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by
RONALD W. SHEPHARD
and
KING-TIM MAK

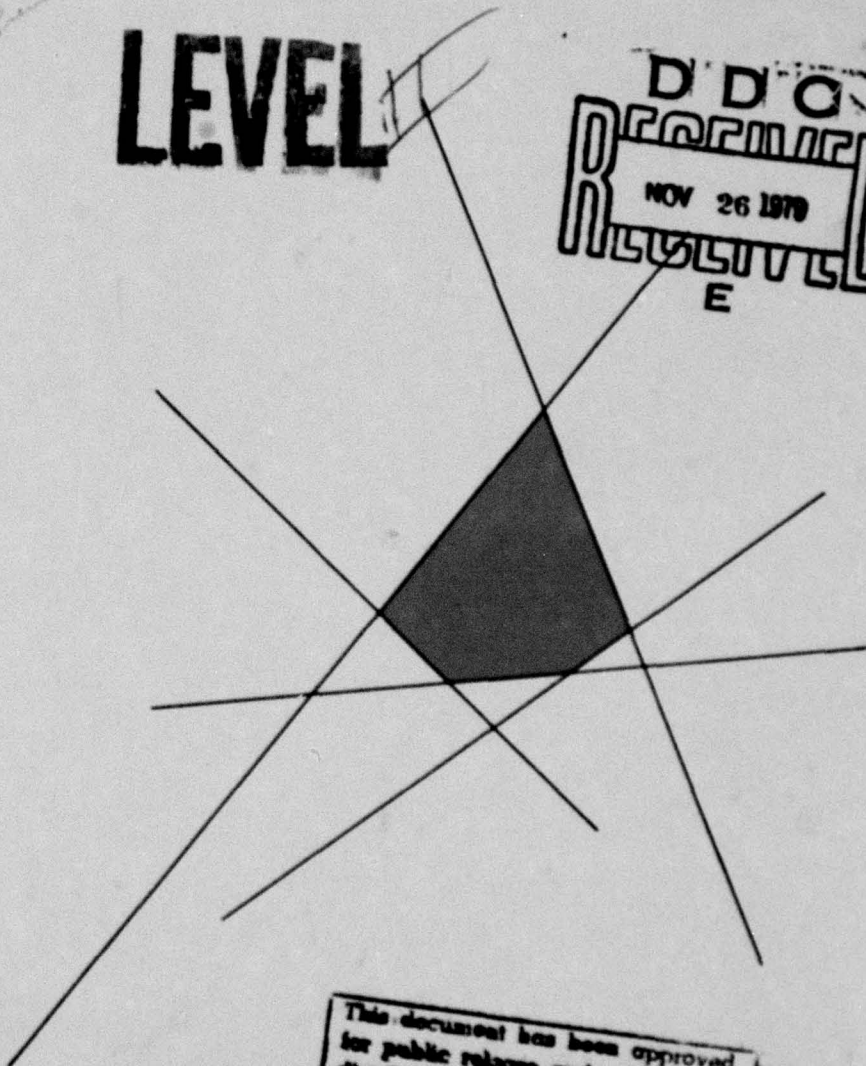
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6 DYNAMIC THEORY OF PRODUCTION
CORRESPONDENCES: LAWS OF RETURN[†] Revised

9 Research rept.,

by

10 Ronald W. Shephard
Department of Industrial Engineering
and Operations Research
University of California, Berkeley

and

King-Tim Mak
Operations Research Center
University of California, Berkeley

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ABSTRACT

Various laws of return are developed in a dynamic production context for both maximal output rate and period of production in the case of exhaustible resources.

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CHAPTER 3

LAWS OF RETURN

The laws to be considered here are suggested by Turgot's law of diminishing returns, on the intensive margin, so called. Loosely worded, in the context of static economic analysis, this law states: as equal quantities of capital and labor are applied successively to a given plot of land, the output resulting from these applications will increase monotonically at first up to a certain point, after which further applications will result in steadily decreasing product increments tending to zero. This expression of a law of return postulates smooth behavior in fine structure which cannot be justified by generally applicable properties (axioms) for production structure. As the law is often reflected by the production functions used in neoclassical economic analysis, unbounded increase in output rate is possible for any bound upon the input rates of essential factors when the other factors are increased indefinitely. As found in the literature such behavior is merely that implied by the concave production functions used and it has no particular significance for the phenomenology of production.

The original motivation for such laws of return was the recognition of the restraint to agricultural output arising from the scarcity of land as a primary factor. What then appeared to be obvious regarding land as an input is no longer so simple, since the techniques of farming have greatly expanded, yielding great increases in output to nourish even larger populations of humans, but at the price of increased dependence upon manufactured products. For modern technology, both farming and manufacturing, the limiting role of inputs is considerably

more complicated, where some may be complete substitutes for others as well as being complementary to still others, and institutionally any factor may be restricted whether or not it is a primary factor.

As one views such possibilities today, limitations of world resources present serious limits on some inputs. Energy is increasingly more dear and environmental aims present serious limits on production to preserve habitat. Thus laws of return are a central issue in the economic theory of production, both as to output rates, and span of possible output as in the case of exhaustible resources.

For steady state (static) models of production, i.e., the case of constant input and output rates, (see Section 2.6) laws of return have been established in (Shephard, 1970:b) for technologies with single output. There, it has been shown for an *essential* subset of the factors that there exist bounds upon the constant input rate of these factors such that the constant output rate obtainable is bounded no matter how much the constant rates of the other factors are increased. Further it was shown under the axioms for such structures (see Section 2.6), by counter-example, that constant output rate need not be bounded for all bounds upon the constant input rates of an essential subset of the factors. The extension of these results to the case of multiple outputs was made in (Shephard and Färe, 1974).

A law of return so expressed for the static model of production is one of a law of bounded output rate. It is suggested for input and output rates which are not constant, i.e., for the dynamic structure of production, that a law of bounded output rate may hold, i.e., if time histories for *essential* factors are subject to an upper bound on input rate, the related output rate histories will be bounded in some way under

unlimited increase in the maximal time rate of the input rate histories of the other factors. In this connection the axiom on disposability of outputs plays a role, and there are three cases of a law of bounded output rate to consider, corresponding to L.6, L.6S and L.6SS. One may also express different laws of bounded output rate depending upon the input rate histories permitted for consideration. The most all inclusive case is one where the entire space $(L_{\infty})_+^n$ is permitted for vectors of input histories, and vectors of output rate histories are taken from $(L_{\infty})_+^m$. Here the stronger axioms E.S and E are needed, with a weak* topology used in the latter case. On the other hand one might restrict consideration of output rate history vectors to the subset $(\tilde{L}_{\infty})_+^m$ of vectors u from $(L_{\infty})_+^m$ with component histories which are summable, since infinite total production may not be of interest, but the time extension of positive output rate may be permitted to be unbounded as a possibility for time substitution. Then the same norms and topologies used for the case $x \in (L_{\infty})_+^n$, $u \in (L_{\infty})_+^m$, may be applied to express somewhat weaker laws of return using the axioms $\tilde{E}.S$ and \tilde{E} .

A third case is one where the dynamic production correspondence is taken as $u \in (L_1)_+^m \rightarrow \mathbb{L}_1(u) \in 2^{(L_1)_+^n}$, i.e., input and output histories are summable with the norms

$$||u_i|| = \int_0^{\infty} |u_i(t)| dv_i(t) \quad i \in \{1, 2, \dots, m\}$$

$$||u|| = \text{Max}_i \{ ||u_i|| \}$$

$$\|x_i\| = \int_0^{\infty} |x_i(t)| d\mu_i(t), \quad i \in \{1, 2, \dots, n\}$$

$$\|x\| = \text{Max}_i (\|x_i\|)$$

for the L_1 -spaces involved. With these norms, a law of return obtained by following an analysis similar to the other two cases is a law of bounded total output, using the weak axioms E_1S and E_1 .

There is still another dimension to consider. The time spans over which essential input rates may be or are applied positively need not be infinite, that is the support of an input may be bounded, and unbounded time substitutions for resources may not be permitted. Then the question arises how outputs may be limited by limitations on the intervals of time over which essential factors may be applied. Propositions of this type are laws of return for bounded intervals of application of essential factors.

Hence two general types of laws of return will be considered:

(1) Laws of Return for bounded input rates of essential factors, (2) Laws of Return for bounded intervals of essential factor application. Before taking up the details concerning these laws of return, it is useful to discuss briefly in the next two sections some concepts related to essentiality of inputs and jointness of outputs. It is possible that some inputs may be completely substituted for by another. Then any bound whatsoever on the input rate of some of these factors may have no limitation on output rates as the input rates of the other factors are increased indefinitely. Further, some outputs may be linked or jointly involved in production, so that limiting the output rate of one may limit that of

another. These interactions need to be clarified before laws of return are formulated.

3.1 Essentiality of Factors

The factors of production may be essential for outputs in several ways. Let $u = (v, w) \in (L_{\infty})_+^m$ be a vector of output histories with subvector v not empty and possibly being the entire output vector u with w empty. A subset $\{v_1, v_2, \dots, v_k\}$ of n factors, $(1 \leq k < n)$, may be essential for: (a) output subvectors $v \in (L_{\infty})_+^l$, $1 \leq l \leq m$, (b) scaled versions $(\theta_1 v_1, \dots, \theta_l v_l)$ of a given output subvector $v \in (L_{\infty})_+^l$, (c) for homogeneously scaled forms (θv) , $\theta \in (0, +\infty)$, of a given subvector $v \in (L_{\infty})_+^l$. These three kinds of essentiality correspond to different situations for disposal of outputs. The third case follows from the second merely by taking $\theta_1 = \theta_2 = \dots = \theta_l = \theta$ and will not be considered separately.

In the formation of the axioms a mixture of the disposal allowed by L.6S and L.6 was not considered, because all such possibilities between L.6, L.6S and L.6SS are matters of greater detail. For the purpose of the discussion at hand, it is sufficient to use the axiom L.6S.

As notation, let

$$D(v_1, v_2, \dots, v_k) = \left\{ x \in (L_{\infty})_+^n : x_{v_i} = 0, i \in \{1, 2, \dots, k\} \right\}.$$

The following definitions will be used.

Definition (3.1-1):

A proper subset $\{v_1, v_2, \dots, v_k\}$ of the n input factors is *Globally Essential* for a subset $\{\alpha_1, \alpha_2, \dots, \alpha_l\}$, $1 \leq l \leq m$, of the m net output rate histories if for every $u \in (L_\infty)_+^m$ with $L(u) \neq \emptyset$ and $(u_{\alpha_1}, u_{\alpha_2}, \dots, u_{\alpha_l}) > 0$, $L(u) \cap D(v_1, v_2, \dots, v_k) = \emptyset$.

Definition (3.1-2):

A proper subset $\{v_1, v_2, \dots, v_k\}$ of the n input factors is *Homogeneously Essential* for a subset $\{\alpha_1, \alpha_2, \dots, \alpha_l\}$, $1 \leq l \leq m$, of the m net output rate histories if for every $u \in (L_\infty)_+^m$ with $L(u) \neq \emptyset$, $(u_{\alpha_1}, u_{\alpha_2}, \dots, u_{\alpha_l}) > 0$ and $L(u) \cap D = \emptyset$, $L(\theta_1 u_{\alpha_1}, \theta_2 u_{\alpha_2}, \dots, \theta_l u_{\alpha_l}, w) \cap D(v_1, v_2, \dots, v_k) = \emptyset$ for all $\theta_i \in (0, +\infty)$, $i \in \{1, 2, \dots, l\}$ and all $w \in (L_\infty)_+^{n-l}$ with $L(\theta_1 u_{\alpha_1}, \theta_2 u_{\alpha_2}, \dots, \theta_l u_{\alpha_l}, w) \neq \emptyset$.

It is convenient to use the efficient subsets $E(u)$ instead of the entire set $L(u)$. The following justifies this procedure.

Proposition (3.1-1): (Färe, 1972)

For any $u \in (L_\infty)_+^m$, $L(u) \cap D(v_1, v_2, \dots, v_k) = \emptyset$ if and only if $E(u) \cap D(v_1, v_2, \dots, v_k) = \emptyset$.

