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NONLINEAR SCHRÖDINGER EVOLUTION EQUATIONS

H. Brezis<sup>1</sup> and T. Gallouet

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ABSTRACT

We consider the nonlinear Schrödinger equation

$$\begin{cases} i \frac{\partial u}{\partial t} - \Delta u + k|u|^2 u = 0 & \text{in } \Omega \times [0, \infty) \\ u(x, t) = 0 & \text{in } \partial\Omega \times [0, \infty) \\ u(x, 0) = u_0(x) & \text{in } \Omega \end{cases}$$

where  $\Omega$  is a bounded domain or an exterior domain of  $\mathbb{R}^2$ . Such an equation has been extensively studied when  $\Omega = \mathbb{R}^2$ , but the methods do not apply if  $\Omega \neq \mathbb{R}^2$ . We prove that there exists a unique global smooth solution if  $k \geq 0$  or if  $k < 0$  and  $|k| \int |u_0|^2 < 4$ . The proof relies on a new interpolation-embedding inequality:

$$\|u\|_{L^\infty} \leq C[1 + \sqrt{\log(1 + \|u\|_{H^2})}] \quad \text{for every } u \in H^2 \quad \text{with } \|u\|_{H^1} \leq 1.$$

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SIGNIFICANCE AND EXPLANATION

→ The nonlinear Schrödinger equation occurs in the study of some problems in nonlinear optics (propagation of laser beams through the atmosphere or in a plasma). ~~It has been considered by many authors.~~ → The main novelty of the present result is that it applies to the propagation of beams in channels (instead of the whole space). The proof relies on a new Sobolev-Orlicz embedding inequality which could be useful in other situations. ↖

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NONLINEAR SCHRÖDINGER EVOLUTION EQUATIONS

H. Erezis<sup>1</sup> and T. Gallouet

Let  $\Omega$  be a domain in  $\mathbb{R}^2$  with compact smooth boundary  $\Gamma$  ( $\Omega$  could be for example a bounded domain or an exterior domain). Consider the equation

$$(1) \quad \begin{cases} i \frac{\partial u}{\partial t} - \Delta u + k|u|^2 u = 0 & \text{in } \Omega \times [0, \infty) \\ u(x, t) = 0 & \text{in } \Gamma \times [0, \infty) \\ u(x, 0) = u_0(x) \end{cases},$$

where  $u(x, t)$  is a complex valued function and  $k \in \mathbb{R}$  is a constant. Problem (1) which occurs in nonlinear optics when  $\Omega = \mathbb{R}^2$  has been extensively studied in this case (see [1], [2], [3], [5], [8]), but we are not aware of any known result when  $\Omega \neq \mathbb{R}^2$ .

Our main result is the following:

Theorem 1. Let  $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ . Assume that one of the following conditions holds

- (a) either  $k \geq 0$ ,
- (b) or  $k < 0$  and  $|k| \int |u_0(x)|^2 dx < 4$ .

Then there exists a unique solution of (1) such that

$$u \in C([0, \infty); H^2(\Omega)) \cap C^1([0, \infty); L^2(\Omega)).$$

The proof of Theorem 1 relies on several Lemmas. The first Lemma is of interest for its own sake; it is a new interpolation-embedding inequality.

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In what follows we denote by  $C$  various constants depending only on  $\Omega$ .

Lemma 2. We have

$$(2) \quad \|u\|_{L^\infty} \leq C(1 + \sqrt{\log(1 + \|u\|_{H^2})})$$

for every  $u \in H^2(\Omega)$  with  $\|u\|_{H^1} \leq 1$ .

Proof. It is well known that an  $H^2$  function on  $\Omega$  can be extended by an  $H^2$  function on  $\mathbb{R}^2$ . More precisely one can construct an extension operator  $P$  such that:

$P$  is a bounded operator from  $H^1(\Omega)$  into  $H^1(\mathbb{R}^2)$

$P$  is a bounded operator from  $H^2(\Omega)$  into  $H^2(\mathbb{R}^2)$

$Pu|_{\Omega} = u$  for every  $u \in H^1(\Omega)$ .

