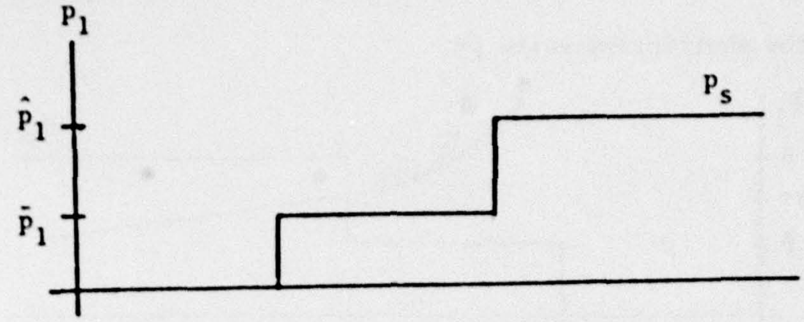

Figure 2. **Supplier's Price vs. Quantity Curve p_S**

The consumer's demand curve p_D is obtained by fitting historical data to prespecified functions (see FEA [6], Halvorsen [8]). In particular, a linear relationship between the logarithm $\ln(q_i)$ and the logarithms $\ln(p_i)$ is postulated. Thus, for the case of n consumable goods, the demand relation is determined by the following set of equations:

$$\ln(q_1/q_1^0) = e_{11}\ln(p_1/p_1^0) + \dots + e_{1n}\ln(p_n/p_n^0)$$

$$\vdots$$

$$\ln(q_n/q_n^0) = e_{n1}\ln(p_1/p_1^0) + \dots + e_{nn}\ln(p_n/p_n^0) ,$$

where e is the elasticity matrix and q^0, p^0 are fixed reference values of p_D, that is, $p_D(q^0) = p^0$.

The market equilibrium for n=2 is illustrated in Figure 3. It should be noted that p*, the equilibrium price vector, is not at a value given by an extreme point of the supplier's dual linear program. Thus, if q were assigned the value q* in the supplier's cost minimization problem and the simplex method was used to solve the resulting linear program, one would obtain either $\hat{p}$ or $\bar{p}$ as optimal dual variables, and not the equilibrium value p^*.

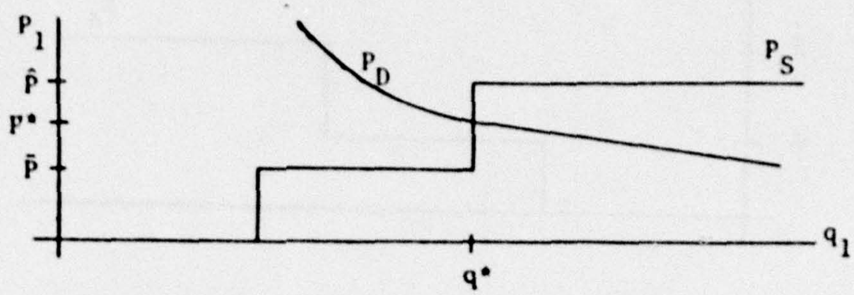

Figure 3. Market Equilibrium

For the case of n consumable goods, the equilibrium quantity q^* and price p^* and factor levels x^* for the competitive market with supply modeled as a linear program

minimize $\langle c,x \rangle$	total cost of production
$Ax \geqq q$	demand requirements
$Bx \geqq b$	non-demand constraints
$x \geqq 0$	non-negative factor levels

and a log-linear consumer demand model

$$\ln(q_i/q_i^0) = \sum_{j=1}^{n} e_{ij} \ln(p_j/p_j^0) \text{ , } i=1,\ldots,n$$

is a triple $(q^*,p^*,x^*) \geqq 0$ satisfying the following equilibrium conditions.

(EQ. 1) x^* solves: minimize $\langle c,x \rangle$ subject to

$$Ax \geqq q^* \text{ , } Bx \geqq b \text{ , } x \geqq 0 \text{ .}$$

(EQ. 2) For some multiplier s^*, (p^*,s^*) solve the dual problem: maximize $\langle q^*,p \rangle + \langle b,s \rangle$ subject to

$$A^T p + B^T s \leqq c \text{ , } (p,s) \geqq 0 \text{ .}$$

(EQ. 3) $$\ln(q_i^*/q_i^0) = \sum_{j=1}^{n} e_{ij} \ln(p_j^*/p_j^0) \ , \ i=1,\ldots,n \ .$$

In non-technical terms, the equilibrium conditions put a constraint, the demand relation, on the optimal dual solution. The constraint relates the optimal dual solution to the cost coefficients of the dual linear program. We note that it is incorrect to solve the equilibrium problem by including the demand constraints in the supplier's linear program. The following example illustrates this point.