Proof:

In case $L(u)$ is empty, $E(u) \subset L(u)$ is likewise empty. Hence suppose $L(u)$ is not empty. Then $L(u) \cap D(v_1, v_2, \dots, v_k)$ empty implies $E(u) \cap D(v_1, v_2, \dots, v_k)$ empty, since

CLOSURE $\mathbb{E}(u) \subset \mathbb{L}(u)$. Conversely, suppose $\mathbb{L}(u)$ not empty with
 CLOSURE $\mathbb{E}(u) \cap D(v_1, v_2, \dots, v_k)$ empty. Then

$$\text{CLOSURE } \mathbb{E}(u) \subset \left((L_{\infty}^n)_+ \sim D(v_1, v_2, \dots, v_k) \right) .$$

By Proposition (2.2.4-2),

$$\mathbb{L}(u) \subset \left(\text{CLOSURE } \mathbb{E}(u) + (L_{\infty}^n)_+ \right) .$$

Hence

$$\mathbb{L}(u) \subset \left((L_{\infty}^n)_+ \sim D(v_1, v_2, \dots, v_k) \right)$$

and $\mathbb{L}(u) \cap D(v_1, v_2, \dots, v_k)$ is empty.

By exactly similar arguments one obtains:

Sub-Proposition (3.1-2):

For any $u = (v, w) \in (L_{\infty}^m)_+$, $v \in (L_{\infty}^l)_+$, $1 \leq l \leq m$, for all
 $\theta_i \in (0, +\infty)$, $i \in \{1, 2, \dots, l\}$, $\mathbb{L}(\theta_1 v_1, \dots, \theta_l v_l, w) \neq \emptyset$ and
 $w \in (L_{\infty}^{m-l})_+$, $\mathbb{L}(\theta_1 v_1, \dots, \theta_l v_l, w) \cap D(v_1, v_2, \dots, v_k) = \emptyset$,
 if and only if $\text{CLOSURE } \mathbb{E}(\theta_1 v_1, \dots, \theta_l v_l, w) \cap D(v_1, v_2, \dots, v_k) = \emptyset$.

3.2 Jointness of Outputs

The strongest, and perhaps the most familiar case, of jointness for multiple output rate histories is one where

$$P(x) = \left\{ u \in (L_{\infty})_+^m : u = \theta \alpha(x) \cdot u^0, u^0 \in (L_{\infty})_+^m, u_i^0 \geq 0 \forall i, \theta \in [0,1] \right\}$$

and $\alpha(x)$ is a functional $\alpha : x \in (L_{\infty})_+^n \rightarrow \alpha(x) \in R_+$, consistent with the axioms P.1, ..., P.6 and u^0 satisfies P.T.1, P.T.2. Then each output rate history bears a fixed relation to the others for $t \in [0, +\infty)$.

Here the jointness of output is complete, strict, and symmetric with respect to jointness of null output rate histories, i.e.,

$$u_i = 0 \iff u_j = 0 \text{ for all } j \in (\{1,2, \dots, m\} \sim i), i \in \{1,2, \dots, m\}.$$

At another extreme there can be no jointness of null output rate histories required, if the correspondence $x \rightarrow P(x)$ satisfies the axiom P.6SS, e.g., where

$$P(x) = \left\{ u \in (L_{\infty})_+^m : u \leq \alpha(x) \cdot u^0, u^0 \in (L_{\infty})_+^m, u_i^0 \geq 0 \forall i \right\}.$$

For the purpose of the laws of return to be discussed, all special relationships of how output rate histories may vary jointly with respect to each other over time is not of interest. Rather, jointness of null output rate history, i.e., *Null Jointness*, alone is of interest in this context, and this kind of joint relationship need not be symmetric.

Let $\beta, \gamma \subset \{1,2, \dots, m\}$ be two subsets of the net output histories, $\beta \neq \emptyset$ and $\gamma \neq \emptyset$, $\beta \cap \gamma = \emptyset$.

Definition (3.2-1):

A subset β of output histories is *Null Joint* with another subset γ if and only if for all $u \in (L_{\infty})_+^m$ with $\mathbb{L}(u) \neq 0$, $\bar{t}_{\beta} \leq \bar{t}_{\gamma}$ where $\bar{t}_{\beta} := \text{Min} \left\{ \bar{t}_{u_i} : i \in \beta \right\}$ and $\bar{t}_{\gamma} := \text{Min} \left\{ \bar{t}_{u_j} : j \in \gamma \right\}$.

The following proposition relates null jointness of subsets of output rate histories to essentiality of input factors.

Proposition (3.2-1):

If a subset $\{v_1, v_2, \dots, v_k\}$ of input factors is globally essential for a subset β , and β is null joint with another subset γ , then $\{v_1, v_2, \dots, v_k\}$ is globally essential for the subset $(\beta \cup \gamma)$.

For arbitrary $u \in (L_{\infty})_+^m$ with $\mathbb{L}(u) \neq \emptyset$ and $u_i \geq 0$, $i \in \beta$, it is true that $\bar{t}_{\beta} > 0$. Then β being null joint with γ implies $\bar{t}_{\gamma} \geq \bar{t}_{\beta} > 0$, and it follows that $\{u_j, j \in \gamma\} \geq 0$.

Hence, if $\mathbb{L}(u \mid u_i \geq 0, i \in \beta) \cap D(v_1, v_2, \dots, v_k)$ is empty, $\mathbb{L}(u \mid u_i \geq 0, i \in \beta; \{u_j, j \in \gamma\} \geq 0) \cap D(v_1, v_2, \dots, v_k)$ is empty. And it follows that $\mathbb{L}(u \mid u_i \geq 0, i \in \beta; u_j \geq 0, j \in \gamma) \cap D(v_1, v_2, \dots, v_k)$ is empty. Thus, $\{v_1, v_2, \dots, v_k\}$ is essential for $\beta \cup \gamma$ if it is globally essential for β and β is null joint with γ .

Note that the same argument can be applied to the case of "output rate history homogenous essentiality," and will not be repeated here.

In this way, outputs may be linked concerning essentiality of subsets of the factors. For the purpose of the text to follow, a single subset $\{\alpha_1, \alpha_2, \dots, \alpha_l\}$, $1 \leq l \leq m$, of the net output rate histories will suffice for consideration of the laws of return.

3.3 Limitation of Output Rates by Input Rates of Essential Factors of Production

Global limitation of output rates is expressed by the following two definitions:

Definition (3.3-1):

A proper subset $\{v_1, v_2, \dots, v_k\}$ of n input factors is *Globally Output Rate Weak Limitational* for a subset $\{\alpha_1, \alpha_2, \dots, \alpha_\ell\}$, $1 \leq \ell \leq m$, of m net output rate histories if for every $u^0 \in (L_\infty)_+^m$ with $v^0 := (u_{\alpha_1}^0, u_{\alpha_2}^0, \dots, u_{\alpha_\ell}^0) > 0$, and $L(u^0) \neq \emptyset$, there exists a positive scalar $B(u^0)$ such that $\left\{ x \in (L_\infty)_+^n : \|x_{v_1}, x_{v_2}, \dots, x_{v_k}\| \leq B(u^0) \right\} \cap L(u) = \emptyset$ for all $u \in (L_\infty)_+^m$ with subvector $v := (u_{\alpha_1}, u_{\alpha_2}, \dots, u_{\alpha_\ell}) \geq v^0$, and $L(u) \neq \emptyset$.

Definition (3.3-2):

A proper subset $\{v_1, v_2, \dots, v_k\}$ of n input factors is *Globally Output Rate Strong Limitational* for a subset $\{\alpha_1, \alpha_2, \dots, \alpha_\ell\}$, $1 \leq \ell \leq m$ of m output rate histories, if for each positive bound $B \in \mathbb{R}_{++}$, and for every $u^0 \in (L_\infty)_+^m$ with $v^0 := (u_{\alpha_1}^0, u_{\alpha_2}^0, \dots, u_{\alpha_\ell}^0) > 0$ and $L(u^0) \neq \emptyset$, there exists a positive scalar $\theta(B, u^0)$ such that $\left\{ x \in (L_\infty)_+^n : \|x_{v_1}, x_{v_2}, \dots, x_{v_k}\| \leq B \right\} \cap L(u) = \emptyset$ for all $u \in (L_\infty)_+^m$ with $v := (u_{\alpha_1}, u_{\alpha_2}, \dots, u_{\alpha_\ell}) \geq \theta(B, v^0) \cdot v^0$, $L(u) \neq \emptyset$.

These first two definitions apply to the situation where outputs are freely disposable, i.e., axiom L.6SS applies. For the situation where L.6S is used, the following two definitions apply.

Definition (3.3-3):

A proper subset $\{v_1, v_2, \dots, v_k\}$ of n input factors is *Homogeneously Output Rate Weak Limitational* for a subset $\{\alpha_1, \alpha_2, \dots, \alpha_\ell\}$, $1 \leq i \leq m$, of m net output rate histories, if for every $u^0 \in (L_\infty)_+^m$ with $L(u^0) \neq \emptyset$ and $v^0 := (u_{\alpha_1}^0, u_{\alpha_2}^0, \dots, u_{\alpha_\ell}^0) > 0$, there exists a positive bound $B(u^0)$ such that $\left\{ x \in (L_\infty)_+^n : \|x_{v_1}, x_{v_2}, \dots, x_{v_k}\| \leq B(v^0) \right\} \cap L(u)$ is empty for all $u \in (L_\infty)_+^m$ with $u_{\alpha_i} = \theta_i u_{\alpha_i}^0$, $\theta_i \in [1, +\infty)$, ($i = 1, 2, \dots, \ell$), and $L(u) \neq \emptyset$.

Definition (3.3-4):

A proper subset $\{v_1, v_2, \dots, v_k\}$ of n input factors is *Homogeneously Output Rate Strong Limitational* for a subset $\{\alpha_1, \alpha_2, \dots, \alpha_\ell\}$, $1 \leq i \leq m$, of m output rate histories, if for each positive bound $B \in R_{++}$ and for every $u^0 \in (L_\infty)_+^m$, $L(u^0) \neq \emptyset$, with $v^0 := (u_{\alpha_1}^0, u_{\alpha_2}^0, \dots, u_{\alpha_\ell}^0) > 0$, there exists positive scalars $\theta_i(B, u^0)$, $i = 1, 2, \dots, \ell$, such that $\left\{ x \in (L_\infty)_+^n : \|x_{v_1}, x_{v_2}, \dots, x_{v_k}\| \leq B \right\} \cap L(u) = \emptyset$ for all $u \in (L_\infty)_+^m$, $L(u) \neq \emptyset$, with $u_{\alpha_i} = \theta_i u_{\alpha_i}^0$, $\theta_i \geq \theta_i(B, u^0)$, ($i = 1, 2, \dots, \ell$), $L(u) \neq \emptyset$.

Definitions (3.3-3) and (3.3-4) refer to bounding the scaling of individual output rate histories. The time distributions of the output rate histories are preserved by this scaling. If one restricts $\theta_1 = \theta_2 = \dots = \theta_\ell$, the relation between output histories is preserved as well as their time distributions.

To start with, the case of fewest exceptions will be considered by invoking the stronger axioms E.S, E for output vectors $u \in (L_{\infty})_+^m$ not restricted to be summable, with a globally essential subset of factors $\{v_1, v_2, \dots, v_k\}$ for output components $\{a_1, a_2, \dots, a_\ell\}$, $1 \leq \ell \leq m$. Also, the strong axiom L.6SS will be taken for the input correspondence. The relationship between essentiality of factors and limitationality of output is given by the following several propositions.

Proposition (3.3-1):

A proper subset $\{v_1, v_2, \dots, v_k\}$ of n input factors is globally output rateweak limitational for a subset $\{a_1, a_2, \dots, a_\ell\}$ of m output histories, $1 \leq \ell \leq m$, if it is globally essential for $\{a_1, a_2, \dots, a_\ell\}$, and axioms E.S and L.6SS apply.