Let  $u \in H^2(\Omega)$  with  $\|u\|_{H^1} \leq 1$ . Let  $v = Pu$  and denote by  $\hat{v}$  the Fourier transform of  $v$ . We clearly have

$$(3) \quad \|(1 + |\xi|) \hat{v}\|_{L^2(\mathbb{R}^2)} \leq C$$

$$(4) \quad \|(1 + |\xi|^2) \hat{v}\|_{L^2(\mathbb{R}^2)} \leq C\|u\|_{H^2(\Omega)}$$

$$(5) \quad \|u\|_{L^\infty(\Omega)} \leq \|v\|_{L^\infty(\mathbb{R}^2)} \leq C\|\hat{v}\|_{L^1(\mathbb{R}^2)}$$

For  $R > 0$  we write

$$\begin{aligned} \|\hat{v}\|_{L^1} &= \int_{|\xi| < R} |\hat{v}(\xi)| d\xi + \int_{|\xi| \geq R} |\hat{v}(\xi)| d\xi = \int_{|\xi| < R} (1 + |\xi|) |\hat{v}(\xi)| \frac{1}{1 + |\xi|} d\xi \\ &\quad + \int_{|\xi| \geq R} (1 + |\xi|^2) |\hat{v}(\xi)| \frac{1}{1 + |\xi|^2} d\xi \\ &\leq C \left[ \int_{|\xi| < R} \frac{1}{(1 + |\xi|)^2} d\xi \right]^{1/2} + C\|u\|_{H^2} \left[ \int_{|\xi| \geq R} \frac{1}{(1 + |\xi|^2)^2} d\xi \right]^{1/2} \end{aligned}$$

by Cauchy-Schwarz, (3) and (4). A straightforward computation leads to

$$\|\hat{v}\|_{L^1} \leq C[\log(1+R)]^{1/2} + C\|u\|_{H^2}(1+R)^{-1}$$

by every  $R \geq 0$ . We obtain (2) by choosing  $R = \|u\|_{H^2}$ .

Lemma 3. We have

$$(6) \quad \| |u|^2 u \|_{H^2} \leq C \|u\|_{L^\infty}^2 \|u\|_{H^2} \quad \text{for every } u \in H^2(\Omega).$$

Proof of Lemma 3. Let  $D$  denote any first order differential operator. For  $u \in H^2$  we have

$$|D^2(|u|^2 u)| \leq C(|u|^2 |D^2 u| + |u| |Du|^2),$$

and so

$$(7) \quad \| |u|^2 u \|_{H^2} \leq C \|u\|_{L^\infty}^2 \|u\|_{H^2} + C \|u\|_{L^\infty} \|u\|_{W^{1,4}}^2.$$

On the other hand an inequality of Gagliardo-Nirenberg (see [6]) implies that

$$(8) \quad \|u\|_{W^{1,4}} \leq C \|u\|_{L^\infty}^{1/2} \|u\|_{H^2}^{1/2}.$$

Combining (7) and (8) we obtain (6).

Finally we recall the following well known result essentially due to Segal [7]:

Lemma 4. Assume  $H$  is a Hilbert space and  $A : D(A) \subset H \rightarrow H$  is an  $m$ -accretive linear operator. Assume  $F$  is a mapping from  $D(A)$  into itself which is Lipschitz on every bounded set of  $D(A)$ . Then for every  $u_0 \in D(A)$ , there exists a unique solution  $u$  of the equation

$$\begin{cases} \frac{du}{dt} + Au = Fu \\ u(0) = u_0 \end{cases}$$

defined for  $t \in [0, T_{\max})$  such that

$$u \in C^1([0, T_{\max}); H) \cap C([0, T_{\max}); D(A))$$

with the additional property that

$$\begin{cases} \text{either } T_{\max} = \infty \\ \text{or } T_{\max} < \infty \text{ and } \lim_{t \uparrow T_{\max}} \|u(t)\| + \|\Delta u(t)\| = \infty. \end{cases}$$

Proof of Theorem 1. We apply Lemma 4 in  $H = L^2(\Omega)$  to  $\Delta u = i\Delta u$ ,  $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ ,  $Fu = ik|u|^2u$ . We shall show that  $T_{\max} = \infty$  by proving that  $\|u(t)\|_{H^2}$  remains bounded on every finite time interval.

First we multiply (1) by  $\bar{u}$  and consider the imaginary part. This leads to

$$(9) \quad \|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2.$$

Next we multiply (1) by  $\frac{\partial \bar{u}}{\partial t}$  and consider the real part. This leads to

$$(10) \quad \frac{1}{2} \int |\nabla u(x,t)|^2 dx + \frac{k}{4} \int |u(x,t)|^4 dx \equiv E_0$$

where

$$E_0 = \frac{1}{2} \int_{\Omega} |\nabla u_0(x)|^2 dx + \frac{k}{4} \int_{\Omega} |u_0(x)|^4 dx.$$

We claim that  $\|u(t)\|_{H^1}$  remains bounded for  $t > 0$ . Indeed, this is clear when  $k \geq 0$ . While if  $k < 0$  we have

$$(11) \quad \int |\nabla u(x,t)|^2 \leq \frac{|k|}{2} \int |u(x,t)|^4 dx + 2E_0.$$