Example 2. Consider the supply linear program minimize x subject to $x \geqq q$, $x \geqq 0$, and its associated dual program

$$\max\ p\ q \quad \text{subject to } p \leqq 1 \ , \ p \geqq 0 \ .$$

The optimal dual price is $p^* = 1$ for any $q > 0$. Suppose the demand relation is given by

$$p_D(q) = \begin{cases} \frac{4}{3}(1-q) & \text{if} \quad 0 \leqq q \leqq 1 \\ 0 & \text{if} \quad q > 1 \end{cases} .$$

The equilibrium, as shown in Figure 4 is $p^* = 1$, $q^* = \frac{1}{4}$.

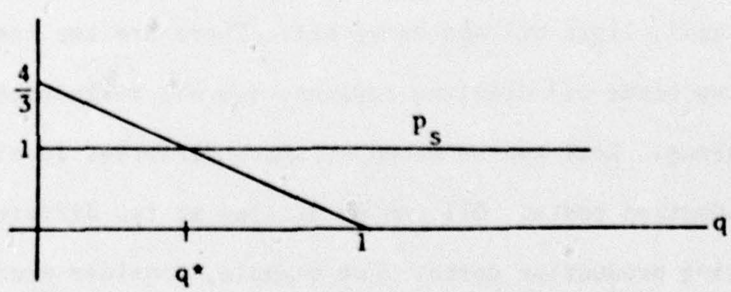

Figure 4. Equilibrium for Example 2.

If we incorporate the demand relation into the dual linear program, we obtain the problem

$$\text{maximize } p(1-\tfrac{3}{4}p) \quad \text{subject to} \quad 0 \leq p \leq 1 ,$$

whose solution is $p^* = \frac{2}{3}$ with corresponding $q^* = \frac{1}{2}$, which is not the equilibrium point of the original problem. The equilibrium point is characterized by finding that value q^* of q which, when q is fixed at that constant value in the supplier's linear program yields a value p^* of p which satisfies the demand relation at q^*. Incorporating the demand relation into the supply linear program has the effect of changing q into a variable in the supplier's problem, which is incorrect if an equilibrium is desired.

3. The Hogan PIES Example

As an illustration of PIES, Hogan [9] described a simplified situation which included the major aspects of the PIES model. We will use this example to test the feasibility of Newton's method as a solution technique for the market equilibrium problem. Figure 5 illustrates the structure of Hogan's example. The factors of production are coal, crude oil, steel and capital. The consumable goods are coal, light oil and heavy oil. There are two coal mining regions, two crude oil drilling regions, two oil refineries, and two demand regions. Coal can be mined at three different levels for differing production costs. Oil can be drilled at two different levels for differing production costs. For example, consider production of coal in region 1. To mine up to 300 tons of coal will consume