Proof of Proposition (3.3-1):

Consider an arbitrary $u^0 \in (L_{\infty})_+^m$ with $L(u^0) \neq \emptyset$ and $v^0 := (u_{a_1}^0, u_{a_2}^0, \dots, u_{a_\ell}^0) > 0$. Denote by $w^0 := (u_{a_{\ell+1}}^0, u_{a_{\ell+2}}^0, \dots, u_{a_m}^0)$ the remaining subvector of u^0 , and let the order of the components be chosen so that $u^0 = (v^0, w^0)$. Since $\{v_1, v_2, \dots, v_k\}$ is globally essential for $\{a_1, a_2, \dots, a_\ell\}$, by supposition, $L(u^0, w^0) \cap D(v_1, v_2, \dots, v_k) = \emptyset$. Accordingly, $\text{CLOSURE } E(v^0, w^0) \cap D(v_1, v_2, \dots, v_k) = \emptyset$ (see Proposition (3.1-1)). Since $D(v_1, v_2, \dots, v_k)$ is a closed nonempty set of $(L_{\infty})_+^n$, the "distance" of $x \in \text{CLOSURE } E(v^0, w^0)$ from $D(v_1, v_2, \dots, v_k)$, defined by

$$d(x, D(v_1, v_2, \dots, v_k)) := \text{Inf} \{ \|x - y\| : y \in D(v_1, v_2, \dots, v_k) \},$$

is strictly positive and continuous in $x \in \text{CLOSURE } E(v^0, w^0)$. (See Berge, 1963, p. 86.) In the norm topology for $(L_{\infty})_+^m$, axiom E.S implies

that CLOSURE $E(v^0, w^0)$ is compact, establishing the existence of an input vector x^* of input rate histories belonging to CLOSURE $E(v^0, w^0)$ such that

$$0 < d(x^*, D(v_1, v_2, \dots, v_k)) = \min_x \{d(x, D(v_1, v_2, \dots, v_k)) : x \in \text{CLOSURE } E(v^0, w^0)\}.$$

Consider the bound

$$B(u^0) := \frac{d(x^*, D(v_1, v_2, \dots, v_k))}{2}.$$

Since axiom L.6SS is taken to apply, $L(v, w^0) \subset L(v^0, w^0) = L(u^0)$

for all $v \geq v^0$. Moreover, by Proposition (2.2.4-2),

$$L(u) \cap \left\{ x \in (L_\infty)_+^n : \|x_{v_1}, x_{v_2}, \dots, x_{v_k}\| \leq B(u^0) \right\} \text{ is empty for } u = (v, w^0),$$

$v \geq v^0$, and $\{v_1, v_2, \dots, v_k\}$ is globally weak limitational for the subset $\{\alpha_1, \alpha_2, \dots, \alpha_\ell\}$, $1 \leq \ell \leq m$, of m output rate histories.

Note that since $L(v^0, w^0) \subset L(v^0, 0)$ for all $w^0 \in (L_\infty)_+^{m-\ell}$, a

bound $B(u^0)$ as constructed can be made independent of the particular w^0 involved in u^0 .

Sub-Proposition (3.3-1):

Proposition (3.3-1) holds under axiom E if a weak* topology is used for $(L_\infty)_+^n$.

Proposition (3.3-2):

A proper subset $\{v_1, v_2, \dots, v_k\}$ of n factors is globally output rate weak limitational for a subset $\{\alpha_1, \alpha_2, \dots, \alpha_\ell\}$, $(1 \leq \ell \leq m)$, of output rate histories if and only if it is globally essential for

$\{\alpha_1, \alpha_2, \dots, \alpha_\ell\}$.

Proof:

The "if" part was proven in Proposition (3.3-1) using either E.S., or \mathbb{E} with the weak* topology.

To prove the "only if" part, assume that $\{v_1, v_2, \dots, v_k\}$ is not globally essential for $\{\alpha_1, \alpha_2, \dots, \alpha_l\}$. Then for some $u^0 \in (L_\infty)_+^m$, $v^0 := \left(u_{\alpha_1}^0, u_{\alpha_2}^0, \dots, u_{\alpha_l}^0 \right) > 0$ there exists $x^0 \neq 0$ with $x^0 \in \mathbb{L}(u^0) \cap D(v_1, v_2, \dots, v_k)$. By the axiom L.4.2 (Section 2.2), there exists for all $\theta \in (0, +\infty)$ a scalar λ_θ such that both $\lambda_\theta \cdot x^0 \in D(v_1, v_2, \dots, v_k)$ and $\lambda_\theta x^0 \in \mathbb{L}(\theta u^0)$. Thus $\{v_1, v_2, \dots, v_k\}$ cannot be globally output rate weak limitational for $\{\alpha_1, \alpha_2, \dots, \alpha_l\}$.

The arguments for Proposition (3.3-1) and (3.3-2) may be carried out in an analogous way to establish the following propositions where the proper subset $\{v_1, v_2, \dots, v_k\}$ of input factors is output history homogenously essential and axiom L.6S holds, with (E.S, Norm Topology) or (\mathbb{E} , Weak* Topology).

Proposition (3.3-3):

A proper subset of factors $\{v_1, v_2, \dots, v_k\}$ is homogenously output rate weak limitational for a subset $\{\alpha_1, \alpha_2, \dots, \alpha_l\}$ of output rate histories, if and only if it is homogenously essential for $\{\alpha_1, \alpha_2, \dots, \alpha_l\}$ and axioms E.S and L.6S apply, when the Norm Topology is invoked for $(L_\infty)_+^n$.

For the proof, the fact that $\mathbb{L}(\theta_1 u_{\alpha_1}^0, \dots, \theta_l u_{\alpha_l}^0, w^0) \subset \mathbb{L}(u_{\alpha_1}^0, \dots, u_{\alpha_l}^0, w^0)$ for $\theta_i \in [1, +\infty)$, $i \in \{1, 2, \dots, l\}$ is used to show that $(\theta_1 u_{\alpha_1}^0, \dots, \theta_l u_{\alpha_l}^0, w^0) \notin \mathbb{P}(x)$ for $\theta_i \in [1, +\infty)$,

$i \in \{1, 2, \dots, l\}$, when $\|x_{v_1}, x_{v_2}, \dots, x_{v_k}\| \leq B(u^0)$.

As in the case of Proposition (3.3-1), there is a sub-proposition related to (3.3-3).

Sub-Proposition (3.3-3):

Proposition (3.3-3) holds under axiom E if a weak* topology is used for $(L_\infty)_+^n$.

Now essentiality will not imply strong limitationality. To see this consider the following counterexample which is a modified version of the expression in (2.3-20) (dynamic C.E.S. production function) for two input rate histories $(x_1, x_2) \in X$ (see Section 2.3-5).

$$F(x_1, x_2) := 0 \in (L_\infty)_+ \text{ for } (x_1, x_2) \notin X,$$

$$F(x_1, x_2) := u \in (L_\infty)_+ \text{ for } (x_1, x_2) \in X,$$

where $F(x_1, x_2)$ at v is denoted by $F(x_1, x_2, v)$ and

$$u(v) = F(x_{1v}, x_{2v}, v) \text{ for } t \in [(v-1)v], v = 1, 2, \dots$$

$$F(x_{1v}, x_{2v}, v) := \begin{cases} \left[\beta_{1v} x_{1v}^{-\rho} + \beta_{2v} x_{2v}^{-\rho} \right]^{-\frac{1}{\rho}} & \text{for } (x_{2v} - A_v) \geq 0 \\ 0 & \text{for } (x_{2v} - A_v) < 0, \end{cases}$$

with $A_v = A > 0$ for $v = 1, 2, \dots$ and $\rho \in (-1, 0)$. This simplified neoclassical dynamic production function satisfies the same axiom structure as that satisfied by (2.3-20). The second factor for $F(x_1, x_2)$ is essential, since $F(x_1, 0) = 0 \in (L_\infty)_+$ for $(x_1, 0) \in X$. For any

bound on x_2 such that $(x_{2v} - A_v) < 0$ for all $v = 1, 2, \dots$, the output rate history generated is bounded, since $F(x_{1v}, x_{2v}, v) = 0$ for $v = 1, 2, \dots$. However for bounds on x_2 such that $(x_{2v} - A_v) \geq 0$, $F(x_{1v}, x_{2v}, v)$ is not bounded for unbounded increase of x_1 .

A sufficient condition can be derived following the approach of Färe [1972] in the steady state case.

Proposition (3.3-4):

Let $u^0 \in (L_\infty)_+^m$ be an arbitrary vector of output rate histories with $L(u^0) \neq \emptyset$ and $v^0 := (u_{\alpha_1}^0, u_{\alpha_2}^0, \dots, u_{\alpha_k}^0) > 0$, $1 \leq k \leq m$.

Order the components of u^0 so that $u^0 = (v^0, w^0)$, $w^0 :=$

$(u_{\alpha_{k+1}}^0, u_{\alpha_{k+2}}^0, \dots, u_{\alpha_m}^0)$. Define

$$\bar{D}(v_1, v_2, \dots, v_k) := \left\{ x \in (L_\infty)_+^n : x \neq 0, x_{v_i} = 0, i = 1, 2, \dots, k \right\},$$

$$K(v^0, w^0) := \text{CLOSURE} \left(\bigcup_{\theta \in (0, +\infty)} k(\theta v^0, w^0) \right),$$

$$k(\theta v^0, w^0) := \left\{ x \in (L_\infty)_+^n : x = \lambda y, \lambda \in (0, +\infty), y \in \text{CLOSURE } E(\theta v^0, w^0) \right\}.$$

Then, if $K(v^0, w^0) \cap \bar{D}(v_1, v_2, \dots, v_k)$ is empty for all u with $v^0 > 0$, the subset of factors $\{v_1, v_2, \dots, v_k\}$ is globally output rate strong limitational for the subset $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ of output rate histories, under axiom L.6SS.

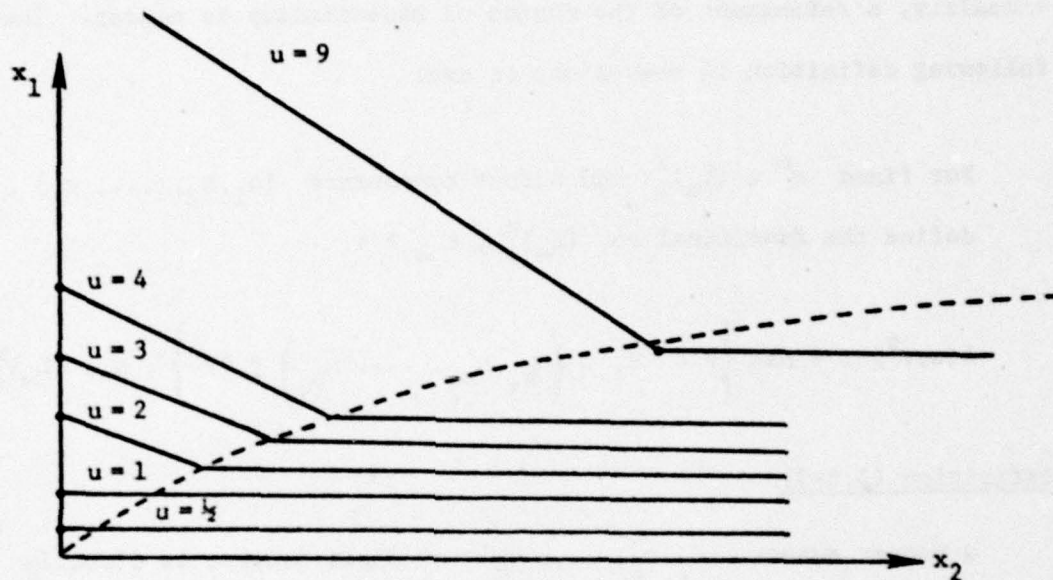
Unfortunately, Proposition (3.3-4) does not also give us a necessary condition for strong limitationality.

Consider the following counterexample, which, for simplicity of exposition, is taken in a steady state framework:

$$u \in \mathbb{R}_+^1 \rightarrow L(u) \in 2^{\mathbb{R}_+^2}.$$

Define, for $u \in \mathbb{R}_+^1$, $y(u) := (\sqrt{u}, u)$, $w(u) := (u, 0)$ and
 $L(u) := \{z \in \mathbb{R}_+^2 : z \geq (\theta \cdot y(u) + (1 - \theta)w(u)) \text{ for } \theta \in [0, 1]\}$.