On the other hand an inequality of Gagliardo and Nirenberg ([6]) shows that<sup>(1)</sup>

(1) In order to obtain the constant  $1/2$  one proceeds as follows. For

$\varphi \in C_0^\infty(\mathbb{R}^2)$  we have  $|\varphi(x_1, x_2)| \leq \frac{1}{2} \int_{-\infty}^{+\infty} |\varphi_{x_1}(t, x_2)| dt$ ,  
 $|\varphi(x_1, x_2)| \leq \frac{1}{2} \int_{-\infty}^{+\infty} |\varphi_{x_2}(x_1, s)| ds$ . Thus  $\int_{\mathbb{R}^2} |\varphi|^2 dx \leq \frac{1}{4} \int_{\mathbb{R}^2} |\varphi_{x_1}| dx \int_{\mathbb{R}^2} |\varphi_{x_2}| dx$ .

Choosing  $\varphi = |u|^2$  leads to

$\int |u|^4 dx \leq \int |u|^2 dx (\int |u_{x_1}|^2 dx)^{1/2} (\int |u_{x_2}|^2 dx)^{1/2} \leq \frac{1}{2} \int |u|^2 dx \int |\nabla u|^2 dx$ .

$$(12) \quad \int |u|^4 dx \leq \frac{1}{2} \int |u|^2 dx \int |\nabla u|^2 dx \\ = \frac{1}{2} \int |u_0|^2 dx \int |\nabla u|^2 dx .$$

Combining (11), (12) and assumption (b) in Theorem 1 we see that

$$(13) \quad \|u(t)\|_{H^1} \leq C$$

where  $C$  is independent of  $t$ .

We now denote by  $S(t)$  the  $L^2$  isometry group generated by  $-A$ . From (1) we have

$$u(t) = S(t)u_0 + ik \int_0^t S(t-s)|u(s)|^2 u(s) ds$$

and so

$$Au(t) = S(t)Au_0 + ik \int_0^t S(t-s)A[|u(s)|^2 u(s)] ds .$$

Thus

$$(14) \quad \|Au(t)\|_{L^2} \leq \|Au_0\|_{L^2} + |k| \int_0^t \|A[|u(s)|^2 u(s)]\|_{L^2} ds .$$

Lemma 3 implies that

$$\|A[|u(s)|^2 u(s)]\|_{L^2} \leq C \|u(s)\|_{L^\infty}^2 \|u(s)\|_{H^2} .$$

From Lemma 2 and estimate (13) we deduce that

$$\|u(s)\|_{L^\infty} \leq C(1 + \sqrt{\log(1 + \|u(s)\|_{H^2})}) .$$

Hence (14) leads to

$$(15) \quad \|u(t)\|_{H^2} \leq C + C \int_0^t \|u(s)\|_{H^2}^2 [1 + \log(1 + \|u(s)\|_{H^2})] ds .$$

We denote by  $G(t)$  the RHS in (15); thus

$$G'(t) = C \|u(t)\|_{H^2} [1 + \log(1 + \|u(t)\|_{H^2})] \leq CG(t) [1 + \log(1 + G(t))] .$$

Consequently

$$\frac{d}{dt} \log[1 + \log(1 + G(t))] \leq C$$

and we find an estimate for  $\|u(t)\|_{H^2}$  of the form

$$\|u(t)\|_{H^2} \leq e^{\alpha} e^{\beta t}$$

for some constants  $\alpha$  and  $\beta$ . Therefore  $\|u(t)\|_{H^2}$  remains bounded on every finite time interval and so we must have  $T_{\max} = \infty$ .

Remarks. 1) The proof of Theorem 1 leads to an estimate of the form

$\|u(t)\|_{L^\infty} \leq \alpha e^{\beta t}$ . We do not know whether  $\|u(t)\|_{L^\infty}$  remains actually bounded as  $t \rightarrow \infty$ .

2) When  $k < 0$  and  $|k| \int |u_0|^2 > 4$ , it is known (see [4],[2]) if  $\Omega = \mathbb{R}^2$  that the solution of (1) corresponding to some initial conditions may blow up in finite time. A similar phenomenon presumably occurs when  $\Omega \neq \mathbb{R}^2$ .

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20. ABSTRACT - Cont'd.

where  $\Omega$  is a bounded domain or an exterior domain of  $\mathbb{R}^2$ . Such an equation has been extensively studied when  $\Omega = \mathbb{R}^2$ , but the methods do not apply if  $\Omega \neq \mathbb{R}^2$ . We prove that there exists a unique global smooth solution if  $k \geq 0$  or if  $k < 0$  and  $|k| \int |u_0|^2 < 4$ . The proof relies on a new interpolation-embedding inequality:

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