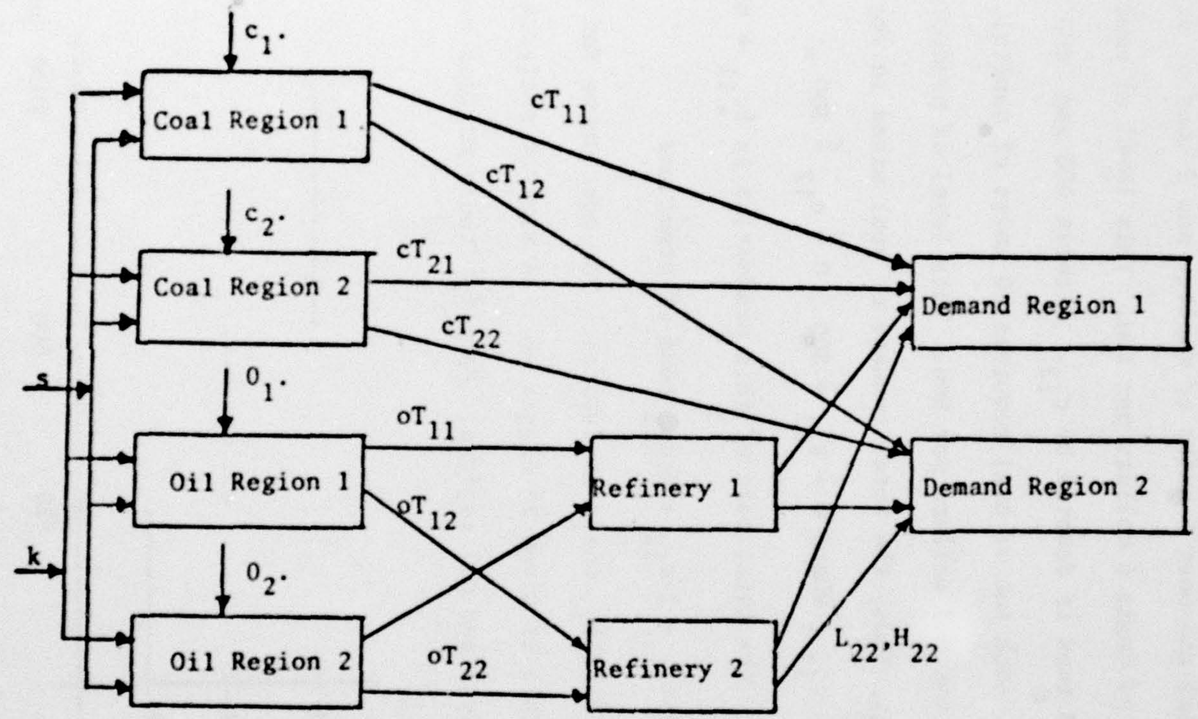

Figure 5. The Hogan PIES Example.

1 unit of capital and 1 ton of steel per ton of coal, and will cost 5 dollars per ton of coal. The variable c_{11} represents this level of coal production. Additional production above 300 tons up to 600 tons of coal consumes 5 units of capital and 2 tons of steel per ton of coal, and costs 6 dollars per ton. This level of coal production above 300 tons is denoted by c_{12}. Between 600 and 1000 tons of coal mined, each ton of coal requires 10 units of capital, 3 units of steel, and costs 8 dollars per ton. This level of production is denoted by c_{13}. Thus, the total tonnage of coal mined in region 1 is $c_{11} + c_{12} + c_{13}$, where $0 \leq c_{11} \leq 300$, $0 \leq c_{12} \leq 300$, $0 \leq c_{13} \leq 400$. The total cost of this production is $5c_{11} + 6c_{12} + 8c_{13}$, and requires $1 \cdot c_{11} + 2 \cdot c_{12} + 3 \cdot c_{13}$ tons of steel and $1 \cdot c_{11} + 5 \cdot c_{12} + 10 \cdot c_{13}$ units of capital. The cost curve for coal mined in region 1 is given in Figure 6. A similar analysis applies to the other coal and oil regions. Data for those regions are given in the appendix.

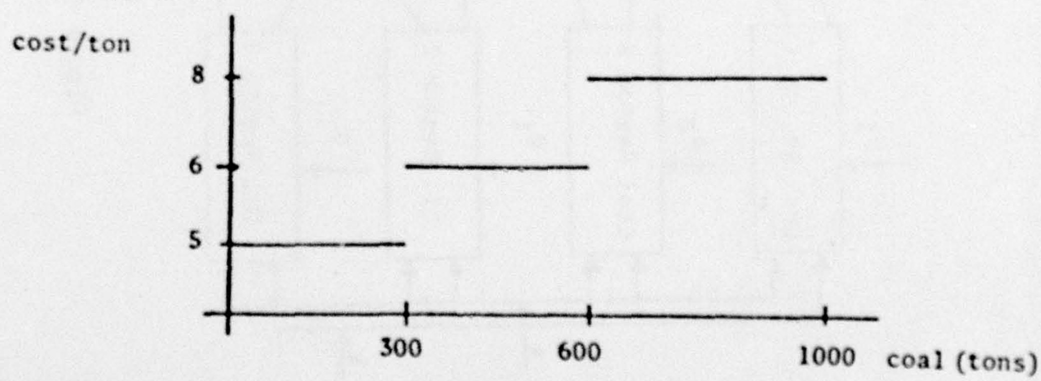

Figure 6. Cost Curve for Coal Region 1.

Coal is transported at level cT_{ij} from coal region i to demand region j at a cost per ton given in the appendix. Crude oil is transported from oil region i to refinery j in a quantity oT_{ij}, where it is refined into a fixed proportion of light and heavy oil. Refinery 1 converts 60% of its crude oil into light oil, while refinery 2 converts 50% of its crude oil into light oil. Light oil L_{ij} and heavy oil H_{ij} are transported from refinery i to demand region j. The transportation costs for crude oil oT and refined oil L, H, are given in the appendix.