Schematically,



It should be clear that x_1 is globally essential and globally input strong limitational for the single output. Now, take $u^0 := 1$ and consider the infinite sequence $\{y(\theta^j \cdot u^0), \theta^j \cdot u^0\}$ with $\theta^j \rightarrow +\infty$. By definition $y(\theta^j \cdot u^0) \in E(\theta^j \cdot u^0)$. Denote $\rho^j := (\lambda \cdot y(\theta^j \cdot u^0), \lambda > 0)$. Also denote $(y_1^j, y_2^j) := y(\theta^j \cdot u^0)$. Since

$$\lim_j \frac{\|y_1^j\|}{\|y_2^j\|} = \lim_j \frac{\sqrt{\theta^j}}{\theta^j} = 0,$$

it follows that $\lim_j \rho^j \in \{x \in \mathbb{R}_+^2 : x_1 = 0\}$. That is,

CLOSURE $K(u^0) \cap \{x \in \mathbb{R}_+^2 : x_1 = 0\}$ is not empty.

Since constant-value functions can be taken as a special case, the above steady state example is sufficient as a counter-example.

To derive a necessary and sufficient condition for strong limitationality, a refinement of the notion of essentiality is needed. The following definition is convenient to use:

For fixed $v^0 \in (L_\infty)_+^l$ and output components $\{a_1, a_2, \dots, a_l\}$, define the functional on $(L_\infty)_+^m$, $m \geq l$:

$$\lambda(u, v^0) := \text{Max} \left\{ \theta \in \mathbb{R}_+ : \left(u_{a_1}, u_{a_2}, \dots, u_{a_l} \right) \geq \theta v^0 \right\}, u \in (L_\infty)_+^m.$$

Definition (3.3-5):

A proper subset $\{v_1, v_2, \dots, v_k\}$ of input factors is *Globally Strong Essential* for a subset $\{a_1, a_2, \dots, a_l\}$ of the output histories if for every $v^0 > 0$ with $L(v^0, w) \neq \emptyset$ for some $w \in (L_\infty)_+^{m-l}$, an arbitrary infinite sequence $\{x^j, u^j\}$, $x^j \in L(u^j)$, $\lambda(u^j, v^0) \rightarrow +\infty$ implies $\|x_{v_1}^j, x_{v_2}^j, \dots, x_{v_k}^j\| \rightarrow +\infty$.

Proposition (3.3-5):

A proper subset $\{v_1, v_2, \dots, v_k\}$ of n input factors is globally output rate strong limitational for a subset $\{a_1, a_2, \dots, a_l\}$ of output histories, $1 \leq l \leq m$, if and only if it is globally strong essential for $\{a_1, a_2, \dots, a_l\}$.

Proof:

Suppose $\{v_1, v_2, \dots, v_k\}$ is not globally output rate strong limitational for $\{\alpha_1, \alpha_2, \dots, \alpha_l\}$. Then there is an input bound $B \in \mathbb{R}_+$, $u^0 = (v^0, w^0)$ and $v^0 = (u_{\alpha_1}^0, u_{\alpha_2}^0, \dots, u_{\alpha_l}^0) > 0$ with $\mathbb{L}(u^0) \neq \emptyset$ for which an infinite sequence $\{x^j, u^j\}$ exists with $x^j \in \mathbb{L}(u^j)$, $\lambda(u^j, v^0) \rightarrow +\infty$ and $\|x_{v_1}^j, x_{v_2}^j, \dots, x_{v_k}^j\| \leq B$. That is, $\{v_1, v_2, \dots, v_k\}$ is not globally strong essential for $\{\alpha_1, \alpha_2, \dots, \alpha_l\}$; establishing the "if" part of the proposition.

To show the converse, assume $\{v_1, v_2, \dots, v_k\}$ not globally strong essential for $\{\alpha_1, \alpha_2, \dots, \alpha_l\}$. Then there exists $v^0 > 0$ and infinite sequence $\{x^j, u^j\}$ with $x^j \in \mathbb{L}(u^j)$, $\lambda(u^j, v^0) \rightarrow +\infty$ while $\limsup \|x_{v_1}^j, x_{v_2}^j, \dots, x_{v_k}^j\| < +\infty$. Consider an input bound $B > \limsup \|x_{v_1}^j, x_{v_2}^j, \dots, x_{v_k}^j\|$. For this input bound B , there cannot exist a scalar $\theta \in \mathbb{R}_+$ such that $\mathbb{L}(u) \cap \left\{ x \in (L_\infty)_+^n : \right.$
 $\left. \|x_{v_1}, x_{v_2}, \dots, x_{v_k}\| \leq B \right\} = \emptyset$ for all u with $(u_{\alpha_1}, u_{\alpha_2}, \dots, u_{\alpha_l}) > \theta \cdot v^0$ since $\lambda(u^j, v^0) \rightarrow +\infty$. Consequently, $\{v_1, v_2, \dots, v_k\}$ is not globally input rate strong limitational for $\{\alpha_1, \alpha_2, \dots, \alpha_l\}$; establishing the "only if" part.

For the case of homogeneously output rate strong limitational factors, the following definition is made analogous to Definition (3.3-5).

Definition (3.3-6):

A proper subset $\{v_1, v_2, \dots, v_k\}$ of input factors is *Homogeneously Output Rate Strong Essential* for a subset $\{a_1, a_2, \dots, a_l\}$ of output histories if for every $u^0 \in (L_\infty)_+^m$ with $L(u^0) \neq 0$ and $v^0 := (u_{a_1}^0, u_{a_2}^0, \dots, u_{a_l}^0) > 0$, and arbitrary infinite sequence $\{x^j, u^j\}$, with $x^j \in L(u^j)$, where $(u_{a_1}^j, u_{a_2}^j, \dots, u_{a_l}^j) = (\theta_1^j v_1^0, \theta_2^j v_2^0, \dots, \theta_l^j v_l^0)$, $\theta_i^j \in (1, +\infty)$, $i \in \{1, 2, \dots, l\}$, $j = 1, 2, \dots$, it holds that $\{\text{Max}(\theta_i^j : i \in \{1, 2, \dots, l\})\} \rightarrow +\infty$ implies $\|x_{v_1}^j, x_{v_2}^j, \dots, x_{v_k}^j\| \rightarrow +\infty$.

Using exactly the same argument as in Proposition (3.3-5), the following proposition can be easily established.

Proposition (3.3-6):

A proper subset $\{v_1, v_2, \dots, v_k\}$ of input factors is homogeneously output rate strong limitational for a subset $\{a_1, a_2, \dots, a_l\}$ of m output rate histories if and only if it is homogeneously output rate strong essential for $\{a_1, a_2, \dots, a_l\}$.

3.4 Limitation of Summable Output Rate Histories by Input Rates of Essential Factors of Production

In the previous section vectors of output rate histories have been taken from the space $(L_{\infty})_{+}^m$, including those which do not correspond to a finite total amount over the time span $t \in [0, +\infty)$. Such total-amount-unbounded planning may seem too expansive. One would expect that most planning would be for a bounded total amount of each kind of output, probably over a finite subinterval of $[0, +\infty)$. In the treatment of such cases one need not select bounded subintervals of planning, which could hardly be done with generality. Rather the vectors u of output histories may be selected from $(L_{\infty})_{+}^m$ so that each component history is summable.

For this purpose let

$$(3.4-1) \quad (\tilde{L}_{\infty})_{+}^m := \left\{ u \in (L_{\infty})_{+}^m : \int_0^{\infty} u_i(t) dv_i(t) < +\infty, i \in \{1, 2, \dots, m\} \right\}.$$

Then output histories which have bounded support (finite planning period) are merely special cases. For vectors $u \in (\tilde{L}_{\infty})_{+}^m$ the supports of the summable component histories u_i need not be bounded.

Correspondingly, there would be little purpose in including for the sets $L(u)$, $u \in (\tilde{L}_{\infty})_{+}^m$, input vectors x with component histories x_i which are not summable. Thus, define

$$(3.4-2) \quad (\tilde{L}_{\infty})_{+}^n := \left\{ x \in (L_{\infty})_{+}^n : \int_0^{\infty} x_i(t) d\mu_i(t) < +\infty, i \in \{1, 2, \dots, n\} \right\}.$$

Then the sets of vectors of input histories of interest are:

$$(3.4-3) \quad \tilde{\mathbb{L}}(u) = \left\{ x \in (L_{\infty})_{+}^n : x \in \mathbb{L}(u) \cap (\tilde{L}_{\infty})_{+}^n \right\}, \quad u \in (\tilde{L}_{\infty})_{+}^m$$

with nonempty efficient subsets:

$$(3.4-4) \quad \tilde{\mathbb{E}}(u) = \left\{ x \in (L_{\infty})_{+}^n : x \in \tilde{\mathbb{L}}(u), \quad y \leq x \Rightarrow y \notin \tilde{\mathbb{L}}(u) \right\}.$$

See Proposition (2.2.3-1).

The definitions previously given, i.e., (3.1-1), (3.1-2), for $x \in (L_{\infty})_{+}^n$, $u \in (L_{\infty})_{+}^m$, may be applied here, since $\tilde{\mathbb{L}}(u) \subset \mathbb{L}(u)$ and $\mathbb{L}(u) \cap D(v_1, v_2, \dots, v_k)$ empty implies $\tilde{\mathbb{L}}(u) \cap D(v_1, v_2, \dots, v_k)$ empty. A subset $\{v_1, v_2, \dots, v_k\}$ which is essential in the sense of any of the two types previously defined is likewise essential when $u \in (\tilde{L}_{\infty})_{+}^m$, $x \in (\tilde{L}_{\infty})_{+}^n$.

Propositions (3.3-1), (3.3-3) and Sub-Propositions (3.3-1), (3.3-3) carry over when output vectors are restricted to $(\tilde{L}_{\infty})_{+}^m$. It suffices to consider Proposition (3.3-1) for verification of these facts.

For arbitrary u^0 with $\mathbb{L}(u^0) \neq \emptyset$ and $v^0 = \left(u_{\alpha_1}^0, u_{\alpha_2}^0, \dots, u_{\alpha_l}^0 \right) > 0$, $1 \leq l \leq m$, global essentiality of input factors $\{v_1, v_2, \dots, v_l\}$ for output components $\{\alpha_1, \alpha_2, \dots, \alpha_l\}$ implies for $w^0 := \left(u_{\alpha_{l+1}}^0, u_{\alpha_{l+2}}^0, \dots, u_{\alpha_m}^0 \right)$ that

$$\text{CLOSURE } \mathbb{E}(v^0, w^0) \cap D(v_1, v_2, \dots, v_k) = \emptyset,$$

as before. Since $\text{CLOSURE } \tilde{\mathbb{E}}(v^0, w^0) \subset \text{CLOSURE } \mathbb{E}(v^0, w^0)$, it follows that

$$\text{CLOSURE } \tilde{\mathbb{E}}(v^0, w^0) \cap D(v_1, v_2, \dots, v_k) = \emptyset.$$

Then using CLOSURE $\tilde{E}(v^0, w^0)$ in place of CLOSURE $E(v^0, w^0)$, the previous argument for Proposition (3.3-1) may be carried out to obtain a positive bound $\tilde{B}(u^0)$ and show that

$$\tilde{L}(u) \cap \left\{ x \in (\tilde{L}_\infty)_+^n : \left\| \begin{pmatrix} x_{v_1} \\ x_{v_2} \\ \dots \\ x_{v_k} \end{pmatrix} \right\| \leq \tilde{B}(u^0) \right\}$$

is empty for all $u \in (\tilde{L}_\infty)_+^m$ with $\begin{pmatrix} u_{\alpha_1} \\ u_{\alpha_2} \\ \dots \\ u_{\alpha_l} \end{pmatrix} \geq v^0$. For these arguments weak or strong axioms \tilde{E} , $\tilde{E}.S$ may be used for compactness of (CLOSURE $\tilde{E}(u)$) under a weak* or norm topology for $(L_\infty)_+^n$, respectively. The advantages of considering only summable output histories is that the axioms \tilde{E} and $\tilde{E}.S$ are weaker than the axioms E and $E.S$ used for the case where histories are not restricted to be summable. Also, only a finite argument is required to show that $\tilde{E}(u)$ is not empty when u has summable components.

In a similar way, Propositions (3.3-5) and (3.3-6) carry over for the situation where output and input vectors are restricted to have summable output and input rate histories for the definitions of $\tilde{L}(v, w)$ and $\tilde{E}(v, w)$.

To approach this topic from another viewpoint, the original production correspondence may be altered. Since only summable input and output rate histories are of interest for the foregoing discussion, one might ask why not restrict $x \in (L_1)_+^n$ and $u \in (L_1)_+^m$. In effect, why not define the production structure as a correspondence

$$(3.4-5) \quad \begin{aligned} x \in (L_1)_+^n &\rightarrow P_1(x) \in 2^{(L_1)_+^m} \\ u \in (L_1)_+^m &\rightarrow L_1(u) \in 2^{(L_1)_+^n} \end{aligned}$$

mapping vectors of summable input rate histories into subsets of vectors of summable output rate histories. But then the norms for $(L_1)_+^n$ and $(L_1)_+^m$ are changed to:

$$(3.4-6) \quad ||x|| = \text{Max}_i ||x_i|| ; ||x_i|| = \int_0^\infty |x_i(t)| d\mu_i(t) , i \in \{1, \dots, n\}$$

$$(3.4-7) \quad ||u|| = \text{Max}_i ||u_i|| ; ||u_i|| = \int_0^\infty |u_i(t)| dv_i(t) , i \in \{1, \dots, m\} ,$$

and measure by components the total amount of the good or service involved. With this approach the laws of return are expressed in total amounts of some good or service involved.

The axiom structure of Chapter 2 carries over naturally for the norm topologies of $(L_1)_+^n$, $(L_1)_+^m$. Only those axioms involving the norm need to be reinterpreted. P.1 has the same meaning. Each vector $u \in P_1(x)$ of output histories has bounded total amount for each output component. By itself this does not mean that all norms $||u||$ for $u \in P_1(x)$ are uniformly bounded, and axioms P.2, P.2S assure boundedness and total boundedness respectively for the set $P_1(x)$, implying that for all $u \in P_1(x)$ each output is uniformly bounded in total amount. There is no difference in the interpretation of properties P.3, P.3S and P.3SS for $x \in (L_1)_+^n$, $u \in (L_1)_+^m$ as opposed to the case where $x \in (L_\infty)_+^n$, $u \in (L_\infty)_+^m$. Likewise there is no difference in the interpretation of the properties P.4.1, P.4.2, P.6, P.6S, P.6SS. However for property P.5 convergence is only defined by neighborhoods of the norm topologies for $(L_1)_+^m$, $(L_1)_+^n$.

With the foregoing changes of meaning for certain parts of the axiom structure for production correspondences, essentiality and limitationality of outputs can be considered.

The definitions (3.1-1), (3.1-2) for essentiality of a proper subset $\{v_1, v_2, \dots, v_k\}$ of n factors of production still apply for the correspondence (3.4-5). All one need to do is to replace $(L_\infty)_+^m$, $(L_\infty)_+^l$ by $(L_1)_+^m$ and $(L_1)_+^l$.

By replacing $\mathbb{E}.S$ by $\mathbb{E}_1.S$, as well, Proposition (3.3-1) carries over by symmetry. Similarly Sub-Proposition (3.3-1) holds. Likewise Proposition (3.3-2), (3.3-3), Sub-Proposition (3.3-3) and Propositions (3.3-5), (3.3-6) follow by symmetry.

Thus, provided one interprets the bounding of a component of an output vector or input vector to mean bounding the total amount of that input or output over $[0, +\infty)$ as opposed to bounding input or output rates, the related laws of return are of the same form.

3.5 Laws of Return in the Norm

The various laws of return in Sections 3 and 4 are expressed by exclusions of output history vectors with subvectors v equal to or greater than a particular subvector v^0 of an output vector u^0 . In each case, the bound involved depends upon the particular reference vector u^0 . Nothing is implied uniformly about these bounds for all $u^0 \in (L_\infty)_+^m$. A limitationality in terms of the norm $\|v\|$ for $u = (v, w)$ can provide a uniform bounding of subvectors v , where the norm may be sup norm, weak* or the L_1 norm.

To fix ideas, the following definition is made.

Definition (3.5-1):

A proper subset $\{v_1, v_2, \dots, v_k\}$ of the n factors of production is *globally output rate weak norm limitational* for a subset $\{a_1, a_2, \dots, a_\ell\}$, $1 \leq \ell \leq m$, of the m net output rate histories if for every positive bound $A \in \mathbb{R}_{++}$ on the norm of output components $\{a_1, a_2, \dots, a_\ell\}$, there exists a positive bound $B(A) \in \mathbb{R}_{++}$ such that for all $u \in (L_\infty)_+^m$ with $\|u_{a_1}, u_{a_2}, \dots, u_{a_\ell}\| \geq A$, $\mathbb{L}(u) \cap \left\{ x \in (L_\infty)_+^n : \|x_{v_1}, x_{v_2}, \dots, x_{v_k}\| \leq B(A) \right\} = \emptyset$.

Note that the above definition is open to many interpretations, according as the norm used on the input and output space is sup, weak^{*} norm, or L_1 -norm in the case where only summable input and output histories are considered.

To derive the laws of return in the norm, the following refinement of the notion of essentiality is needed:

Definition (3.5-2):

A proper subset $\{v_1, v_2, \dots, v_k\}$ of the n factors of production is *globally totally essential* for a subset $\{a_1, a_2, \dots, a_\ell\}$, $1 \leq \ell \leq m$, of the m net output rate histories if for every component of output histories a_i , $a_i \in \{a_1, a_2, \dots, a_\ell\}$, there exists a subset $\{v_{i1}, v_{i2}, \dots, v_{iq}\} \subset \{v_1, v_2, \dots, v_k\}$ of input factors which is globally essential for $\{a_i\}$.

Consider a proper subset $\{v_1, v_2, \dots, v_k\}$ of the n input factors which is globally totally essential for output components $\{a_1, a_2, \dots, a_\ell\}$. Under strong disposability of output (L.6SS), it is sufficient for our purpose to consider only subvectors $v \in (L_\infty)_+^k$ because $\mathbb{L}(v, 0) \supset \mathbb{L}(v, w)$,

$w \in (L_{\infty}^m)_+$. Also as will become apparent why later on, a weak* topology will be used for $(L_{\infty}^m)_+$.

For arbitrary $A \in \mathbb{R}_{++}$ and $1 \leq l \leq m$, define the set

$$S(A) := \left\{ v \in (L_{\infty}^l)_+ : \|v\|^* = A \right\}, \|\cdot\|^* \text{ being a weak}^* \text{ norm}$$

and define the following functional on $S(A)$,

$$d(v, D(v_1, \dots, v_k)) :=$$

$$\inf_{x,y} \{ \|x - y\| : x \in \text{CLOSURE } \mathbb{E}(v, 0), y \in D(v_1, v_2, \dots, v_k) \}.$$

First, consider $v \in (L_{\infty}^l)_+$ with $\mathbb{L}(v, 0) \neq \emptyset$. If $\{v_1, v_2, \dots, v_k\}$ is globally totally essential for $\alpha := (\alpha_1, \alpha_2, \dots, \alpha_l)$, it is clear that it is also globally essential for α . Then, it follows from Proposition (3.1-1) that $\text{CLOSURE } \mathbb{E}(v, 0) \cap D(v_1, v_2, \dots, v_k) = \emptyset$. Since $D(v_1, v_2, \dots, v_k)$ is a closed set, $d(v, D(v_1, v_2, \dots, v_k))$ is well defined and positive (refer to Proof of Proposition (3.3-1)) if axiom $\mathbb{E}.S$ holds or axiom \mathbb{E} holds with the weak* topology taken for the input space $(L_{\infty}^n)_+$.

Now, the functional d is not necessarily finite for all $v \in S(A)$ since it is possible that $\mathbb{L}(v, 0)$ is empty. However, d is finite for some $v \in S(A)$ since otherwise, in view of axiom $\mathbb{L}.4.2$, it would follow that $\mathbb{L}(u) = \emptyset$ for all output histories u with subvector $\begin{pmatrix} u_{\alpha_1}, u_{\alpha_2}, \dots, u_{\alpha_l} \end{pmatrix} \neq 0$, contradicting axiom $\mathbb{L}.4.1$. In order to accommodate the possibility of $d(v, D(v_1, \dots, v_k))$ being unbounded, the extended real line will be used for assigning a value $+\infty$ to the functional d when $\mathbb{L}(v, 0) = \emptyset$.

The following lemma is useful for the proposition to follow:

Lemma (3.5-1):

If the weak^{*} topology is taken for the output space $(L_\infty)_+^m$ and either axiom E.S holds or axiom E holds taken with the weak^{*} topology for the input space $(L_\infty)_+^n$, then $\text{Inf} \{d(v, D(v_1, \dots, v_k)) : v \in S(A)\}$ is positive if $\{v_1, v_2, \dots, v_k\}$ is globally totally essential for $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$. (L.5 is taken to be stated in weak^{*} topology.)

Proof:

Suppose the contrary of the lemma holds. Then there exists an infinite sequence $\{v^j\} \subset S(A)$ with $\{d(v^j, D(v_1, \dots, v_k))\} \rightarrow 0$. Since, as a consequence of the Banach-Alaoglu theorem, $S(A)$ is a compact set under the weak^{*} topology, there exists an infinite subsequence $\{v^p\} \subset \{v^j\}$ converging weak^{*} to $\bar{v} \in S(A)$. Then $\|\bar{v}\| = A$, with the corresponding sequence $\{d(v^p, D(v_1, v_2, \dots, v_k))\} \rightarrow 0$. For notation, let $\delta^p := d(v^p, D(v_1, v_2, \dots, v_k))$. Of course, $\delta^p \rightarrow 0$. By the definition of functional d , we know that $\mathbb{L}(v^p, 0) \cap \left\{x \in (L_\infty)_+^n : \|x_{v_1}, x_{v_2}, \dots, x_{v_k}\| \leq \delta^p\right\} \neq \emptyset$. Then as a consequence of L.5 as stated under the weak^{*} topology, $v^p \xrightarrow{\text{weak}^*} \bar{v}$, and $\delta^p \rightarrow 0$ imply $\mathbb{L}(\bar{v}, 0) \cap D(v_1, v_2, \dots, v_k) \neq \emptyset$. This then contradicts the hypothesis that $\{v_1, v_2, \dots, v_k\}$ is globally totally essential.

With this lemma, the following proposition due to Mak may be established.

Proposition (3.5-1): (Mak, 1979)

If axiom $\mathbb{L}.6SS$ and the hypothesis of Lemma (3.5-1) hold, then a proper subset of input factors $\{v_1, v_2, \dots, v_k\}$ is globally output rate weak norm limitational for a subset $\{\alpha_1, \alpha_2, \dots, \alpha_\ell\}$ of output histories if and only if $\{v_1, v_2, \dots, v_k\}$ is globally totally essential for $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$.

Note that here output norm is weak^{*} norm and input norm is sup norm.

Proof:

By Lemma (3.5-1), $\text{Inf} \{d(v, D(v_1, \dots, v_k)) : v \in S(A)\}$ is positive. Then there exists a bound $B(A) \in \mathbb{R}_{++}$ defined by

$$B(A) := \frac{1}{2} \text{Inf} \{d(v, D(v_1, v_2, \dots, v_k)) : v \in S(A)\}$$

such that for each $v \in S(A)$, $(v, 0) \notin \mathbb{P}(x)$ if $\|x_{v_1}, x_{v_2}, \dots, x_{v_k}\| \leq B(A)$. Moreover, since $\mathbb{L}.6SS$ applies, for all $(\tilde{v}, \tilde{w}) \in (L_\infty)_+^m$ with $\tilde{v} \geq v$ for some $v \in S(A)$ and $\tilde{w} \geq 0$, we have $\mathbb{L}(\tilde{v}, \tilde{w}) \subset \mathbb{L}(v, 0)$, hence $\mathbb{L}(\tilde{v}, \tilde{w}) \cap \left\{x \in (L_\infty)_+^n : \|x_{v_1}, x_{v_2}, \dots, x_{v_k}\| \leq B(A)\right\} = \emptyset$.