The linear program representing the minimum cost of production and delivery of consumable goods to meet a demand $q \in R^6_+$ is of the form

min $\langle r,x \rangle$	subject to
$Ax \geqq q$	demand requirements
$Bx = 0$	transportation flow balance
$Ex \leqq a$	factor of production upper bounds
$Fx \leqq b$	steel and capital resource constraint
$x \geqq 0$	non-negitive factors of production

The components of x represent the levels of production of coal and crude oil, levels of coal transported to demand regions, levels of crude oil transported to refineries, and levels of refined oil transported to the demand regions. The coefficient r_i is the unit cost of the activity associated with x_i. The vector Ax consists of levels of consumable goods delivered to the demand regions. Bx represents the conservation of materials transported through a region or refinery. Fx is the vector of steel and capital consumed in the processing of factors x. The equations and inequalities given by this linear program are detailed in the appendix.

The demand relationship between the desired quantity of a consumable good in a particular demand region, q_i, and the prices consumers are willing to pay for such goods, $\{p_j\}$ $j=1\ldots,6$, is given by

$$\ln(q_i/q_i^0) = \sum_{j=1}^{6} e_{ij} \ln(p_j/p_j^0), \quad i=1,\ldots,6$$

The values of the data q^0, p^0 and e_{ij} are given in the appendix.

4. FEA Solution Algorithm

The motivation for the FEA's solution algorithm for computing the market equilibrium in the PIES model (Hogan [9], FEA [7]) follows from the observation that finding a zero of a gradient mapping can, under suitable conditions, be replaced by minimizing a scalar-valued potential function defined in terms of the line integral of the gradient mapping (Ortega and Rheinboldt [15]). This observation can be applied to the market equilibrium problem in the following manner. The Kuhn-Tucker conditions for the suppliers linear program (P) with perceived demand q^*,

$$\text{(P)} \qquad \min \langle c,x\rangle \text{ subject to } Ax \geq q^* ,\ Bx \geq b ,\ x \geq 0 .$$

are the classical complementarity conditions (Luenberger [14]): there exists x^*, p^*, s^*, such that

(1) $c - A^T p^* - B^T s^* \geqq 0$, $x^* \geqq 0$, $\langle x^*, c - A^T p^* - B^T s^* \rangle = 0$

(2) $Ax^* - q^* \geqq 0$, $p^* \geqq 0$, $\langle p^*, Ax^* - q^* \rangle = 0$

(3) $Bx^* - b \geqq 0$, $s^* \geqq 0$, $\langle s^*, Bx^* - b \rangle = 0$

The demand constraint is given by

(4) $p^* = p_D(q^*)$

Note that the equilibrium conditions (EQ. 1) and (EQ. 2) of the previous section are equivalent to conditions (1), (2), and (3), while equilibrium condition (EQ. 3) is equilvalent to equation (4). Thus, the four conditions (1) - (4) determine equilibrium values of consumption quantity q^*, price p^*, production factor levels x^*, and non-demand constraint multipliers s^*. Suppose p_D is a gradient mapping, that is, for some function $g: R^n \to R$,

(5) $p_D(q) = \nabla g(q)$.

The Kuhn-Tucker conditions for the non-linear minimization problem (NLP):

(NLP) minimize $\langle c,x \rangle - g(q)$ subject to

$Ax - q \geqq 0$, $Bx \geqq b$, $x \geqq 0$,

are the following:

there exists x^*, p^*, s^*, q^* such that

(1') $c - A^T p^* - B^T s^* \geqq 0$, $x^* \geqq 0$, $\langle x^*, c - A^T p^* - B^T s^* \rangle = 0$

(2') $Ax^* - q^* \geqq 0$, $p^* \geqq 0$, $\langle p^*, Ax^* - q^* \rangle = 0$

(3') $\quad Bx^* - b \geqq 0 \ , \ s^* \geqq 0 \ , \ \langle s^*, Bx^*-b \rangle = 0$

(4') $\quad -\nabla g(q^*) + p^* = 0$

Using (5) to substitute $p_D(q^*)$ for $\nabla g(q^*)$ in (4'), we see that (1'), (2'), (3'), and (4') are identical to (1), (2), (3), and (4). Thus, under the assumption that p_D is a gradient mapping, we can replace the equilibrium problem with finding a Kuhn-Tucker point for the nonlinear programming problem (NLP). However, p_D is a gradient mapping in a neighborhood of a point q^* if and only if the derivative $p_D(q)$ is symmetric for q in that neighborhood (Ortega and Rheinboldt [15], p. 95). This condition does not hold for the PIES model p_D function. To circumvent this difficulty, the algorithm used in FEA ([7]) employs a zero cross-elasticity approximation. That is, the original demand relation