Since \tilde{v} has weak^{*} norm not less than A if and only if there exists $v \in S(A)$ with $\tilde{v} \geq v$, and because $A \in \mathbb{R}_{++}$ was arbitrarily chosen, the "if" part of the proposition is established.

The converse is trivial. Assume $\{v_1, v_2, \dots, v_k\}$ not globally totally essential for $\{\alpha_1, \alpha_2, \dots, \alpha_\ell\}$. Then $\{v_1, v_2, \dots, v_k\}$ is not globally essential for some $\{\alpha_1\} \subset \{\alpha_1, \alpha_2, \dots, \alpha_\ell\}$. It then follows from Proposition (3.3-2) that $\{v_1, v_2, \dots, v_k\}$ is not globally output rate weak limitational for $\{\alpha_1\}$, and consequently not globally output rate weak norm limitational for $\{\alpha_1\}$ or $\{\alpha_1, \alpha_2, \dots, \alpha_\ell\}$.

There are several extensions of the above propositions:

(3.5-3) If the functional d is defined by a weak^{*} norm:

$$d(v, D(v_1, \dots, v_k)) :=$$

$$\inf_{x,y} \{ \|x - y\|^{*} : x \in \text{CLOSURE } \mathbb{E}(v, 0), y \in D(v_1, v_2, \dots, v_k) \}$$

the arguments establishing Proposition (3.5-1) carry over as long as we interpret Definition (3.5-1) with weak^{*} norm for both input and output histories.

(3.5-4) If we are only interested in summable input and output histories, and an L_1 -norm $\|\cdot\|_1$ is used, redefine

$$S(A) := \left\{ v \in (L_{\infty})_+^k : \|v\|_1 = A \right\},$$

and

$$d(v, D(v_1, \dots, v_k)) :=$$

$$\inf_{x,y} \{ \|x - y\|_1 : x \in \text{CLOSURE } \tilde{\mathbb{E}}(v, 0), y \in D(v_1, v_2, \dots, v_k) \}.$$

For arbitrary $A \in \mathbb{R}_+$, consider a neighborhood V of 0 in $(L_1)_+^m$ where $V := \left\{ y \in (L_1)_+^m : \|y\|_1 \leq 1/A + \epsilon \right\}$, ϵ being an arbitrary small positive scalar. Then by the Banach-Alaoglu theorem, the polar of V ,

$$\tilde{V} := \left\{ x \in (L_{\infty})_+^m : \langle x, y \rangle \leq 1 \text{ for every } y \in V \right\}$$

is weak^{*} compact. Since the set $S(A)$ is a closed subset of \tilde{V} ,

it is also weak* compact.

Then, again, Proposition (3.5-1) can be established where Definition (3.5-1) is interpreted with L_1 -norm for both the input and output histories. In the same vein, Proposition (3.5-1) is also valid when L_1 -norm is used for output, and the input norm is either sup-norm or weak* norm.

Up to now, the sup norm has not been used for the output space, because the set $S(A)$ defined as $v \in (L_\infty)_+^l : ||v|| = A$ is weak* compact but not closed, i.e., a weak* convergent sequence in $S(A)$ need not converge in the sup norm to a point \bar{v} with $||\bar{v}|| = A$. Thus, the argument in Lemma (3.5-1) fails to apply when the sup norm is used; if we are interested in bounding the maximum output rates, the sup-norm has to be used. To make Lemma (3.5-1) valid, in this case, let us make the following modified definition of the set $S(A)$:

$$(3.5-5) \quad \begin{aligned} T(v_i) &:= v_i\{t \in [0, +\infty) : \text{Sup } |v_i(t)| = ||v||\}, \quad i = 1, 2, \dots, l \\ S(A, \epsilon) &:= \left\{ v \in (L_\infty)_+^l : ||v|| = A, \text{Max}_i T(v_i) \geq \epsilon \right\}, \quad \epsilon \in \mathbb{R}_{++} \end{aligned}$$

That is, we restrict attention to the subset of output vectors with supremal output rate prevailing over a period with measure at least as large as a positive quantity ϵ . Since $S(A, \epsilon) \subset S(A)$, the subset $S(A, \epsilon)$ is weak* compact. Thus, for an infinite sequence $\{v^j\} \subset S(A, \epsilon)$ with $v^j \rightarrow \bar{v}$, weak*, it is implied that

$$\begin{aligned} ||v^j||^* &= \text{Max}_i \left\{ \text{Sup}_{||r_i|| \leq 1} \left(\int_0^\infty r_i(t) v_i^j(t) dv_i(t) \right) \right\} \\ &\geq A \cdot \epsilon > 0. \end{aligned}$$

Consequently, $\bar{v} \neq 0$. Thus, the argument of Lemma (3.5-1) can be carried over to establish Proposition (3.5-1) with the understanding that "globally output rate weak-norm limitationality" is understood as:

(3.5-6) For every $\epsilon > 0$ and positive bound $A \in \mathbb{R}_{++}$, there

exists a positive bound $B(A, \epsilon) \in \mathbb{R}_{++}$ such that for

all $u \in (L_{\infty})_+^m$ with $\|u_{\alpha_1}, u_{\alpha_2}, \dots, u_{\alpha_l}\| \geq A$, and

$$\max_i \mu_i \{t \in [0, +\infty) : \max_i |u_{\alpha_i}| \geq A\} \geq \epsilon,$$

$$\mathbb{L}(u) \cap \left\{ x \in (L_{\infty})_+^n : \|x_{v_1}, x_{v_2}, \dots, x_{v_k}\| \leq B(A, \epsilon) \right\} = \emptyset.$$

With respect to strong limitationality, the following definition is made:

Definition (3.5-3):

A proper subset $\{v_1, v_2, \dots, v_k\}$ of the n factors of production is *globally output rate strong norm limitational* for a subset

$\{\alpha_1, \alpha_2, \dots, \alpha_l\}$, $1 \leq l \leq m$, of m net output rate histories if for

every positive bound $B \in \mathbb{R}_{++}$ on the norm of input factors

$\{v_1, v_2, \dots, v_k\}$, there exists a positive bound $A(B) \in \mathbb{R}_{++}$ such that

for all $u \in (L_{\infty})_+^m$ with $\|u_{\alpha_1}, u_{\alpha_2}, \dots, u_{\alpha_l}\| \geq A(B)$, $\mathbb{L}(u) \cap$

$$\left\{ x \in (L_{\infty})_+^n : \|x_{v_1}, x_{v_2}, \dots, x_{v_k}\| \leq B \right\} = \emptyset.$$

A stronger notion of essentiality is also needed.

Definition (3.5-4):

A proper subset $\{v_1, v_2, \dots, v_k\}$ of input factors is *Globally Strong Norm Essential* for a subset $\{a_1, a_2, \dots, a_l\}$ of the output histories if for every infinite sequence $\{x^j, u^j\}$, $x^j \in \mathbb{L}(u^j)$, $\|u_{a_1}^j, u_{a_2}^j, \dots, u_{a_l}^j\| \rightarrow +\infty$ implies $\|x_{v_1}^j, x_{v_2}^j, \dots, x_{v_k}^j\| \rightarrow +\infty$.

Again, both Definitions (3.5-3) and (3.5-4) are subjected to various interpretation as different norms can be used for the input and output spaces.

Using identical argument as in Proposition (3.3-5), the following proposition can be easily established (proof is omitted).

Proposition (3.5-2): (Mak, 1979)

A proper subset $\{v_1, v_2, \dots, v_k\}$ of input factors is globally output rate strong-norm limitational for a subset $\{a_1, a_2, \dots, a_l\}$ of output histories if and only if it is globally strong-norm essential for $\{a_1, a_2, \dots, a_l\}$.

3.6 Limitation of Output Rates by Intervals of Time Over Which Essential Factors are Applied

Here one is concerned with the limitations upon output rates imposed when essential factors of production have limited supports, i.e., limited periods of time with positive rate of application. Only the case of entire output vectors $u \in (L_\infty)_+^m$ will be considered.

For the analysis to follow certain definitions are useful:

$$(3.6-1) \quad S(u^0) := \left\{ t \in \mathbb{R}_+^n : t = \left(\bar{t}_{x_1}, \bar{t}_{x_2}, \dots, \bar{t}_{x_n} \right), x \in \text{CLOSURE } \mathbb{E}(u^0) \right\}.$$

$$(3.6-2) \quad S(D(v_1, v_2, \dots, v_k)) := \left\{ t \in \mathbb{R}_+^n : t = \left(\bar{t}_{x_1}, \bar{t}_{x_2}, \dots, \bar{t}_{x_n} \right), \right. \\ \left. x \in D(v_1, v_2, \dots, v_k) \right\}.$$

Definition (3.6-1):

A proper subset of the n factors $\{v_1, v_2, \dots, v_k\}$ is *globally output support weak limitational* if for every $u^0 \in (L_\infty)_+^m$, $L(u^0) \neq \emptyset$ with $\text{Max}_i \left\{ \bar{t}_{u_i}^0 \right\} < +\infty$, there exists a positive bound $T(u^0) \in \mathbb{R}_{++}$ such that for all $u \geq u^0$,

$$L(u) \cap \left\{ x \in (L_\infty)_+^n : \text{Max} \left(\bar{t}_{x_{v_1}}, \bar{t}_{x_{v_2}}, \dots, \bar{t}_{x_{v_k}} \right) \leq T(u^0) \right\}$$

is empty.

Definition (3.6-2):

A proper subset of the n factors $\{v_1, v_2, \dots, v_k\}$ is *globally output support strong limitational* if for every positive time bound $T \in \mathbb{R}_{++}$, and every $u^0 \geq 0$ with $L(u^0) \neq \emptyset$, there exists a bound $\tau(T, u^0) \in \mathbb{R}_{++}$ on output supports such that for all $u \in (L_\infty)_+^m$, with $u \geq u^0$ and $\text{Max}_i \left\{ \bar{t}_{u_i} \right\} \geq \tau(T, u^0)$

$$L(u) \cap \left\{ x \in (L_\infty)_+^n : \text{Max} \left(\bar{t}_{x_{v_1}}, \bar{t}_{x_{v_2}}, \dots, \bar{t}_{x_{v_k}} \right) \leq T \right\}$$

is empty.

Then the following proposition characterizes a limitation upon vectors due to limitations on the span of positive input rates for essential factors of production.

Proposition (3.6-1):

A proper subset of the n factors $\{v_1, v_2, \dots, v_k\}$ is globally output support weak limitational if and only if it is globally essential for the whole output space.

Consider arbitrary $u^0 \in (L_\infty)_+^m$, $\mathbb{L}(u^0) \neq \emptyset$ and $\text{Max}_i \left\{ \bar{t}_{u_i^0} \right\} < +\infty$.

Note first that $S(u^0)$ is bounded, since $\text{Supp } \mathbb{E}(u^0)$ is bounded by virtue of axiom $\mathbb{L.T.2}$. Next it is to be shown that

$$(3.6-3) \quad (\text{CLOSURE } S(u^0)) \cap S(D(v_1, v_2, \dots, v_k))$$

is empty. Note that $\text{CLOSURE } S(u^0)$ is compact since it is a bounded and closed subset of R_+^n . Assume that (3.6-3) is not empty. Then there exists a vector $\hat{t} \in R_+^n$ such that $\hat{t} \in \text{CLOSURE } S(u^0)$ and $\hat{t} \in S(D(v_1, v_2, \dots, v_k))$, and there exists an infinite sequence $\{t^\alpha\} \subset \text{CLOSURE } S(u^0)$ with $t^\alpha \rightarrow \hat{t}$. Corresponding to $\{t^\alpha\}$ there exists an infinite sequence $\{x^\alpha\} \subset \text{CLOSURE } \mathbb{E}(u^0)$. Now, either by axiom $\mathbb{E.S}$ with norm topology for $(L_\infty)_+^n$ or by axiom \mathbb{E} with weak* topology for $(L_\infty)_+^n$, $\text{CLOSURE } \mathbb{E}(u^0)$ is compact. Then there exists a subsequence $\{x^{\alpha_k}\} \subset \text{CLOSURE } \mathbb{E}(u^0)$ such that $x^{\alpha_k} \rightarrow x^0$ and $x^0 \in \text{CLOSURE } \mathbb{E}(u^0)$. Further let $t^0 \in S(u^0)$ for x^0 . Now $t^0 = \hat{t}$, otherwise $x^{\alpha_k} \not\rightarrow x^0$. Since $x^0 \notin D(v_1, v_2, \dots, v_k)$ due to the essentiality of $\{v_1, v_2, \dots, v_k\}$ for u^0 , it follows that $t^0 \notin S(D(v_1, v_2, \dots, v_k))$, a contradiction.

Now since $\text{CLOSURE } S(u^0)$ is compact and $S(D(v_1, v_2, \dots, v_k))$ is a closed set, the distance, defined for $t \in \text{CLOSURE } S(u^0)$, and given by

$$\Delta(t, S(D(v_1, v_2, \dots, v_k))) := \inf_{\sigma} \{ \|\sigma - t\| : \sigma \in S(D(v_1, v_2, \dots, v_k)) \},$$

is continuous in t , and strictly positive since $\text{CLOSURE } S(u^0) \cap S(D(v_1, v_2, \dots, v_k))$ is empty. Then, since $\text{CLOSURE } S(u^0)$ is a nonempty compact set, there exists an input vector with $t^* \in \text{CLOSURE } S(u^0)$ such that

$$0 < \delta := \min_t \{ \Delta(t, S(D(v_1, v_2, \dots, v_k))) : t \in \text{CLOSURE } S(u^0) \}.$$

Let $T(u^0) := \delta/2$ be a bound upon the norm of the subvector

$$\left(\bar{t}_{x_{v_1}}, \bar{t}_{x_{v_2}}, \dots, \bar{t}_{x_{v_k}} \right). \text{ Define } \bar{t}_x := \left\{ \bar{t}_{x_1}, \bar{t}_{x_2}, \dots, \bar{t}_{x_n} \right\}. \text{ Then for}$$

$$\bar{t}_x \in \left\{ t \in \mathbb{R}_+^n : \|\bar{t}_{x_{v_1}}, \bar{t}_{x_{v_2}}, \dots, \bar{t}_{x_{v_k}}\| \leq T(u^0) \right\}, \text{ CLOSURE } \mathbb{E}(u^0) \cap$$

$D(v_1, v_2, \dots, v_k)$ is empty, since $(\text{CLOSURE } S(u^0)) \cap S(D(v_1, v_2, \dots, v_k))$ is empty and input vectors x with \bar{t}_x so bounded cannot belong to those of $\text{CLOSURE } \mathbb{E}(u^0)$. Further, since, $\mathbb{L}(u^0) \subset (\text{CLOSURE } \mathbb{E}(u^0) + (L_\infty)_+^n)$, see Proposition (2.2.4-2), it follows that

$$\left\{ t \in \mathbb{R}_+^n : t = \left(\bar{t}_{x_1}, \bar{t}_{x_2}, \dots, \bar{t}_{x_n} \right), x \in \mathbb{L}(u^0) \right\} \subset \text{CLOSURE } (S(u^0) + \mathbb{R}_+^n).$$

Moreover, if axiom $\mathbb{L}.6SS$ applies, $\mathbb{L}(u) \subset \mathbb{L}(u^0)$ for $u \geq u^0$. Then, it is shown that, if $\{v_1, v_2, \dots, v_k\}$ is an essential proper subset of factors for an output vector $u^0 \in (L_\infty)_+^m$, there exists a positive bound $T(u^0)$ such that $u \notin \mathbb{P}(x)$ for $\max_i \left\{ \bar{t}_{x_{v_i}} \right\} \leq T(u^0)$ and $u \geq u^0$.

The converse is trivially true.

Essentiality of a proper subset of factors therefore engenders limitation upon output rate histories relative to a given output vector u^0 when input histories of essential factors have supports not exceeding a bound $T(u^0)$ depending upon u^0 and $\bar{t}_{u_i}^0$ is bounded for all $i \in \{1, 2, \dots, m\}$.

For each vector $u \geq 0$, $L(u) \neq \emptyset$, of output histories with bounded supports, output rate histories may be so bounded by restrictions on the supports of factors essential for u . For any given bound $T(u^0)$ not all output vectors $u \in (L_\infty)_+^m$ need, by the axioms, have bounded supports of output rate histories. Only when $\text{Supp } u_i \supset \text{Supp } u_i^0$, $i \in \{1, 2, \dots, m\}$, and $u \geq u^0$, does $\text{Max} \left\{ \bar{t}_{x_{v_1}}, \bar{t}_{x_{v_2}}, \dots, \bar{t}_{x_{v_k}} \right\} \leq T(u^0)$ exclude such output vectors u . Even in the case $m = 1$ and $u^0 \in (L_\infty)_+$, the same issue arises, i.e., $v \in (L_\infty)_+$ need not be restricted in support when $v \not\geq u^0$.

Following the treatment given earlier for limitation by input rates when L.6S applies, one may define

Definition (3.6-3):

A proper subset of factors $\{v_1, v_2, \dots, v_k\}$ is homogeneously output support weak limitational if for every $u^0 \in (L_\infty)_+^m$ with $L(u^0) \neq \emptyset$ and $\text{Max}_i \left\{ \bar{t}_{u_i}^0 \right\} < +\infty$, there exists a positive bound $T(u^0)$ such that $(\theta_1 u_1^0, \dots, \theta_m u_m^0) \notin P(x)$ for $\theta_i \in [1, +\infty)$, $i \in \{1, 2, \dots, m\}$ when $\text{Max} \left\{ \bar{t}_{x_{v_1}}, \bar{t}_{x_{v_2}}, \dots, \bar{t}_{x_{v_k}} \right\} \leq T(u^0)$ while $\bar{t}_{x_{v_j}}$ is unrestricted for $j \in \{(k+1), (k+2), \dots, n\}$.

Then the following proposition is a consequence of the arguments of Proposition (3.6-1):

Proposition (3.6-2):

A proper subset of the n factors $\{v_1, v_2, \dots, v_k\}$ is homogeneously output support weak limitational if and only if it is output history homogeneously essential for the whole output space.

Here, by the very scaling of output histories permitted by the axioms L.6S for disposal of outputs, the supports of the output vectors so scaled and compared are identical, and the supports of the output vectors $(\theta_1 u_1^0, \dots, \theta_m u_m^0)$ and θu^0 , respectively are trivially bounded for $\theta_i \in [1, +\infty)$, $i \in \{1, 2, \dots, m\}$ and $\theta \in [1, +\infty)$.

In the same fashion as Definition (3.3-5), the following definition is made in order to derive an equivalent statement of output support strong limitationality.

Definition (3.6-4):

A proper subset $\{v_1, v_2, \dots, v_k\}$ of input factors is *globally support essential* if for every $u^0 > 0$, $L(u^0) \neq \emptyset$, and arbitrary infinite sequence $\{x^j, u^j\}$ with $x^j \in L(u^j)$ and $u^j \geq u^0$, $\text{Max}_i \left\{ \bar{t}_{u_i}^j \right\} \rightarrow +\infty$ implies

$$\text{Max} \left\{ \bar{t}_{x_{v_1}}^j, \bar{t}_{x_{v_2}}^j, \dots, \bar{t}_{x_{v_k}}^j \right\} \rightarrow +\infty .$$

Proposition (3.6-3):

A proper subset of the n factors $\{v_1, v_2, \dots, v_k\}$ is globally output support strong limitational if and only if it is globally support essential.

The proof is trivial.

A stronger form of support limitationality than that of (3.6-1) is as follows:

Definition (3.6-5):

A proper subset $\{v_1, v_2, \dots, v_k\}$ of the n factors of production is *output support weak limitational* if for every $\tau \in \mathbb{R}_{++}$, there exists a positive bound $T(\tau) \in \mathbb{R}_{++}$ on input supports such that for all $u \in (L_\omega)_+^m$ with $\text{Max}_i \left\{ \bar{t}_{u_i} \right\} \geq \tau$,

$$\mathbb{L}(u) \cap \left\{ x \in (L_\omega)_+^n : \text{Max} \left(\bar{t}_{x_{v_1}}, \bar{t}_{x_{v_2}}, \dots, \bar{t}_{x_{v_k}} \right) \leq T(\tau) \right\}$$

is empty.

Proposition (3.6-4): (Mak, 1979)

Under $\mathbb{L}.6SS$, if the weak^{*} topology is taken for the output space and either axiom $\mathbb{E}.S$ holds or axiom \mathbb{E} taken with the weak^{*} topology for input space, then input factors $\{v_1, v_2, \dots, v_k\}$ are output support weak limitational if and only if they are globally totally essential for the whole output space.

Proof:

Assume $\{v_1, v_2, \dots, v_k\}$ is not output support weak limitational. That is, there exists $\tau \in \mathbb{R}_{++}$ such that for every positive input support bound $T \in \mathbb{R}_{++}$, $\mathbb{L}(u) \cap \left\{ x \in (L_\omega)_+^n : \text{Max} \left(\bar{t}_{x_{v_1}}, \bar{t}_{x_{v_2}}, \dots, \bar{t}_{x_{v_k}} \right) \leq T \right\}$ is not empty for some u with $\text{Max}_i \left\{ \bar{t}_{u_i} \right\} \geq \tau$.

For notation, let $\bar{t}(x, \nu) := \text{Max} \left\{ \bar{t}_{x_{\nu_1}}, \bar{t}_{x_{\nu_2}}, \dots, \bar{t}_{x_{\nu_k}} \right\}$ and $\bar{t}(u) := \text{Max}_1 \left\{ \bar{t}_{u_i} \right\}$.

Consider an infinite sequence $\{T^j\} \subset \mathbb{R}_{++}$, $T^j \leq \tau$ for $j = 1, 2, \dots$, and $T^j \rightarrow 0$. According to the above assumption, there is an infinite sequence $\{(x^j, u^j)\}$ with $x^j \in \mathbb{L}(u^j)$, $\bar{t}(x^j, \nu) \leq T^j$ and $\bar{t}(u^j) \geq \tau$.

For each (x^j, u^j) pair, there is scalar $\theta^j \in (0, +\infty)$ which gives $\|\theta^j \cdot \nu^j\| = 1$. By axiom L.4.2, there exists a scalar $\lambda(\theta^j) \in (0, +\infty)$ such that $\lambda(\theta^j) \cdot x^j \in \mathbb{L}(\theta^j \cdot u^j)$. Next, construct a sequence $\{w^j\}$ as follows:

Let

$$\sigma^j := \text{Inf} \left\{ t \in [\tau, +\infty) : \text{Max}_1 \left[\int_{\tau}^t \theta^j |u_i(s)| d\nu_1(s) \right] \geq 1 \right\}$$

$$w_i^j(t) := \begin{cases} 0, & t \in [0, \tau), \\ \theta^j \cdot u_i^j(t), & t \in [\tau, \sigma^j], \\ 0, & t \in (\sigma^j, +\infty) \end{cases}, \quad i \in \{1, 2, \dots, m\}.$$

It is clear that $w^j \leq \theta^j \cdot u^j$. Then, according to L.6SS, $\lambda(\theta^j) \cdot x^j \in \mathbb{L}(\theta^j \cdot u^j) \subset \mathbb{L}(w^j)$. Furthermore, the L_1 -norm obeys $\|w^j\|_1 \leq 1$, $j = 1, 2, \dots$. Consequently, $\{w^j\}$ is weak* compact.

Then there is a subsequence $\{w^p\} \subset \{w^j\}$ with w^p converging weak* to $\bar{w} \geq 0$. And $\lambda(\theta^j) \cdot x^j \in \mathbb{L}(w^p) \cap \left\{ x \in (L_{\infty}^n)_+ : \bar{t}(x, \nu) \leq T^p \right\}$. Then by L.5 as stated in weak* topology, since $T^p \rightarrow 0$, $\mathbb{L}(\bar{w}) \cap \left\{ x \in (L_{\infty}^n)_+ : \bar{t}(x) = 0 \right\}$ is not empty.