(6) $$\ell n q_i = \ell n q_i^0 + \sum_{j=1}^{n} e_{ij} \ \ell n (p_j/p_j^0) \qquad i=1,\ldots,n$$

is replaced by

(6') $$\ell n q_i = \ell n q_i^0 + e_{ii} \ \ell n (p_i/p_i^0) \qquad i=1,\ldots,n$$

This yields the consumer demand function p_D with components

(7) $$p_D(q)_i = p_i^0 \cdot (q_i/q_i^0)^{(e_{ii}^{-1})} \qquad i=1,\ldots,n$$

which has a symmetric derivative. We are now faced with solving problem (NLP), with g given by

(8) $$g(q) = \sum_{i=1}^{n} \int_{q_i^0}^{q_i} p_D(a)_i \, da$$

Since the elasticities e_{ii}, $i=1,\ldots,n$ are negative, each $p_D(q)_i$ is monotone non-increasing, and we can thus approximate $(-g)$ by a piecewise linear, convex function whose value at any q can be represented as the solution of a linear program with q appearing as a constant in the constraints. Thus, replacing $(-g)$ in (NLP) by a linear program, we arrive at a large linear program of the form

(LP) $$\text{minimize } \langle c,x\rangle - \langle a,z\rangle \quad \text{subject to}$$
$$Bx \geq b \ , \ Ax - Dz \geq d \ , \ Fz + f = q \ , \ x \geq 0 \ , \ z \geq 0 \ ,$$

which is an approximation to (NLP), whose solution yields an equilibrium point of the original market equilibrium problem. The details of the approximations and formulations discussed above can be found in Hogan [9], and FEA [7].

The FEA solution algorithm proceeds as follows. The linear programming problem (LP) is solved to obtain an estimate of the equilibrium price vector p^*. If p^* is not "close" to p^0, a new "base point", q^1, is computed by setting $p = p^*$ in (6). A new zero cross-elasticity approximation is made about (q^1,p^*): that is, (7) is replaced by

(7') $$p_D(q)_i = p_i^* \cdot (q_i/q_i^1)^{(e_{ii}^{-1})} \quad , \ i=1,\ldots,n \quad .$$

The process of approximating g in (8) and solving (LP) is then repeated. The algorithm can be written as follows:

i) input q^0, p^0 and (e_{ij}) ; set $t = 0$.

ii) replace q^0, p^0 in (7) by (q^t, p^t)

iii) use (7) and (8) to form (LP) and solve for (x,z), then compute p^*.

iv) if p^* is "close" to p^t, then "solution" has been found; if not, set $p = p^*$ in (6), compute q, set $q^{t+1} = q$, $p^{t+1} = p^*$, set t to t+1, and return to (ii).

Let (q_0, p_0) in Equation (6) be called the center of the demand curve. Step (ii) at iteration t has the effect of shifting the demand curve to a new center (q^t, p^t), where the zero cross-elasticity approximation (7) is then made. The point (q^t, p^t) is always chosen to lie on the original demand curve centered at (q^0, p^0). This is illustrated in Figure 7. Conditions under which this algorithm converges can be found in Ahn [1].

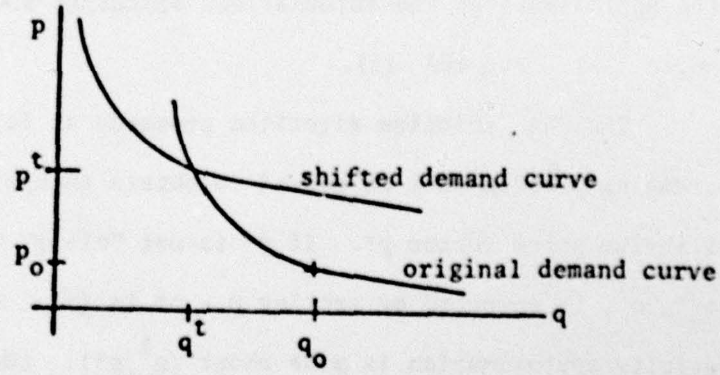

Figure 7. Shifted Demand Curve

5. Newton's Method for PIES Equilibrium

The market equilibrium for the PIES model is a quadruple (x^*,p^*,q^*,s^*) satisfying conditions (1), (2), (3), and (4). Let $F: R^{\ell+n} \rightarrow R^{\ell+n}$ be defined by

$$(9) \quad F(x,p,s,q) := \begin{bmatrix} c \\ o \\ -b \\ -p_D(q) \end{bmatrix} + \begin{bmatrix} 0 & -A^T & -B^T & 0 \\ A & 0 & 0 & -I \\ B & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ p \\ s \\ q \end{bmatrix}$$

and $K: = R^{\ell}_{+} \times R^n$, where $\ell = m+n+v$, $x \in R^m$, $p \in R^n$, $s \in R^v$, and $q \in R^n$. Then the equilibrium conditions can be written, using the notation of Josephy [11], as x^*,p^*,s^*,q^* solving

$$(10) \quad 0 \in F(x,p,s,q) + N_K(x,p,s,q) \quad .$$

The linearization of (10) about $\bar{x},\bar{p},\bar{s},\bar{q}$ is

$$(11) \quad 0 \in \begin{bmatrix} c \\ o \\ -b \\ -p_D(\bar{q})+p_D'(\bar{q})\bar{q} \end{bmatrix} + \begin{bmatrix} 0 & -A^T & -B^T & 0 \\ A & 0 & 0 & -I \\ B & 0 & 0 & 0 \\ 0 & I & 0 & -p_D'(\bar{q}) \end{bmatrix} \cdot \begin{bmatrix} x \\ p \\ s \\ q \end{bmatrix} + N_k(x,p,s,q)$$

This corresponds to the linear complementarity problem obtained from conditions (1), (2), (3), and (4) by replacing (4) with

(12) $p^* = (Lp_D)_{\bar{q}}(q^*)$, where $(Lp_D)_{\bar{q}}(q) := p_D(\bar{q}) + p_D'(\bar{q})(q-\bar{q})$

Thus, Newton's method iteratively solves the linear complementarity problem (1), (2), (3), and (12).

6. Computational Results

It has been shown in Josephy [11] that Newton's method for generalized equations, when applied to the nonlinear complementarity problem, results in a sequence of linear complementarity problems. We have shown how the economic equilibrium problem of the PIES energy model results in a nonlinear complementarity problem. Thus, to compute equilibrium solutions for PIES, we solve linear complementarity problems. The solution technique we will use is Lemke's complementary pivot algorithm (Cottle and Dantzig [4]). Not all linear complementarity problems are solvable by Lemke's algorithm. However, it is shown in Josephy [13] that for the data given by Hogan, the linear complementarity problems resulting from applying Newton's method for generalized equations to Hogan's example are solvable by Lemke's algorithm.

Table 1 summarizes the equilibrium quantities and prices for Hogan's example. Column 1 contains Hogan's published equilibrium values. Column 2 contains the equilibrium values computed by Newton's method as developed here, with Lemke's algorithm applied to the linear complementarity problem. The equilibrium values in Column 3 are computed as were those of Column 2, except that the off-diagonal elements of the elasticity matrix are set to zero. In each case in which Newton's method was applied to the equilibrium nonlinear complementarity problem, convergence occurred in three iterations

TABLE 1

EQUILIBRIUM QUANTITIES AND PRICES

	Hogan	Newton	Newton with Zero Cross-Elasticities
q_1	996	1016	1029
q_2	910	912	913
q_3	1205	1201	1212
q_4	1229	1223	1205
q_5	996	998	1012
q_6	1020	1011	992
p_1	11.3	11.7	11.6
p_2	13.4	13.7	13.6
p_3	15.4	15.8	15.7
p_4	15.6	16.0	15.9
p_5	11.5	11.9	11.7
p_6	12.0	12.4	12.7

with an error less than 10^{-6}. The difference in the values between column 1 and column 2 are due to the error introduced in the FEA algorithm when a linear programming approximation (LP) is made to the nonlinear programming problem (NLP). Ahn [1] has shown that, under suitable conditions, the FEA algorithm will converge to an equilibrium if (NLP) rather than (LP) is solved at each iteration.

Table 2 lists the equilibrium values of the problem variables. The notation used in both tables is defined in the appendix.

Rows 1 and 2 of Table 3 list the optimal dual prices for the supplier's linear program when the fixed demand q is set to the Hogan equilibrium and to the Newton-computed equilibrium. The third row of Table 3 lists the dual prices for the supplier's linear program with fixed demand q set to the Newton equilibrium with q_1 decreased by .1 percent. These numbers were obtained by applying the simplex method to the supplier's linear program.