Consequently, $\{v_1, v_2, \dots, v_k\}$ cannot be globally totally essential for the whole output space.

The converse is trivial.

For strong limitation on output supports, we define

Definition (3.6-6):

A proper subset $\{v_1, v_2, \dots, v_k\}$ of the n factors of production is *output support strong limitational* if for every positive bound $T \in \mathbb{R}_{++}$ on input support, there exists a positive bound $\tau(T) \in \mathbb{R}_{++}$ on the output support such that for all $u \in (L_{\infty})_+^m$ with $\bar{t}(u) \geq \tau(T)$, $L(u) \cap \{x \in (L_{\infty})_+^m, \bar{t}(x, v) \leq T\}$ is empty.

Definition (3.6-7):

A proper subset $\{v_1, v_2, \dots, v_k\}$ of input factors is *output support strong essential* if for arbitrary sequence $\{x^j, u^j\}$ with $x^j \in L(u^j)$, $\bar{t}(u^j) \rightarrow +\infty$ implies $\bar{t}(x^j, v) \rightarrow +\infty$.

Proposition (3.6-5):

A proper subset $\{v_1, v_2, \dots, v_k\}$ of the n factors of production is output support strong limitational if and only if it is output support strong essential.

See below, Proposition (3.7-1) for further consideration of output support strong limitational.

3.7 Restriction of Output by Nondisposability of Factors

In this section, all input factors are assumed to be nondisposable. That is to say, all input histories considered are actual inputs to the production process. Thus, our attention will be focused on the isoquants of input sets. Also, for ease of exposition, single output is assumed.

In some production processes, certain inputs must be present in minimal proportions. The following definition makes this notion precise.

Definition (3.7-1):

A proper subset $\{v_1, v_2, \dots, v_k\}$ of the n input factors is *relatively essential* if for all $u \geq 0$, $L(u) \neq \emptyset$, $\text{Inf} \left\{ \|x_{v_1}, x_{v_2}, \dots, x_{v_k}\| / \|x\| : x \in \text{ISOQ } L(u) \right\} = \varepsilon(u) > 0$.

The notion of relative essentiality is equivalent to the following:

Definition (3.7-2): (Färe and Jansson, 1976)

A proper subset $\{v_{k+1}, v_{k+2}, \dots, v_n\}$ of the n input factors is *Null Joint* with $\{v_1, v_2, \dots, v_k\}$ if for all $u > 0$, $L(u) \neq \emptyset$;
 $\left(x_{v_1}, x_{v_2}, \dots, x_{v_k} \right) = 0$ implies $\left(x_{v_{k+1}}, \dots, x_{v_n} \right) = 0$ where $x \in C(u) :=$
 CLOSURE $\left\{ x \in (L_\omega)_+^n : x = \lambda y, \lambda > 0, y \in \text{ISOQ } L(u) \right\}$.

The equivalence of null-jointness with relative essentiality can be easily established; proof will be omitted. Färe [1976] demonstrated that null-jointness is a sufficient condition for strong limitationality. However, relative essentiality is used here since it is more in line with the investigation of the implications of various notions of essentiality.

Proposition (3.7-1): (Mak, 1979)

If a proper subset $\{v_1, v_2, \dots, v_k\}$ of input factors is relatively essential, then it is globally output rate strong limitational. (L.6SS assumed to hold.)

Proof:

To use contra-positive argument, assume $\{v_1, v_2, \dots, v_k\}$ not globally input rate strong limitational. Then for some $u^0 \geq 0$ with $L(u^0) \neq \emptyset$, there exists a positive bound on the input $B \in \mathbb{R}_{++}$ and an infinite sequence $\{u^j\}$ with $\lambda(u^j, u^0) \rightarrow +\infty$ (refer to 3.3 for definition of λ) and $L(u^j) \cap \left\{ x \in (L_\infty)_+^n : \|x_{v_1}, x_{v_2}, \dots, x_{v_k}\| \leq B \right\} \neq \emptyset$.

For simplicity, denote $\lambda^j := \lambda(u^j, u^0)$.

By L.6SS, ISOQ $L(\lambda^j \cdot u^0) \cap \left\{ x \in (L_\infty)_+^n : \|x_{v_1}, x_{v_2}, \dots, x_{v_k}\| \leq B \right\} \neq \emptyset$. Let x^j belong to the intersection, $j = 1, 2, \dots$. Since $\lambda^j \rightarrow +\infty$ and $\|x_{v_1}^j, x_{v_2}^j, \dots, x_{v_k}^j\|$ is bounded, it follows from P.2 that $\|x_{v_{k+1}}^j, \dots, x_{v_n}^j\| \rightarrow +\infty$. Consequently, $\|x_{v_1}^j, x_{v_2}^j, \dots, x_{v_k}^j\| / \|x^j\| \rightarrow 0$.

According to L.4.2, there exists for each j with $\lambda^j \geq 1$, a $y^j \in \text{ISOQ}(u^0)$ such that $y^j = \theta^j x^j$ for some $\theta^j \in (0, +\infty)$. Thus there exists an infinite

sequence $\{y^j\} \subset \text{ISOQ } L(u^0)$ with $\frac{\|y_{v_1}^j, y_{v_2}^j, \dots, y_{v_k}^j\|}{\|y^j\|} \rightarrow 0$, and therefore $\{v_1, v_2, \dots, v_k\}$ cannot be relatively essential.

There is a stronger consequence of relative essentiality, namely, that of congestion of input factors.

Definition (3.7-3):

A proper subset $\{v_{k+1}, v_{k+2}, \dots, v_n\}$ is *potentially congesting* if for arbitrary positive bound $(x_{v_1}^0, x_{v_2}^0, \dots, x_{v_k}^0)$ on input factors $\{v_1, v_2, \dots, v_k\}$, and arbitrary $u^0 \geq 0$ with $\mathbb{L}(u^0) \neq \emptyset$, there exists a positive number N depending on u^0 and $(x_{v_1}^0, x_{v_2}^0, \dots, x_{v_k}^0)$ such that

$$\text{ISOQ } \mathbb{L}(u) \cap \left\{ x \in (L_{\infty})_+^n : \right. \\ \left. (x_{v_1}, x_{v_2}, \dots, x_{v_k}) \leq (x_{v_1}^0, x_{v_2}^0, \dots, x_{v_k}^0) \text{ and } \|x\| \geq N \right\} = \emptyset$$

for all $u \geq u^0$.

Proposition (3.7-2):

If a proper subset $\{v_1, v_2, \dots, v_k\}$ of input factors is relatively essential, then its complement $\{v_{k+1}, \dots, v_n\}$ is potentially congesting, L.6SS taken to hold.

Consider arbitrary $u^0 \geq 0$ with $\mathbb{L}(u^0) \neq \emptyset$. Define $C(u^0) :=$ CLOSURE $\{x \in (L_{\infty})_+^n : x = \lambda y, \lambda > 0, y \in \text{ISOQ } \mathbb{L}(u^0)\}$.

Suppose $C(u^0) \cap D(v_1, v_2, \dots, v_k) \neq \emptyset$. This implies the existence of either $y \in \text{ISOQ } \mathbb{L}(u^0)$ with $\frac{\|y_{v_1}, y_{v_2}, \dots, y_{v_k}\|}{\|y\|} = 0$ or an infinite sequence of rays $\rho^j \in C(u^0)$ with $\lim_j \rho^j \in D(v_1, v_2, \dots, v_k)$ which in turn implies the existence of $\{y^j\} \subset \text{ISOQ } \mathbb{L}(u^0)$ with

$\lim_j \frac{\|y_{v_1}^j, y_{v_2}^j, \dots, y_{v_k}^j\|}{\|y^j\|} \rightarrow 0$. This contradicts the intended hypothesis that $\{v_1, v_2, \dots, v_k\}$ is relatively essential. Next, we will establish

the following:

Lemma:

Let $K \subset (L_{\infty})_+^n$, $K \neq \{0\}$, $0 \in K$ be a closed cone such that $K \cap D(v_1, v_2, \dots, v_k)$ is empty. The intersection set $K \cap \left\{ x \in (L_{\infty})_+^n : \begin{pmatrix} x_{v_1}, x_{v_2}, \dots, x_{v_k} \end{pmatrix} \leq \begin{pmatrix} x_{v_1}^0, x_{v_2}^0, \dots, x_{v_k}^0 \end{pmatrix} \right\}$ is closed and bounded for each positive $\begin{pmatrix} x_{v_1}^0, x_{v_2}^0, \dots, x_{v_k}^0 \end{pmatrix}$ on the input factors $\{v_1, v_2, \dots, v_k\}$.

Proof:

Let $\begin{pmatrix} x_{v_1}^0, x_{v_2}^0, \dots, x_{v_k}^0 \end{pmatrix}$ be any positive bound on the input factors $\{v_1, v_2, \dots, v_k\}$. Denote $S^0 := K \cap \left\{ x \in (L_{\infty})_+^n : \begin{pmatrix} x_{v_1}, x_{v_2}, \dots, x_{v_k} \end{pmatrix} \leq \begin{pmatrix} x_{v_1}^0, x_{v_2}^0, \dots, x_{v_k}^0 \end{pmatrix} \right\}$. Since S^0 is the intersection of two closed sets it is closed. In order to show that S^0 is bounded, assume that $\{x^\alpha\} \subset S^0$ is an infinite sequence with $\|x^\alpha\| \rightarrow +\infty$ for $\alpha \rightarrow \infty$. The subvectors $\begin{pmatrix} x_{v_{k+1}}^\alpha, x_{v_{k+2}}^\alpha, \dots, x_{v_n}^\alpha \end{pmatrix}$ must be such that

$$\| \begin{pmatrix} x_{v_{k+1}}^\alpha, x_{v_{k+2}}^\alpha, \dots, x_{v_n}^\alpha \end{pmatrix} \| \rightarrow +\infty.$$

Let $\rho^\alpha := \{\lambda x^\alpha : \lambda \in (0, +\infty)\}$. Since the subvectors $\begin{pmatrix} x_{v_1}^\alpha, x_{v_2}^\alpha, \dots, x_{v_k}^\alpha \end{pmatrix}$ are uniformly bounded for $\alpha = 1, 2, \dots$, $\lim_{\alpha \rightarrow \infty} \rho^\alpha \in D(v_1, v_2, \dots, v_k)$.

Then since $K(v, w)$ is closed, $K(v, w) \cap D(v_1, v_2, \dots, v_k)$ is not empty, a contradiction.

It follows from the above discussion and lemma that: if $\{v_1, v_2, \dots, v_k\}$ is relatively essential, then for all $u^0 \geq 0$, $L(u^0) \neq \emptyset$ and all $(x_{v_1}^0, \dots, x_{v_k}^0) > 0$, there exists positive number N such that $C(u^0) \cap \left\{ x \in (L_\infty)_+^n : (x_{v_1}, x_{v_2}, \dots, x_{v_k}) \leq (x_{v_1}^0, x_{v_2}^0, \dots, x_{v_k}^0) \text{ and } \|x\| \geq N \right\}$ is empty. Then by L.6SS and L.4.2, for $u \geq u^0$, $L(u) \subset L(u^0) \subset C(u^0)$, consequently $\text{ISOQ } L(u) \cap \left\{ x \in (L_\infty)_+^n : (x_{v_1}, x_{v_2}, \dots, x_{v_k}) \leq (x_{v_1}^0, x_{v_2}^0, \dots, x_{v_k}^0) \text{ and } \|x\| \geq N \right\}$ is empty. Proposition (3.7-2) is thus established.

It is from Proposition (3.7-2) that the so-called laws of variable return may be deduced by considering production correspondences $u \rightarrow L(u)$ with further assumption on fine structures. Proposition (3.7-2) states in a general way that relatively essential input factors are so linked with the other factors that beyond a certain size (norm) of input vector, all output vectors larger than a given mix are impossible. How, under increases of input rates, an output vector may increase and decrease to zero is a matter of fine structure not considered here.

Finally, we note that the sup-norm is used in this section, and it could easily be replaced by the weak* or L_1 -norm as appropriate.

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