The numbers in Table 3 indicate that the dual feasible region has extreme points whose price components, as listed in rows 1 and 2 of the Table, are not close. However, a small change in only one component of the dual program objective coefficient vector causes the optimal prices to jump between the two extreme points. Also note that neither dual solution is the equilibrium price vector. This type of occurrence is discussed in Section 2.

We remark in closing that the generalized equation framework is eminently capable of modeling equilibrium problems containing equations or unconstrained variables. An example of a more general equilibrium problem than that discussed in this paper is the following.

TABLE 2

EQUILIBRIUM VALUES OF PROBLEM VARIABLES

	c_{11}	c_{12}	c_{13}	c_{21}	c_{22}	c_{23}	o_{11}	o_{12}	o_{21}	o_{22}	cT_{11}	cT_{12}	cT_{21}	cT_{22}
Hogan	300	300	206	200	300	600	1100	989	1300	1063	0	806	996	104
Newton	300	300	228	200	300	600	1100	975	1300	1058	0	828	1016	84

	oT_{11}	oT_{12}	oT_{21}	oT_{22}	L_{11}	L_{12}	L_{21}	L_{22}	H_{11}	H_{12}	H_{21}	H_{22}
Hogan	2089	0	0	2363	24	1229	1181	0	0	835	996	185
Newton	2075	0	0	2358	22	1223	1179	0	0	830	998	180

TABLE 3

DUAL LINEAR PROGRAM PRICES

	P_1	P_2	P_3	P_4	P_5	P_6
Hogan	8.5	10.5	16.4	16.6	2.6	3.1
Newton	37.6	39.6	45.4	45.6	41.6	42.1
Newton with .1% q_i Change	8.5	10.5	16.4	16.6	2.6	3.1

It is of interest to gauge the effect of the energy sector of the United States economy on the economy as a whole (see Askin [2,3] and Hogan and Manne [10]). One approach to measuring this effect is to include, in the microeconomic energy model, macroeconomic equations relating the level of consumer purchases both to prices of goods and to macroeconomic variables such as gross national product. These additional equations can be incorporated into the equilibrium formulation and the techniques discussed in Josephy [11] can be applied to the generalized equation formulation of the enlarged equilibrium problem.

APPENDIX

DATA FOR HOGAN'S EXAMPLE

The following notation is used:

c_{ij} : coal in coal region i at increment j.

o_{ij} : oil in oil region i at increment j.

cT_{ij}: coal transported from coal region i to demand region j.

oT_{ij}: oil transported from oil region i to refinery j.

L_{ij} : light oil transported from refinery i to demand region j.

H_{ij} : heavy oil transported from refinery i to demand region j.

TABLE 1

QUANTITY AND PRICE NOTATION

Quantity and Price	Good	Region
q_1 , p_1	Coal	1
q_2 , p_2	Coal	2
q_3 , p_3	Light Oil	1
q_4 , p_4	Light Oil	2
q_5 , p_5	Heavy Oil	1
q_6 , p_6	Heavy Oil	2

The following data specify the supply side of the economic equilibrium problem.

Coal production level limits:

coal region 1: $0 \leq c_{11} \leq 300$, $0 \leq c_{12} \leq 300$, $0 \leq c_{13} \leq 400$;

coal region 2: $0 \leq c_{21} \leq 300$, $0 \leq c_{22} \leq 300$, $0 \leq c_{23} \leq 600$.

Oil production level limits:

oil region 1: $0 \leq 0_{11} \leq 1100$, $0 \leq 0_{12} \leq 1200$;

oil region 2: $0 \leq 0_{21} \leq 1300$, $0 \leq 0_{22} \leq 1100$.

Transportation flow balance:

coal regions: $c_{i1} + c_{i2} + c_{i3} - cT_{i1} - cT_{i2} - cT_{i3} = 0 \quad i=1,2$;

oil regions: $o_{i1} + o_{i2} - oT_{i1} - oT_{i2} = 0 \quad i=1,2$.

Refinery yields:

	Refinery 1	Refinery 2
Light Oil	.6	.5
Heavy Oil	.4	.5

Transportation flow balance:

light oil, refinery 1: $L_{11} + L_{12} - .6\ (oT_{11} + oT_{21}) = 0$

light oil, refinery 2: $L_{21} + L_{22} - .5\ (oT_{12} + oT_{22}) = 0$

heavy oil, refinery 1: $H_{11} + H_{12} - .4\ (oT_{11} + oT_{21}) = 0$

heavy oil, refinery 2: $H_{21} + H_{22} - .5\ (oT_{12} + oT_{22}) = 0$

Demand Constraints:

coal: $cT_{11} + cT_{21} \geq q_1$

$cT_{12} + cT_{22} \geq q_2$

light oil: $L_{11} + L_{21} \geq q_3$

$L_{12} + L_{22} \geq q_4$

heavy oil: $H_{11} + H_{21} \geq q_5$

$H_{12} + H_{22} \geq q_6$

Resource constraints

$$\text{steel (S): } 12{,}000. \geq c_{11}+2c_{12}+3c_{13}+c_{21}+4c_{22}+5c_{23}+4o_{12}+2o_{22}$$

$$\text{capital (K): } 35{,}000. \geq c_{11}+5c_{12}+10c_{13}+c_{21}+5c_{22}+6c_{23}+10o_{12}+15o_{22}$$

Production cost function:

$$5c_{11}+6c_{12}+8c_{13}+4c_{21}+5c_{22}+7c_{23}+o_{11}+1.5o_{12}+1.25o_{21}+1.5o_{22}+$$

$$+cT_{11}+2.5cT_{12}+0.75cT_{21}+2.75cT_{22}+8.5oT_{11}+8oT_{12}+$$

$$+10.5oT_{21}+7oT_{22}+L_{11}+1.2L_{12}+L_{21}+1.5L_{22}+H_{11}+1.2H_{12}+$$

$$+H_{21}+1.5H_{22}$$

The following data specify the demand side of the economic *equilibrium problem.*

The initial center point of the consumer demand curve:

q^o = 1000, 1000, 1200, 1200, 1000, 1000 ;

p^o = 12, 12, 16, 16, 12, 12 .

The elasticity matrix e:

$$e = \begin{bmatrix} -.75 & 0 & .1 & 0 & .2 & 0 \\ 0 & -.75 & 0 & .1 & 0 & .2 \\ .1 & 0 & -.5 & 0 & .2 & 0 \\ 0 & .1 & 0 & -.5 & 0 & .2 \\ .2 & 0 & .1 & 0 & -.5 & 0 \\ 0 & .2 & 0 & .1 & 0 & -.5 \end{bmatrix}$$

Acknowledgements

Parts of the research reported in this paper are based on the author's doctoral thesis submitted to the Department of Industrial Engineering at the University of Wisconsin-Madison, written under the direction of Professor Stephen M. Robinson.

I wish to express my sincere gratitude to Professor Stephen M. Robinson for his advise and support. His continued encouragement, guidance and understanding throughout my graduate studies have been invaluable to me.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

(14) MRC-TSR

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: 1971
2. GOVT ACCESSION NO.:
3. RECIPIENT'S CATALOG NUMBER: (9) Technical
4. TITLE (and Subtitle): (6) A NEWTON METHOD FOR THE PIES ENERGY MODEL.
5. TYPE OF REPORT & PERIOD COVERED: Summary Report - no specific reporting period
6. PERFORMING ORG. REPORT NUMBER:
7. AUTHOR(s): (10) Norman H. Josephy
8. CONTRACT OR GRANT NUMBER(s): (15) ~~DCR 74-20584~~ DAAG29-75-C-0024; NSF-MCS 74-20584 A02
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 5 - Mathematical Programming and Operations Research
11. CONTROLLING OFFICE NAME AND ADDRESS: See Item 18 below.
12. REPORT DATE: (11) June 1979
13. NUMBER OF PAGES: 29
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): (12) 33
15. SECURITY CLASS. (of this report): UNCLASSIFIED
15a. DECLASSIFICATION/DOWNGRADING SCHEDULE:
16. DISTRIBUTION STATEMENT (of this Report): Approved for public release; distribution unlimited.
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report):
18. SUPPLEMENTARY NOTES:

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D.C. 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

PIES Energy Model
Economic Equilibrium
Newton's Method

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Newton's method is a well known and often applied technique for computing a zero of a nonlinear function. By using the theory of generalized equations, a Newton method has been developed to solve problems arising in both mathematical programming and mathematical economics (see Josephy [11]).

We show how Newton's method for generalized equations can be applied to the economic equilibrium problem of the Project Independence Evaluation System (PIES) Energy Model. Solutions to a simplified version of PIES are obtained using a Newton method, and comparisons are made to solutions which have appeared in the literature (Hogan [9]).

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